Certain extremal problems are classified as Turán type problems. Here we shall consider the Zarankiewicz problem, which aims to maximize the number of edges in a graph under the condition that the graph does not contain any copy of a complete bipartite graph $K_{t,r}$. We say that $G$ is a $K_{t,r}$-free if, in addition to the classical version of the Zarankiewicz problem, we shall discuss it under a spectral viewpoint, since it is possible to derive bounds for the classical version from inequalities of Spectral Graph Theory.

In this work, we consider two matrix representations of graphs:

### Adjacency Matrix
Given a graph $G = (V, E)$ with vertex set $\{v_1, \ldots, v_n\}$ the adjacency matrix $A(G) = (a_{ij})$ is given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

### Signless Laplacian Matrix
$Q(G) = A(G) + D(G)$, where $D(G)$ is the diagonal matrix whose $i$-th entry is given by the degree of vertex $v_i$.

The largest eigenvalues associated with the matrices $A(G)$ and $Q(G)$ are called the spectral radius $\lambda(G)$ and the $q$-index $q(G)$, respectively.

#### Bounds on the spectral radius
The spectral radius $\lambda$ satisfies the following inequality

$$\frac{2q(G)}{n} \leq \lambda.$$ 

Therefore, bounds on the spectral radius lead to bounds on the edges number. Consequently, we may state a spectral question associated with the Zarankiewicz problem:

**Introduction**

Since the spectral radius and the $q$-index are related as follows

$$2\lambda \leq q$$

bounds on the $q$-index imply bounds on the spectral radius, and thus on the number of edges. This leads to the following spectral question: *What is the maximum $q$ of a graph $G$ of order $n$ with $K_{t,r}$-free property?*

**Future Work**

- Verify whether the above conditions may be relaxed in some instances.
- Look for better upper and lower bounds on the $q$-index.
- Consider other spectral parameters related with the Zarankiewicz problem.
- Consider other classical problems and their relation with spectral parameters.

### References


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### Bounds for the Zarankiewicz problem

<table>
<thead>
<tr>
<th>Reference</th>
<th>Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kővári, Sós, and Turán [4]</td>
<td>$2q(G)/n \leq (s-1)^{1/2}n^{1/2} + t - 1$</td>
</tr>
<tr>
<td>Füredi [4]</td>
<td>$2q(G)/n \leq (s-t+1)^{1/2}n^{1/2} + t^2/2 + t$</td>
</tr>
</tbody>
</table>

Table 1: Bounds on the average degree of a $K_{t,r}$-free graph $G$ with $n$ vertices, where $2 \leq t \leq s$.

### Bounds for the $q$-index

The spectral radius $q$ satisfies the following inequality

$$q(G) \geq n/2 + 1 + \sqrt{n/2 + 1} \geq n.$$

Therefore, bounds on the spectral radius lead to bounds on the edges number. Hence, we may state a spectral question associated with the Zarankiewicz problem:

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