The inverse problem of variational calculus and the problem of mixed endpoint conditions

Pedro Gonçalves Henriques

Abstract. P. A. Griffiths established the so-called mixed endpoint conditions for variational problems with non-holonomic constraints. We will present some results in this context and discuss the inverse problem of calculus of variations.

Keywords: Inverse problem of calculus of variations.

1. Introduction

The study of Calculus of Variations for multiple integrals was first developed by Caratheodory [1929], while Weil-De Donder [1936], [1935] advanced a different theory later. The two approaches were unified by Lepage [1936-1942], Dedecker [1953-1977] and Liesen [1967] in a framework using the $n$-Grassmannian manifold of a $C^\infty$ manifold. Important contributions in the Calculus of Variations on smooth manifolds were made by R. Hermann [1966], H. Goldschmidt and S. Sternberg [1973] with their Hamilton-Cartan formalism, as well as by Ouzilou [1972], D. Krupka [1970-1975] and I. M. Anderson [1980]. The symplectic approach of P. L. Garcia and A. Pérez-Rendón [1969-1978], the multisymplectic version of Kijowski and Tulczyjew [1979] based on the theory of Dedecker, the polysymplectic approach of C. Günther [1987], Edelen [1961] and Rund [1966] are also important references in this field. Here we deal with the broader problem of finding extrema of a functional on a set of $n$-dimensional integral manifolds of a Pfaffian differential system.

In 1983, Griffiths proposed a new approach to variational problems based on techniques from the theory of exterior differential systems. His work dealt with the problem of finding extrema for a functional $\phi$ defined on the set of one-dimensional integral manifolds of a differential system $(T^*, L^*)$.

Supported by the Center for Mathematical Analysis, Geometry, and Dynamical Systems.
This approach was established using intrinsic entities. In this work we present a general setting based on [25] (sections 2 to 8), and we deal with the inverse problem (section 9).

In 1887 Helmholtz addressed the following problem: given

$$P_i = P_i(x, u^j, u_x^j, u_{xx}^j),$$

is there a Lagrangian $L(x, u^j, u_x^j)$ such that

$$E_i(L) = \frac{\partial L}{\partial u^i} - D_x \frac{\partial L}{\partial u_x^i} = P_i$$

where

$$D_x = \frac{\partial}{\partial x} + u_i^j \frac{\partial}{\partial u^i} + u_{x}^j \frac{\partial}{\partial u_x^i}.$$ He found necessary conditions for $P_i$ to form an Euler-Lagrange system of equations (see (9.1), (9.2) and (9.3)). Some years later, these conditions were proved to be locally sufficient. I. M. Anderson [1992; 1980], P. J. Olver [1986], F. Takens [1979], W. M. Tulczyjew [1980] and A. M. Vinogradov [1984] generalized Helmholtz’s conditions for both higher order systems of partial differential equations and multiple integrals.

2. Integral manifolds of a differential system and valued differential systems

We assume that a Pfaffian differential system $(I^*, L^*)$ is given on a real-manifold $X$ by:

1) a subbundle $I^* \subset T^*X$,
2) another subbundle $L^* \subset T^*X$ with $I^* \subset L^* \subset T^*X$,

such that the rank $(L^*/I^*) = n$ (with $n$ being a natural number).

An integral manifold of $(I^*, L^*)$ is given by an oriented connected compact $n$-dimensional smooth manifold $N$ (possibly with a piecewise smooth boundary $\partial N$) together with a smooth mapping

$$f : N \to X$$

satisfying

$$I_{f(x)}^* \perp = L_{f(x)}^* \perp + f_*(TN),$$  \hspace{1cm} (2.1)

for all $x \in N$, where $f_* : T_xN \to T_{f(x)}X$ is the differential of $f$ at $x$.

We denote by $V(I^*, L^*)$ the collection of integral manifolds $f$ of $(I^*, L^*)$.

A valued differential system is a triple $(I^*, L^*, \varphi)$, where $(I^*, L^*)$ is a Pfaffian differential system and $\varphi$ is an $n$-form on $X$.

We define the functional $\phi$ associated with $(I^*, L^*, \varphi)$ in $V(I^*, L^*)$ by:

$$\phi : V(I^*, L^*) \to R,$$
Inverse problem of variational calculus and problem of mixed endpoint conditions

3. Local embeddability

The following definition is a general setting for the study of problems in the Calculus of Variations. In [25] we proved that there exist locally defined mappings that induce $(I^*, L^*)$ from the canonical system in $J^1(R^n, R^s)$ possibly with some constraints, establishing a local correspondence between these differential systems. Let us assume that $d(C^\infty(X, L^*)) \subset C^\infty(X, L^* \wedge T^*X)$, and let $d' = \dim X; \quad s = \text{rank} I^* (d(C^\infty(X, L^*)))$ is the set of images produced by the exterior derivative of $C^\infty(X, L^*)$. Using the Frobenius theorem, we can set for every $p \in X$ a chart coordinate system \( \{u^1, ..., u^{s+n}, v^1, ..., v^{d'-s-n}\} \) so that

\[
\begin{align*}
(i) & \quad L^* = \text{span}\{du^\alpha|1 \leq \alpha \leq s+n\}, \\
(ii) & \quad L^{*\perp} = \text{span}\{\partial \partial v^i|1 \leq i \leq d'-s-n\}
\end{align*}
\]

for an open subset $U$ of $X$ with $p \in U$.

**Definition 3.1.** Let $(I^*, L^*)$ be a Pfaffian differential system with $d(C^\infty(X, L)) \subset C^\infty(X, L^* \wedge T^*X)$.

We say that $(I^*, L^*)$ is locally embeddable if for every $p \in X$ there exist an open neighborhood $U$ of $p$ and local coframes $\text{CF} = \{\theta_1, ..., \theta_s\}$ (3.3)

for $I^*$ and $\text{CF}' = \{\theta_1, ..., \theta_s, du'^n + 1, du'^n + n\}$ (3.4)

for $L^*_U$, satisfying the following conditions:

(i) \( \delta(I^*_U \wedge \Omega) \subset T^* \wedge \Lambda^n(L^*_U)/(T^*U \wedge I^*_U \wedge \Lambda^{n-1}(L^*)) \) (3.5)

(ii) Ker $\delta$ is a constant rank subbundle of $I^* \wedge \Omega$,

where $\Omega = \text{span}\{du'^{n+1} \wedge ... \wedge du'^{n+s} \wedge ... \wedge du'^{s+n}\}; \quad du'^{n+s} - \text{means deletion of the s + b factor (for n = 1, du'^{n+1} = 1)}. \quad \text{We use u'' since we may have to reorder these coordinates.}$
The map \( \delta : I^* \wedge \Omega \to \Lambda^{n+1}(T^*U)/I_u^* \wedge (\Lambda^n(T_* U)) \) is induced by
\[
d : C^\infty(U, I^* \wedge \Omega) \to C^\infty(U, \Lambda^{n+1}(T^*U))
\]
on \( I^* \wedge \Omega \).

This definition means that if \( I^* \) has no Cauchy characteristics, the structure equations are locally:
\[
d\theta_i \equiv \pi^i_{\alpha j} \wedge \theta^\alpha + B_i^{\alpha \beta} \wedge \theta^\alpha \bmod I \quad (3.6)
\]
\( 1 \leq i, i', \alpha \leq s, 1 \leq j, j', \beta \leq n, I = C^\infty(X, I^*) \).

4. The Cartan system of \( \Psi \)

Let \((I^*, L^*, \varphi)\) be a valued differential system on \( X \), and \( W \) be the total space of \( I^* \). Let \( \chi \) be the canonical form on \( T^*X \), and \( i \) the inclusion map \( W \hookrightarrow T^*X \).

Let us assume that there exists a local \( n \)-form \( \omega \) inducing a nonzero section of \( \Lambda^n(L^*/I^*) \) and has the following form:
\[
\omega = \omega^1 \wedge \ldots \wedge \omega^n. \quad (4.1)
\]
We define:
\[
\omega_i = (-1)^{i-1} \omega^1 \wedge \ldots \wedge \hat{\omega}_i \ldots \wedge \omega^n. \quad (4.2)
\]

Let \( W^n \) be the \( n \)-Cartesian power of \( W \), and \( Z \) be a subset of \( W^n \) defined by \( Z = \{ z \in W^n : \pi^i(z) \in \Delta X^n \} \), where \( \pi^i \) is the natural projection \( \pi^i : W^n \to X^n \), and \( \Delta X^n \) is the diagonal submanifold of \( X^n \). The subset \( Z \) is a vector subbundle over \( X \) and \( \dim Z = d + sn \). We define
\[
\Psi = d\psi \quad (4.3)
\]
where \( \psi \) is given by
\[
\psi = \pi^* \varphi + (\pi^j \circ i')^* \{ i^* (\chi) \} \wedge \pi^* \omega_j. \quad (4.4)
\]
\( \pi^j \) is the natural projection into the \( j \)th component \( \pi^j : W^n \to W \), \( i' \) is the inclusion map \( Z \to W^n \) and \( \pi \) is the natural projection \( \pi : Z \to X \).

