Triangular matrix categories and recollements

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Motivation

We recall that given a unitary ring R we can construct the preadditive category $\mathcal{C}_R$ defined as follows:

(a) $\mathrm{Obj}(\mathcal{C}_R) := \{*\}$

(b) $\mathrm{Hom}_{\mathcal{C}_R}(*, *) := R$.

It is well known that there exists an isomorphism

$$(\mathcal{C}_R, \mathbf{Ab}) \longrightarrow \mathrm{Mod}(R).$$

where $(\mathcal{C}_R, \mathbf{Ab})$ denotes the category of additive covariant functors $F : \mathcal{C}_R \longrightarrow \mathbf{Ab}$ and $\mathrm{Mod}(R)$ is the category of left R-modules.

Following Mitchell's philosophy, given a small preadditive category $\mathcal{C}$ we can think $\mathcal{C}$ as a ring with several objects. So, we can construct its category of left $\mathcal{C}$-modules as follows:

$$\mathrm{Mod}(\mathcal{C}) := (\mathcal{C}, \mathbf{Ab}) := \{F : \mathcal{C} \longrightarrow \mathbf{Ab} \mid F \text{ additive and covariant}\}$$

We recall that

(a) $\mathrm{Mod}(\mathcal{C})$ is a Grothendieck abelian category.

(b) $\{\mathrm{Hom}_{\mathcal{C}}(C, -)\}_{C\in\mathcal{C}}$ is a set of projective generators.

Let T and U be rings and M a T-U-bimodule ($M \in {}_U\mathrm{Mod}_T$). We can construct the **triangular matrix ring**

$$\Lambda = \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}.$$

The elements of Λ are 2×2 matrices $\begin{bmatrix} t & 0 \\ m & u \end{bmatrix}$ with $t \in T$, $u \in U$ and $m \in M$. Addition and multiplication are given by the ordinary operations on matrices as follows:

1. $\begin{bmatrix} t_1 & 0 \\ m_1 & u_1 \end{bmatrix} + \begin{bmatrix} t_2 & 0 \\ m_2 & u_2 \end{bmatrix} = \begin{bmatrix} t_1+t_2 & 0 \\ m_1+m_2 & u_1+u_2 \end{bmatrix}$ and

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2. $\begin{bmatrix} t_1 & 0 \\ m_1 & u_1 \end{bmatrix} \begin{bmatrix} t_2 & 0 \\ m_2 & u_2 \end{bmatrix} = \begin{bmatrix} t_1 t_2 & 0 \\ m_1 t_2 + u_1 m_2 & u_1 u_2 \end{bmatrix}$

In order to construct the triangular matrix category I need to recall some basic constructions.

If $\mathcal{C}$ and $\mathcal{D}$ are preadditive categories, we can define the **tensor product** $\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{D}$ of two preadditive categories:

(a) Objects: pairs (C, D) with $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

(b) Morphisms:

$$\mathrm{Hom}_{\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{D}}\Big((C, D), (C', D')\Big) := \mathrm{Hom}_{\mathcal{C}}(C, C') \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(D, D').$$

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$$\mathrm{Hom}_{\mathcal{C}\otimes_{\mathbb{Z}}\mathcal{D}}\Big((C,D),(C',D')\Big) := \mathrm{Hom}_{\mathcal{C}}(C,C') \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(D,D').$$

The composition in $\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{D}$ is given as follows: given

$$f_1 \otimes g_1 \in \mathcal{C}(C,C') \otimes \mathcal{D}(D,D') = \mathrm{Hom}_{\mathcal{C}\otimes_{\mathbb{Z}}\mathcal{D}}\Big((C,D),(C',D')\Big)$$

and

$$f_2 \otimes g_2 \in \mathcal{C}(C',C'') \otimes \mathcal{D}(D',D'') = \mathrm{Hom}_{\mathcal{C}\otimes_{\mathbb{Z}}\mathcal{D}}\Big((C',D'),(C'',D'')\Big).$$

we define the composition

$$(C, D) \xrightarrow{(f_1 \otimes g_1)} (C', D') \xrightarrow{f_2 \otimes g_2} (C'', D'')$$

as follows:

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) := (f_2 \circ f_1) \otimes (g_2 \circ g_1)$$

With this $\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{D}$ becomes an preadditive category.

So we can consider its category of modules

$$\operatorname{Mod}(\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{D}).$$

We recall that if U and T are rings, then M is a $T-U$ bimodule if and only if $M \in \text{Mod}(U \otimes_{\mathbb{Z}} T^{op})$.

Definition

Let $\mathcal{U}$ and $\mathcal{T}$ be preadditive categories. We say that M is a $\mathcal{U}-\mathcal{T}$-bimodule if $M \in \text{Mod}(\mathcal{U} \otimes_{\mathbb{Z}} \mathcal{T}^{op})$.

