τ-tilting theory and stratifying systems

Plan of the talk:

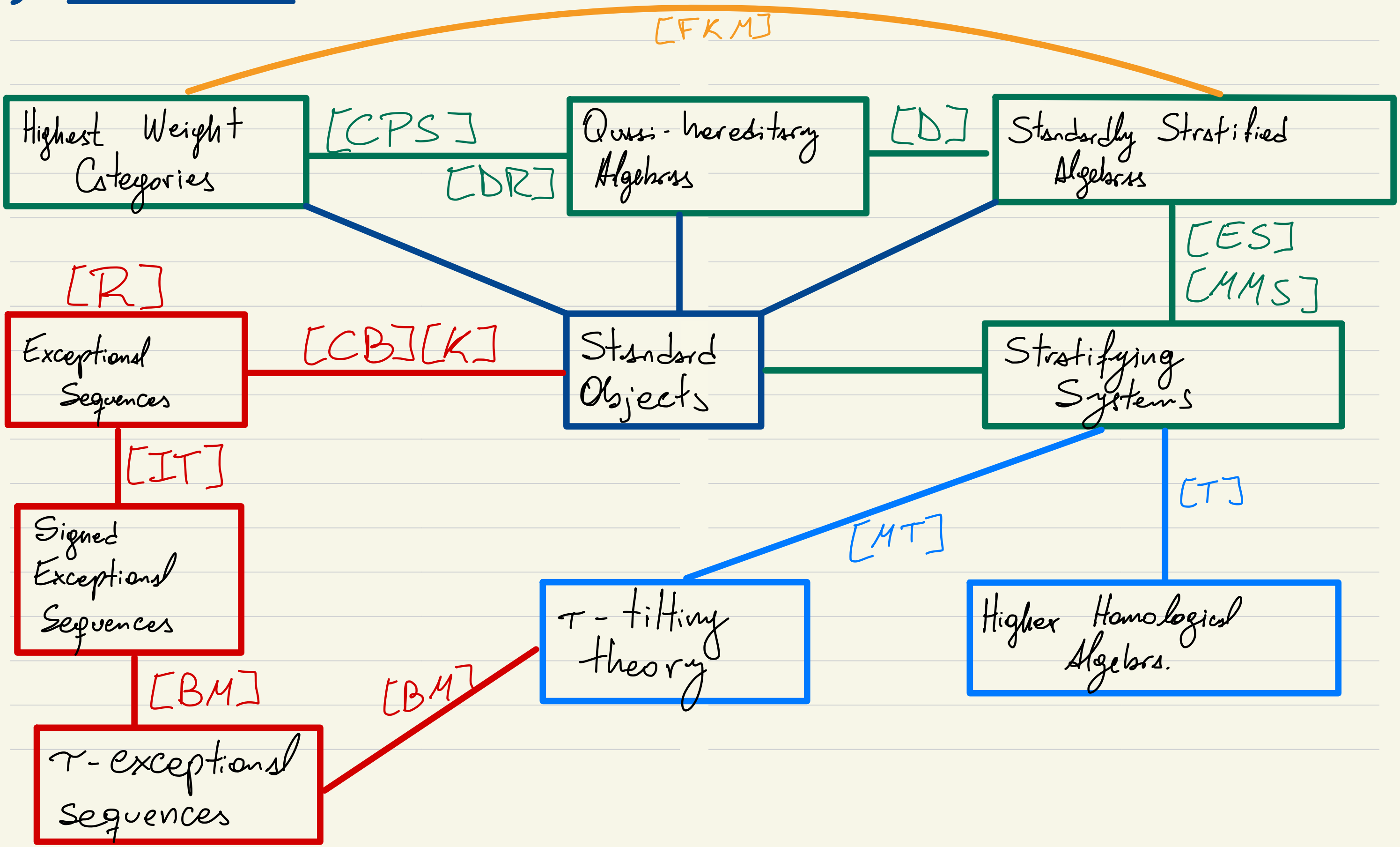
§1 Introduction

§2 τ-tilting theory

§3 Stratifying systems

§4 Results

§1 Introduction:



Setting:

* A is an artinian algebra (basic)
* $\operatorname{mod} A$ f.g. (right) A-modules
* $\tau : \operatorname{mod} A \to \operatorname{mod} A$ is the Auslander-Reiten translation in $\operatorname{mod} A$.
* $|M| = \#$ isoclasses of ind. direct summands of M.
* $A = \bigoplus_{i=1}^{n} P(i)$ $\qquad |A| = n$

Def:
A torsion pair in $\operatorname{mod} A$ is a pair of subcategories $(\mathcal{T}, \mathcal{F})$ such that

