Topology and dynamics on the boundary of two-dimensional domains

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Joint work with
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Basic Problem

- $f : S \to S$ homeomorphism of an orientable surface;
- $U \subset S$ invariant domain;
- Describe the dynamics in the boundary of U.
  - Existence of periodic points in ∂U
  - Topological restrictions imposed by the dynamics of $f|_{\partial U}$.

Simplest setting

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ orientation-preserving homeomorphism;
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∂U is a circle (so $\overline{U} \simeq \overline{\mathbb{D}}$)
$\implies$ $f|_{\partial U}$ is a circle homeomorphism
$\implies$ Poincaré Theory. Key: **Rotation number!**

Theorem (Poincaré)

$\exists$ *periodic point* $\iff$ *rotation number of* $f|_{\partial U}$ *is rational.*

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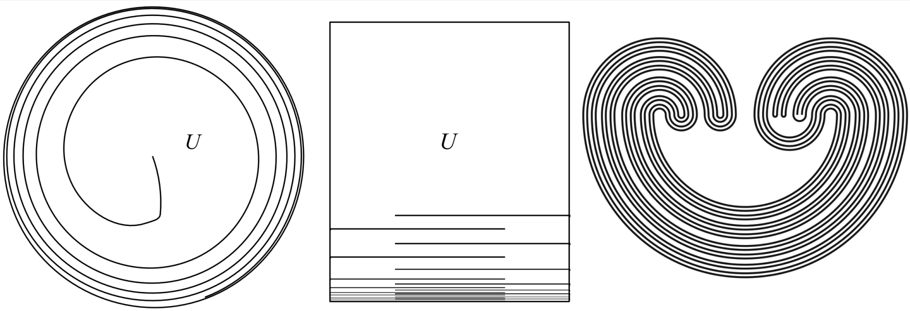
∂U can have very very complicated topology!



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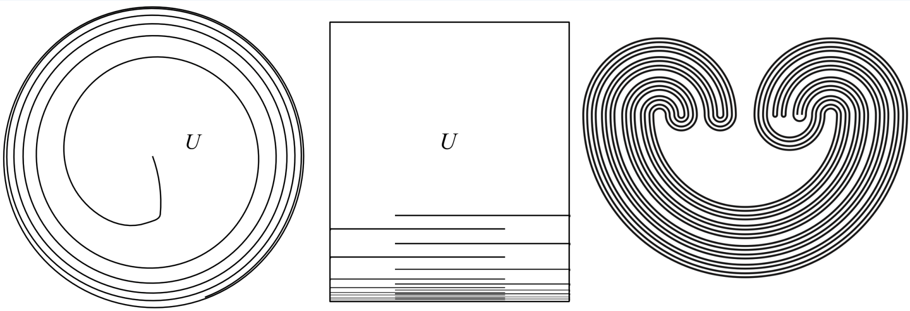

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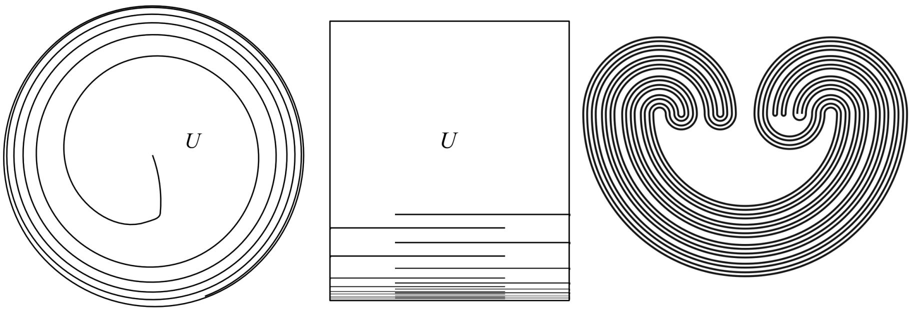

- may have points inaccessible from U,
- can be nowhere locally connected,
- worse things (e.g. an hereditarily indecomposable continuum)
- these are not isolated or infrequent, independently of regularity.

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- Compactify U by adding an "ideal" circle (in a sensible way)

$$\widehat{U} := U \sqcup \mathbb{S}^1$$

with a suitable topology such that $\widehat{U} \simeq \overline{\mathbb{D}}$.