We have the necessary ingredients to construct the triangular matrix category.

Given $\mathcal{U}$ and $\mathcal{T}$ preadditive categories and $M \in \mathrm{Mod}(\mathcal{U} \otimes \mathcal{T}^{op})$ we have the following.

Definition

We define the ***triangular matrix category*** $\mathbf{\Lambda} = \left[\begin{smallmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{smallmatrix}\right]$ *as follows.*

(a) *Objects: are matrices of the form*

$$\left[\begin{smallmatrix} T & 0 \\ M & U \end{smallmatrix}\right]$$

with $T \in \mathcal{T}$ *and* $U \in \mathcal{U}$.

(b) *Given a pair of objects in* $\left[\begin{smallmatrix} T & 0 \\ M & U \end{smallmatrix}\right], \left[\begin{smallmatrix} T' & 0 \\ M & U' \end{smallmatrix}\right]$ *in* $\mathbf{\Lambda}$ *we define*

$$\mathsf{Hom}_{\mathbf{\Lambda}}\left(\left[\begin{smallmatrix} T & 0 \\ M & U \end{smallmatrix}\right], \left[\begin{smallmatrix} T' & 0 \\ M & U' \end{smallmatrix}\right]\right) := \left[\begin{smallmatrix} \mathsf{Hom}_{\mathcal{T}}(T,T') & 0 \\ M(U',T) & \mathsf{Hom}_{\mathcal{U}}(U,U') \end{smallmatrix}\right].$$

(We recall that $(U', T) \in \mathrm{Obj}(\mathcal{U} \otimes \mathcal{T}^{op})$ *and* $M : \mathcal{U} \otimes \mathcal{T}^{op} \longrightarrow \mathbf{Ab}$*).*

Given two morphisms in $\mathbf{\Lambda} = \left[\begin{smallmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{smallmatrix}\right]$:

$$\left[\begin{smallmatrix} t_1 & 0 \\ m_1 & u_1 \end{smallmatrix}\right] : \left[\begin{smallmatrix} T & 0 \\ M & U \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} T' & 0 \\ M & U' \end{smallmatrix}\right] \quad (m_1 \in M(U', T))$$

$$\left[\begin{smallmatrix} t_2 & 0 \\ m_2 & u_2 \end{smallmatrix}\right] : \left[\begin{smallmatrix} T' & 0 \\ M & U' \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} T'' & 0 \\ M & U'' \end{smallmatrix}\right] \quad (m_2 \in M(U'', T'))$$

we want to define

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But this must be an element in

$$\mathsf{Hom}_{\mathbf{\Lambda}}\left(\left[\begin{smallmatrix} T & 0 \\ M & U \end{smallmatrix}\right], \left[\begin{smallmatrix} T'' & 0 \\ M & U'' \end{smallmatrix}\right]\right) := \left[\begin{smallmatrix} \mathsf{Hom}_{\mathcal{T}}(T,T'') & 0 \\ M(U'',T) & \mathsf{Hom}_{\mathcal{U}}(U,U'') \end{smallmatrix}\right].$$

We define

$$\left[\begin{smallmatrix} t_2 & 0 \\ m_2 & u_2 \end{smallmatrix}\right] \circ \left[\begin{smallmatrix} t_1 & 0 \\ m_1 & u_1 \end{smallmatrix}\right] := \left[\begin{smallmatrix} t_2 \circ t_1 & 0 \\ \mathbf{?} & u_2 \circ u_1 \end{smallmatrix}\right]$$

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Then we have that

$$\mathbf{?} \in M(U'', T).$$

If this were the matrices from linear algebra, we have that

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This is the right definition.

Since $t_1 : T \longrightarrow T'$ we have $1_{U''} \otimes t_1^{op} : (U'', T') \longrightarrow (U'', T)$ in $\mathcal{U} \otimes \mathcal{T}^{op}$. Now, we recall that $M \in \mathrm{Mod}(\mathcal{U} \otimes \mathcal{T}^{op})$ (i.e, $M : \mathcal{U} \otimes \mathcal{T}^{op} \longrightarrow \mathbf{Ab}$) then we have a morphism of abelian groups

$$M(1_{U''} \otimes t_1^{op}) : M(U'', T') \longrightarrow M(U'', T)$$

Since $m_2 \in M(U'', T')$ we have that $M(1_{U''} \otimes t_1^{op})(m_2) \in M(U'', T)$. So, we define

$$m_2 \bullet t_1 := M(1_{U''} \otimes t_1^{op})(m_2).$$

Similarly, we set

$$u_2 \bullet m_1 := M(u_2 \otimes 1_T)(m_1)$$

where $M(u_2 \otimes 1_T) : M(U', T) \longrightarrow M(U'', T)$.