(a) $\operatorname{Hom}_A(X, Y) = 0 \quad \forall X \in \mathcal{T}, Y \in \mathcal{F}$

(b) For every object $M \in \operatorname{mod} A$ there exists a short exact sequence

$$0 \to \underset{\in \mathcal{T}}{tM} \to M \to \underset{\in \mathcal{F}}{fM} \to 0 \quad (*)$$

We say that $\mathcal{T}$ is a torsion class, $\mathcal{F}$ is a torsion-free class and $(*)$ is the canonical short exact sequence of M with respect to $(\mathcal{T}, \mathcal{F})$

§2. τ-tilting theory

Def: [Adachi-Iyama-Reiten]

Let $M \in \operatorname{mod} A$ and P projective A-module.

(1) M is τ-rigid if $\operatorname{Hom}_A(M, \tau M) = 0$ $\Rightarrow$ $\operatorname{Ext}^1_A(M,M)=0$

(2) The pair (M,P) is τ-rigid if M is τ-rigid and $\operatorname{Hom}_A(P,M)=0$

(3) A τ-rigid pair (M,P) is τ-tilting if $|M|+|P| = n = |A|$

$*$ $\operatorname{Fac} M = \{ X \in \operatorname{mod} A : \exists t \in \mathbb{N}$
$M^t \longrightarrow X \longrightarrow 0 \} = \operatorname{Gen} M$

Theorem: [Auslander-Smalø] [AIR]

There is a surjective map of sets

$$\Phi : \{\tau\text{-rigid pairs}\} \longrightarrow \{\text{ff torsion classes}\}$$
$$(M,P) \longmapsto \operatorname{Fac} M.$$

Moreover Φ is bijective if we restrict it to the set of τ-tilting pairs.

$M^\perp := \{ X : \mathrm{Hom}_A(M,X)=0 \}$

${}^\perp M := \{ Y : \mathrm{Hom}_A(Y,M)=0 \}$

Rmk. Let (M,P) be τ-rigid. Then $\underline{\mathrm{Fac}\, M}$ and $\underset{\substack{\rotatebox{90}{$\in$}\\ M}}{\underline{{}^\perp(\tau M)\cap P^\perp}}$ are torsion classes in mod A

Def:
Let (M,P) be a τ-rigid pair.
A τ-tilting pair $(\tilde{M},\tilde{P})$ is a <u>completion</u> of (M,P) if there exist a τ-rigid pair (M',P') such that

$$\tilde{M} = M\oplus M' \quad \text{and} \quad \tilde{P} = P\oplus P'.$$

* **Rmk:** Let (M,P) is a τ-rigid pair.

(A) $\Phi^{-1}(\mathrm{Fac}\, M)$ is a completion of (M,P)
co-Bongartz completion.

(B) $\Phi^{-1}({}^\perp(\tau M)\cap P^\perp)$ is a completion of (M,P).
Bongartz completion

Theorem: [AIR]
Let (M,P) be a τ-rigid pair and let $(\tilde{M},\tilde{P})$ be a completion of (M,P).
Then

$$\underset{\substack{\updownarrow \\ \{0\}\subset \operatorname{mod} B_{(M,P)}}}{\mathrm{Fac}\, M} \subset \mathrm{Fac}\, \tilde{M} \subset \underset{\substack{\updownarrow \\ \operatorname{mod} B_{(M,P)}}}{{}^\perp(\tau M)\cap P^\perp}$$

Question: Can we describe all completions of a τ-rigid pair (M,P)?

Def:
The perpendicular subcategory $J(M,P)$ of a τ-rigid pair (M,P) is

$(\mathrm{Fac}\, M, M^{\perp})$

$$J(M,P) = M^{\perp} \cap {}^{\perp}(\tau M) \cap P^{\perp}$$

Theorem [Jasso]
Let (M,P) be τ-rigid. Then there exists an algebra $B_{(M,P)}$ and an equivalence of categories

$$F : J(M,P) \longrightarrow \mathrm{mod}\, B_{(M,P)}$$

Moreover $|B_{(M,P)}| = |A| - |M| - |P|$.

