A counterexample to the existence of a Poisson structure on a twisted group algebra

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Abstract. Crawley-Boevey [1] introduced the definition of a noncommutative Poisson structure on an associative algebra $A$ that extends the notion of the usual Poisson bracket. Let $(V, \omega)$ be a symplectic manifold and $G$ be a finite group of symplectomorphisms of $V$. Consider the twisted group algebra $A = \mathbb{C}[V] \# G$. We produce a counterexample to prove that it is not always possible to define a noncommutative poisson structure on $\mathbb{C}[V] \# G$ that extends the Poisson bracket on $\mathbb{C}[V]^G$.

1. Introduction

Crawley-Boevey [1] defined a noncommutative Poisson structure on an associative algebra $A$ over a ring $K$ as a Lie bracket $\langle - , - \rangle$ on $A/[A,A]$ such that for each $a \in A$ the map $(\bar{a}, -) : A/[A,A] \to A/[A,A]$ is induced by a derivation $d_a : A \to A$; i.e. $\langle \bar{a}, \bar{b} \rangle = d_a(b)$ where the map $a \mapsto \bar{a}$ is the projection $A \to A/[A,A]$. When $A$ is commutative a noncommutative Poisson structure is the same as a Poisson bracket.

Let $(V, \omega)$ be a symplectic manifold, with the usual Poisson bracket $\{ - , - \}$ on $\mathbb{C}[V]$. Let $G$ be a finite group of symplectomorphisms of $V$. Consider the twisted group algebra $A = \mathbb{C}[V] \# G$. The algebra of $G$-invariant polynomials $\mathbb{C}[V]^G$ is contained in $A/[A,A]$. We produce a counterexample to prove that it is not always possible to define a noncommutative poisson structure on $\mathbb{C}[V] \# G$ that extends the Poisson bracket on $\mathbb{C}[V]^G$.

¹The author is grateful to William Crawley-Boevey for a careful review of this paper, and to Victor Ginzburg for posing the question.
2. Twisted group algebra and derivations

From now on, let $A = \mathbb{C}[V] \#_G (\mathbb{C} \text{ can be replaced by any field of characteristic } 0)$. We use the symbol $^g\psi$ to denote the left action of $g \in G$ on $\psi \in \mathbb{C}[V]$. For every $g \in G$ we denote $(-)_g$ the projection $A \to \mathbb{C}[V]$ into the $g$-part, i.e., $(\psi h)_g = \psi \delta_{g,h}$ if $\psi \in \mathbb{C}[V], h \in G$. Let $G = C_0 \cup C_1 \cup \cdots$ be the conjugacy classes of $G$, with $C_0 = \{1\}$.

It is proved in [4] that $A_{[A,A]} = HH_0(A) = (HH_0(\mathbb{C}[V], \mathbb{C}[V] \#_G))^G$, therefore

$$\frac{A}{[A,A]} = \left( \bigoplus_{g \in G} HH_0(\mathbb{C}[V], \mathbb{C}[V]_g) \right)^G$$

$$= \left( \bigoplus_{g \in G} \frac{\mathbb{C}[V]}{\langle \varphi - g \varphi : \varphi \in \mathbb{C}[V] \rangle} \right)^G$$

$$= \bigoplus_i \left( \frac{\mathbb{C}[V]}{\langle \varphi - g_i \varphi : \varphi \in \mathbb{C}[V] \rangle} \right)^{G_{g_i}}$$

where $g_i$ is an arbitrary element of $C_i$ and $G_g = \{ h \in G | gh = hg \}$. The first summand is precisely $\mathbb{C}[V]^G$. Let $P_i$ be the projection

$$A \to \left( \frac{\mathbb{C}[V]}{\langle \varphi - g_i \varphi : \varphi \in \mathbb{C}[V] \rangle} \right)^{G_{g_i}}$$

The Poisson bracket gives us a family of derivations $d_\psi : \mathbb{C}[V]^G \to \mathbb{C}[V]^G, \phi \mapsto \{ \psi, \phi \}$ for $\psi \in \mathbb{C}[V]^G$; and we want to extend it to a larger family. The following Lemma restricts the possibilities.