Definition 4.1. Given the \( n + 1 \)-form \( \Psi \), the Cartan system \( C(\Psi) \) is the ideal generated by the set of \( n \)-forms
\[ \{ v \wedge \Psi \ \text{where} \ v \in C^\infty(Z, TZ) \} \]

An integral manifold of \((C(\Psi), \omega)\) is given by an oriented connected compact \( n \)-dimensional smooth manifold \( N \) (possibly with a piecewise smooth boundary \( \partial N \)) together with a smooth mapping
\[ f : N \to X \]
satisfying:

Inverse problem of variational calculus and problem of mixed endpoint conditions

\[ f^* \theta = 0 \quad \text{for every} \quad \theta \in C(\Psi) \quad (4.5) \]

and

\[ f^*(\omega) \neq 0. \quad (4.6) \]

A solution of \((C(\Psi), \omega)\) projected in \(X\) will give an extremum of \(\phi\).

5. The momentum space, prolongation of \((C(\Psi), R^*\omega)\) in the momentum space, non-degeneracy

The momentum space is constructed in the following way. Suppose we are given on \(\mathbb{Z}\) (see section 4):

(i) a closed \((n+1)\)-form \(\Psi\) with the associated Cartan system \(C(\Psi)\),

(ii) \(R^*\) the pull back to \(\mathbb{Z}\) of the \(\omega\) \(n\)-form which induces a nonzero section on \(\Lambda^n(L^*/R^*)\).

Integral elements of \((C(\Psi), R^*\omega)\) are defined in a similar way as the integral elements of \((R^*, L^*)\). The set of integral elements \([x_0, E^n]\) gives a subset \(\mathbb{V}_n(C(\Psi), R^*\omega) \subset G_n(\mathbb{Z})\) (\(G_n(\mathbb{Z})\) is the \(n\)-Grassmanian).

Denoting by \(R^*\) the projection \(G_n(\mathbb{Z}) \to \mathbb{Z}\) and assuming regularity at each step, one inductively defines:

\[ Z_1 = \pi''(V_n(C(\Psi), R^*\omega), V'_n(C(\Psi), R^*\omega)) = \{ E \in V_n(C(\Psi), R^*\omega) : E \text{ tangent to } Z_1 \}, \quad (5.1) \]

\[ Z_2 = \pi''(V'_n(C(\Psi), R^*\omega), V''_n(C(\Psi), R^*\omega)) = \{ E \in V'_n(C(\Psi, R^*\omega)) : E \text{ tangent to } Z_2 \}. \quad (5.2) \]

**Definition 5.1.** Suppose \((R^*, L^*, \varphi)\) is a valued differential system, with \((R^*, L^*)\) being a locally embeddable differential system and \(\omega = \omega^1 \wedge \ldots \wedge \omega^n\).

If there exists a \(k_0 \in \mathbb{N}\) such that \(Z_{k_0} = Z_{k_0+1} = \ldots = Z_{k_0+n'(n' \in \mathbb{N})}\) in the above construction, with

(i) \(Z_{k_0}\) being a manifold of dimension \((n+1)m + n\) for \(m \in \mathbb{N}\), and

(ii) \((C(\Psi), R^*\omega)_{Z_{k_0}}\) being a differential system in \(Z_{k_0}\) with \(r_n = 0\) (Cartan number in Cartan-Kähler Theorem) for all \(V_{n-1}(C(\Psi), R^*\omega)\);

(for \(n = 1\) we follow \([23]\) and replace this condition by \(\psi \wedge \Psi^n \neq 0\) on \(Z_{k_0}\)).

Then \((R^*, L^*, \varphi)\) is a non-degenerate valued differential system, and \(Z = Y\) is called the momentum space.

We call \((C(Ψ), π^∗ω)_Y\) the prolongation of \((C(Ψ), π^∗ω)\) in the momentum space. By construction, the differential system \((C(Ψ), π^∗ω)_Y\) satisfies:

(i) the projection \((C(Ψ), π^∗ω) → Y\) is surjective,
(ii) the integral manifolds of \((C(Ψ), π^∗ω)\) on \(Z\) coincide with those of \((C(Ψ), π^∗ω)\) on \(Y\).

6. Well-posed valued differential systems

**Definition 6.1.** \((I^∗, L^∗, ϕ, P^∗, M^∗)\) is a well-posed valued differential system, if the following conditions are satisfied:

(i) \((I^∗, L^∗, ϕ)\) is a non-degenerate valued differential system (with \(\text{dim} Y = (n + 1)m + n\)) and \(ϕ = Lω\) for a smooth function \(L\) on \(X\);
(ii) there exists a subbundle \(P^∗\) of \(I^∗\) of rank \(m\) and a subbundle \(M^∗\) of \(L^∗\) of rank \(m + n\), such that:

\[ I^∗ \subset L^∗ \subset T^∗X \]

(a) \(\bigcup P^∗ \subset M^∗\),
(b) the locally given \(n\)-form \(ω\) also induces a nonzero section on \(Λ^n(M^∗/P^∗)\),
(c) \(Y \subset (P^∗)^n|_{ΔX^n}\), with \(Y\) a subbundle of \((P^∗)^n|_{ΔX^n}\),
(iii) \(π^∗M^∗ = \text{span}\{π^∗θ | C^∞(X, M^∗)\}\) is completely integrable on \(Y\), where \(π^* = π \circ i\). As before \(i\) denotes the inclusion mapping \(Y \to Z\) and \(π\) the projection \(Z \to X\).

Let us assume that there exists a coframe \(CF = \{θ^α, du^s+j, π^∗_j, π^∗_j | 1 ≤ α ≤ s, 1 ≤ j ≤ L'_v, s_{t+1} ≤ j' ≤ s, 1 ≤ j ≤ n\}\) for \(T^∗X\) with \(L'_v \subset \{k ∈ N, 1 ≤ k ≤ n\}\) such that

(i) \(I^∗ = \text{span}\{θ^α | 1 ≤ α ≤ s\}\); \hspace{1cm} (6.1)
(ii) \(L^∗ = \text{span}\{θ^α, du^s+j | 1 ≤ α ≤ s, 1 ≤ j ≤ n\}\); \hspace{1cm} (6.2)
(iii) \(T^∗X = L^∗ \oplus R^∗\) (\(\oplus\) denotes a direct sum) with \(R^∗ = \text{span}\{π^∗_j, π^∗_j | 1 ≤ j' ≤ s_t, s_{t+1} ≤ j'' ≤ s, 1 ≤ j ≤ n\}\);
(iv) \(dθ^∗_{j'} ≡ 0 \mod I, \text{ for } j'' ∉ L'_v\); \hspace{1cm} (6.3)
(v) \(dθ^∗_{j'} ≡ π^∗_j \wedge ω \mod I, \text{ for } j' \in L'_v\); \hspace{1cm} (6.4)
(vi) \(dθ^∗_j ≡ π^∗_j \wedge ω \mod I, \text{ when } 1 ≤ j ≤ n\); \hspace{1cm} (6.5)
Inverse problem of variational calculus and problem of mixed endpoint conditions

(vii) $\pi_j^i, \pi_j''$ are linearly independent mod $L$.

We define $\theta_j^\alpha = \theta^\alpha \wedge \omega_j$.

Let $d\varphi \equiv L_{i,j}^\alpha \pi_i^\alpha + L_{j,i}^\alpha \pi_j^\alpha \mod I$ and $dL_{i,j}^\alpha \equiv L_{i,j}^{\alpha\nu} \pi_i^\nu \mod \pi L^*$.

Quadratic form $A$: Let $(I^*, L^*, \varphi, P^*, M^*)$ be a well-posed valued differential system and $A$ be a quadratic form defined in $T^*X$ given by $A(v, w) = L_{i,j}^{\alpha\nu} v^\alpha \pi_i^\nu w^\alpha \pi_j^\nu$, where $v = v^\alpha \theta^\alpha \partial / \partial \theta^\alpha + v^\nu \pi^\nu \partial / \partial \pi^\nu$ and $w = w^\alpha \theta^\alpha \partial / \partial \theta^\alpha + w^\nu \pi^\nu \partial / \partial \pi^\nu$. This quadratic form plays an important role in establishing necessary conditions for a local extremum.