With this we have that

$$\mathbf{?} := m_2 \bullet t_1 + u_2 \bullet m_1 \in M(U'', T).$$

Now, for

$$\left[\begin{smallmatrix} t_1 & 0 \\ m_1 & u_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} r_1 & 0 \\ n_1 & v_1 \end{smallmatrix}\right] \in \mathsf{Hom}_{\mathbf{\Lambda}}\left(\left[\begin{smallmatrix} T & 0 \\ M & U \end{smallmatrix}\right], \left[\begin{smallmatrix} T' & 0 \\ M & U' \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \mathsf{Hom}_{\mathcal{T}}(T,T') & 0 \\ M(U',T) & \mathsf{Hom}_{\mathcal{U}}(U,U') \end{smallmatrix}\right]$$

we define

$$\left[\begin{smallmatrix} t_1 & 0 \\ m_1 & u_1 \end{smallmatrix}\right] + \left[\begin{smallmatrix} r_1 & 0 \\ n_1 & v_1 \end{smallmatrix}\right] := \left[\begin{smallmatrix} t_1+r_1 & 0 \\ m_1+n_1 & u_1+v_1 \end{smallmatrix}\right]$$

Then, it is clear that $\mathbf{\Lambda}$ is a preadditive category since $\mathcal{T}$ and $\mathcal{U}$ are preadditive categories and $M(U',T)$ is an abelian group.

Definition

The (Jacobson) ***radical*** *of a preadditive category* $\mathcal{C}$ *is the two-sided ideal* $\operatorname{rad}_{\mathcal{C}}$ *in* $\mathcal{C}$ *defined by the formula*

$$\operatorname{rad}_{\mathcal{C}}(X,Y) = \{h \in \mathcal{C}(X,Y) \mid 1_X - gh \textit{ is invertible } \forall g \in \mathcal{C}(Y,X)\}$$

for all objects X *and* Y *of* $\mathcal{C}$.

Now, we compute the radical in $\mathbf{\Lambda}$.

Proposition

$$\operatorname{rad}_{\mathbf{\Lambda}}\left(\left[\begin{smallmatrix} T & 0 \\ M & U \end{smallmatrix}\right], \left[\begin{smallmatrix} T' & 0 \\ M & U' \end{smallmatrix}\right]\right) = \begin{bmatrix} \mathsf{rad}_{\mathcal{T}}(T,T') & 0 \\ M(U',T) & \mathsf{rad}_{\mathcal{U}}(U,U') \end{bmatrix}$$

In the classical setting: U and T rings and M a $U-T$-bimodule. We have the following comma category:

$$\Big(\mathrm{Mod}(T), \mathbf{Hom}(\mathbf{M}, \mathrm{Mod}(U))\Big)$$

(a) Objects: are morphisms in $\mathrm{Mod}(T)$ of the form

$$f : A \longrightarrow \mathrm{Hom}_U(M, B)$$

with $A \in \mathrm{Mod}(T)$ and $B \in \mathrm{Mod}(U)$.

(b) Morphisms: a morphism

$$\begin{array}{ccc} A & & A' \\ \downarrow & \xrightarrow{\theta} & \downarrow \\ \mathrm{Hom}_U(M,B) & & \mathrm{Hom}_U(M,B') \end{array}$$

consists of a pair of morphisms (α, β) where $\alpha : A \longrightarrow A'$ in $\mathrm{Mod}(T)$, $\beta : B \longrightarrow B'$ in $\mathrm{Mod}(U)$ such that the following commutes

$$\begin{array}{ccc} A & \xrightarrow{\quad\alpha\quad} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathrm{Hom}_U(M,B) & \xrightarrow{\mathrm{Hom}_U(M,\beta)} & \mathrm{Hom}_U(M,B') \end{array}$$

Theorem

There exists an equivalence

$$\mathrm{Mod}\left(\left[\begin{smallmatrix} T & 0 \\ M & U \end{smallmatrix}\right] \right) \simeq \Big(\mathrm{Mod}(T), \mathbf{Hom}(\mathbf{M}, \mathrm{Mod}(U))\Big).$$

In order to have the same result for rings with several objects we need to define the analogous of the functor $\mathrm{Hom}_U(M,-)$. First, we note that if $M \in \mathrm{Mod}(\mathcal{U} \otimes_{\mathbb{Z}} \mathcal{T}^{op})$ and $T \in \mathcal{T}^{op}$ then $M(-,T) : \mathcal{U} \longrightarrow \mathbf{Ab}$ (i.e, $M(-,T) \in \mathrm{Mod}(\mathcal{U})$).

