Complemented copies of $c_0(\tau)$ in tensor products of Banach spaces

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Let X be a Banach space. A subspace Y of X is *complemented* in X if there exists a projection P from X onto Y, that is, a bounded linear operator $P : X \to X$ such that $P(X) = Y$ and $P \circ P = P$.

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Let X and Y be Banach spaces.

- We say that X *contains a copy of* Y, and write $Y \hookrightarrow X$, if X contains a subspace isomorphic to Y.
- We say that X *contains a complemented copy of* Y, and write $Y \overset{c}{\hookrightarrow} X$, if X contains a complemented subspace isomorphic to Y.

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We are interested in (complemented) copies of the spaces $c_0(I)$, where I is an infinite set. Recall that $c_0(I)$ is the Banach space of all families of scalars $(\alpha_i)_{i\in I}$ such that, for every $\varepsilon > 0$, the set $\{i \in I : |\alpha_i| \geq \varepsilon\}$ is finite, equipped with the supremum norm. When $I = \mathbb{N}$, this space will be denoted simply by c_0 (the classical space of all scalar sequences that converge to zero).

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Since $c_0(I)$ and $c_0(J)$ are isomorphic whenever I and J have the same cardinality, it is enough to consider the spaces $c_0(\tau)$, where τ is an infinite cardinal.

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This is an active field of research in Banach space theory, with roots in Bessaga-Pełczyński's fundamental work in the fifties.

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A family $(x_i)_{i \in I}$ in a Banach space X is *unconditionally summable* if for every $\varepsilon > 0$ there exists a finite subset F of I such that

$$\left\| \sum_{i \in G} x_i \right\| < \varepsilon,$$

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A family $(x_i^*)_{i\in I}$ in X^* is *weak*-null* if $(x_i^*(x))_{i\in I}$ belongs to $c_0(I)$ for every $x \in X$.

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A family $(x_i)_{i\in I}$ is *equivalent* to the canonical basis of $c_0(I)$ if there exist constants $0 < \delta \leq M$ such that

$$\delta \sup_{i\in F} |\alpha_i| \leq \left\| \sum_{i\in F} \alpha_i x_i \right\| \leq M \sup_{i\in F} |\alpha_i|,$$

for each finite subset F of I and all families of scalars $(\alpha_i)_{i\in F}$.

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Equivalently, $(x_i)_{i\in I}$ is equivalent to the canonical basis of $c_0(I)$ if there exists $T : c_0(I) \to X$ such that $T(e_i) = x_i$ for every $i \in I$ and T is an isomorphism onto its image.

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Let X be Banach space. Then $c_0 \hookrightarrow X$ if, and only if, there exists a sequence $(x_n)_{n\geq 1}$ in X that is weakly unconditionally summable but not unconditionally summable.

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Theorem (Schlumprecht, 1988)

Let X be Banach space. Then $c_0 \overset{c}{\hookrightarrow} X$ if, and only if, there exist sequences $(x_n)_{n\geq 1}$ in X and $(x_n^*)_{n\geq 1}$ in X^* such that $(x_n)_{n\geq 1}$ is equivalent to the canonical basis of c_0, $(x_n^*)_{n\geq 1}$ is weak*-null and $x_n^*(x_n) \not\to 0$.

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Given X a Banach space and $1 \leq p \leq \infty$, we have

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In the previous theorem, the forward implication is due to Kwapień ($1 \leq p < \infty$) and Díaz ($p = \infty$) and the converse is due to Emmanuele.

Tensor products of Banach spaces

Let X and Y be real or complex vector spaces. Denote by $B(X \times Y)$ the space of all bilinear forms $A : X \times Y \to \mathbb{K}$. Given $x \in X$ and $y \in Y$, let $x \otimes y$ be the linear functional on $B(X \times Y)$ defined by

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The *tensor product* of X and Y, denoted by $X \otimes Y$, is the subspace of the (algebraic) dual of $B(X \times Y)$ spanned by the functionals $x \otimes y$, where $x \in X$ and $y \in Y$.

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$$\|u\|_{\varepsilon} = \sup \left\{ \left| \sum_{i=1}^{n} x^*(x_i) y^*(y_i) \right| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\},$$

where $u = \sum_{i=1}^{n} x_i \otimes y_i$ is an arbitrary representation of u.

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The *projective norm* of $u \in X \otimes Y$ is defined by

$$\|u\|_{\pi} = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| \right\},$$

where the infimum is taken over *all* the representations $\sum_{i=1}^{n} x_i \otimes y_i$ of u.