6.1. Generalized Lagrange Problem. Let us describe the following problem:

Generalized Lagrange Problem. Let $X = J^1(R^n, R^m)$ (the 1 jet manifold), with the canonical system $I^*$ defined on $X$ (i.e. $I^* = \text{span}\{\theta^\alpha = dy^\alpha - y^\alpha_i dx^i\}$). Let $\varphi = L\omega$ with $\omega = dx^1 \wedge ... \wedge dx^n$. We choose $x^1, ..., x^n$ to be coordinates for $R^n$, and $y^1, ..., y^m$ to be coordinates for $R^m$.

We proved in [26] that a Lagrange problem for $n = 1$ with $L\det L_{i,j}^{\alpha\nu} \neq 0$, and with constraints not involving more than one variable $y$ in each equation of restriction is a well posed valued differential system.

7. The Euler-Lagrange differential system for a well-posed valued differential system

When we compute the first variation of $\varphi$, we find an integral over $N$ and another over the boundary $\partial N$. The volume integral will vanish for projections of integral manifolds of the Cartan system $(C(\Psi), \pi^*\omega)$ into $X$. Choosing suitably the set of boundary conditions we can make the integral over the boundary to vanish as well, providing stationary integral manifolds for generalized Lagrange problems (see [25]).

7.1. The Euler-Lagrange differential system.

Definition 7.1. Let $(I^*, L^*, \varphi)$ be a valued differential system. The Cartan system $(C(\Psi), \pi^*\omega)$ is called the Euler-Lagrange differential system associated with $(I^*, L^*, \varphi)$.

Assuming that $(I^*, L^*, \varphi)$ is non-degenerate, we now consider the restriction to $Y$ of the Euler-Lagrange differential system associated with $(I^*, L^*, \varphi)$. The following proposition is easy to prove (see [25]):

Proposition 7.1. If $g$ is an integral manifold of $(C(\Psi), \pi^*\omega)$, then $\pi \circ g \in V(I^*, L^*)$, where $\pi$ is the natural projection $\pi : Z \rightarrow X$. 

We denote by \((V(C(\Psi), \pi^*\omega))\) the set of integral manifolds of \((C(\Psi), \pi^*\omega))\).

8. Examples

Example 1. Strings [41], [42]

Let \(X = J^1(N, R^m)\), \(N\) being a two-dimensional manifold. In this case \(I^* = \text{span}\{dx^\alpha - x'^\alpha dt - \dot{x}'^\alpha d\tau | 0 \leq \alpha \leq m - 1, x'^\alpha\}\) are coordinates in \(R^m\), and \(\sigma, \tau\) are coordinates of \(N\), \(x'^\alpha = \frac{\partial x^\alpha}{\partial \sigma}\), \(\dot{x}'^\alpha = \frac{\partial x^\alpha}{\partial \tau}\). In \(R^m\) we take a metric defined in \(TR^m\) by \(g_{00} = -g^{11} = 1, 1 \leq i \leq m\) and \(g^{ij} = 0\) for \(i \neq j\). The set \(X\) is given by: \(X = \{x \in X_0 | (\dot{x} \cdot \dot{x}) \geq 0\} \) (where \((\cdot)\) denotes the inner product with respect to the metric \(g\)). The form \(\omega\) is \(\omega = d\sigma \land d\tau\). We have

\[\varphi = L \omega = [(x' \cdot \dot{x})^2 - (\dot{x} \cdot \dot{x})(x' \cdot x')]^{1/2} d\sigma \land d\tau. \quad (8.1)\]

Note: \(L\) is a function of \(\dot{x}\) and \(x'\) only.

First variation of \(\phi\). Let \(\phi = \int f^*(\varphi)\), where \(f \in V(I^*, L^*)\). Then

\[\delta \phi = \int f^*(v_\sigma d\varphi + d(v_\tau \varphi)), \quad (8.2)\]

where \(v(\sigma, \tau) = F_t(\partial/\partial t)(t, \sigma, \tau)|_{t=0}, (\sigma, \tau) \in N, t \in [0, 1]\) and \(F\) is the one parameter variation of \(f\). i.e., \(F(t, \sigma, \tau)|_{t=1} \in V(I^*, L^*)\) for all \(0 \leq t_1 \leq 1\). Hence the Lie derivative of \(dx^\alpha - x'^\alpha dt - \dot{x}'^\alpha d\tau\) by \(v\) along \(f(N)\) vanishes,

\[d(v_\sigma(dx^\alpha - x'^\alpha dt - \dot{x}'^\alpha d\tau)) + (v_\tau(-dx'^\alpha \land d\sigma - d\dot{x}'^\alpha \land d\tau))|_{f(N)} = 0. \quad (8.3)\]

The form \(\Psi_Z\) is given by

\[\Psi_Z = (L_{x^\alpha} - \dot{\lambda}_\alpha)\pi^*(dx^\alpha \land \omega) + (L_{x'^\alpha} - \lambda'_\alpha)\pi^*(dx'^\alpha \land \omega) + (d\lambda'_\alpha \land \pi^*d\sigma - d\lambda'_\alpha \land \pi^*d\tau) \land \pi^* dx'^\alpha + (\dot{x}'^\alpha d\dot{\lambda}_\alpha - x'^\alpha d\lambda'_\alpha) \land \pi^* \omega \quad (8.3)\]

The Cartan system in \(Z\) is:

(i) \(\partial/\partial \dot{\lambda}_\alpha \Psi_Z = -\pi^*(dx^\alpha - \dot{x}'^\alpha d\tau) \land \pi^* d\sigma) = 0, \quad (8.4)\]

(ii) \(\partial/\partial \lambda'_\alpha \Psi_Z = -\pi^*(dx'^\alpha - x'^\alpha d\tau) \land \pi^* d\sigma) = 0, \quad (8.5)\]

(iii) \(\partial/\partial \dot{x}'^\alpha \Psi_Z = -\pi^*(L_{x'^\alpha} - \dot{\lambda}_\alpha) \omega = 0, \quad (8.6)\]

(iv) \(\partial/\partial x'^\alpha \Psi_Z = -\pi^*(L_{x'^\alpha} - \lambda'_\alpha) \omega = 0, \quad (8.7)\]

(v) \(\partial/\partial x^\alpha \Psi_Z = -\pi^*d\dot{\lambda}_\alpha \land \pi^*d\sigma - d\lambda'_\alpha \land \pi^*d\tau) = 0. \quad (8.8)\]
Hence

\[ Z_1 = Z|L_{2\alpha} - \lambda_0, L_{\alpha} - \lambda^0. \] (8.9)

Note that from (i) and (ii) we have \( \theta^0 = 0 \);
from (iii), (iv) and (v) we have \( E[L]|\omega = (\partial L/\partial x^a - D_\sigma \partial L/\partial x^a - D_{\tau} \partial L/\partial \dot{x}^a)\omega = 0 \) for \( D_\tau = \partial/\partial \tau + \dot{x}^a \partial/\partial x^a + \ddot{x}^a \partial/\partial \dot{x}^a \) and \( D_\sigma = \partial/\partial \sigma + x^a \partial/\partial x^a + x'^a \partial/\partial x^a \).

The generalized momenta are given by

\[ \dot{\lambda}_a = \frac{x^a(x' \cdot \dot{x}) - (x' \cdot x') \dot{x}^a}{(x' \cdot \dot{x})^2 - (\dot{x} \cdot \dot{x})(x' \cdot x')}^{1/2}. \] (8.10)

\[ \lambda' = \frac{\dot{x}^a(x' \cdot \dot{x}) - (x' \cdot \dot{x}) \dot{x}^a}{(x' \cdot \dot{x})^2 - (\dot{x} \cdot \dot{x})(x' \cdot x')}^{1/2}. \] (8.11)

Let \( R^{2m}|(\dot{x} \cdot \dot{x}) \geq 0, (x' \cdot x') \leq 0 \) and \( F' = R^{2m} \) be given by

\[ F'(x^a, x'^a) = (\lambda'_a(x^a, x'^a), \dot{\lambda}_a(x, x')). \]

In this case \( F' \) has an inverse in \( R^{2m}|(\dot{x} \cdot \dot{x}) \geq 0, (x' \cdot x') \leq 0 \) and \( F'^{-1} \) is given by:

\[ x^a = \frac{\lambda'_a(\lambda' \cdot \dot{\lambda}) - (\lambda' \cdot \lambda') \dot{\lambda}}{(\lambda' \cdot \dot{\lambda})^2 - (\dot{\lambda} \cdot \dot{\lambda})(\lambda' \cdot \lambda')^{1/2}}, \] (8.12)

\[ x'^a = \frac{\dot{\lambda}_a(\lambda' \cdot \dot{\lambda}) - (\dot{\lambda} \cdot \dot{\lambda}) \lambda'_a}{(\lambda' \cdot \dot{\lambda})^2 - (\dot{\lambda} \cdot \dot{\lambda})(\lambda' \cdot \lambda')^{1/2}}. \] (8.13)

The Cartan system in \( Z_1 = Z_1|((\dot{\lambda} \cdot \lambda) \geq 0, (\lambda' \cdot \lambda') \leq 0 \) is given by (i),(ii), (iv) and (v) of the Cartan system in \( Z \). Let \( Y = Z_1 \). The prolongation of \( (C(\Psi), \pi^* \omega) \) ends at \( Z_1 \). The dimension of \( Y \) is \( \dim Y = 3m + 2 \). Every point in \( Y \) is a zero-dimensional integral element of \( (C(\Psi), \pi^* \omega) \), and \( r_1 = 2m + 1 \).