Definition

Let $M \in \mathrm{Mod}(\mathcal{U} \otimes_{\mathbb{Z}} \mathcal{T}^{op})$ be, define

$$\mathbb{G} : \mathrm{Mod}(\mathcal{U}) \longrightarrow \mathrm{Mod}(\mathcal{T})$$

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Definition

Let $M \in \operatorname{Mod}(\mathcal{U} \otimes_{\mathbb{Z}} \mathcal{T}^{op})$ be, define

$$\mathbb{G} : \operatorname{Mod}(\mathcal{U}) \longrightarrow \operatorname{Mod}(\mathcal{T})$$

(a) For $B \in \operatorname{Mod}(\mathcal{U})$, we set $\mathbb{G}(B)(T) := \operatorname{Hom}_{\operatorname{Mod}(\mathcal{U})}(M(-,T), B)$ for every $T \in \mathcal{T}$.

Definition

For $\eta : B \longrightarrow B'$ in $\mathrm{Mod}(\mathcal{U})$

$$[\mathbb{G}(\eta)]_T := \mathrm{Hom}_{\mathrm{Mod}(\mathcal{U})}(M(-,T),\eta) : \mathbb{G}(B)(T) \longrightarrow \mathbb{G}(B')(T)$$

So, we have the following

Theorem [GOS]

There exists an equivalence

$$\mathrm{Mod}\Big(\left[\begin{smallmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{smallmatrix}\right] \Big) \simeq \Big(\mathrm{Mod}(\mathcal{T}), \mathbb{G}(\mathrm{Mod}(\mathcal{U}))\Big).$$

Where $\Big(\mathsf{Mod}(\mathcal{T}), \mathbb{G}\mathsf{Mod}(\mathcal{U})\Big)$ is the comma category whose objects are the triples (A, f, B) with $A \in \mathsf{Mod}(\mathcal{T}), B \in \mathsf{Mod}(\mathcal{U})$, and $f : A \longrightarrow \mathbb{G}(B)$ a morphism of $\mathcal{T}$-modules. A morphism between two objects (A, f, B) and (A', f', B') is a pairs of morphism (α, β) where $\alpha : A \longrightarrow A'$ is a morphism of $\mathcal{T}$-modules and $\beta : B \longrightarrow B'$ is a morphism of $\mathcal{U}$-modules such that the diagram commutes

$$\begin{CD} A @>{\alpha}>> A' \\ @V{f}VV @VV{f'}V \\ \mathbb{G}(B) @>{\mathbb{G}(\beta)}>> \mathbb{G}(B') \end{CD}$$

Let us recall the definition of a dualizing R-variety due to Auslander and Reiten.

Definition

Let $\mathcal{C}$ be a category. It is said that $\mathcal{C}$ is a **variety** if $\mathcal{C}$ is preadditive, with coproducts and with splitting idempotents.

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It is said that and additive category $\mathcal{C}$ is a category with **splitting idempotents** if for each idempotent $e = e^2 \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ there are morphisms $\mu : Y \longrightarrow X$ and $\rho : X \longrightarrow Y$ such that $\mu\rho = e$ and $\rho\mu = 1_Y$.

Now, we consider the R a commutative artin ring.

Definition

Let $\mathcal{C}$ be a variety.

(a) It is said that $\mathcal{C}$ is an **R-variety** if $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is an R-module and the composition is R-bilinear.

(a) An R-variety $\mathcal{C}$ is **Hom-finite** if $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is a finitely generated R-module.

Now, let us consider

$$\Big(\mathcal{C}, \operatorname{Mod}(R)\Big) := \{F : \mathcal{C} \longrightarrow \operatorname{Mod}(R) \mid F \text{ covariant}\}$$

We have

$$\Big(\mathcal{C}, \mathrm{mod}(R)\Big) := \{F \in \big(\mathcal{C}, \mathrm{Mod}(R)\big) \mid F(C) \in \mathrm{mod}(R)\ \forall C \in \mathcal{C}\}$$

Remark

$\Big(\mathcal{C}, \mathrm{mod}(R)\Big)$ is an abelian full subcategory of $\Big(\mathcal{C}, \mathrm{Mod}(R)\Big)$.

If $\mathcal{C}$ is an R-variety, there exists an isomorphism

$$\mathrm{Mod}(\mathcal{C}) := \Big(\mathcal{C}, \mathbf{Ab}\Big) \xrightarrow{\simeq} \Big(\mathcal{C}, \mathrm{Mod}(R)\Big).$$

We have a duality

$$\mathbb{D}_{\mathcal{C}} : \Big(\mathcal{C}, \mathrm{mod}(R)\Big) \longrightarrow \Big(\mathcal{C}^{op}, \mathrm{mod}(R)\Big)$$

given by

$$\mathbb{D}_{\mathcal{C}}(M)(C) = \mathrm{Hom}_R\Big(M(C), I(R/\mathrm{rad}(R))\Big) =$$

where $I(R/\mathrm{rad}(R))$ is the injective envelope of $R/\mathrm{rad}(R)$.

