Introduction

We consider complete (and complete bipartite) graphs $G$ whose edges are each coloured with a set of $k$ colours, chosen among $r$ colours in total. That is, we consider functions $\varphi : E(G) \to \{1, \ldots, r\}$. We call any such $\varphi$ an $(r,k)$-colouring (so, the usually considered $r$-colourings for Ramsey problems are $(r,1)$-colourings).

The first problem we consider is the tree covering problem. In the traditional setting, one is interested in the minimum number $t_{r,k}(K_n)$ such that each $(r,k)$-colouring of $E(K_n)$ admits a cover with $t_{r,k}(K_n)$ monochromatic trees (not necessarily of the same colour). The following conjecture has been put forward by Gyárfás:

Conjecture 1 (Gyárfás) [5]. For all $n \geq 2$, we have $t_{r,k}(K_n) \leq r - 1$.

Note that this conjecture becomes trivial if we replace $r - 1$ with $r$, as for any colouring, all monochromatic stars centered at any fixed vertex cover $K_1$. Also, the conjecture is tight when $r = 1$ is a prime power, and holds for $r = 2$. This is due to results from [1, 4, 5].

In our setting, for a given graph $G$ we define $t_{r,k}(G)$ as the minimum number $m$ such that each $(r,k)$-colouring of $E(G)$ admits a cover with $m$ monochromatic trees. In this context, a monochromatic tree in $G$ is a tree $T$ such that there is a colour $r$, which for each $e \in E(T)$, belongs to the set of colours assigned to $e$.

Tree coverings have also been studied for complete bipartite graphs $K_{n,m}$. Chen, Fujita, Gyárfás, Lehel, and Tóth [2] proved the following conjecture:

Conjecture 2 (Chen et al. [2]). If $r \geq 1$ then $t_{r,k}(K_{n,m}) \leq 2r - 2$ for all $n, m \geq 1$.

It is shown in [2] that this conjecture is tight; that is, it is true for $r = 2$ and $k = 1$, and that $t_{r,k}(K_{n,m}) \geq 2r - 2$ for all $r, n, m > 2$.

Also classical Ramsey problems extend to $(r,k)$-colourings. Define the $r$-Ramsey number $r_r(K_h)$ of a graph $H$ as the smallest $n$ such that every $(r,k)$-colouring of $K_n$ contains a monochromatic copy of $H$. (Above, a monochromatic subgraph $H$ of $G$ is a subgraph $H \subseteq G$ such that there is a colour $r$ that appears on each $e \in E(H)$.) So the usual $r$-colour Ramsey number of $H$ equals $r_r(H)$. Note that $r_r(H)$ is increasing in $r$, if $H$ and $r$ are fixed, and decreasing in $k$, if $H$ and $r$ are fixed.

Results

Let $\varphi$ be an $(r,k)$-colouring of a graph $G$. Note that deleting $k - 1$ fixed colours from all the edges, and, if necessary, deleting some more colours from some of the edges, we can produce an $(r-k+1)$-colouring from $\varphi$. So, Conjecture 1, if true, implies that $t_{r,k}(K_n) \geq r - k$.

We confirm this bound, in the case that $r$ is not much larger than $k$.

Theorem 1. If $r \leq 2k + 2$ then $t_{r,k}(K_n) \leq r - k$ for all $n \geq 1$.

Clearly, the bound from Theorem 1 is tight for $r = k = 1$, and it is also tight for $r = k = r - 2$, as Figure 1 shows, but in general, the bound is not tight. The smallest example (in terms of $r$ and $k$) corresponds to $r = 2$ and $k = 1$.

Theorem 2. For all $n \geq 1$, we have $t_{r,k}(K_{n,m}) = 2r - 2$.