- Hopefully, $f|_U$ extends to $\widehat{f} \colon \widehat{U} \to \widehat{U}$.
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Cartwright-Littlewood, 1951

$\widehat{U}$ = Carathéodory's prime ends compactification

$\rho(f, U)$ = Prime ends rotation number.

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How is the relation between two dynamics:

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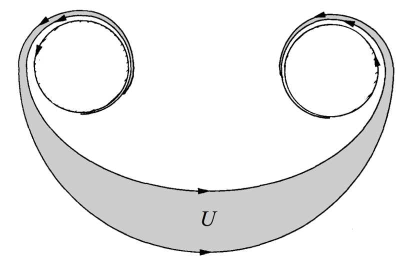
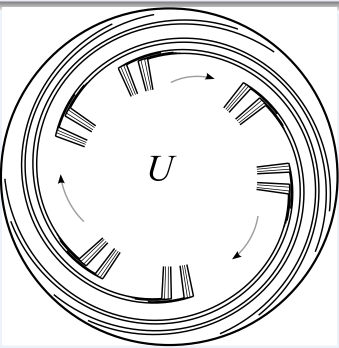


Figure : $\rho = 0$ and $\mathrm{Fix}(f|_{\partial U}) = \emptyset$



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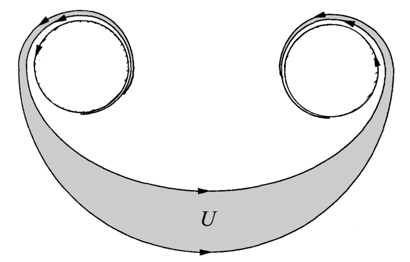

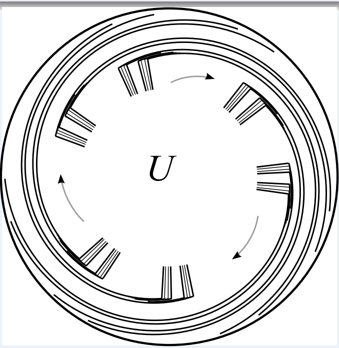

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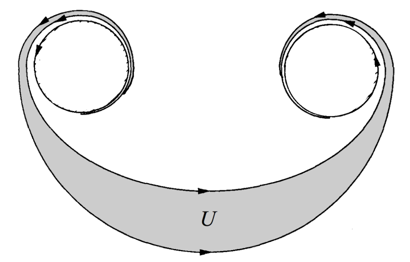

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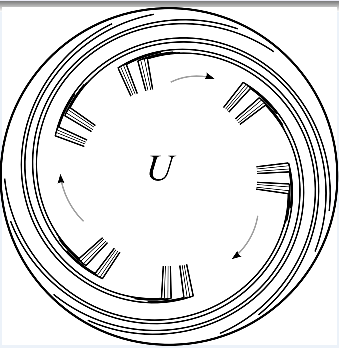

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- Note: Both examples have *attracting* regions near the boundary.
- Not possible if f preserves area (or nonwandering)....

Consequences of the rotation number

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ homeomorphism
- $U \subset \mathbb{R}^2$ bounded, simply connected, open, f-invariant
- f is nonwandering (e.g. area-preserving) in U.

Theorem (Cartwright-Littlewood, 1951)

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Opposite direction? What if $\rho \notin \mathbb{Q}$?

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Theorem A (Converse of [C-L])

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Theorem A (Converse of [C-L])

$\rho(f, U) \notin \mathbb{Q} \implies \nexists$ periodic point in ∂U and ∂U is annular.

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- $f : \mathbb{R}^2 \to \mathbb{R}^2$ homeomorphism
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$$\rho(f, U) \neq 0 \quad \Longrightarrow \quad \nexists \text{ fixed point in } \partial U.$$

Moreover: if U is unbounded,

$$\rho(f, U) \neq 0 \quad \Longrightarrow \quad \nexists \text{ fixed point in } \mathbb{R}^2 \setminus U.$$

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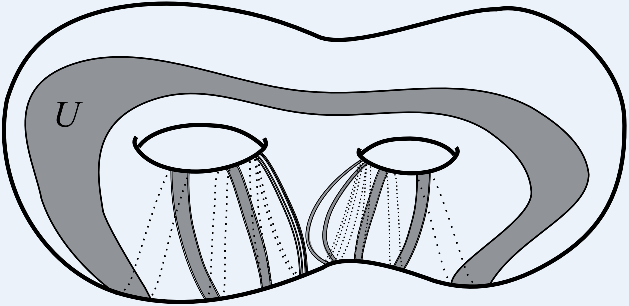

Figure : a simply connected open set

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Still true for an arbitrary surface S?

