A link between representations of peak posets and cluster algebras

Julian Serna

Pedagogical and Technological University of Colombia
Tunja-Colombia

https://www.sciencedirect.com/science/article/abs/pii/S0022404920301353

Joint work with R. Schiffler

Overview

Goal: To give a geometric interpretation (using diagonals and polygons) to the category of representations of peak posets (peak $\mathcal{P}$-spaces) of type $\mathbb{A}$. This establishes a link between the theory of cluster algebras and the the category of peak $\mathcal{P}$-spaces.

Outline

- It is a generalization to the representations of ordinary posets.
- Due Simson in 1991 [8, 10].
- Finiteness criterion is due Simson. Moreover, J. Kosakowska classified the exact posets and its exact representations (finite representation type case) [2–4].
- The tameness is due Kasjan and Simson in [5].
- AR-sequences were studied by Simson and J.A. de la Peña [6].

Given $\mathcal{P}$ a finite poset, $\mathcal{P}$ is an r-peak poset if $|\max \mathcal{P}| = r$.

Definition

A peak $\mathcal{P}$-space U of $\mathcal{P}$ over $\mathbb{k}$ is a system of finite-dimensional $\mathbb{k}$-vector spaces

$$U = (U_x)_{x \in \mathcal{P}},$$

satisfying:

(a) For each $x \in \mathcal{P}$, U_x is a $\mathbb{k}$-subspace of $U^\bullet = \bigoplus_{z \in \max \mathcal{P}} U_z$.

(b) For each $x \prec y$ in $\mathcal{P}$ it holds that $\pi_y(U_x) \subseteq U_y$, where π_y is the natural map

$$U^\bullet \rightarrowtail U_y^\bullet = \bigoplus_{y \preceq z \in \max \mathcal{P}} U_z \hookrightarrow U^\bullet$$

(c) If $z \in \max \mathcal{P}$ and $x \npreceq z$ then $\pi_z(U_x) = 0$.

Definition

A morphism $f : U \to V$ is a collection of $\mathbb{k}$-linear maps

$$f = (f_z : U_z \to V_z)_{z \in \max \mathcal{P}}$$

such that for all $x \in \mathcal{P}$

$$\Big(\bigoplus_{z \in \max \mathcal{P}} f_z \Big)(U_x) \subseteq V_x.$$

$=$ instead $\subseteq$ (isomorphism).

This category is denoted by $\widehat{\mathrm{rep}}\, \mathcal{P}$.

Example

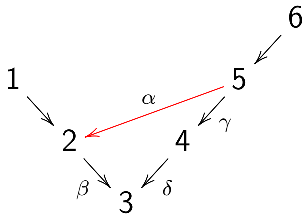
The incidence algebra $\mathbb{k}\mathcal{P}$ is a bound quiver algebra $\mathbb{k}\mathcal{P} = \mathbb{k}Q/I$ induced by the quiver Q whose vertices are the points of $\mathcal{P}$ and there is an arrow $\alpha : x \to y$ for each pair $x, y \in \mathcal{P}$ such that y covers x.

The ideal I is generated by all the commutativity relations $\gamma - \gamma'$ with γ and γ' parallel paths in Q.

We let $\mathrm{mod}(\mathbb{k}\mathcal{P})$ denote the category of finitely generated right $\mathbb{k}\mathcal{P}$-modules.

Example

In our example, the incidence algebra $\Bbbk\mathcal{P}$ of $\mathcal{P}$ is the bound quiver algebra defined by the quiver Q^F



and the ideal $\langle \alpha\beta - \gamma\delta \rangle$.

Definition (socle-projective modules)

Recall that the socle soc M of a module M is the submodule generated by all simple submodules of M. A module M is called socle-projective if soc M is a projective module.

$\mathrm{mod}_{sp}\Bbbk\mathcal{P}$ denotes the category of socle-projective f.g $\Bbbk\mathcal{P}-$modules over the incidence algebra $\Bbbk\mathcal{P}$ of $\mathcal{P}$.

The categories $\widehat{\mathrm{rep}}\,\mathcal{P}$ and $\mathrm{mod}_{sp}\Bbbk\mathcal{P}$ are equivalent.

