RELATIONS BETWEEN TILTING AND STRATIFICATION.

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Abstract. In this work we study the relation between tilting and standard stratification. We recall that for each standardly stratified algebra corresponds a tilting module. We show that the poset given by the different stratifications of one algebra is a subposet of the poset formed by the tilting modules. Also, we show several examples, in particular we see that in the oriented $A_n$ for $n = 2, 3, 4$ all tilting modules are given by stratifications.

Preliminars

In this work all algebras are finite dimensional $K$ - algebras, basic and indecomposables, $K$ is an algebraically closed field and it is known that an algebra $\Lambda$ with these properties is of the form $\Lambda = \frac{KQ}{I}$ where $Q$ is a finite quiver and $I$ an admissible ideal.

Let $v_1, \ldots, v_n$ be the vertices of $Q$ in a fixed order and $S_1, \ldots, S_n$ the corresponding order of simple modules, $P_i$ the projective cover of $S_i$ and $Q_i$ the injective envelope of $S_i$. The standard module $\Delta_i$ is defined as the maximal factor of $P_i$ with composition factors $S_j, j \leq i[R]$. In dual way, it is defined the co-standard $\nabla_i$ as the maximal submodule of $Q_i$ with composition factors $S_j, j \leq i[R]$.

Let $\Delta = \{\Delta_1, \ldots, \Delta_n\}$, consider $F(\Delta)$, the full subcategory of mod $\Lambda$, consisting by $M \in \text{mod } \Lambda$ such that $M$ has a filtration with
factors in $\Delta$, this is, $0 = M_0 \subset M_1 \subset ... \subset M_t = M$ con $\frac{M_i}{M_{i-1}} \cong \Delta_k$.

Dually, it is defined $F(\nabla)$.

There are the following subcategories of $\text{mod} \, \Lambda$:

- $Y(\Delta) = \{Y \in \text{mod} \Lambda / \text{Ext}^1(F(\Delta), Y) = 0\}$
- $F(\Delta) \cap Y(\Delta)$
- $W(\nabla) = \{W \in \text{mod} \Lambda / \text{Ext}^1(W, F(\nabla)) = 0\}$
- $W(\nabla) \cap F(\nabla)$

The algebra $\Lambda$ is called standardly stratified if $\Lambda \in F(\Delta)$.

If also, the endomorphisms ring of each standard module is simple, $\Lambda$ is called quasi - hereditary (see for instance [R] and [X]).

1. **A tilting module associated to the standard stratification**

An $A$ - module $T$ is called tilting (generalized) if:

1. $\text{pd}T < \infty$.
2. $\text{Ext}^i(T, T) = 0, \forall i > 0$
3. There is an exact sequence $0 \to A \to T_0 \to T_1 \to ... \to T_s \to 0$, with $T_i \in \text{add}T, \forall i$.

If the algebra $\Lambda$ is standardly stratified, we have that $F(\Delta)$ is a resolving category ([X]), i. e. is closed under extensions, kernel of surjections and contains the projectives.

Let $\varpi(\Delta)$ be the intersection of the subcategories $F(\Delta)$ and $Y(\Delta)$

There is the following fact, proved in [X], Theor. 4.3:

**Proposition 1.** If $\Lambda$ is standardly stratified. Then there is a tilting module $T$, unique except for the multiplicity of the indecomposable direct summands such that $\text{add}(T) = \varpi(\Delta)$.

2. **A Poset given by the standard stratifications**

For an Artin algebra $\Lambda$, consider the set $\mathcal{T}_\Lambda$ of all tilting modules with direct summands of multiplicity one.
For each tilting module $T \in T_\Lambda$ consider the right perpendicular category $T^\perp = \{ X \in \text{mod} \Lambda/\text{Ext}^i(T, X) = 0, \forall i \}$

In [HU], it is defined a partial order in the class of all tilting modules for an Artin algebra by the following relation $T_1 \leq T_2 \iff T_1^\perp \subseteq T_2^\perp$.

For this relation $T$ is minimal if and only if $P^<\infty$ is contravariantly finite ([HU]).

Using the results of [AR], we see that $Y(\Delta) = T^\perp$.