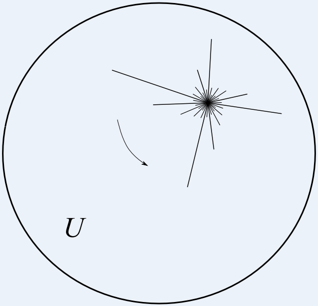


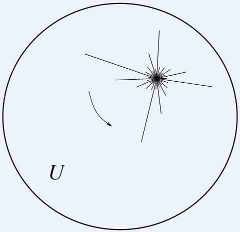
Figure : unique fixed point in ∂U, surface $= \mathbb{S}^2$

Theorem B (on closed surfaces)

- f nonwandering homeomorphism of a closed orientable surface S,
- $U \subset S$ open, f-invariant, simply connected.
- $\rho(f, U) \notin \mathbb{Q}$

One of these two holds:

1. ∂U contains a unique fixed point and no other periodic points
   $S =$ Sphere, U is dense in S, $\partial U = S \setminus U$ cellular continuum, or
2. ∂U is aperiodic contractible annular continuum.



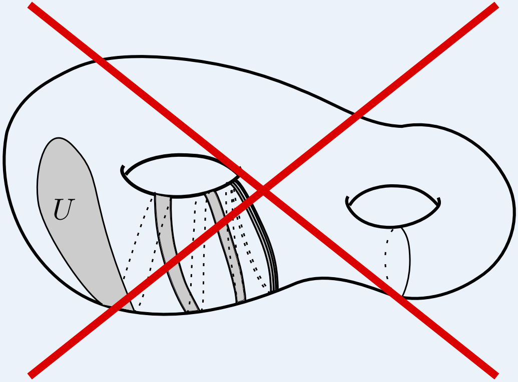


Figure : Impossible! if $\rho(f, U) \notin \mathbb{Q}$.

Theorem B'

Theorem B extends to

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Remark

The ∂-nonwandering condition holds if f is a holomorphic diffeomorphism.
$\Longrightarrow$ consequences in one-dimensional holomorphic dynamics.

Application for generic area-preserving diffeos

Theorem [Mather '81]

- f a C^r-generic area preserving diffeomorphism ($r \geq 16$),
- U periodic complementary domain,
- $\Longrightarrow$ prime ends rotation numbers of U are irrational at each end.

Example: $p \in \mathrm{Per}_h(f)$, $U =$ connected component of $S \setminus \overline{W^s(p)}$.

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Dynamical consequences?

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Remarks

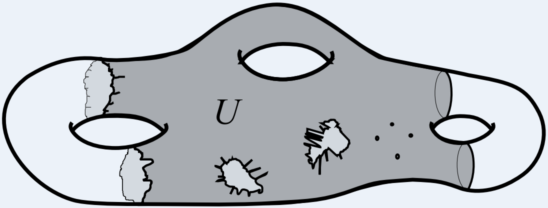
1. Mather [1981] proved $\rho \notin \mathbb{Q}$, assuming r large ($r \geq 16$, KAM & KS)
2. For $\mathbb{S}^2$ and $\mathbb{T}^2$, can be proved using Mather + Pixton-Oliveira.
3. Generic condition is explicit.

Application for generic area-preserving diffeos

Theorem C'

- f a C^r-generic area preserving diffeomorphism of a closed surface ($r \geq 1$)
- U periodic open set with finitely many topological ends.

Then $\partial U = \{$aperiodic annular continua$\} \sqcup \{$periodic points$\}$ (finitely many of each)



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Corollary C' completes the proof of:

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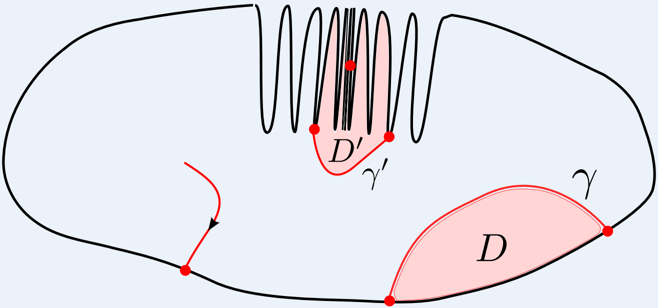
- For $S = \mathbb{S}^2$, $r \geq 16$: done by Franks and Le Calvez [ETDS, 2003]
- For any genus: proof of J. Xia [CMP, 2006] relies in Corollary C' (gap).