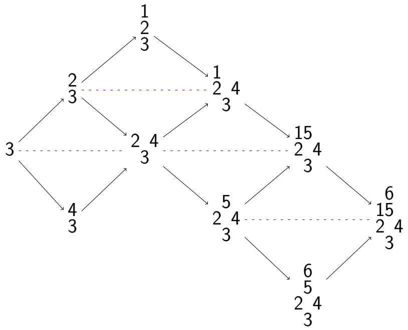
Each f.g. right $\Bbbk\mathcal{P}$-module M is identified with a quiver representation $M = (M_x, {}_y h_x)_{x,y \in \mathcal{P}}$ of $\Bbbk\mathcal{P}$, where M_x is a f.d. $\Bbbk$-vector space and ${}_y h_x : M_x \longrightarrow M_y$ is a $\Bbbk$-linear map, one for each relation $x \preceq y$ in $\mathcal{P}$, satisfying: [10].

(a) ${}_x h_x$ is the identity of M_x for all $x \in \mathcal{P}$ and ${}_w h_y \cdot {}_y h_x = {}_w h_x$ for all $x \preceq y \preceq w$ in $\mathcal{P}$.

This is a socle-projective representation if and only if additionally satisfies: [10]:

(b) $I_x = \bigcap_{z \in x^{\nabla} \cap \max \mathcal{P}} \ker {}_z h_x = 0$ for all $x \in \mathcal{P}^- = \mathcal{P} \setminus \max \mathcal{P}$.

The AR-quiver of $\text{mod}_{sp}(\Bbbk\mathcal{P})$

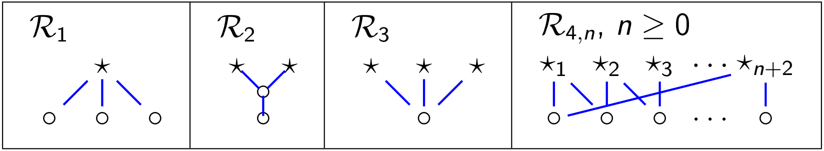


Posets of type $\mathbb{A}$

Roughly speaking, the posets of type $\mathbb{A}$ are posets with $n \geq 1$ elements whose category of socle-projective representations is embedded in the category of representations of a Dynkin quiver of type $\mathbb{A}_n$.

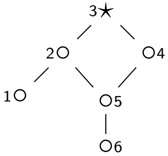
Definition

A full subposet $\mathcal{P}'$ of $\mathcal{P}$ is a peak-subposet if $\max \mathcal{P}' \subseteq \max \mathcal{P}$.
A finite connected poset $\mathcal{P}$ is of type $\mathbb{A}$ if $\mathcal{P} \not\supseteq \mathcal{R}_1 - \mathcal{R}_{4,n}$ as a peak-subposet.



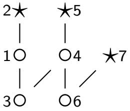
Examples

1. The one-peak poset



   Although $\{2, 4, 5, 6\}$ is a subposet of type $\mathcal{R}_2$ it is not a peak-subposet.

2. The three-peak poset



   This poset can be viewed as a Dynkin quiver of type $\mathbb{E}_7$.

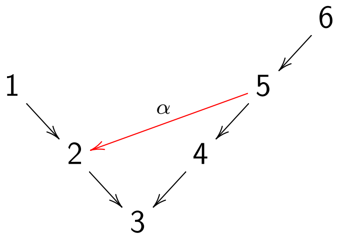
Alien set of new arrows for Q

Q is a Dynkin quiver of type $\mathbb{A}$. A set $F = \{\alpha_1, \ldots, \alpha_t\}$ of new arrows for Q is an Alien set for Q if the following conditions hold.

1. $\forall \alpha \in F$ there exists a sink vertex $z \in Q_0$ s.t $s(\alpha), t(\alpha) \in \text{Supp } I(z)$. But,
2. $t(\alpha)$ isn't a source vertex in Q unless it's an extremal vertex in Q.
3. $\forall \alpha \in F$, the arrow α is the unique path from $s(\alpha)$ to $t(\alpha)$ in Q^F, where Q^F is s.t $Q_0^F = Q_0$ and $Q_1^F = Q_1 \cup F$.
4. The quiver Q^F is acyclic.

Example

If Q is $1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5 \longleftarrow 6$ and $F = \{\ 5 \xrightarrow{\alpha} 2\}$
then Q^F is



Definition

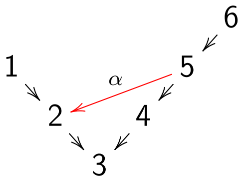
If Q is an acyclic quiver, the poset $\mathcal{P}_Q = (Q_0, \preceq)$, where

$$x \preceq y \text{ if and only if there exists a path from } x \text{ to } y \text{ in } Q$$

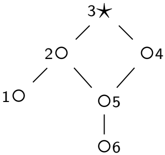
is the poset associated to Q.