The Cartan system is in involution at \( \tau \) if \( \det C(v)|_{X_0} \neq 0 \), and

\[ C(v) = \begin{bmatrix} < v, d\tau > I & < v, d\sigma > I \\ m \times m & m \times m \\ A & B \end{bmatrix} \] (8.14)

for every \( v \neq 0 \) along \( E^1 \), with \([x_0, E^1]\) being any integral element of \( (C(\Psi), \pi^* \omega) \), where

\[ A = < v, d\sigma > L_{x^a \dot{x} \beta} - < v, d\tau > L_{x^a \dot{x} \beta} \] (8.15)

and

\[ B = < v, d\sigma > L_{x^a x^b \beta} - < v, d\tau > L_{x^a x^b \beta}, \] with \( 0 \leq \beta \leq m - 1. \] (8.16)
Let us define the energy momentum current $P = (P^0, ..., P^{m-1})$ on the surface $\gamma = \{x^\alpha(\sigma, \tau), \sigma, \tau | 0 \leq \alpha \leq m-1\}$ by

$$P^\alpha = \int P^\alpha d\sigma + P^\alpha d\tau$$

where $\dot{P}^\alpha = -L_{\dot{x}^\alpha}, P^\alpha = -L_{x^\alpha}$.

**Case 1. Open strings.** Let $N = [0, \pi] \times [t_1, t_2], (t_1, t_2) \in R^2, t_1 < t_2$. We will impose the following constraints on variations of $f \in \mathcal{V}(I^*, L^*)$:

a) $g^*(v \cdot \pi^* \omega)_{\partial N} = 0,$

b) $g^*(v \cdot \pi^*(dx^\alpha - \dot{x}^\alpha d\tau - x^\alpha d\sigma))_{B} = 0$

where $B = [0, \pi] \times t_1 \cup [0, \pi] \times t_2$.

c) $\lambda'_{\alpha} = 0$ on $g(A)$ where $A = N \setminus B.$

In this case, $G$ is any smooth lift of $F$ to $Y$ with $G|_{t=0} = g, (\pi \circ g = f)$, and $v$ is a vector field defined along $g$ with $v = G_{\alpha}(\partial / \partial t)|_{t=0}$. The constraint c) forces the boundary term in the first variation of $\phi(f)$ vanish.

**Case 2. Closed strings.** Let $N = S_1 \times [t_1, t_2]$, with $S_1$ being the unit circle. Its coordinate $\sigma \in [0, 2\pi]$, and $(t_1, t_2) \in R^2, t_1 < t_2$. We will replace the constraints on variations of $f \in \mathcal{V}(I^*, L^*)$ of the previous case with the following:

a) $g^*(v \cdot \pi^* \omega)_{\partial N} = 0,$

b) $g^*(v \cdot \pi^*(dx^\alpha - \dot{x}^\alpha d\tau - x^\alpha d\sigma))_{B} = 0$

where $B = S_1 \times t_1 \cup [0, \pi] \times t_2$.

The quadratic form $A$. The cone $X' = X |(\dot{x} \cdot \dot{x}) \geq 0, (x' \cdot x') \leq 0$ is convex. $F'$ has an inverse in $X'$ with $F' : X'' F'^{-1} R^{2m}$ where $X'' = R^{2m}|(\dot{\lambda} \cdot \dot{\lambda}) \geq 0, (\lambda' \lambda') \leq 0$. Hence the matrix

$$A' = \begin{bmatrix} L_{\dot{x}^\alpha \dot{x}^\beta} & L_{\dot{x}^\alpha x^\beta} \\ L_{x^\alpha \dot{x}^\beta} & L_{x^\alpha x^\beta} \end{bmatrix}$$

has an inverse. Therefore, the eigenvalues of $A'$ do not vanish on $X'$. Thus, it suffices to know the eigenvalues of $A'$ at an interior point of $X'$ to determine the number of positive eigenvalues of $A'$ in every point of $X'$.
Let
\[ a = \{ x^0 = 1, x^i = 0, x'^1 = 1, x'^j = 0 \quad \text{with} \quad 1 \leq i \leq m - 1, j = 0 \]
\[ \text{or} \quad 2 \leq j \leq m - 1 \}. \]
Then
\[ L_{x^0 x'^1}(a) = -L_{x'^1 x^0}(a) = -L_{x^i x'^i}(a) = 1, 2 \leq i \leq m - 1, \]
(8.24)
and all the other elements of \( A' \) are zero. We conclude that the matrix has \( m \)-positive eigenvalues and \( m \)-negative eigenvalues in \( X' \) and the quadratic form \( A \) is neither positive nor negative definite.

**Example 2.** Let \( X_0 = J^1(\mathbb{R}^2, \mathbb{R}^m), N \subset \mathbb{R}^2, \) with \( N \) being a two-dimensional manifold with boundary. Let also
\[ \mathcal{I}^* = \text{span}\{ dx^\alpha - x'^\alpha d\sigma - \dot{x}'^\alpha d\tau | 1 \leq \alpha \leq m \}, \]
\( x^\alpha \) are coordinates in \( \mathbb{R}^m \) and \( x'^\alpha = \frac{\partial x^\alpha}{\partial \sigma}, \dot{x}'^\alpha = \frac{\partial x^\alpha}{\partial \tau}. \)
Moreover, let
\[ \varphi = L\omega = \sum_{\alpha=1}^{m} (x'^\alpha)^2 + (\dot{x}'^\alpha)^2 d\sigma \wedge d\tau. \]
(8.25)

The Cartan system in \( Z \) is

(i) \[ \partial/\partial \dot{\lambda}_\alpha \cdot \Psi_Z = -\pi^* ((dx^\alpha - \dot{x}'^\alpha) \wedge \pi^* d\sigma) = 0, \]
(8.26)

(ii) \[ \partial/\partial \lambda'_\alpha \cdot \Psi_Z = -\pi^* ((dx^\alpha - x'^\alpha d\tau) \wedge \pi^* d\sigma) = 0, \]
(8.27)

(iii) \[ \partial/\partial \dot{x}'^\alpha \cdot \Psi_Z = -\pi^* (2\dot{x}'^\alpha - \dot{\lambda}_\alpha) \omega = 0, \]
(8.28)

(iv) \[ \partial/\partial x'^\alpha \cdot \Psi_Z = -\pi^* (2x'^\alpha - \lambda'_\alpha) \omega = 0, \]
(8.29)

(v) \[ \partial/\partial x^\alpha \cdot \Psi_Z = -\pi^* d\dot{\lambda}_\alpha \wedge \pi^* d\sigma - d\dot{\lambda}'_\alpha \wedge \pi^* d\tau = 0. \]
(8.30)

Hence
\[ Z_1 = Z| L_{x'^\alpha} = \dot{\lambda}_\alpha, L_{x'^\alpha} = \lambda'_\alpha. \]
(8.31)
The prolongation ends at \( Z_1 \) with \( (C(\Psi), \pi^* \omega) \) on \( Z_1 \) given by (8.26), (8.27) and (8.30). It is easy to prove that \( (C(\Psi), \pi^* \omega) \) in \( Y \) is in involution and \( (\mathcal{I}^*, L^*, \varphi, \mathcal{I}, L^*) \) is a well-posed valued differential system.
Boundary conditions. The constraints on one-parameter variations $F$ of $f$ in $V(I^*, L^*)$ are:

a)  
\[ g^*(v_{,\pi}^*\omega)_{\partial N} = 0, \]  
(8.32)

b)  
\[ g^*(v_{,\pi}^*(dx^\alpha - \dot{x}^\alpha d\tau - x''^{\alpha}d\sigma))_{\partial N} = 0. \]  
(8.33)

In this case, too, $G$ is any smooth lift of $F$ to $Y$ with $G|_{t=0} = g, (\pi \circ g = f)$, and $v$ is a vector field defined along $g$ with $v = G|_{t=0} (\partial/\partial t)$. 

