Khovanskii's Theorem

Ricardo Bianconi

IME-USP

September, 2019

Definition (Pfaffian Chains)

A finite sequence of C^1 functions (they are actually real analytic), $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ is a *Pfaffian chain* if there are real polynomials $P_{i,j}(X_1, \ldots, X_n, Y_1, \ldots, Y_j)$, $1 \leq j \leq k$, such that

$$\frac{\partial f_j}{\partial x_i} = P_{i,j}(\boldsymbol{x}, f_1, \ldots, f_j),$$

for all $1 \leq i \leq n$ and $1 \leq j \leq k$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$.

Definition (Pfaffian Chains)

A finite sequence of C^1 functions (they are actually real analytic), $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ is a *Pfaffian chain* if there are real polynomials $P_{i,j}(X_1, \ldots, X_n, Y_1, \ldots, Y_j)$, $1 \leq j \leq k$, such that

$$\frac{\partial f_j}{\partial x_i} = P_{i,j}(\boldsymbol{x}, f_1, \ldots, f_j),$$

for all $1 \leq i \leq n$ and $1 \leq j \leq k$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$.

Note that these equations can be written as

- $dy_1 = \sum_{i=1}^n P_{i1}(\boldsymbol{x}, y_1) dx_i$
- $dy_2 = \sum_{i=1}^n P_{i2}(\boldsymbol{x}, y_1, y_2) dx_i$
- $\vdots \qquad\qquad \vdots$
- $dy_k = \sum_{i=1}^n P_{ik}(\boldsymbol{x}, y_1, \ldots, y_k) dx_i.$

Definition (Pfaffian Functions)

Given a Pfaffian chain $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$, a *Pfaffian function* is a function of the form $g(\boldsymbol{x}) = Q(\boldsymbol{x}, f_1, \ldots, f_k)$.

It is easy to see that a Pfaffian function also belongs to a Pfaffian chain, but this definition is easier to use.

Definition (Pfaffian Functions)

Given a Pfaffian chain $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$, a *Pfaffian function* is a function of the form $g(\boldsymbol{x}) = Q(\boldsymbol{x}, f_1, \ldots, f_k)$.

It is easy to see that a Pfaffian function also belongs to a Pfaffian chain, but this definition is easier to use.

Definition (Their Complexity)

The *complexity* of g is the tuple containing the degrees of Q, $P_{i,j}$'s, and the number of variables.

Example

1. any rational function is Pfaffian: if $f = p/q$, a Pfaffian chain for f is $f_1 = 1/q$ and $f_2 = f$: $f_1' = -q' f_1^2$, $f_2' = (p'q - pq')f_1^2$;
2. $f(x) = \exp x$: $f' = f$;
3. $f_1(x) = (1+x)^{-1}$, $f_2(x) = \arctan x$: $f_1' = 2xf_1^2$, $f_2' = f_1$.

Example

1. any rational function is Pfaffian: if $f = p/q$, a Pfaffian chain for f is $f_1 = 1/q$ and $f_2 = f$: $f_1' = -q' f_1^2$, $f_2' = (p'q - pq')f_1^2$;
2. $f(x) = \exp x$: $f' = f$;
3. $f_1(x) = (1+x)^{-1}$, $f_2(x) = \arctan x$: $f_1' = 2xf_1^2$, $f_2' = f_1$.

Example

Some more elaborate examples come from the real and imaginary parts of elliptic integrals.

Example

1. any rational function is Pfaffian: if $f = p/q$, a Pfaffian chain for f is $f_1 = 1/q$ and $f_2 = f$: $f_1' = -q' f_1^2$, $f_2' = (p'q - pq')f_1^2$;
2. $f(x) = \exp x$: $f' = f$;
3. $f_1(x) = (1+x)^{-1}$, $f_2(x) = \arctan x$: $f_1' = 2xf_1^2$, $f_2' = f_1$.

Example

Some more elaborate examples come from the real and imaginary parts of elliptic integrals.

Example

The composition of Pfaffian functions is a Pfaffian function, for instance: $f = \exp\arctan x$, $f' = f/(1+x^2)$.

Theorem (Khovanskii, 1980)

Let $F_1, \dots, F_m : \mathbb{R}^n \to \mathbb{R}$ be Pfaffian functions (with $m \leq n$), obtained from a Pfaffian chain $g_1, \dots, g_k$, $F_j(\boldsymbol{x}) = Q_j(\boldsymbol{x}, \boldsymbol{g})$. The sum of the Betti numbers of the zero set of $F = (F_1, \dots, F_m)$ is bounded by a (computable) function on the complexity of $F_1, \dots, F_n$.