For example when we have an artin algebra A and consider the category $\mathcal{C} = \mathcal{C}_A$ with just one object, then

$$\mathbb{D}_{\mathcal{C}} : \Big(\mathcal{C}, \operatorname{mod}(R)\Big) \longrightarrow \Big(\mathcal{C}^{op}, \operatorname{mod}(R)\Big)$$

becomes the usual duality in artin algebras

$$D : \operatorname{mod}(A) \longrightarrow \operatorname{mod}(A^{op})$$

Definition

Let $\mathcal{C}$ be a Hom-finite R-variety. We denote by $\operatorname{mod}(\mathcal{C})$ the full subcategory of $\operatorname{Mod}(\mathcal{C})$ whose objects are the **finitely presented functors**. That is, $M \in \operatorname{mod}(\mathcal{C})$ if and only if, there exists an exact sequence in $\mathsf{Mod}(\mathcal{C})$

$$\operatorname{Hom}_{\mathcal{C}}(C_1,-) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(C_0,-) \longrightarrow M \longrightarrow 0,$$

We have that $\operatorname{mod}(\mathcal{C}) \subseteq \Big(\mathcal{C}, \operatorname{mod}(R)\Big)$.

Definition

An Hom-finite R-variety $\mathcal{C}$ is **dualizing**, if the functor

$$\mathbb{D}_{\mathcal{C}} : (\mathcal{C}, \mathrm{mod}(R)) \to (\mathcal{C}^{op}, \mathrm{mod}(R)) \qquad (1)$$

induces a duality between the categories $\mathrm{mod}(\mathcal{C})$ and $\mathrm{mod}(\mathcal{C}^{op})$:

$$\begin{array}{ccc} (\mathcal{C}, \mathrm{mod}(R)) & \xrightarrow{\mathbb{D}_{\mathcal{C}}} & (\mathcal{C}^{op}, \mathrm{mod}(R)) \\ \uparrow & & \uparrow \\ \mathrm{mod}(\mathcal{C}) & \longrightarrow & \mathrm{mod}(\mathcal{C}^{op}) \end{array}$$

Example

Let Λ an artin algebra, then $\mathrm{mod}(\Lambda)$ is dualizing variety and also $\mathrm{mod}(\mathrm{mod}(\Lambda))$ is a dualizing variety.

Theorem [GOS]

Let $\mathcal{T}$ and $\mathcal{U}$ be dualizing R-varieties and $M \in \mathrm{Mod}(\mathcal{U} \otimes_R \mathcal{T}^{op})$ such that

1. $M(U,-) \in \mathrm{mod}(\mathcal{T}^{op})$ for all U and
2. $M(-,T) \in \mathrm{mod}(\mathcal{U})$ for all $,T$.

Theorem [GOS]

Let $\mathcal{T}$ and $\mathcal{U}$ be dualizing R-varieties and $M \in \mathrm{Mod}(\mathcal{U} \otimes_R \mathcal{T}^{op})$ such that

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2. $M(-, T) \in \mathrm{mod}(\mathcal{U})$ for all $, T$.

(a) Then $\mathbf{\Lambda} = \left[\begin{smallmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{smallmatrix}\right]$ is a dualizing R-variety.

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(a) Then $\mathbf{\Lambda} = \left[\begin{smallmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{smallmatrix}\right]$ is a dualizing R-variety.

(b) In particular, $\mathrm{mod}(\mathbf{\Lambda})$ has AR-sequences.

Proposition

Let $\mathcal{C}$ be a dualizing K-variety with duality $\mathbb{D}_{\mathcal{C}} : \mathrm{mod}(\mathcal{C}) \longrightarrow \mathrm{mod}(\mathcal{C}^{op})$. Then the following statements hold.

(a) *The triangular matrix category* $\begin{bmatrix} \mathcal{C} & 0 \\ \widehat{\mathbb{H}\mathrm{om}} & \mathcal{C} \end{bmatrix}$ *is dualizing.*

(b) *Suppose that $\mathcal{C}$ is an abelian category with enough projectives. Then the triangular matrix category* $\begin{bmatrix} \mathcal{C} & 0 \\ \widehat{\mathbb{E}\mathrm{xt}^1} & \mathcal{C} \end{bmatrix}$ *is dualizing.*

The maps category maps($\mathcal{C}$)

Assume that $\mathcal{C}$ is an R-variety. The maps category, maps($\mathcal{C}$) is defined as follows.