The quadratic form $A$. A simple computation yields 

\[ L\dot{x}^\alpha \dot{x}^\beta = 2\delta^\alpha_\beta, \quad L\dot{x}^\alpha x'^\beta = 0, \quad Lx'^\alpha x'^\beta = 2\delta^\alpha_\beta. \]  
(8.34)

Thus, the quadratic form $A$ is positive definite.

Example 3. Let $X_0 = J^1(R^2, R^m)$. We associate coordinates $\sigma, \tau$ to $R^2, x^i$, $1 \leq i \leq m$ to $R^m$, and $x'^i = \frac{\partial x^i}{\partial \sigma}, \dot{x}^i = \frac{\partial x^i}{\partial \tau}$. Let $X = X_0|_{g_1 = 0}$, where $g_1(\dot{x}^1, x^2) = \dot{x}^1 - x^2 = 0$. Let $N = B_1$ be a ball with radius 1 centered at $(0, 0)$. Then 

\[ x^1(t, b) - x^1(a, b) = \int_a^t \frac{\partial x^1}{\partial \tau} d\tau = \int_a^t x^2 d\tau, \]  
(8.35)

where $a \leq 0$ and $a^2 + b^2 = 1$.

Boundary condition $h_{A'}$. We have the following system for $v = F_*(\partial/\partial t)(t, x)|_{t=0}$ where $F$ is a one-parameter variation of $f$:

\[ \frac{\partial v_{x^1}}{\partial \tau} - v_{x^1} = 0, \]  
(8.36)

\[ \frac{\partial v_{x^1}}{\partial \sigma} - v_{x^1} = 0, \]  
(8.37)

\[ \frac{\partial v_{x^1}}{\partial \tau} - v_{x^2} = 0, \]  
(8.38)

\[ \frac{\partial v_{x^1}}{\partial \sigma} - v_{x^1} = 0. \]  
(8.39)

Let $A' = \{ (\tau, \sigma) \in R^2 | (\tau)^2 + (\sigma)^2 = 1 \quad \text{and} \quad \tau \leq 0 \}$. $A'$ is nowhere characteristic for (8.38) and the values of $v_{x^1}$ at $A'$ and $v_{x^2}$ in $N$ determine uniquely a solution in $N$ for the system of equations. Let $h_{A'}: A' \to R$ and $h_{\partial N}^j: \partial N \to R$ ($2 \leq j \leq m$) be a smooth function. Assume $f \in V(I^*, L^*)$, and let $I^*, L^*$ be as before. Then, $f$ satisfies the boundary condition $[h_{A'}]$ if 

\[ x_{A'}^1 = h_{A'}^1 \quad \text{and} \quad x_{\partial N}^j = h_{\partial N}^j. \]  
(8.40)
In this case,

\[ \phi[f] = \int f^* \varphi, \quad \text{where } f \in V(I^*, L^*, [h_A]), \quad (8.41) \]

and

\[ \varphi = Lw = [(x^i)^2 + \sum_j (\dot{x}^j)^2 + \sum_j (x^j)^2] d\sigma \land d\tau. \quad (8.42) \]

**Momentum space.** The Cartan system in \( Z \) is:

(i) \[ \partial/\partial \dot{\lambda}_i \Psi_Z = -\pi^*(dx_i - \dot{x}_i d\tau) \land \pi^* d\sigma = 0, \quad (8.43) \]

(ii) \[ \partial/\partial \dot{x}^j \Psi_Z = -\pi^*(2\dot{x}^j - \dot{\lambda}_j) \omega = 0, \quad (8.45) \]

(iii) \[ \partial/\partial x_i \Psi_Z = -\pi^*(d\dot{\lambda}_i \land \pi^* d\sigma + d\lambda_i \land \pi^* d\tau) = 0. \quad (8.47) \]

From (8.46) and (8.47) we also have

\[ Y = Z_1 = Z \vert_{2\dot{x}^i = \dot{\lambda}_i, 2x^i = \lambda'_i}. \]

This Cartan system \((C(\Psi), \pi^* \omega)\) is non-degenerate. Let us transfer the boundary condition to \( Q_i = Y \vert_{\pi^* L_i}, \) where \( L_i^* = \text{span}\{dx^i - \dot{x}^i d\tau - x^i d\sigma, d\sigma, d\tau\}. \) Then, \( f \in V(I^*, L^*) \) satisfies the boundary condition \( h_A', \) if for any lift \( g \) of \( f \) to \( Y \) we have:

\[ (\omega'_1 \circ g) \vert_{A'} = h^1_{A'} \quad \text{and} \quad (\omega'_j \circ g) \vert_{\partial N} = h^j_{\partial N}, \quad (8.48) \]

where \( h^1_{A'}: A' \rightarrow Q_1 \) with \( \pi_1 \circ h^1_{A'} = h^1_{A'} \) and the projection \( \pi_1: Q_1 \rightarrow R \) given by \( \pi_1(g) = x^i(q). \)

Furthermore, \( g \) is a solution to the Euler-Lagrange system satisfying the mixed boundary condition \( [h_{A'}], \) if \( g \) satisfies (8.43), (8.44) and (8.47), and

\[ \nu \cdot \dot{\lambda}_i \pi^*[dx^i - \dot{x}^i d\tau - x^i d\sigma] \land d\tau + \lambda'_i \pi^*[dx^i - \dot{x}^i d\tau - x^i d\sigma] \land d\sigma \vert_\partial(\partial N \setminus A') = 0 \quad (8.49) \]

for any element, \( v = F_*(\partial/\partial t)(t, x) \vert_{t=0} \) where \( F \) is a one parameter variation of \( \pi \circ g \) satisfying \( \nu_1 \vert_{A'=0} \) and \( \nu_2 \vert_{N=0}. \)

Finally, the quadratic form \( A \) is positive definite.
9. Inverse problem for calculus of variations

Example 4. In 1887, Helmholtz solved the following problem:

It is given \( P_i = P_i(x, u^i, u_x^i, u_{xx}^i) \). Is there a Lagrangian \( L(x, u^i, u_{xx}^i) \) such that \( E_i(L) = \partial L/\partial u^i - D_x \partial L/\partial u_x^i = P_i \), where \( D_x = \partial/\partial x + u_x^i \partial/\partial u^i + u_{xx}^i \partial/\partial u_{xx}^i \)? He found the following necessary conditions for \( P_i \):

(i) \[ \partial P_i / \partial u_{xx}^i = \partial P_j / \partial u_x^j, \] (9.1)

(ii) \[ \partial P_i / \partial u_x^i = \partial P_j / \partial u_x^j + 2D_x \partial P_j / \partial u_{xx}^j, \] (9.2)

(iii) \[ \partial P_i / \partial u^i = \partial P_j / \partial u^j - D_x \partial P_j / \partial u_x^j + D_{xx} \partial P_j / \partial u_{xx}^j. \] (9.3)

This problem led to the following studies ([2]):

(i) - the derivation and analysis of Helmholtz conditions as necessary and (locally) sufficient conditions for a differential operator to coincide with the Euler-Lagrange operator for some Lagrangian;

(ii) - the characterization of the obstructions to the existence of global variational principles for different operators defined on manifolds;

(iii) - the invariant inverse problem for different operators with symmetry;

and

(iv) - the variational multiplier problem wherein variational principles are found, not for a given differential operator, but rather for the differential equations determined by that operator.

That is: find a matrix \( B = [B^j_i] \) such that \( B^j_i P_j = E_i(L) \) for some \( L \) with \( B \) being non-singular.

Let \( E \to M \) be a fibered manifold. \( J^\infty(E) \) is the infinite jet of \( E \).

Let

\[ \theta^i = du^i - u_x^i dx \] (9.4)
\[ \theta_x^i = du_x^i - u_{xx}^i dx \] (9.5)

and

\[ \Omega_P = P_i \theta^i \wedge dx + 1/2 [\partial P_i / \partial u_x^i - D_x \partial P_i / \partial u_{xx}^i] \theta^i \wedge \theta_x^i \]
\[ + 1/2 [\partial P_i / \partial u_x^i + \partial P_j / \partial u_{xx}^i] \theta^i \wedge \theta_x^j. \] (9.6)

If \( P \) satisfies the Helmholtz conditions, then \( d\Omega_P = 0 \). If the \( H^{n+1}(E) - n+1 \) de Rham cohomology group of \( E \) is trivial, then \( \Omega_P \) is exact. This fact implies that \( P_i \) is globally variational. If \( \theta_L = L dx + \partial L / \partial u_x^i \theta^i \),

Let of one independent variable and to equations

\[ \text{Theorem 9.1.} \]

Then we have a global solution for the inverse problem in the case of one independent variable and to equations \( P_1 = 0 \) of second order.