**Lemma 1.** Let $d : A \to A$ be any derivation. If $x \in \mathbb{C}[V]^g \neq \mathbb{C}[V]$ then $(d(x))_g = 0$.

**Proof.** Let $y \notin \mathbb{C}[V]^g$. The equality $d(xy) = d(yx)$ implies $d(x)y + xd(y) = d(y)x + yd(x)$. The $g$-part of this equality is

$$(d(x))_g y + x (d(y))_g = (d(y))_g x + y (d(x))_g$$

or

$$(d(x))_g y + x (d(y))_g = (d(y))_g x + y (d(x))_g.$$  

Since $^g x = x, ^g y \neq y$ we conclude $(d(x))_g (^g y - y) = 0$, so $(d(x))_g = 0$.
Therefore if the action of $G$ on $V$ is faithful and $g \neq 1$, the $g$-part of the derivative an element of $\mathbb{C}[V]^g$ is zero. This implies that for every $\psi \in \mathbb{C}[V]^G$, $d(\psi) \in \mathbb{C}[V] < A$.

The condition $\langle \psi g, \phi h \rangle = -\langle \phi h, \psi g \rangle$ implies $d_{\phi g}(\phi h) = -d_{\phi h}(\psi g)$. Consider the case $\phi, \psi \in \mathbb{C}[V]^G, \ h = 1$ and $g \in C_i, i \neq 0$. Since $P_i(d\phi(\phi)) = 0$, we must have $0 = P_i(d\phi(\psi)g + \psi d\phi(g))$. The only terms that must be taken into account are $d\phi(\psi)g + \psi \sum (d\phi(g))_{hgh^{-1}} hgh^{-1}$. Modulo $[A, A]$ this is equal to
\[
\left( d\phi(\psi) + \sum_h h^{-1} \left( (d\phi(g))_{hgh^{-1}} \right) \right) g = (d\phi(\psi) + \psi \sigma_{\phi, g}) g
\]
where $\sigma_{\phi, g} = \sum_h h^{-1} \left( (d\phi(g))_{hgh^{-1}} \right)$ does not depend on $\psi$.

We want $0 = P_i((d\phi(\psi) + \psi \sigma_{\phi, g}) g) = P_i((\{\phi, \psi\} + \psi \sigma_{\phi, g}) g)$ since we want a Poisson structure extending the usual Poisson bracket on $\mathbb{C}[V]^G$. Therefore a necessary condition for the existence of the Poisson structure is the existence of $\sigma_{\phi, g} \in \mathbb{C}[V]$ so that
\[
P_i((\{\phi, \psi\} + \psi \sigma_{\phi, g}) g) = 0
\]
for every $\psi \in \mathbb{C}[V]$. We will see that this is not always possible.

### 3. The counterexample

Let $V = \mathbb{C}^4$ with linear coordinates $\{x_1, x_2, x_3, x_4\}$ and the symplectic form $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$, so $\mathbb{C}[V] = \mathbb{C}[x_1, x_2, x_3, x_4]$, and
\[
\{\phi, \psi\} = \frac{\partial \phi}{\partial x_1} \frac{\partial \psi}{\partial x_2} - \frac{\partial \phi}{\partial x_2} \frac{\partial \psi}{\partial x_1} + \frac{\partial \phi}{\partial x_3} \frac{\partial \psi}{\partial x_4} - \frac{\partial \phi}{\partial x_4} \frac{\partial \psi}{\partial x_3}.
\]

Let $G = \mathbb{Z}_2 \times (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. Let $e, b, c$ be the generators of the three copies of $\mathbb{Z}_2$ (in that order). $G$ acts on $V$ as follows: $b$ and $c$ act as $diag(-1, -1, 1, 1)$ and $diag(1, 1, -1, -1)$, respectively, on $\{x_1, x_2, x_3, x_4\}$ and $e$ interchanges $x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4$. Clearly $\mathbb{C}[V]^G$ is the set of all polynomials $\sum \lambda_{i_1,i_2,i_3,i_4} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ such that $\lambda_{i_1,i_2,i_3,i_4} \neq 0$ implies $i_1 + i_2, i_3 + i_4$ are even and $\lambda_{i_1,i_2,i_3,i_4} = \lambda_{i_3,i_4,i_1,i_2}$.