(a) The objects in maps($\mathcal{C}$) are morphisms $(f_1, A_1, A_0) : A_1 \xrightarrow{f_1} A_0$

(b) the maps are pairs $(h_1, h_0) : (f_1, A_1, A_0) \to (g_1, B_1, B_0)$, such that the following square commutes

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & A_0 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_0} \\ B_1 & \xrightarrow[g_1]{} & B_0. \end{array}$$

Proposition

Let $\mathcal{C}$ be a K-variety and consider the category $\mathbf{\Lambda} = \begin{bmatrix} \mathcal{C} & 0 \\ \widehat{\mathbb{H}\mathrm{om}} & \mathcal{C} \end{bmatrix}$.

(i) *There is an equivalence of categories*

$$\mathrm{Mod}(\mathbf{\Lambda}) \xrightarrow{\sim} \mathrm{maps}(\mathrm{Mod}(\mathcal{C}))$$

(ii) *If $\mathcal{C}$ is dualizing, there is an equivalence of categories*

$$\mathrm{mod}(\mathbf{\Lambda}) \xrightarrow{\sim} \mathrm{maps}(\mathrm{mod}(\mathcal{C}))$$

Some AR-sequences of $\mathrm{maps}(\mathrm{mod}(\mathcal{C}))$ can be computed from those of $\mathrm{mod}(\mathcal{C})$.

Proposition

Let $\mathcal{C}$ be a dualizing K-variety.

(1) Let $0 \to \tau M \xrightarrow{j} E \xrightarrow{\pi} M \to 0$ be an almost split sequence of $\mathcal{C}$-modules. Then the exact sequence in maps(mod($\mathcal{C}$)):

$$0 \to (\tau M, 0, 0) \xrightarrow{(j,\ 0)} (E, \pi, M) \xrightarrow{(\pi,\ 1_M)} (M, 1_M, M) \to 0$$

which is represented by the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau M & \xrightarrow{j} & E & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle 1_M} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow[1_M]{} & M & \longrightarrow & 0 \end{array}$$

is an AR-sequence.

Proposition

Let $\mathcal{C}$ be a dualizing K-variety.

(1) Let $0 \to \tau M \xrightarrow{j} E \xrightarrow{\pi} M \to 0$ be an almost split sequence of $\mathcal{C}$-modules. Then the exact sequence in maps(mod($\mathcal{C}$)):

$$0 \to (\tau M, 1_{\tau M}, \tau M) \xrightarrow{(1_{\tau M},\ j)} (\tau M, j, E) \xrightarrow{(0,\ \pi)} (0, 0, M) \to 0,$$

which is represented by the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau M & \xrightarrow{1} & \tau M & \xrightarrow{0} & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \tau M & \xrightarrow[j]{} & E & \xrightarrow[\pi]{} & M & \longrightarrow & 0 \end{array}$$

is an AR-sequence.

Given a finite dimensional K-algebra $\Lambda := KQ/I$. Let i a source in Q and $\overline{e}_i$ the corresponding idempotent in Λ.

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So $\Lambda = KQ/I$ is obtained from $\Lambda' := KQ'/I'$ by adding one vertex i, together with arrows and relations starting in i.

Then $\Lambda := \begin{bmatrix} K & 0 \\ (1-\overline{e}_i)\Lambda\overline{e}_i & \Lambda' \end{bmatrix}$. So Λ is the **one-point extension** of Λ'.

Now, in order to give a generalization of the previous construction, we consider the following setting. Let $\mathcal{C}$ be a Krull-Schmidt category and $(\mathcal{U}, \mathcal{T})$ a pair of additive full subcategories of $\mathcal{C}$. It is said that $(\mathcal{U}, \mathcal{T})$ is a **splitting torsion pair** if

(i) For all $X \in \mathrm{ind}(\mathcal{C})$, then either $X \in \mathcal{U}$ or $X \in \mathcal{T}$.

(ii) $\mathrm{Hom}_{\mathcal{C}}(X, -)|_{\mathcal{T}} = 0$ for all $X \in \mathcal{U}$.

We get the following result that tell us that we can obtain a category as extension of two subcategories.

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Proposition (GOS)

Let $(\mathcal{U}, \mathcal{T})$ be a splitting torsion pair. Then we have a equivalence of categories

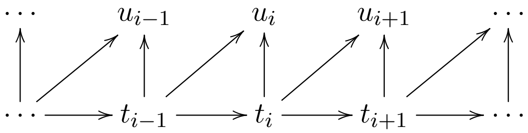
$$\mathcal{C} \cong \begin{bmatrix} \mathcal{T} & 0 \\ \widehat{\mathrm{Hom}_\circ} & \mathcal{U} \end{bmatrix}.$$

Here without danger to cause confusion $\widehat{\mathrm{Hom}_\circ}$ denotes the restriction of $\widehat{\mathrm{Hom}_\circ} : \mathcal{C} \otimes \mathcal{C}^{op} \to \mathbf{Ab}$ to the subcategory $\mathcal{U} \otimes \mathcal{T}^{op}$ of $\mathcal{C} \otimes \mathcal{C}^{op}$.