Vaingberg [1969] generalized this result; however his Lagrangian is usually of a much higher order than necessary.

In [2] we can find the following theorem.

**Theorem 9.1.** Let \( P_1 \) be a differential operator of order \( 2k \)

\[ P_1 = P_1(x, u^1, u^1_1, \ldots, u^1_{2k}). \]

Then \( P_1 \) is the Euler-Lagrange operator of a \( k \)-th order Lagrangian \( L = L(x, u^1, u^1_1, \ldots, u^1_{2k}) \) if and only if the functions \( P_1 \) satisfy the higher order Helmholtz conditions, and the functions

\[ p_1(t) = P_1(x, u^1, u^1_1, \ldots, u^1_{2k}) \]

are polynomials in \( t \) of degree less or equal to \( k \).

**Example 5.** Let us now look to another example where we have a function of three independent variables \( x, y \) and \( z \), with a single dependent variable \( u \). Let \( T = T(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, u_{xz}, \ldots, u_{zz}) \) be a second order operator.

\[ E[L] = \partial L/\partial u - D_x\partial L/\partial u_x - D_y\partial L/\partial u_y - D_z\partial L/\partial u_z \]

Let \( \psi = \psi^* \omega + (\psi^* \chi)^*[i^*(\chi)] \) and \( \pi^* \omega_j \) by \( v \) is

\[ v_\pi d\psi + d(v_\pi \psi) = E[L](u)\pi^*(dx \wedge dy \wedge dz) \]

\[ + d(\partial L/\partial u_x v^1 \pi^*(dy \wedge dz) - \partial L/\partial u_y v^1 \pi^*(dx \wedge dz) + \partial L/\partial u_z v^1 \pi^*(dx \wedge dy)). \]

Suppose that for some vector \( w \) with \( \pi^*w \in T \mathcal{V}(T^*, L^*, \varphi, [h]) \) (i.e. \( w_\pi d\theta + d(w_\pi \theta) \) for \( \theta = du - u_x dx - u_y dy - u_z dz \) and \( w_\pi \theta |_{\partial N} = 0 \) we have \( v_\pi d\psi + d(v_\pi \psi) = T[u]v^1 \pi^*(dx \wedge dy \wedge dz) + d(\partial L/\partial u_x w^1 \pi^*(dy \wedge dz) - \partial L/\partial u_y w^1 \pi^*(dx \wedge dz) + \partial L/\partial u_z w^1 \pi^*(dx \wedge dy)). \]

Then we have \( T[u] = E[L](u) \)

If we identify \( e_1 \) with \( dy \wedge dz \), \( e_2 \) with \( dz \wedge dx \) and \( e_3 \) with \( dx \wedge dy \) at each point of the integral manifold of \( (C(\psi), \pi^*\omega) \), we can write

\[ d(\partial L/\partial u_x v^1 \pi^*(dy \wedge dz) - \partial L/\partial u_y v^1 \pi^*(dx \wedge dz) \]
We have
\[ V[u] = \partial L/\partial u_x v^1 + \partial L/\partial u_y v^2 + \partial L/\partial u_z v^3. \]  
(9.15)

The divergence operator is defined in terms of the total derivatives \( D_x, D_y \) and \( D_z \).

We can conclude that \( v \cdot d\psi + d(v \cdot \psi) = (E[L](u) v + \text{Div}V[u]) \pi^*(dx \wedge dy \wedge dz) \).

We have
\[ E[L](u) = 0 \] whenever \( L[u] = \text{Div}W[u] \).

Suppose \( T[u] = E[L](u) \). Then the first variation formula is
\[ v \cdot d\psi + d(v \cdot \psi) = (T[u] v^1 + \text{Div}W[u]) \pi^*(dx \wedge dy \wedge dz). \]  
(9.17)

By applying the Euler-Lagrange operator (i.e. \( E[\alpha[u] \pi^*(dx \wedge dy \wedge dz)] = E[\alpha[u] \pi^*(dx \wedge dy \wedge dz)] \)), we obtain
\[ E[v \cdot d\psi + d(v \cdot \psi)] = E[T[u] v] \pi^*(dx \wedge dy \wedge dz), \] since \( E(\text{Div}W)(u) = 0 \).

We have
\[ E[v \cdot d\psi + d(v \cdot \psi)] = (v \cdot dE[L](u) + d(v \cdot dE[L](u))) \pi^*(dx \wedge dy \wedge dz) \]  
(9.19)
\[ = (v \cdot dT + d(v \cdot dT)) \pi^*(dx \wedge dy \wedge dz). \]  
(9.20)

Therefore
\[ E[T[u] v] \pi^*(dx \wedge dy \wedge dz) = (v \cdot dT + d(v \cdot dT)) \pi^*(dx \wedge dy \wedge dz). \]  
(9.21)

Let
\[ \psi' = \pi^* T \omega + (\pi^* \alpha')^* [\pi^* (\chi)] \pi^* \omega_j, \]  
(9.22)
and
\[ v \cdot d\psi' + d(v \cdot \psi') = E[T[u] v] \pi^*(dx \wedge dy \wedge dz). \]  
(9.23)

If we define
\[ H[T[v]] \pi^*(dx \wedge dy \wedge dz) = v \cdot d\psi' + d(v \cdot \psi') - E[T(u) v] \pi^*(dx \wedge dy \wedge dz), \]  
then \( H(T) = 0 \) if \( T[u] \) is Euler-Lagrange. Helmholtz equations are:

(i) \[ \partial T/\partial u_x = D_x \partial T/\partial u_{xx} + 1/2D_y \partial T/\partial u_{xy} + 1/2D_z \partial T/\partial u_{xz}, \]  
(9.25)

(ii) \[ \partial T/\partial u_y = D_y \partial T/\partial u_{yy} + 1/2D_x \partial T/\partial u_{yx} + 1/2D_z \partial T/\partial u_{yz}, \]  
(9.26)

(iii) \[ \partial T/\partial u_z = D_z \partial T/\partial u_{zz} + 1/2D_x \partial T/\partial u_{zx} + 1/2D_y \partial T/\partial u_{zy}. \]  
(9.27)
We have a sequence of spaces

\[
0 \rightarrow \mathbb{R} \rightarrow F[u] \rightarrow V(u) \rightarrow F(u) \rightarrow F(u) \rightarrow V(u)
\]

that is a cochain complex, the Euler-Lagrange complex. Since this complex is exact, the inverse problem is globally solved in this second example.

9.1. Variational Bicomplex. Let us introduce now a very important tool for a globalization of the inverse problem.

**Definition 9.1.** A \( p \) form \( \omega \) on \( J^\infty(E) \) is said to be of type \((r,s)\), where \( r + s = p \), if at each point \( x \) of \( J^\infty(E) \)

\[
\omega(X_1, X_2, \ldots, X_p) = 0,
\]

whenever either

(i) more than \( s \) of the vectors \( X_1, X_2, \ldots, X_p \) are \( \pi^\infty_M \) vertical, or

(ii) more than \( r \) of the vectors \( X_1, X_2, \ldots, X_p \) annihilate all contact one forms.

Note: \( \Omega^{r,s} \) denotes the space of type \((r,s)\) forms on \( J^\infty(E) \).

(i) \( \pi : E \rightarrow M \) be a fiber bundle.

(ii) Let us assume that there exists a transformation group \( G \) acting on \( E \), and

(iii) that there exists a set of differential equations on sections of \( E \).

\[
d = d_H + d_V,
\]

\[
d_H : \Omega^{r,s}(J^\infty(E)) \rightarrow \Omega^{r+1,s}(J^\infty(E)), \quad (9.30)
\]

\[
d_V : \Omega^{r,s}(J^\infty(E)) \rightarrow \Omega^{r,s+1}(J^\infty(E)), \quad (9.31)
\]

\[
d_H^2 = 0, \quad d_H d_V = - d_V d_H, \quad d_V^2 = 0. \quad (9.32)
\]

In local coordinates

\[
d_H f = [\partial f / \partial x^i + u\alpha_i \partial f / \partial u^\alpha + u^\alpha_j \partial f / \partial u^\alpha_j + \ldots] dx^i \quad (9.33)
\]

\[
d_V f = \partial f / \partial u^\alpha \theta^\alpha + \partial f / \partial u^\alpha \theta^\alpha + \ldots \quad (9.34)
\]

The sequences of spaces

\[
0 \rightarrow \Omega^{0,0} \rightarrow \Omega^{1,0} \rightarrow \Omega^{2,0} \rightarrow \Omega^{3,0} \rightarrow \ldots
\]

\[
\uparrow d_V \quad \uparrow d_V \quad \uparrow d_V \quad \uparrow d_V \quad \uparrow d_V \quad \uparrow d_V
\]

\[
0 \rightarrow \Omega^{0,1} \rightarrow \Omega^{1,1} \rightarrow \Omega^{2,1} \rightarrow \Omega^{3,1} \rightarrow \ldots
\]

\[
\uparrow d_V \quad \uparrow d_V \quad \uparrow d_V \quad \uparrow d_V \quad \uparrow d_V \quad \uparrow d_V
\]

\[
0 \rightarrow R \rightarrow \Omega^{0,0} \rightarrow \Omega^{1,0} \rightarrow \Omega^{2,0} \rightarrow \Omega^{3,0} \rightarrow \ldots
\]

\[
\uparrow d_V \quad \uparrow d_V \quad \uparrow d_V \quad \uparrow d_V \quad \uparrow d_V \quad \uparrow d_V
\]
is the Variational Bicomplex.