Theorem (Khovanskii, 1980)

Let $F_1, \ldots, F_m : \mathbb{R}^n \to \mathbb{R}$ be Pfaffian functions (with $m \leq n$), obtained from a Pfaffian chain $g_1, \ldots, g_k$, $F_j(\boldsymbol{x}) = Q_j(\boldsymbol{x}, \boldsymbol{g})$. The sum of the Betti numbers of the zero set of $F = (F_1, \ldots, F_m)$ is bounded by a (computable) function on the complexity of F_1, ..., F_n.

Actually, the bound is exponential on the numbers appearing in their complexity.

Theorem (Khovanskii, 1980)

Let $F_1, \dots, F_m : \mathbb{R}^n \to \mathbb{R}$ be Pfaffian functions (with $m \leq n$), obtained from a Pfaffian chain $g_1, \dots, g_k$, $F_j(\boldsymbol{x}) = Q_j(\boldsymbol{x}, \boldsymbol{g})$. The sum of the Betti numbers of the zero set of $F = (F_1, \dots, F_m)$ is bounded by a (computable) function on the complexity of $F_1, \dots, F_n$.

Actually, the bound is exponential on the numbers appearing in their complexity.

Example

Note that this theorem rules out sin and cos as Pfaffian functions. But if we drop out the restrictions on the domains to be the whole $\mathbb{R}$, then $f = \tan x$ is Pfaffian in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$: $f' = 1 + f^2$. Then both sine and cossine can be made into Pfaffian functions in this interval because

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}, \text{ and } \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}.$$

Proof.

The idea of the proof is to reduce the Theorem to the case of $m = n$ via Morse Theory and then compute a bound to the number of non singular zeros of $(F_1, \ldots, F_n)$. A simple Morse function is included as a Pfaffian function.

To find such a bound, we proceed by induction on the length of the corresponding Pfaffian chain, reducing the chain by increasing the number of functions and of variables until we end with a system of polynomial equations.

Finally, Bézout Theorem then gives the desired bound (which is the product of the total degrees of those polynomials). □

Example

Some applications:

1. Real Algebraic Geometry: number of zeros of polynomial equations depending on the number of monomials but not the degree (Khovanskii).

Example

Some applications:

1. Real Algebraic Geometry: number of zeros of polynomial equations depending on the number of monomials but not the degree (Khovanskii).
2. Dynamical Systems: Dulac's Problem (Moussu-Roche).

Example

Some applications:

1. Real Algebraic Geometry: number of zeros of polynomial equations depending on the number of monomials but not the degree (Khovanskii).
2. Dynamical Systems: Dulac's Problem (Moussu-Roche).
3. Model Theory: o-minimality (Wilkie, Speissegger,etc).

Example

Some applications:

1. Real Algebraic Geometry: number of zeros of polynomial equations depending on the number of monomials but not the degree (Khovanskii).
2. Dynamical Systems: Dulac's Problem (Moussu-Roche).
3. Model Theory: o-minimality (Wilkie, Speissegger,etc).

Example (Noetherian Functions)

Noetherian functions compose a class containing the Pfaffian functions. Noetherian chains satisfy $\dfrac{\partial f_j}{\partial x_i} = P_{i,j}(\boldsymbol{x}, f_1, \ldots, f_k)$, (no restriction to depend only on previous functions). **Open Problem:** find a bound to the number of connected components of their zero sets (restricted to a convenient region). Because of Cauchy-Riemann equations, some harmonic functions are Noetherian.

- A. Gabrielov. N. Vorobjov. Complexity of Computations with Pfaffian and Noetherian Functions. In: *Normal Forms, Bifurcations and Finiteness Problems in Differential Equations*, 211-250, Kluwer, 2004.
- A. G. Hovanskii. On a Class of Systems of Transcendental Equations. *Soviet Math. Dokl.*, 22, 762–765 (1980).
- A. G. Khovanskii, *Fewnomials*. Translations of Mathematical Monographs, vol. 88 (American Mathematical Society, Providence, 1991).
- D. Marker, Khovanskii's theorem. *Algebraic model theory* (Toronto, ON, 1996), 181–193, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 496, Kluwer Acad. Publ., Dordrecht, 1997.
- R. Moussu, C. Roche, Théorie de Hovanskii et problème de Dulac. *Invent. Math.* 105, 431441 (1991).
- J.-J. Risler. Complexité et géométrie réelle (d'après A. Khovansky). Seminar Bourbaki, Vol. 1984/85. Astérisque No. 133-134 (1986), 89–100.

P. Speissegger, The Pfaffian closure of an o-minimal structure. *J. Reine Angew. Math.* 508, 189211 (1999).

A.J. Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *J. Am. Math. Soc.* 9, 10511094 (1996).