$(B_{(M,P)}, 0)$

Theorem: [Jasso]
Let (M,P) be a τ-rigid pair.

$$\left\{ \begin{matrix} \text{Completions of} \\ (M,P) \end{matrix} \right\} \xleftrightarrow{1-1} \left\{ \begin{matrix} \tau\text{-tilting pairs} \\ \text{in } \mathrm{mod}\, B_{(M,P)} \end{matrix} \right\}$$

Cor: [AIR]
Let (M,P) be a τ-rigid pair. Then (M,P) is τ-tilting pair if and only if

$$\mathrm{Fac}\, M = {}^{\perp}(\tau M) \cap P^{\perp}$$

$(M,P) \longrightarrow$ comp de Bongartz $(M \oplus M', P)$

$C = \mathrm{End}_A(M \oplus M')$ $\quad Ce_M \cong \mathrm{Hom}_A(M \oplus M', M)$

$$B_{(M,P)} = \frac{C}{Ce_M C}$$

§3. Stratifying systems

$TA = 0$

Def: [Erdmann-Saenz]
Let $\Theta = \{\theta_1, \theta_2, \dots\}$ be a set of indecomposable modules in mod A. We say that Θ is a <u>stratifying system</u> if

(a) $\mathrm{Hom}_A(\theta_i, \theta_j) = 0$ if $i > j$;
(b) $\mathrm{Ext}^1_A(\theta_i, \theta_j) = 0$ if $i \geq j$.

<u>Rmk/Thm:</u> [T] There exists a stratifying system of infinite size

Def:
$M, N \in \mathrm{mod}\, A$

$$Tr_M N = \sum_{f \in \mathrm{Hom}_A(M,N)} \mathrm{Im} f \hookrightarrow N$$

Ex: $A = \bigoplus_{i=1}^{n} P(i)$

$$0 \to Tr_{\bigoplus_{j>i} P(j)} P(i) \to P(i) \to \Delta(i) \to 0 \qquad \Delta(i) \neq 0$$

The set $\Delta = \{\Delta(1), \Delta(2), \dots, \Delta(n)\}$ is a stratifying system. Standard objects.

* $\mathcal{X} \subset \mathrm{mod}\, A$

$$\mathcal{F}ilt(\mathcal{X}) = \{ Y : 0 \subset Y_1 \subset Y_2 \subset \dots \subset Y_k = Y \quad \frac{Y_i}{Y_{i-1}} \in \mathcal{X} \}$$

Def: [Dlab]
An algebra A is <u>standardly stratified</u> if there exists a decomposition $A = \bigoplus_{i=1}^{n} P(i)$ such that $A = \mathcal{F}ilt(\Delta)$.

§4. Results

Prob: Stratifying systems are difficult to find in nature.

Sol: Use τ-tilting theory

Def: Let M be τ-rigid. We say that a decomposition $M = \bigoplus_{i=1}^{t} M_i$ is TF-admissible if

$$M_i \notin \mathrm{Fac}\Big(\bigoplus_{i<j} M_j\Big).$$

$M_1, M_2, \dots, M_i, M_{i+1}, \dots, M_t$

Lemma: [MT]
Every τ-rigid module M admits a TF-admissible decomposition.

Theorem: [MT]
Let M be a τ-rigid module with TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$ and take $\Big(\mathrm{Fac}\big(\bigoplus_{j>i} M_j\big), -^{\perp}\Big)$

$$0 \to T_{r(\bigoplus_{j>i} M_j)} M_i \to M_i \to \Theta_i \to 0.$$

$\in \mathrm{Fac}\, M$

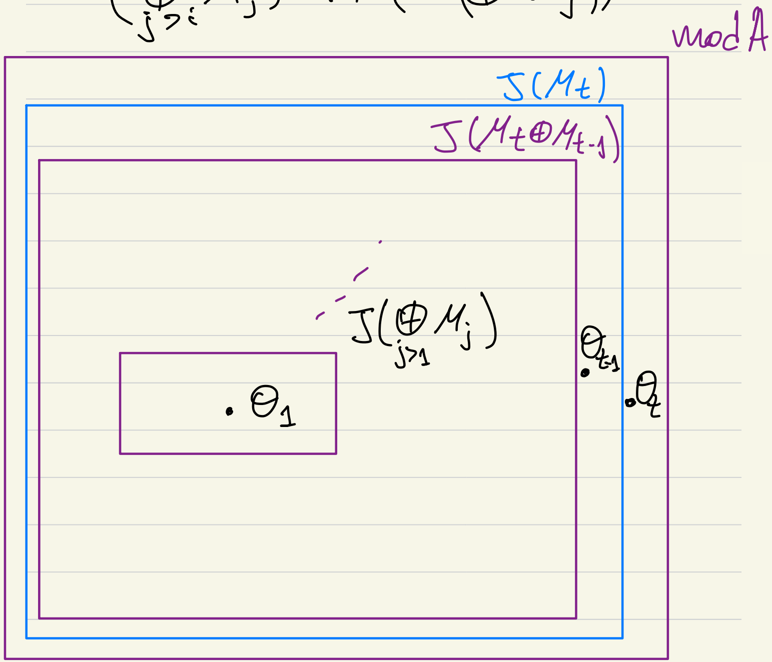
$$0 \to t_{i+1} M_i \to M_i \to \Theta_i \to 0$$

Then $\{\Theta_1, \Theta_2, \dots, \Theta_t\}$ is a stratifying system.