Prime ends

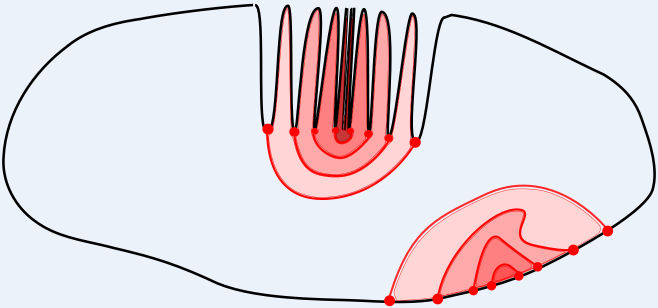
Definition

- **cross-cut**: a simple arc γ in U with endpoints in ∂U.
- **cross-section**: any one of the two components of $U \setminus \gamma$.



Chains

A **chain** in U is a decreasing sequence of cross sections (D_n) bounded by cross-cuts (γ_n) such that $\overline{\gamma_n} \cap \overline{\gamma_{n+1}} = \emptyset$.



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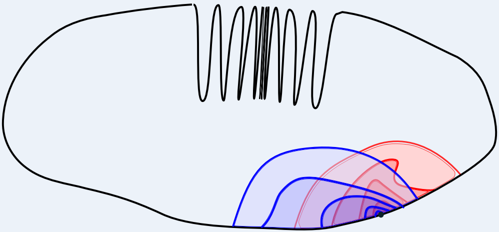
If (D'_n) is another chain, we say that (D_n) divides (D'_n) if for each $n > 0$ there is m such that $D_m \subset D'_n$.



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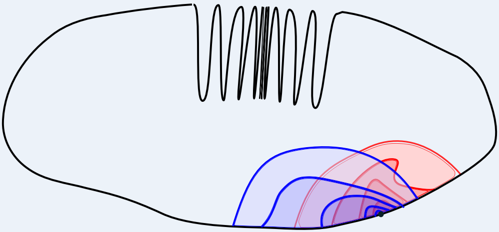

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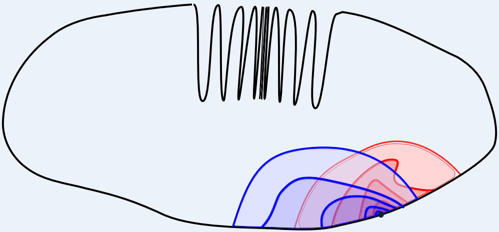

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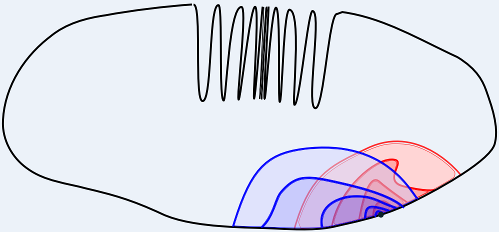

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Prime ends of $U = \mathcal{PE}(U) :=$ {prime chains} / equivalence

Prime chain

If $\overline{U}$ is compact, then we may define in this way:
A **prime chain** in U is a decreasing sequence of cross sections (D_n) bounded by cross-cuts (γ_n) such that

- $\mathrm{diam}(\gamma_n) \to 0$ as $n \to \infty$
- $\overline{\gamma_n} \cap \overline{\gamma_{n+1}} = \emptyset$

Prime ends

Prime ends compactification (Carathéodory)

$$\mathcal{PE}(U) \simeq \mathbb{S}^1$$

$$\widehat{U} := U \sqcup \mathcal{PE}(U) \simeq \overline{\mathbb{D}}$$

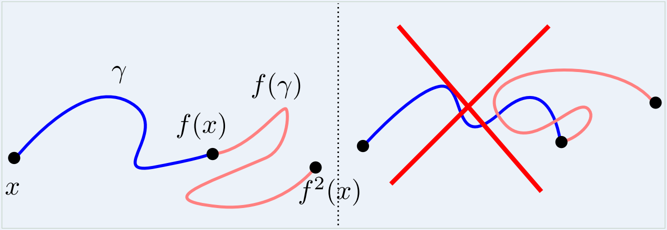
Prime ends rotation number

f extends to a **homeomorphism** $\widehat{f} : \widehat{U} \to \widehat{U}$

$\rho(f, U) =$ Poincaré rotation number of $\widehat{f}|_{\mathbb{S}^1}$

N-translation arc

- γ simple arc from x to $f(x) \neq x$,
- $\Gamma = \gamma \cup f(\gamma) \cup \cdots \cup f^N(\gamma)$ is also a simple arc.