Using Magma (http://magma.maths.usyd.edu.au) we find that the ring of invariant polynomials is generated, as an algebra, by $f_1 = x_1^2 + x_3^2, f_2 = x_2^2 + x_4^2, f_3 = x_1^4 + x_3^4, f_4 = x_2^4 + x_4^4, h_1 = x_1 x_2 + x_3 x_4, h_2 = x_1^2 x_3^2 + x_2^2 x_4^2, h_3 = x_1^2 x_3 x_4 + x_1 x_2 x_4^2, h_4 = x_1 x_2 x_3^2 + x_2^2 x_3 x_4$; with relations...
\[-f_1 f_2 h_1 + f_1 h_4 + f_2 h_3 - h_1^3 + 2 h_1 h_2,\]
\[-\frac{1}{2} f_1^2 f_2 + \frac{1}{2} f_1 h_1^2 - \frac{1}{2} f_1 h_2 - \frac{1}{2} f_2 f_3 - h_1 h_3,\]
\[-\frac{1}{2} f_1 f_2^2 - \frac{1}{2} f_1 f_4 + \frac{1}{2} f_2 h_1^2 - \frac{1}{2} f_2 h_2 - h_1 h_4,\]
\[-\frac{1}{2} f_2^2 f_4 + f_1 f_2 h_2 - \frac{1}{2} f_2^2 f_3 + f_3 f_4 - h_2^2,\]
\[-\frac{1}{2} f_1^2 h_4 + \frac{1}{2} f_1 f_2 h_3 + \frac{1}{2} f_1 h_1 h_2 - \frac{1}{2} f_2 h_3 h_1 + f_3 h_4 - h_2 h_3,\]
\[-\frac{1}{2} f_1 f_2 h_4 - \frac{1}{2} f_1 f_4 h_1 - \frac{1}{2} f_1^2 h_3 + \frac{1}{2} f_2 f_1 h_2 + f_4 h_3 - h_2 h_4,\]
\[-\frac{1}{2} f_1^2 f_2 + \frac{1}{2} f_1^2 h_1^2 - f_1^2 h_2 - \frac{1}{2} f_1 f_2 f_3 - \frac{1}{2} f_2 h_3^2 + f_3 h_2 - h_3^2,\]
\[-\frac{1}{2} f_1^2 f_2^2 - 3/4 f_1^2 f_4 + \frac{1}{2} f_1 f_2 h_1^2 - 3/4 f_2^2 f_3 + f_3 f_4 - \frac{1}{2} h_2^2 h_2 - h_3 h_4,\]
\[-\frac{1}{2} f_1 f_3 - \frac{1}{2} f_1 f_2 f_4 + \frac{1}{2} f_2 h_1^2 - f_2 h_2 - \frac{1}{2} f_4 h_1^2 + f_4 h_2 - h_4^2.\]

**Proposition 2.** The Poisson bracket on \( \mathbb{C}[V]^G \) cannot be extended to a Poisson structure on \( \mathbb{C}[V] \# G \) for \( V \) and \( G \) as defined above.

**Proof.** Take \( \phi = x_1^2 + x_3^2, \psi = x_1 x_2 + x_3 x_4 \in \mathbb{C}[V]^G \) and \( g = b. \) In this case \( \{\phi, \psi\} = 2x_1^2 + 2x_3^2, \langle \varphi - g \varphi : \varphi \in \mathbb{C}[V] \rangle = \langle x_1, x_2 \rangle \) and \( G_b = \{1, b, c, bc\}. \) Hence,
\[
\left( \frac{\mathbb{C}[V]}{\langle \varphi - g \varphi : \varphi \in \mathbb{C}[V] \rangle} \right)^{G_g} = \mathbb{C}[x_3, x_4]^\{1, b, c, bc\} = \mathbb{C}[x_3^2, x_3 x_4, x_4^2],
\]
so \( P_1(\{\phi, \psi\}) = 2x_3^2 b. \)

On the other hand, \( P_1((\psi \sigma_{\phi, g}) b) = P_1(((x_1 x_2 + x_3 x_4) \sigma_{\phi, g}) b) = \)
\( P_1((x_3 x_4) \sigma_{\phi, g}) b \) and none of the terms here can be equal to \(-2x_3^2\) since they all contain \( x_4. \) This contradicts (1). \( \square \)

**References**


