Chromatic index, treewidth and maximum degree

Revisiting an old result of Vizing

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Abstract

We conjecture that any graph G with treewidth k and maximum degree \(\Delta(G)\) ≥ \(k + 3\sqrt{k}\) satisfies \(\chi'(G) = \Delta(G)\). In support of the conjecture we prove its fractional version.

Background

An edge colouring is an assignment of colours to the edges of a graph G such that no two adjacent edges receive the same colour. The least number of colours necessary to the edge colouring is termed the chromatic index of G and denoted by \(\chi'(G)\). A classic theorem of Vizing states that a graph G has either chromatic index \(\Delta(G)\) or \(\Delta(G) + 1\). But to determine whether \(\Delta\) is or \(\Delta + 1\) colours are needed is a difficult algorithmic problem.

Theorem (Vizing, 1964):

Any graph of treewidth \(k\) and maximum degree \(\Delta \geq 2k\) has chromatic index \(\Delta\).

This is tight! No, it turns out. Using two recent adjacency lemmas ([2], [5]), we obtain:

Proposition 1 (Bruhn, Gellert, L., 2016):

For any graph G of treewidth \(k \geq 1\) and maximum degree \(\Delta \geq 2k\), it holds that \(\chi'(G) = \Delta\).

This immediately suggests the question: how much further can the maximum degree be lowered? We conjecture:

Conjecture (Bruhn, Gellert, L., 2016):

Any graph of treewidth \(k\) and maximum degree \(\Delta \geq k + \sqrt{\Delta}\) has chromatic index \(\Delta\).

The bound is close to best possible: we construct for infinitely many \(k\), graphs with treewidth \(k\), maximum degree \(\Delta = k + \sqrt{\Delta} < k + \sqrt{\Delta}\) and chromatic index \(\Delta + 1\), see Figure 2.

Figure 2: Removing the dashed edges from a \(P^k\) gives a graph of treewidth \(k\) and maximum degree = \(10\). Due to overcolouring, its chromatic index is \(11\).

The problem of edge colouring can be relaxed as follows. A fractional edge colouring is an assignment of weights \(w_M\) to each matching \(M\) in \(G\) such that \(\sum_{e\in M} w_M(e) = 1\) for every \(e\in E(G)\). The fractional chromatic index \(\chi'_f(G)\) is defined as the minimum of \(\sum_{e\in E(G)} w_M(e)\) for all fractional edge colourings \(w\). Since an edge colouring can be interpreted as 0.5-valued fractional edge colouring, it follows that \(\chi'(G) = \chi'_f(G)\). The converse, however, does not always hold as illustrated in Figure 1. Moreover, in contrast to the chromatic index, the fractional chromatic index can be computed efficiently.

In support of the conjecture we prove its fractional version.

Theorem 1 (Bruhn, Gellert, L., 2016):

Any graph of treewidth \(k\) and maximum degree \(\Delta \geq k + \sqrt{\Delta}\) has fractional chromatic index \(\Delta\).

The theorem follows from a new upper bound on the number of edges for these graphs, whose proof is quite intriguing.

Proposition 1:

A graph G of treewidth \(k\) and maximum degree \(\Delta\) satisfies \(\Delta \leq \chi'_f(G) \leq (\Delta - k)(\Delta - k + 1)\).

A graph G is overfull if it has an odd number n of vertices and strictly more than \(\Delta(G)(n - 1)/2\) edges, a subgraph B of G is an overfull subgraph if it is overfull and satisfies \(\Delta(B) = \Delta\). Proposition 2 implies quite directly that no graph with treewidth \(k\) and maximum degree \(\Delta \geq k + 4\) can be overfull. It follows from Edmonds’ matching polytope theorem that \(\chi'_f(G) = \Delta\). If the graph G does not contain any overfull subgraph of maximum degree \(\Delta\), see [3, Ch. 28.5].

As the treewidth of a subgraph is never larger than the treewidth of the original graph, Theorem 1 is a consequence of Proposition 2.

Two strange tree inequalities

We use the following two lemmas about trees in the proof of Proposition 2. We denote the number of vertices in a tree T by \(|T|\).

Lemma 1:

For a tree T and a positive integer \(s \leq |T|\) it holds that:

\[
\max \{d - |T[s]|, 0\} \geq d|T| - 1.
\]

Remark that Lemma 1 is wrong if we omit \(\max \{., .\}\).

For a subtree \(S \subset T\) let \(d(S)\) be the set of oriented edges leaving S.

\[
\delta^+(S) = \{(f_s, f_t) : (f_s, f_t) \in d(S)\}.
\]

Lemma 2:

Let T be a tree and let \(s \leq |T|\) be a positive integer. Then for any subintervals \(S \subset T \) it holds that:

\[
\max \{d - |T[s]|, 0\} \geq \max \{d|T| - 1, 0\}.
\]

Treewidth

For a graph G a tree-decomposition \((T, B)\) consists of a tree T and a collection \(B = \{B_t : t \in V(T)\}\) of bags \(B_t \subset V(G)\) such that

(a) each vertex of G is in at most \(|T|\) bags,
(b) the vertices of each edge of G are in some common bag and
(c) if \(v \in B_s \cap B_t\) then \(v \in B_u\) for each vertex \(v\) on the path connecting \(s, t\) in T.

A tree-decomposition \((T, B)\) has width \(k\) if each bag has a size of at most \(k + 1\). The treewidth of G is the smallest integer \(k\) for which there is a width \(k\) tree-decomposition of G. A tree-decomposition \((T, B)\) of width \(k\) is smooth if

(d) each bag has size exactly \(k + 1\), and
(e) \(\delta^+(B_s) \subset B_t\) for all \(s \in T\).

A graph of treewidth at most \(k\) always admits a smooth tree-decomposition of width \(k\), see [1].

References


Figure 3: The graph G has a smooth tree-decomposition \((T, B)\) of width 2. Since \(v_6 \in V(B_t)\) it follows that \(v_e \in B_1\) and \(v_k \in B_k\) as well.

Figure 4: The vertex \(v_e\) induces the subtree \(T(v_e)\) in \(T\).

Figure 5: A k-degenerate graph of maximum degree 10. Thick grey edges indicate that the two vertex sets are not complete to each other. The elimination order of the \(v_i\) is drawn in dashed lines.