Example

Given the quiver



the associated poset is



Posets of type $\mathbb{A}$ are coming from quivers Q^F

Lemma

A poset $\mathcal{P}$ is of type $\mathbb{A}$ iff $\mathcal{P} = \mathcal{P}_{Q^F}$, for any Dynkin quiver Q of type $\mathbb{A}$ and an alien set F of new arrows for Q.

Lemma

Let $\mathcal{P} = \mathcal{P}_{Q^F}$ be a poset of type $\mathbb{A}$. The following statements hold:

(a) *$mod_{sp}\Bbbk\mathcal{P}$ is finite representation type.*

(b) *Up to isomorphism, any indecomposable $\Bbbk\mathcal{P}-$module $M = (M_x, {}_y h_x)_{x,y\in\mathcal{P}}$ in $mod_{sp}(\Bbbk\mathcal{P})$ is s.t $M_x = k$ and ${}_y h_x = id_k$ for all $x \preceq y$ in $SuppM$.*

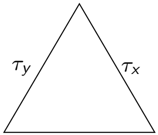
(c) *$SuppM$ is connected as a subset of the quiver Q.*

Category of diagonals not in T

Q ($\mathbb{A}$ type with n vertices), Π_{n+3} a regular polygon with $n+3$ vertices. We associate to Q a triangulation (set of n non-crossing diagonals)

$$T = \{\tau_1, \ldots, \tau_n\}$$

such that: there is an arrow $x \to y$ in Q_1 precisely if the diagonals τ_x and τ_y bound a triangle in which τ_y lies clockwise from τ_x (see Figure 1).

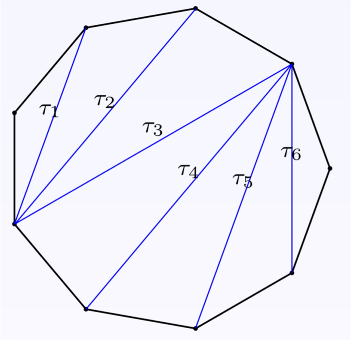


$\rightsquigarrow \; x \to y$

Figure: τ_y is counter-clockwise from τ_x

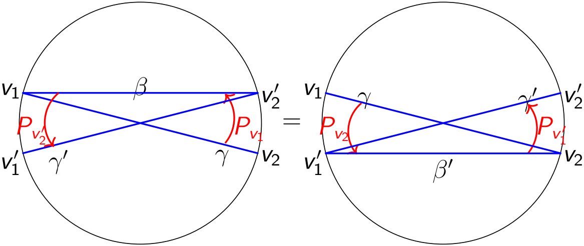
Example

In our example, $Q = \ 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5 \longleftarrow 6$ and T is



The additive category of diagonals C_T is s.t

- Indecomposable objects: the diagonals that are not in T.
- Morphisms: the compositions of pivoting elementary moves modulo mesh relations.



$$P_{v_2'}P_{v_1} = P_{v_1'}P_{v_2}$$

According to [CCS] rep$Q$ is equivalent to the category C_T.

In particular, there is a bijection

$$\{\text{diagonals not in } T\} \leftrightarrows \{\text{conected subsets of } Q\}$$

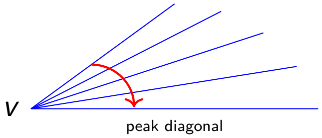
What kind of diagonals correspond to the socle-projective modules?

Definition

A fan in T is a maximal subset $\Sigma_v \subseteq T$, $|\Sigma_v| \geq 2$, s.t all of the diagonals in Σ_v share the vertex v.

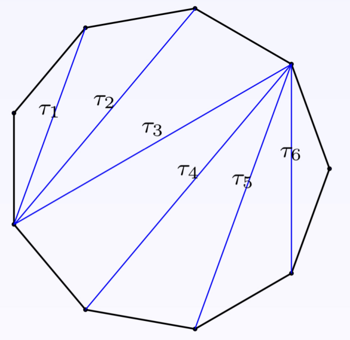
Definition

The peak diagonal of a fan Σ_v is the diagonal that can be obtained from each other diagonal in Σ_v by a clockwise rotation around the vertex v.



Example

In our example, $Q = \; 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5 \longleftarrow 6$ and T is



There are two fans $\Sigma = \{\tau_1, \tau_2, \tau_3\}$ and $\Sigma' = \{\tau_3, \tau_4, \tau_5, \tau_6\}$. τ_3 is the peak diagonal in both fans.