As an application of the last result, we consider $Q = (Q_1, Q_0)$ be a quiver. Recall that the **path category** KQ is an additive category, with indecomposable objects the vertices, and given $a, b \in Q_0$, the set of the maps $\mathrm{Hom}_{KQ}(a, b)$ is given by the K-vector space with basis the set of all paths from a to b. The composition of maps is induced from the usual composition of paths. Let $U = \{x \in Q_0 | x \text{ is a sink }\}$ and let $T = Q_0 - U$, and consider $\mathcal{U} = \text{add } U$ and $\mathcal{T} = \text{add } T$. We consider the triangular matrix category $\left[\begin{smallmatrix} \mathcal{T} & 0 \\ \mathsf{Hom}_{KQ} & \mathcal{U} \end{smallmatrix}\right]$. Then we have a equivalence of categories

$$KQ \cong \begin{bmatrix} \mathcal{T} & 0 \\ \widehat{\mathrm{Hom}_{\circ}} & \mathcal{U} \end{bmatrix}.$$

As a concrete example, consider the following quiver $Q = (Q_0, Q_1)$ with set of vertices $Q_0 = \{u_i, t_i : i \in \mathbb{Z}\}$. As above, if $U = \{u_i : i \in \mathbb{Z}\}$ and $T = \{t_i : i \in \mathbb{Z}\}$, and we consider $\mathcal{U} = \text{add } U$ and $\mathcal{T} = \text{add } T$, then we have an equivalence of categories $KQ \cong \begin{bmatrix} \mathcal{T} & 0 \\ \widehat{\text{Hom}} & \mathcal{U} \end{bmatrix}$,



Definition

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories

(a) *The diagram*

$$\mathcal{C} \underset{i_*}{\overset{i^*}{\leftrightarrows}} \mathcal{A} \underset{j^!}{\overset{j_!}{\leftrightarrows}} \mathcal{B}$$

is a called a ***left recollement*** *if the additive functors $i^*, i_*, j_!$ and $j^!$ satisfy the following conditions:*

(LR1) *(i^*, i_*) and $(j_!, j^!)$ are adjoint pairs;*

(LR2) *$j^! i_* = 0$;*

(LR3) *$i_*, j_!$ are full embedding functors.*

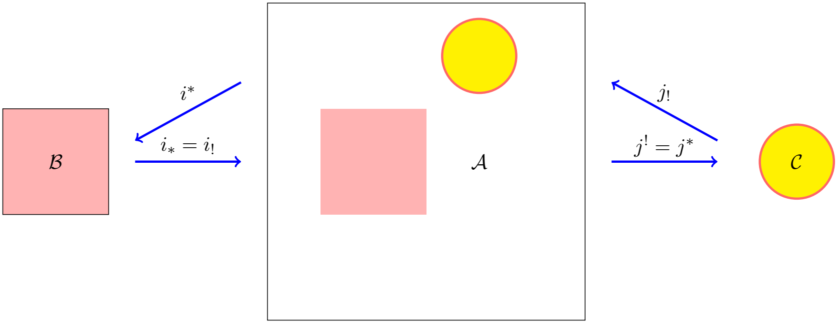


Figura: Left Recollement

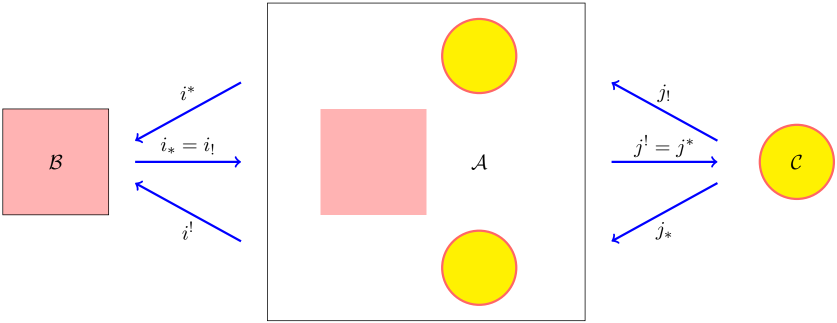


Figura: Recollement

Our purpose in this section is to prove a generalization of the following Theorem given by Q. Chen and M. Zheng in [4, Theo. 4.4].

Theorem [Chen-Zheng]

Let R, S, C and T be rings. For any $M \in \mathrm{Mod}(R \otimes T^{op})$, consider the matrix rings $\mathbf{\Lambda} := \left(\begin{smallmatrix} T & 0 \\ M & R \end{smallmatrix}\right)$ $\mathbf{\Lambda}^! := \begin{pmatrix} T & 0 \\ j_!(M) & S \end{pmatrix}$, $\mathbf{\Lambda}^* := \begin{pmatrix} T & 0 \\ j_*(M) & S \end{pmatrix}$.