Therefore the generalization of (9.28) is:

\[
0 \rightarrow R \rightarrow \Omega^{0,0} \rightarrow \Omega^{1,0} \rightarrow \Omega^{2,0} ... \rightarrow \Omega^{n-1,0} \rightarrow \Omega^{n,0} \rightarrow E \delta \nu \delta \nu \rightarrow F^1 \rightarrow F^2 \rightarrow F^3.
\]

9.2. Lagrange problem with non-holonomic constraints. Let us recall from [26] the Lagrange problem with non-holonomic constraints. We showed that a well-posed variational problem with mixed endpoint conditions for \( n = 1 \) is locally a Lagrange problem with non-holonomic constraints.

**Proposition 9.1.** Let us assume that a Lagrange problem with non-holonomic constraints \( g^\sigma(x, u, \dot{u}) = 0 \), with \( \text{rank}[\partial \theta^\sigma/\partial \dot{u}^\alpha] = m - l \) with \( 1 \leq j \leq m \) and \( 1 \leq \rho \leq m - l, l \geq 0 \) is given. If \( \det[L_{\mu\nu}] \neq 0 \) and \( L \det[A_{\mu\nu}] \neq 0 \) for all \( (\lambda_1, \ldots, \lambda_{m-1}) \in \mathbb{R}^{m-1} \), then \( (I^*, L^*, \varphi, L^*) \) is a well-posed valued differential system, where \( I^* = \text{span} \{\theta^\alpha|1 \leq \alpha \leq m\} \), and \( L^* = \text{span} \{\theta^\alpha, dx|1 \leq \alpha \leq m\} \)

\[
\theta^\mu = d\theta^\mu \leq u^\mu\sigma dx + g^\mu_{\sigma\lambda} (u^\mu - u^\mu_0 dx) \quad 1 \leq \sigma \leq m - l, \quad (9.35)
\]

\[
\theta^\alpha = u^\alpha dx \quad m - l + 1 \leq \mu, \nu \leq m. \quad (9.36)
\]

In this setting we have

\[
\theta^\mu = -du^\mu_0 \wedge dx, \quad (9.37)
\]

\[
d\theta^\nu \equiv -A^\nu_{\mu\rho} du^\mu_0 \wedge \theta^\rho - B^\nu_{\sigma\alpha} dx \wedge \theta^\alpha \mod\{\theta^\alpha \wedge \theta^\alpha | 1 \leq \alpha, \alpha' \leq m\}, \quad (9.38)
\]

\[
A^\rho_{\mu\rho'} = g_{u^\mu_0 u^\rho_0} a^\rho_{\mu'} a^\rho_{\rho'} g^\mu_{\rho\rho'} + g_{u^\mu_0 u^\rho_0} a^\rho_{\rho'}, \quad (9.39)
\]

\[
A^\nu_{\mu\nu'} = g_{u^\mu_0 u^\nu_0} a^\nu_{\mu'} a^\nu_{\nu'} \beta^\rho_{\mu\rho'} g^\rho_{\nu\nu'} - g_{u^\mu_0 u^\nu_0} a^\nu_{\mu'} a^\nu_{\nu'} \beta^\rho_{\nu\rho'} g^\rho_{\mu\mu'} + g_{u^\mu_0 u^\nu_0} a^\nu_{\nu'}, \quad (9.40)
\]

\[
B^\rho_{\sigma} = g_{u^\rho_0 u^\rho_0} a^\rho_{\sigma} + g_{u^\rho_0 u^\rho_0} g^\rho_{\sigma} (g^\rho_{\rho_0} - g^\rho_{u^\rho_0 u^\rho_0}) + g_{u^\rho_0 u^\rho_0} a^\rho_{\sigma} u^\rho_0 - g_{u^\rho_0 u^\rho_0} a^\rho_{\sigma} x^\rho_0 + g_{u^\rho_0 u^\rho_0} a^\rho_{\sigma} u^\rho_0, \quad (9.41)
\]

\[
B^\rho_{\mu} = -g_{u^\rho_0 u^\rho_0} a^\rho_{\mu} g^\rho_{u^\rho_0 u^\rho_0} g^\rho_{\mu} (g^\rho_{\rho_0} - g^\rho_{u^\rho_0 u^\rho_0}) - g_{u^\rho_0 u^\rho_0} a^\rho_{\mu} g^\rho_{u^\rho_0 u^\rho_0} u^\rho_0 x^\rho_0 + g_{u^\rho_0 u^\rho_0} a^\rho_{\mu} g^\rho_{u^\rho_0 u^\rho_0} u^\rho_0 + g_{u^\rho_0 u^\rho_0} a^\rho_{\mu} u^\rho_0.
\]
\[-g^\rho_\sigma \alpha^\rho_{\omega^\tau} \gamma^\tau_{\omega^\rho} + g^\rho_\sigma \alpha^\rho_{\omega^\tau} \alpha^\rho_{\omega^\rho} (g^\rho_\sigma - g^\rho_\omega \alpha^\rho_{\omega^\rho}) + g^\rho_\sigma \alpha^\rho_{\omega^\rho} \alpha^\rho_{\omega^\rho} - g^\rho_\omega \gamma^\tau_{\omega^\rho} \alpha^\rho_{\omega^\rho} \gamma^\tau_{\omega^\rho} \]

\[L_\mu = \frac{\partial}{\partial u^\rho_\mu} - a^\rho_{\omega^\rho} g^\rho_\omega \frac{\partial}{\partial u^\rho_\omega} L, \quad L_{\mu \nu} = \frac{\partial}{\partial u^\rho_\mu} - a^\rho_{\omega^\rho} g^\rho_\omega \frac{\partial}{\partial u^\rho_\omega} L_{\mu}, \]

and

\[A_{\mu \nu} (\lambda_1, \ldots, \lambda_{m-l}) = \rho \in \rho, \rho', \rho'', \sigma, \sigma' \leq m - l \text{ and } m - l + 1 \leq \mu, \nu \leq m.\]

\[\psi \equiv (L_\mu - \lambda_\mu) \pi^*(du^\mu \wedge dx) + (d\lambda_\mu - (A_\mu + \lambda_\mu B_\mu) \pi^* dx + \lambda_\mu A_\mu \pi^* du^\mu) \wedge \pi^* \theta^\mu + (d\lambda_\sigma - (A_\sigma + \lambda_\sigma B_\sigma) \pi^* dx + \lambda_\sigma A_\sigma \pi^* du^\mu) \wedge \pi^* \theta^\sigma \]

\[\mod \{\pi^*(\theta^\rho \wedge \theta^\sigma)|1 \leq \alpha, \alpha' \leq m\}, \]

with

\[A_\mu = L_{\alpha} - \omega_{\omega^\rho} a^\rho_{\omega^\rho} g^\rho_\omega + L_{\omega^\rho} a^\rho_{\omega^\rho} g^\rho_\omega + g^\rho_\omega a^\rho_{\omega^\rho} g^\rho_\omega - L_{\omega^\rho} a^\rho_{\omega^\rho} g^\rho_\omega \]

\[A_\sigma = L_{\alpha} \sigma^\rho_{\omega^\rho} - L_{\omega^\rho} a^\rho_{\omega^\rho} g^\rho_\omega a^\rho_{\omega^\rho}. \]