Definition

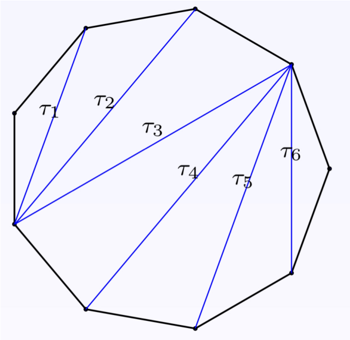
A diagonal $\gamma \notin T$ is an sp-diagonal if it satisfies the following conditions:

(a) If γ crosses $\tau \in T$ then γ crosses the peak diagonal of a fan Σ s.t $\tau \in \Sigma$.

(b) If γ crosses $\tau_{s(\alpha)}$ and τ_z, where $\alpha \in F$ is s.t $s(\alpha), t(\alpha) \in \mathrm{Supp}I(z)$, then γ crosses $\tau_{t(\alpha)}$.

Example

In our example, $F = \{\ 5 \xrightarrow{\alpha} 2\}$ and T is



The sp-diagonals γ are s.t γ crosses τ_3 and if γ crosses τ_5 then γ crosses τ_2.

Category of sp-diagonals

$\mathcal{C}_{(T,F)}$:=full subcategory of the category of diagonals $\mathcal{C}_T$ generated by all sp-diagonals. Here, the irreducible morphisms are pivoting

sp-moves between sp-diagonals, i.e, a composition

$$P : \gamma = \gamma_0 \xrightarrow{P_v^{(1)}} \gamma_1 \xrightarrow{P_v^{(2)}} \dots \xrightarrow{P_v^{(s)}} \gamma_s = \gamma'$$

of pivoting elementary moves with the same pivot v s.t $\gamma_1, \dots, \gamma_{s-1}$ are not sp-diagonals in $\mathcal{C}_T$.

Geometrically

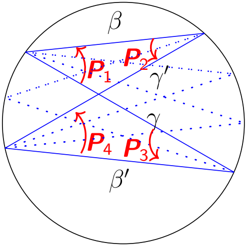


Figure: Mesh relations in $\mathcal{C}_{(T,F)}$.

Definition

The functor

$$\Omega : \mathcal{C}_{(T,F)} \longrightarrow \mathrm{mod}_{sp}(\Bbbk\mathcal{P})$$

is s.t for any sp-diagonal γ, $\Omega(\gamma) = M^\gamma = (M^\gamma_x, {}_y h^\gamma_x)$, where

$$M^\gamma_x = \begin{cases} k, & \text{if } \gamma \text{ crosses } \tau_x, \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{If } x \preceq y \in \mathcal{P} \text{ then } {}_y h^\gamma_x = \begin{cases} \mathrm{id}_k, & \text{if } M^\gamma_x = M^\gamma_y = k, \\ 0, & \text{otherwise.} \end{cases}$$

For any pivoting sp-move $\gamma \xrightarrow{P} \gamma'$ in $\mathcal{C}_{(T,F)}$, the morphism $M^\gamma \xrightarrow{\Omega(P)} M^{\gamma'}$ is defined by

$$\Omega(P)_x = \begin{cases} \mathrm{id}_k, & \text{if } M_x^\gamma = M_x^{\gamma'} = k, \\ 0, & \text{otherwise.} \end{cases}$$

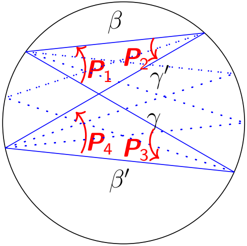
Theorem

Ω is a categorical equivalence.

Corollary

(a) *The irreducible morphisms of $\mathcal{C}_{(T,F)}$ are direct sums of the generating morphisms given by pivoting sp-moves.*

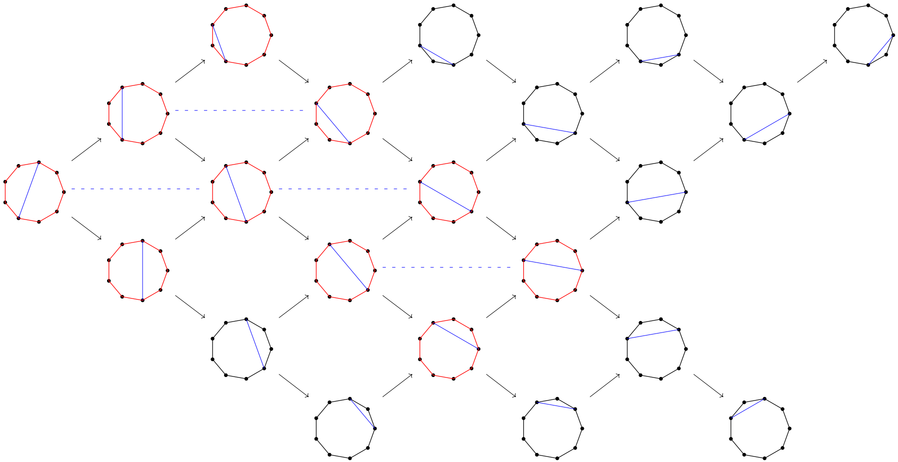
(b) *Let $\gamma \xrightarrow{P_1} \beta \xrightarrow{P_2} \gamma'$ be a composition of two pivoting sp-moves between sp-diagonals as in the figure*