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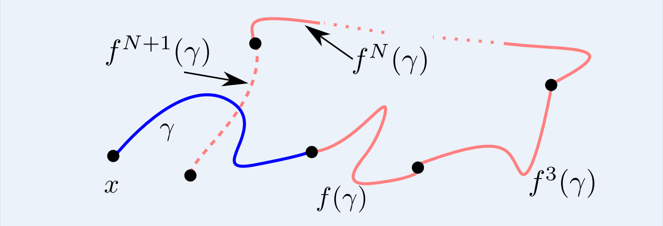

Theorem E (Arc Lemma)

- $f\colon S \to S$ homeomorphism, S surface of genus g.
- $U \subset S$ invariant open topological disk ($S \setminus U \supsetneq$ one point)
- f is nonwandering in U,
- $\rho(f, U) = \alpha \neq 0$

$\Longrightarrow \exists N = N_{\alpha,g}$ s.t every N-translation arc in S is disjoint from ∂U.

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Case $N = \infty$ and "transverse": easy. "non-transverse" ($\subset \partial U$): hard.

Remark (Brouwer theory)

Assuming $S = \mathbb{R}^2$:

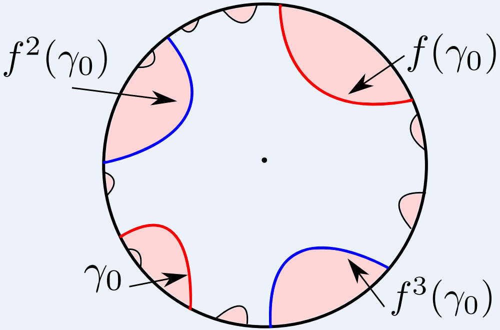
- Every non-fixed point belongs to an 1-translation arc γ.
- If γ is not an N-translation arc, then $\Gamma = \gamma \cup f(\gamma) \cup \cdots \cup f^N(\gamma)$ surrounds a fixed point.

Arc Lemma: idea of the proof

Assume $S = \mathbb{R}^2$. In this case, $N = 3$

- Let γ be a 3-translation arc intersecting ∂U
- $\exists$ **maximal cross-cut γ_0 defined by γ**
- Cyclic order of iterations of γ_0 by rotation number
- Linear order of iterations of γ_0 by 3-translation arc γ
- **Construct a pair of simple closed curves with intersection number = 1**
- $\implies$ genus of $S > 0$. **Contradiction !**

Arc Lemma: idea of the proof (heuristics)



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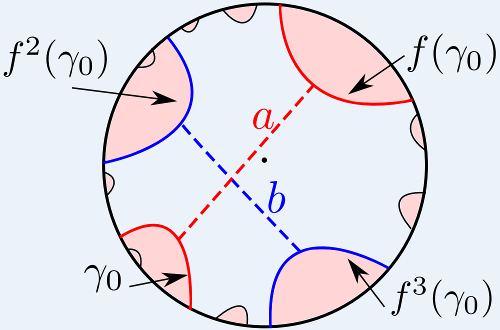

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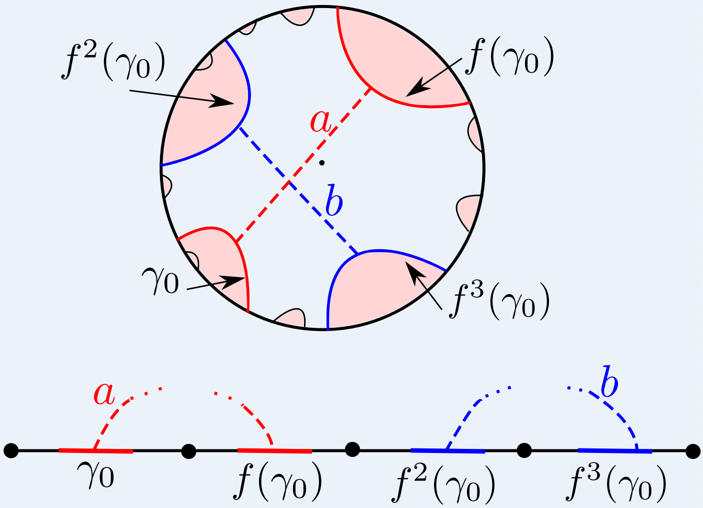