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The *injective tensor product* and the *projective tensor product* of X and Y are the completions of $X \otimes Y$ endowed with the injective and projective norms, respectively. These spaces will be denoted by $X \widehat{\otimes}_\varepsilon Y$ and $X \widehat{\otimes}_\pi Y$.

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- Similarly, $L_1[0,1] \widehat{\otimes}_\pi X \equiv L_1([0,1], X)$.

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Theorem (Cembranos-Freniche, 1984)

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The map $S : X\widehat{\otimes}_\varepsilon Y \to \mathcal{K}(X^*, Y)$ defined by

$$S(u)(x^*) = \sum_{i=1}^{n} x^*(x_i)y_i,$$

for all $x^* \in X^*$ and $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, is a well defined linear isometry onto its image. This allows us to identify the injective tensor product with a subspace of $\mathcal{K}(X^*, Y)$.

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Theorem (Ryan, 1991)

Let X and Y be Banach spaces. If X is infinite dimensional and $c_0 \hookrightarrow Y$, then $X \widehat{\otimes}_\varepsilon Y$ contains a subspace isomorphic to c_0 that is complemented in $\mathcal{K}(X^*, Y)$ (and therefore complemented in $X \widehat{\otimes}_\varepsilon Y$).

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Theorem (Rosenthal, 1970)

Let X be Banach space and τ be an infinite cardinal. The following are equivalent:

1. $c_0(\tau) \hookrightarrow X$;
2. There exists a bounded linear operator $T : c_0(\tau) \to X$ such that $\inf_{i\in\tau} \|T(e_i)\| > 0$;
3. There exists a weakly unconditionally summable family $(x_i)_{i\in\tau}$ such that $\inf_{i\in\tau} \|x_i\| > 0$.

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Let X and Y be Banach spaces, τ be an infinite cardinal and α denote either the injective or projective norm. Under which conditions do we have

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- $\ell_1 \overset{c}{\hookrightarrow} \ell_2\widehat{\otimes}_\pi\ell_2$
- (Pisier, 1983) There exist Banach spaces X and Y such that $c_0 \overset{c}{\hookrightarrow} X\widehat{\otimes}_\pi Y$ but $c_0 \not\hookrightarrow X$ and $c_0 \not\hookrightarrow Y$.

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Since $C(\beta\mathbb{N}) \equiv \ell_\infty$, by the Cembranos-Freniche theorem we have

$$c_0 \overset{c}{\hookrightarrow} C(\beta\mathbb{N}, X) \equiv \ell_\infty \widehat{\otimes}_\varepsilon X,$$

whenever X is an infinite dimensional Banach space.

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Let's begin by looking at a similar, simpler problem.

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Let X be a Banach space and K be a compact Hausdorff space. It is not difficult to show that

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Sketch: if K is infinite, then $C(K)$ contains a copy of c_0. If K is finite, then $C(K)$ is linearly isometric to X^n equipped with the supremum norm, where $n = |K|$, and it is well-known that

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Theorem (Galego-Hagler, 2012)

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Theorem (Galego-Hagler, 2012)

Let X be a Banach space, K be a compact Hausdorff space and τ be an infinite cardinal. If $\mathrm{cf}(\tau) > \mathrm{dens}(K)$, then

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Theorem (C, 2017)

Let X be a Banach space, K be a compact Hausdorff space and τ be an infinite cardinal. If $\mathrm{cf}(\tau) > \mathrm{w}(K)$, then

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This theorem is optimal for any τ. The inequality $\mathrm{cf}(\tau) > \mathrm{w}(X)$ cannot be replaced by $\mathrm{cf}(\tau) \geq \mathrm{w}(X)$.

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Definition

A Banach space X has the λ-*bounded approximation property*, $\lambda \geq 1$, if given K a compact subset of X and $\varepsilon > 0$, there exists a finite rank operator $T : X \to X$ satisfying $\|T\| \leq \lambda$ and $\|T(x) - x\| < \varepsilon$, for all $x \in K$. We say that X has the *bounded approximation property* if it has the λ-bounded approximation property for some $\lambda \geq 1$.