There is an interesting connection between Theorem 1 and Ryser’s Conjecture. The latter conjecture states that $\tau(H) \leq (r - 1)\nu(H)$ for each $r$-partite $r$-uniform hypergraph with $r > 1$, where $\tau(H)$ is the size of a smallest transversal of $H$, and $\nu(H)$ is the size of a largest matching in $H$. It is not hard to see that $\tau(H) \leq (r - 1)\nu(H)$, and $\tau(H)$ is the minimum number of stars centered at any fixed vertex cover $K_1$. In particular, for these hypergraphs, Ryser’s conjecture holds.

The stronger bound $t_{r,k}(K_n) \leq r - k$, we obtain a stronger version of Ryser’s conjecture for these hypergraphs:

Corollary 1. We have $\tau(H) \leq r - k$ for all $r$-partite $r$-uniform $k$-intersecting hypergraphs $H$ with $r \leq 2k + 2$.

In the case of $G$ being a complete bipartite graph, we can use the argument from above, deleting $k - 1$ fixed colours, to see that $t_{r,k}(K_{n,m}) \geq 2r - 2k - 1$. However, it is possible to improve the upper bound as the following theorem shows.

Theorem 3. For all $r, n, m$, $t_{r,k}(K_{n,m}) \leq \begin{cases} r - k + 1, & \text{if } r \leq 2k \leq 2r - 2k + 1, \\ 2r - 2k + 1, & \text{if } 2k < r \leq 2k/2, \\ 2r - 2k + 2, & \text{otherwise.} \end{cases}$

For the case $r \leq 2k$, our bound is sharp for large graphs.

Theorem 4. For each $r,k$ with $r > k$ there is $n,m$ such that $n \geq m \geq m_0$ then $t_{r,k}(K_{n,m}) \geq 2r - 2k - 1$.

Considering set-Ramsey numbers, we can bound $r_{r,k}(H)$ with the help of the usual $r$-colour Ramsey number $r_r(H)$. In fact, in the same way as we obtained our trivial bounds on $t_{r,k} \cdot 1$, one can prove (see also [6]) that for every graph $G$ and integers $r > k > 0$, $r_{r,k}(H) \geq r_{r,k}(H)$.

Both bounds are not best possible as already the example of $r = 3$, $k = 2$ and $H = K_2$, or $K_{r,k}$, shows. Namely, it is not difficult to show that $t_{r,k}(K_{2,4}) = 5$, and the value $t_{r,k}(K_{1,5}) = 10$ follows from results of [3].

If $H$ is a cycle $C_k$, we can estimate $r_{r,k}(K_{2,4})$ using Turán’s Theorem.

Theorem 5. Let $t_r(k) = (1 + \sqrt{4r - 1} - 1)/2$. Then $t_{r,k}(K_{2,4}) = \begin{cases} k + 1, & \text{if } k < r - 1. \\ 2k, & \text{otherwise.} \end{cases}$

This bound is sharp if $r - 1 = k - 1$ is a prime power, in which case $t_{r,k}(K_{2,4}) = k^2 + 1$.

We also establish a lower bound for cycles under this setting.

Theorem 6. If $H$ is odd and $k > 2$, then $r_{r,k}(H) \geq \max(2k^2, 2k(r - 1))$.

It is possible to show, by using Theorem 6 and some basic combinatorial arguments, that $r_{r,k}(K_{1,5}) = 9$, being this number another example in which Equation (1) is not sharp.

Open Questions

Related to Theorem 1, the lower bound we know is $t_{r,k}(K_{n,m}) \geq \begin{cases} m + 1, & \text{if } n \geq r - 1. \\ 1, & \text{otherwise.} \end{cases}$

We do not know where in the interval $[\frac{r - 1}{r - 1}, r - k]$ the true value of $t_{r,k}(K_{n,m})$ lies.

Problem 1. Determine $t_{r,k}(K_{n,m})$ for all $r, k, n, m$.

For the case of complete bipartite graphs and for $r > 2k$, we do not know the true value of $t_{r,k}(K_{n,m})$.

Problem 2. Determine $t_{r,k}(K_{n,m})$ for all $r, k, n, m$.

References