$t \leq n$

Prop:
With the notation above $\Theta_i \in J(\bigoplus_{j>i} M_j)$.

$$= \Big(\bigoplus_{j>i} M_j\Big)^{\perp} \cap {}^{\perp}\Big(\tau\big(\bigoplus M_j\big)\Big)$$



Rmk:
Suppose that $\Theta = \{\Theta_1, \dots, \Theta_t\}$ is a stratifying system induced by a τ-rigid module M.
Then we can reconstruct M from Θ

Prop:
Let M be τ-rigid with TF-admissible decomposition $M = \bigoplus_{i=1} M_i$ and let Θ be the stratifying system induced by M.

Then $\operatorname{Fac} M$ is the smallest torsion class containing Θ

Def:

Let A be an algebra

An exceptional sequence is a set $\Delta = \{\Delta_1, \dots, \Delta_t\}$ of indecomposables such that:

(a) $\mathrm{End}_A(\Delta_i)$ is a division ring

(b) $\mathrm{Hom}_A(\Delta_i, \Delta_j) = 0$ if $i > j$

(c) $\mathrm{Ext}^k_A(\Delta_i, \Delta_j) = 0$ if $i \geq j$ $\quad k > 0$

Theorem [Cadavid - Marcos]

Let A be a hereditary algebra. Then a set $\Theta = \{\Theta_1, \dots, \Theta_t\}$ is an exceptional sequence if and only if it is a stratifying system such that $\mathrm{End}_A(\Theta_i)$ is a division ring.

Prueba:

$\mathrm{dp}\, M \leq 1 \Rightarrow \mathrm{Ext}^1_A(M, M) \cong D\mathrm{Hom}_A(M, \tau M)$

Como Θ_t es rígido $\Rightarrow$ Θ_t es τ-rígido

$$
\begin{array}{ccc}
 & & \mathrm{mod}\, A \\
 & & \cup \\
\{\Theta_1, \dots, \Theta_{t-1}\} \subset & \Theta_t^{\perp} \cap {}^{\perp}(\tau \Theta_t) = & J(\Theta_t) \\
 & & \wr\! \wr \\
 & & \mathrm{mod}\, B_{\Theta_t}
\end{array}
$$

$\{\theta_1, \dots, \theta_t\}$

(1) θ_t es τ-rígido y

$$\{\theta_1, \dots, \theta_{t-1}\} \subset J(\theta_t) \cong \operatorname{mod} B_{\theta_t}$$

(2) θ_{t-1} sea "τ-rígido" en $J(\theta_t)$

$$\{\theta_1, \dots, \theta_{t-2}\} \subset J(\theta_{t-1}) \cap J(\theta_t)$$

$$B_d: \quad 1 \underset{\alpha_d}{\overset{\alpha_1}{\rightrightarrows}} 2 \underset{\beta_d}{\overset{\beta_1}{\rightrightarrows}} \cdots \quad d+1$$

Def: [BM]

Una secuencia de pares τ-rígidos indes. $\{(M_1, P_1), \dots, (M_t, P_t)\}$ es una secuencia τ-excepcional si

(*) (M_t, P_t) es τ-rígido en mod A

(**) $\{(M_1, P_1), \dots, (M_{t-1}, P_{t-1})\}$ es una secuencia τ-excepcional en $J(M_t, P_t)$

$d \in \mathbb{N}$

Cat d-ort maxi $\mathcal{M}$ (d-Cluster-tilting)

$$\mathcal{M} = \{ X \in \operatorname{mod} A : \operatorname{Ext}^i_A(X, \mathcal{M}) = 0 \quad 1 \le i \le d-1 \}$$

$$= \{ Y \in \operatorname{mod} A : \operatorname{Ext}^i_A(\mathcal{M}, Y) = 0 \quad 1 \le i \le d-1 \}$$

$$0 \to X_1 \to X_2 \to \dots \to X_d \to X_{d+1} \to X_{d+2} \to 0$$

mod $\mathcal{B}_d$.

$\tau_d^{-1} \longrightarrow$ $\qquad\qquad$ $\longleftarrow \tau_d$

$\mathcal{P}_d \qquad \mathcal{R}_d \qquad \mathcal{I}_d$