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Theorem (C-Galego-Samuel, 2019)

Let X and Y be a Banach spaces and τ be an infinite cardinal. If Y has the bounded approximation property and $\mathrm{cf}(\tau) > \mathrm{dens}(Y)$, then

$$c_0(\tau) \overset{c}{\hookrightarrow} X\widehat{\otimes}_\varepsilon Y \implies c_0(\tau) \overset{c}{\hookrightarrow} X.$$

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Recall that $S : X\widehat{\otimes}_\varepsilon Y \to \mathcal{K}(Y^*, X)$ given by

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- $\max(|A|, |B|) \leq \mathrm{dens}(Y)$;
- Given $u \in X\widehat{\otimes}_\varepsilon Y$ and $\delta > 0$, there exist $y_1, \ldots, y_m \in A$ and $\varphi_1, \ldots, \varphi_m \in B$ such that

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Next, suppose that $c_0(\tau) \overset{c}{\hookrightarrow} X\widehat{\otimes}_\varepsilon Y$. Take families $(u_i)_{i\in\tau}$ in $X\widehat{\otimes}_\varepsilon Y$ and $(\psi_i)_{i\in\tau}$ in $(X\widehat{\otimes}_\varepsilon Y)^*$ such that $(u_i)_{i\in\tau}$ is equivalent to the canonical basis of $c_0(\tau)$, $(\psi_i)_{i\in\tau}$ is weak*-null and $\psi_i(u_i) = 1$ for all $i \in \tau$.

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By construction, for each $i \in \tau$ there exist $y^i_1, \dots, y^i_{m_i} \in A$ and $\varphi^i_1, \dots, \varphi^i_{m_i} \in B$ such that

$$\frac{1}{2} < \sum_{n=1}^{m_i} |\psi_i(S(u_i)(\varphi^i_n) \otimes y^i_n)|.$$

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Since τ has large cofinality, using a "diagonalization" argument, we can extract a subset τ' of τ, a natural number $M \geq 1$ and two elements $y_0 \in A, \varphi_0 \in B$ with the property that $|\tau'| = \tau$ and

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If we define $x_i = S(u_i)(\varphi_0) \in X$ and $x_i^* = \psi_i(\cdot \otimes y_0) \in X^*$, it is easy to show that the families $(x_i)_{i \in \tau'}$ and $(x_i^*)_{i \in \tau'}$ verify the hypotheses of the generalized Schlumprecht theorem. Thus, $c_0(\tau) \overset{c}{\hookrightarrow} X$.

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- $\mathrm{dens}(C(K)) = w(K)$ (K infinite);
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- $\mathrm{dens}(L_p[0,1]) = \aleph_0$ $(1 \leq p < \infty)$.

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Corollary (C-Galego-Samuel, 2018)

Let X be a Banach space, I be an infinite set, τ be an infinite cardinal and $1 \leq p < \infty$. If with $\mathrm{cf}(\tau) > |I|$, then

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Let X be a Banach space, τ be an infinite cardinal and $1 \leq p < \infty$. If with $\mathrm{cf}(\tau) > \aleph_0$, then

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The projective case

Recall that

$$c_0 \overset{c}{\hookrightarrow} \ell_p \widehat{\otimes}_\pi X \iff c_0 \overset{c}{\hookrightarrow} L_q[0,1] \widehat{\otimes}_\pi X \iff c_0 \overset{c}{\hookrightarrow} X,$$

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for all $1 \leq p \leq \infty$. In particular, since $L_1([0,1], X) \equiv L_1[0,1] \widehat{\otimes}_\pi X$, we have

$$c_0 \overset{c}{\hookrightarrow} L_1[0,1] \widehat{\otimes}_\pi \ell_\infty,$$

although $c_0 \not\hookrightarrow L_1[0,1]$ and $c_0 \overset{c}{\not\hookrightarrow} \ell_\infty$.

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$$c_0(\tau) \overset{c}{\hookrightarrow} L_p(\{-1,1\}^I, X) \implies c_0(\tau) \overset{c}{\hookrightarrow} X.$$

In particular, if τ has uncountable cofinality, then

$$c_0(\tau) \overset{c}{\hookrightarrow} L_p([0,1], X) \implies c_0(\tau) \overset{c}{\hookrightarrow} X.$$

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There are other contexts where we can investigate (complemented) copies of $c_0(\tau)$, such as:

- Operator spaces;
- Vector-valued measure spaces;
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We can also study copies of other spaces: ℓ_1, ℓ_∞ (well-studied, with convenient chacterizations), ℓ_p (much more difficult) or even $L_1[0,1]$.

Thank you!