The Cartan system is

\[\pi^* \theta^\alpha \quad (1 \leq \alpha \leq m), \]

\[(L_\mu - \lambda_\mu) \pi^* dx \quad (m - l + 1 \leq \mu \leq m), \]

\[(d\lambda_\mu - (A_\mu + \lambda_\mu B_\mu) \pi^* dx + \lambda_\mu A_\mu \pi^* du^\mu) \quad (m - l + 1 \leq \mu \leq m), \]

\[(d\lambda_\sigma - (A_\sigma + \lambda_\sigma B_\sigma) \pi^* dx + \lambda_\sigma A_\sigma \pi^* du^\mu) \quad (1 \leq \sigma \leq m - l). \]
Proposition 9.2. Let \((I^*, L^*)\) be a locally embeddable differential system defined in \(X = J^1(R, R^m)|g^\alpha(x, w, u^\beta) = 0\), rank \(\partial g^\alpha/\partial u^\beta_i = m - l, 1 \leq j \leq m\) and \(1 \leq \rho \leq m - l, l \geq 0\), where \(I^* = \text{span} \{\theta^\alpha|1 \leq \alpha \leq m\}\) and \(L^* = \text{span} \{\theta^\alpha, dx|1 \leq \alpha \leq m\}\).

\[
\theta^\alpha = g^\alpha_{\nu} (du^\nu - u^\nu_x dx) + g^\alpha_{\nu\mu} (du^\mu - u^\mu_x dx)
\]

\[1 \leq \sigma, \rho \leq m - l,
\]

(9.54)

\[
\theta^\mu = du^\mu - u^\mu_x dx
\]

\[m - l + 1 \leq \mu, \nu \leq m.
\]

(9.55)

Let \(Q_i = Q_i(x, w^j, u^\mu_j, u^\nu_x, \lambda_p \lambda_{\rho_x}), 1 \leq i \leq m\), with \(Q_i(x, w^j, u^\mu_j, tu^\mu, \lambda_p \lambda_{\rho_x})\) being polynomial in \(t\) of degree less or equal to 1, and

\[
P^\mu = Q^\mu + \lambda_p B^\mu_p - \lambda_p A^\mu_p \frac{du^\nu}{dx},
\]

(9.56)

\[
R^\sigma = Q^\sigma - \lambda_{\sigma x} + \lambda_p B^\sigma_p - \lambda_p A^\sigma_p \frac{du^\nu}{dx},
\]

(9.57)

and

\[
R^\mu = P^\mu + D_x (\partial P^\mu/\partial u^\nu_{xx}).
\]

(9.58)

Furthermore, let us assume that the functions \(P^\mu\) satisfy the Helmholtz conditions, that the functions \(R^\mu\) do not depend on \(\lambda_p\) and \((\lambda_p)_x\) coordinates, and the 1-form \(\Theta = R_t(x, w^j, u^\mu_j, u^\nu_x)\theta^\alpha\) is closed mod \(R\), where \(R = C^\infty(Z, R^*)\), \(Z = J^1(R, R^m)|g^\alpha(x, w^j, u^\mu_j) = 0\) with coordinates \(\{x, w^j, u^\mu_j, u^\nu_x\}\) and \(R^* = \text{span} \{dx, du^\mu, du^\nu_x\}\). Then, \(Q_i\) is locally a Euler-Lagrange operator for a Lagrangian \(L(x, w^j, u^\mu_j)\).

Proof: From Theorem 9.1 we know that a function \(F(x, w^j, u^\mu_j)\) can be found that does not depend on \(u^\nu_{xx}\), such that \(E^\mu(F) = \partial F/\partial u^\mu - D_x \partial F/\partial u^\nu_{xx} = F^\mu\) (note that if \(R^\mu\) does not depend on \(\lambda_p\), then neither does \(P^\mu\)).

Therefore,

\[
\partial P^\mu/\partial u^\nu_{xx} = F^\mu,
\]

(9.59)

where

\[
F^\mu_{\nu\nu} = \partial /\partial u^\mu_x - a^\sigma_p g^\rho_{\nu\sigma} \partial /\partial u^\rho_x F^\nu,
\]

(9.60)

and

\[
F^\mu = \partial /\partial u^\mu_x - a^\sigma_p g^\rho_{\nu\sigma} \partial /\partial u^\rho_x F^\nu.
\]

(9.61)

The \(R^\mu\) functions satisfy

\[
R^\mu = \partial /\partial u^\mu - a^\sigma_p g^\rho_{\nu\sigma} \partial /\partial u^\rho_x F^\nu.
\]

(9.62)

Hence, if the \(\Theta\)-form is closed mod \(R\), then locally

\[
R^\sigma = \partial /\partial u^\sigma - a^\rho_p g^\nu_{\sigma\rho} \partial /\partial u^\nu_x F^\mu.
\]

(9.63)

Finally, we make \(F = L\).
In addition, if the domain of the $R_n$ functions is simply connected and

\[
\Omega_P = P\mu \theta^\alpha \wedge dx + 1/2[\partial P_\mu/\partial u^i_x - D_x \partial P_\mu/\partial u_x] \theta^\alpha \wedge \theta^i 
+ 1/2[\partial P_\mu/\partial u_x] + \partial P_\mu/\partial u_x] \theta^\alpha \wedge \theta^i.
\] (9.64)

is exact, then we have a global solution for the inverse problem.

**Example 6.** Let $X$ be the $J^3(R, R^3) g(v, y, z, v_x, y_x, z_x) = 0$, where

\[
g(v, y, z, v_x, y_x, z_x) = mvv_x - mgz_x + R\sqrt{1 + (y_x)^2 + (z_x)^2}.
\] (9.65)

Let

\[
Q_1 = -\lambda_{y_x} - \frac{\sqrt{1 + (y_x)^2 + (z_x)^2}}{mv^3} = 0,
\] (9.66)

and

\[
Q_2 = \frac{Ry_x}{mv^3} - \frac{v(1 + z_x^2)v_{xx} - y_xz_xz_{xx} - v_x^2y_x\sqrt{1 + (y_x)^2 + (z_x)^2}}{v^2(\sqrt{1 + (y_x)^2 + (z_x)^2})^3}
- \lambda_1 \left( \frac{R(1 + z_x^2)v_{xx}}{\sqrt{1 + (y_x)^2 + (z_x)^2}} \right)
- \frac{Rz_x y_x z_{xx}}{\sqrt{1 + (y_x)^2 + (z_x)^2}}
- \frac{v(1 + y_x^2)z_{xx} - y_xz_x v_{xx} - v_x z_x \sqrt{1 + (y_x)^2 + (z_x)^2}}{v^2(\sqrt{1 + (y_x)^2 + (z_x)^2})^3}
- \lambda_1 \left( \frac{R(1 + y_x^2)z_{xx}}{\sqrt{1 + (y_x)^2 + (z_x)^2}} \right) + \frac{Rz_x y_x y_{xx}}{\sqrt{1 + (y_x)^2 + (z_x)^2}} = 0.
\] (9.67)

Hence,

\[
P_2 = -\frac{Ry_x}{mv^3} - \frac{v(1 + z_x^2)v_{xx} - y_xz_xz_{xx} - v_x^2y_x\sqrt{1 + (y_x)^2 + (z_x)^2}}{v^2(\sqrt{1 + (y_x)^2 + (z_x)^2})^3},
\] (9.69)

\[
P_3 = -\lambda_1 \left( \frac{Rz_x y_x z_{xx}}{\sqrt{1 + (y_x)^2 + (z_x)^2}} \right) + \frac{v(1 + y_x^2)z_{xx} - y_xz_x v_{xx} - v_x z_x \sqrt{1 + (y_x)^2 + (z_x)^2}}{v^2(\sqrt{1 + (y_x)^2 + (z_x)^2})^3}.
\] (9.70)
\[
R_1 = -\frac{\sqrt{1 + (y_x)^2 + (z_x)^2}}{mv^3}, \quad (9.71)
\]
\[
R_2 = -\frac{R_{yx}}{mv^3\sqrt{1 + (y_x)^2 + (z_x)^2}}, \quad (9.72)
\]
\[
R_3 = -\frac{\sqrt{1 + (y_x)^2 + (z_x)^2}}{mv^3} \left( mg - \frac{R_{zx}}{\sqrt{1 + (y_x)^2 + (z_x)^2}} \right). \quad (9.73)
\]

It is easy to verify that \( P_2 \) and \( P_3 \) satisfy Helmholtz conditions, and that the 1-form \( \Theta = R_1 \theta^1 + R_2 \theta^2 + R_3 \theta^3 \) is closed mod \( R \), with \( R^* = \text{span} \{ dx, dy_x, dz_x \} \) and \( R = C^\infty(X, R^*) \). The Lagrangian for this example is
\[
L = \frac{\sqrt{1 + (y_x)^2 + (z_x)^2}}{v}. \]

References

Inverse problem of variational calculus and problem of mixed endpoint conditions


[38] —, Applications of Lie groups to differential equations, Springer-Verlag, New York, 1986.