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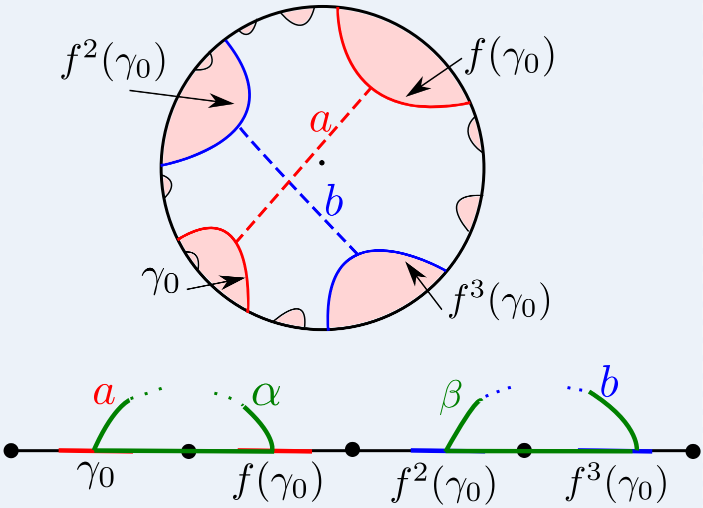

Idea of the proof of Theorem A

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ homeomorphism
- $U \subset \mathbb{R}^2$, simply connected, open, f-invariant
- f is nonwandering in U

Theorem A (Converse of [C-L])

$$\rho(f, U) \neq 0 \implies \nexists \text{ fixed point in } \partial U$$

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Assuming there is a fixed point z_1 in ∂U:

- Find an N-translation arc in a neighborhood of z_1
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General strategy

Assuming there is a fixed point z_1 in ∂U:

- Find an N-translation arc in a neighborhood of z_1
- contradicts Arc Lemma.
- Problem: doesn't work directly!

Idea of the proof of Theorem A

Suppose $\rho(f, U) \neq 0$ but $z_1 \in \mathrm{Fix}(f) \cap \partial U$

Reduce to the case where:

- $\exists$ unique fixed point $z_0 \in U$.
- $\nexists$ accessible fixed point in ∂U
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- Remove maximal unlinked $X \subset \mathrm{Fix}(f)$, $z_0, z_1 \in X$. (using the work of O. Jaulent [2012])
- $M = \mathbb{R}^2 \setminus (X \setminus \{z_0\})$, $\pi : \widetilde{M} \to M$ universal covering map
- Define $\widetilde{f} : \widetilde{M} \to \widetilde{M}$, $\widetilde{U}$ invariant for $\widetilde{f}$, same rotation number.
- $\pi(\mathrm{Fix}(\widetilde{f}))$ far from z_1
- Find N-translation arc $\widetilde{\gamma}$ for $\widetilde{f}$ that projects near z_1.
- Brouwer $\Longrightarrow$ Γ "turns around" a fixed point of $\widetilde{f}$. Contradiction!

a technical problem

In the classic theory of prime ends:

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Theorem

- The theory of prime ends extends to the unbounded case;
- If $U \subset S' \subset S$ open invariant sets and $\partial_{S'} U \neq \emptyset$, then

$$\rho(f, U \subset S) = \rho(f, U \subset S')$$

Further problems and results

Poincaré theory on S^1

- Rotation number is independent of the point used to compute it.
- $\rho(f) = p/q \in \mathbb{Q} \implies \mathrm{Fix}(f^q) \neq \emptyset$
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Cartwright-Littlewod + Theorem A $\implies$ this holds for boundary dynamics (+nonwandering).

Further problems and results

Refinement of Poincaré theory on S^1

- $\rho(f) = p/q \in \mathbb{Q} \implies \mathrm{Fix}(f^q) \neq \emptyset$ and $\alpha(x) \cup \omega(x) \subset \mathrm{Fix}(f^q)$ for all $x \in S^1$.
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Work in progress

The first item holds for boundary dynamics with a nonwandering condition.