**Theorem 2.** The order among the different forms in that an algebra can be standardly stratified, given by inclusion between the respective subcategories $F(\Delta)$, induces an inverse order between the tilting modules corresponding to these stratifications.

**Proof.** If we have two orders of simple modules such that $\Lambda$ is standardly stratified in these orders and $F_1(\Delta) \subset F_2(\Delta) \Rightarrow Y_2(\Delta) \subset Y_1(\Delta)$

(If $Y \in Y_2(\Delta) \Rightarrow \text{Ext}^1(X,Y) = 0, X \in F_2(\Delta)$, as $F_1(\Delta) \subset F_2(\Delta) \Rightarrow \text{Ext}^1(X,Y) = 0, X \in F_1(\Delta) \Rightarrow Y \in Y_1(\Delta)$).

Then we have $Y_2(\Delta) \subset Y_1(\Delta)$, and as $Y_1(\Delta) = T_1^\perp$ then $T_2^\perp \subseteq T_1^\perp$.

We know that $\text{Proj} \subset F(\Delta) \subset \text{mod} A$, also $F(\Delta) \subset P^<\infty$.

If $F(\Delta) = P^<\infty$, that is to say $F(\Delta)$ is maximal then $P^<\infty$ is contravariantly finite, well $F(\Delta)$ it is, then $T$ is minimal.

If $F(\Delta) = \text{Proj}$, that is to say $F(\Delta)$ is minimal then $Y(\Delta) = \{ Y/\text{Ext}^1(X,Y) = 0, X \in F(\Delta) \} = \text{mod} A$, then $F(\Delta) \cap Y(\Delta) = \text{Proj}$, therefore $T = P_1 \oplus \ldots \oplus P_n = A$, then $T^\perp = A^\perp = \text{mod} A$ and we conclude that $T$ is maximal.

If $F(\Delta)$ is maximal (minimal) not necessarily $F(\Delta) = P^<\infty(\text{Proj})$.

**Example 3.** Let $A_m$ be the algebra $\frac{KQ}{\alpha}$ where $Q$ is the quiver

$$
\begin{array}{cccccc}
1 & \beta_1 & 2 & \beta_2 & 3 & \ldots & m-1 & \beta_{m-1} & m \\
\downarrow & \alpha_1 & \downarrow & \alpha_2 & \downarrow & \ldots & \downarrow & \alpha_{m-1} & \downarrow \\
\end{array}
$$

and I the ideal generated by \( \alpha_{i+1}\alpha_i, \beta_i\beta_{i+1}, \alpha_i\beta_i - \beta_{i+1}\alpha_{i+1}, \)

\( 1 \leq i \leq m - 2, \alpha_{m-1}\beta_{m-1}. \)

We can see that this algebra is quasi hereditary, only in this order of simple modules, then \( F(\Delta) \) is maximal and minimal because the poset has only one element and \( F(\Delta) \neq P^{<\infty} \) and \( F(\Delta) \neq \text{Proj} \).

3. Remarks and Examples

**Remark 4.** We have several cases in that the maximal and the minimal are reached for the Poset given by the standard stratifications\(^{[HM1]}\)

1. The Hereditary algebras
2. The quasi hereditary algebras without oriented cycles, except loops
3. The algebras which are standardly stratified in all orders

For the algebras with radical square zero, if it quasi triangular it is reached the minimal and the maximal.\(^{[HM2]}\)

**Remark 5.** For the hereditary algebras given by the quiver \( A_n \) for \( n = 2, 3, 4 \), we can check that all tilting modules are given by stratifications, but for the hereditary algebra given by the quiver

\[
\begin{array}{c}
1\bullet \\
\downarrow \\
\bullet 3 \\
\downarrow \\
2\bullet
\end{array}
\]

the tilting module \( T = P_1 \oplus P_2 \oplus I_3 \) is not associated to stratification.

In the Kronecker algebra, that is to say the hereditary algebra given by the quiver \( 1\bullet \Rightarrow \bullet 2 \) we only have two stratifications: the one given by the projectives and the other given by the injectives and we have infinite tilting modules.

The algebra given by the quiver \( 1\bullet \rightleftharpoons \bullet 2 \) with radical square zero is not standardly stratified in any orden and we have an unique tilting module which is the trivial given by the sum of the projectives.
References