Then,

- If $\beta' \in \mathcal{C}_{(T,F)}$ then $0 \longrightarrow \gamma \longrightarrow \beta \oplus \beta' \longrightarrow \gamma' \longrightarrow 0$ is a AR-sequence.
- If β' is either a boundary edge or a diagonal in T then $0 \longrightarrow \gamma \longrightarrow \beta \longrightarrow \gamma' \longrightarrow 0$ is a AR-sequence.
- If $\beta' \notin \mathcal{C}_{(T,F)}$ then γ' is a indec. projective in $\mathcal{C}_{(T,F)}$ and γ is a indec. injective in $\mathcal{C}_{(T,F)}$.

Example: The AR-quiver of $\mathcal{C}_{(T,F)}$



A link to cluster algebras

- $\mathcal{P}$ is poset of type $\mathbb{A}$ associted to Q^F.
- $\boldsymbol{x} = (x_1, \ldots, x_n)$ initial cluster
- $\mathcal{A}(Q, \boldsymbol{x})$:= Cluster algebra associated to the inicial seed $(\boldsymbol{x}, Q)$.
- $\mathcal{A}(\mathcal{P})$:= the subalgebra of $\mathcal{A}(Q, \boldsymbol{x})$ generated by the cluster variables x_γ s.t γ is an sp-diagonal in the category $\mathcal{C}_{(T,F)}$ together with the cluster variables in the initial cluster $\boldsymbol{x}$.

Under which conditions we have $\mathcal{A}(\mathcal{P}) = \mathcal{A}(Q, \boldsymbol{x})$?

Theorem

Let $\mathcal{P}$ be a poset of type $\mathbb{A}$ associated to $Q^\emptyset$ and let $\mathcal{A}(\mathcal{P})$ be the subalgebra of $\mathcal{A}(Q, \boldsymbol{x})$ associated to $\mathcal{P}$. Then $\mathcal{A}(\mathcal{P}) = \mathcal{A}(Q, \boldsymbol{x})$.

References I

[1] A. Berenstein, S. Fomin, and A. Zelevinsky, *Cluster algebras III: Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), no. 1.

[2] J. Kosakowska, *classification of sincere two-peak posets of finite prinjective type and their sincere prinjective representations*, Colloq. Math. **86** (2001), 27–77.

[3] ______, *Sincere posets of finite prinjective type with three maximal elements and their sincere prinjective representations*, Colloq. Math. **93** (2002), 155–208.

[4] ______, *Indecomposable sincere prinjective modules over multipeak sincere posets of finite prinjective type with at least four maximal elements*, Representations of algebras II (Beijing, 2000), Vol. 2, Beijing Normal University Press, 2002, pp. 253–291.

[5] S. Kasjan and D. Simson, *Varieties of poset representations and minimal posets of wild prinjective type*, Representations of algebras (Ottawa, ON, 1992), CMS Conf. Proc., vol. 14, Amer. Math. Soc., Providence, RI, 1993. MR1206948

[6] J.A. de la Peña and D. Simson, *Prinjective modules, reflection functors, quadratic forms and Auslander-Reiten sequences*, Trans. Amer. Math. Soc. **329** (1992), no. 2, 733–753.

[7] P. Caldero, F. Chapoton, and R. Schiffler, *Quivers with relations arising from clusters (A_n case)*, Trans AMS **358** (May 26,2005), no. 3, 1347–1364.

References II

[8] D. Simson, *Two-peak posets of finite prinjective type*, Proccedings of the Tsukuba International Conference on Representations of Finite-Dimensional Algebras, Canad. Math. Soc. Conf. Proc. **11** (1991), 287–298.

[9] ______, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Gordon and Breach, London, London, 1992.

[10] ______, *Posets of finite prinjective type and a class of orders*, J. Pure Appl. Algebra **90** (1993), 77–103.

Obrigado¡¡¡