(a) If the diagram

$$\mathrm{Mod}(C) \underset{i_*}{\overset{i^*}{\leftrightarrows}} \mathrm{Mod}(S) \underset{j^!}{\overset{j_!}{\leftrightarrows}} \mathrm{Mod}(R)$$

is a left recollement, then there is a left recollement

$$\mathrm{Mod}(C) \underset{\tilde{i_*}}{\overset{\tilde{i^*}}{\leftrightarrows}} \mathrm{Mod}(\mathbf{\Lambda}^!) \underset{\tilde{j^!}}{\overset{\tilde{j_!}}{\leftrightarrows}} \mathrm{Mod}(\mathbf{\Lambda})$$

So we have the result

Theorem [GOS]

Let $\mathcal{R}, \mathcal{S}, \mathcal{C}$ and $\mathcal{T}$ be aditive categories. For any $M \in \mathrm{Mod}(\mathcal{R} \otimes \mathcal{T}^{op})$, consider the matrix categories $\mathbf{\Lambda} := \left(\begin{smallmatrix} \mathcal{T} & 0 \\ M & \mathcal{R} \end{smallmatrix}\right)$ $\mathbf{\Lambda}^! := \begin{pmatrix} \mathcal{T} & 0 \\ j_!(M) & \mathcal{S} \end{pmatrix}$, $\mathbf{\Lambda}^* := \begin{pmatrix} \mathcal{T} & 0 \\ j_*(M) & \mathcal{S} \end{pmatrix}$, where the bimodules $j_!(M)$ and $j_*(M)$ are canonical constructed.

(a) If the diagram

$$\mathrm{Mod}(\mathcal{C}) \; \underset{\xrightarrow[i_*]{\qquad\qquad}}{\xleftarrow{\;\;\; i^* \;\;\;}} \; \mathrm{Mod}(\mathcal{S}) \; \underset{\xrightarrow[j^!]{\qquad\qquad}}{\xleftarrow{\;\;\; j_! \;\;\;}} \; \mathrm{Mod}(\mathcal{R})$$

is a left recollement, then there is a left recollement

$$\mathrm{Mod}(\mathcal{C}) \; \underset{\xrightarrow[\tilde{i_*}]{\qquad\qquad}}{\xleftarrow{\;\;\; \tilde{i^*} \;\;\;}} \; \mathrm{Mod}(\mathbf{\Lambda}^!) \; \underset{\xrightarrow[\tilde{j^!}]{\qquad\qquad}}{\xleftarrow{\;\;\; \tilde{j_!} \;\;\;}} \; \mathrm{Mod}(\mathbf{\Lambda})$$

Theorem [GOS]

Let $\mathcal{R}, \mathcal{S}, \mathcal{C}$ and $\mathcal{T}$ be dualizing. For $M \in \mathrm{Mod}(\mathcal{R} \otimes_K \mathcal{T}^{op})$ such that $M_T \in \mathrm{mod}(\mathcal{R})$ and $M_R \in \mathrm{mod}(\mathcal{T}^{op})$ for all $T \in \mathcal{T}$ and $R \in \mathcal{U}$, consider the matrix categories $\mathbf{\Lambda} := \left(\begin{smallmatrix} \mathcal{T} & 0 \\ M & \mathcal{R} \end{smallmatrix}\right)$ $\mathbf{\Lambda}^! := \begin{pmatrix} \mathcal{T} & 0 \\ j_!(M) & \mathcal{S} \end{pmatrix}$, $\mathbf{\Lambda}^* := \begin{pmatrix} \mathcal{T} & 0 \\ j_*(M) & \mathcal{S} \end{pmatrix}$. Moreover suppose that $j_!(M)_S,\ j_*(M)_S \in \mathrm{mod}(\mathcal{T}^{op})$ for all $S \in \mathcal{S}$.

(a) If the diagram

$$\mathrm{mod}(\mathcal{C}) \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow[\;i_*\;]{} \end{array} \mathrm{mod}(\mathcal{S}) \begin{array}{c} \xleftarrow{\;j_!\;} \\ \xrightarrow[\;j^!\;]{} \end{array} \mathrm{mod}(\mathcal{R})$$

is a left recollement, then there is a left recollement

$$\mathrm{mod}(\mathcal{C}) \begin{array}{c} \xleftarrow{\;\tilde{i^*}\;} \\ \xrightarrow[\;\tilde{i_*}\;]{} \end{array} \mathrm{mod}(\mathbf{\Lambda}^!) \begin{array}{c} \xleftarrow{\;\tilde{j_!}\;} \\ \xrightarrow[\;\tilde{j^!}\;]{} \end{array} \mathrm{mod}(\mathbf{\Lambda})$$

Thank you

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