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Moduli spaces of flags of sheaves: a quiver description

Valeriano Lanza (UFF)

(joint work with R. von Flach and M. Jardim)

"Moduli spaces of framed flags of sheaves on the projective plane",
R. Von Flach, M. Jardim (2017)

"Obstruction theory for moduli spaces of framed flags of sheaves over the projective plane", R. von Flach, M. Jardim, V. L. (2021)

def A **framed flag of sheaves on $\mathbb{P}^2_{\mathbb{C}}$** is a triple (E, F, φ), where:

(·) F is a torsion-free sheaf on $\mathbb{P}^2$

(··) $\varphi : F|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus \operatorname{rk} F}$ ("framing")

(···) $E \subseteq F$ subsheaf, s.t. F/E is supported away from ℓ_∞

$\rightsquigarrow$ moduli space $\to$ $r := \mathrm{rk}\, F = \mathrm{rk}\, E$

$\to$ $n := c_2(F)$

$\to$ $\ell := h^0(F/E)$

(Rmk: $c_1(F) = c_1(E) = 0$ because of the framing)

Observe that $c_2(E) = n + \ell$

$\rightsquigarrow$ $\mathcal{F}(r, n, \ell)$ = moduli space of framed flags of sheaves

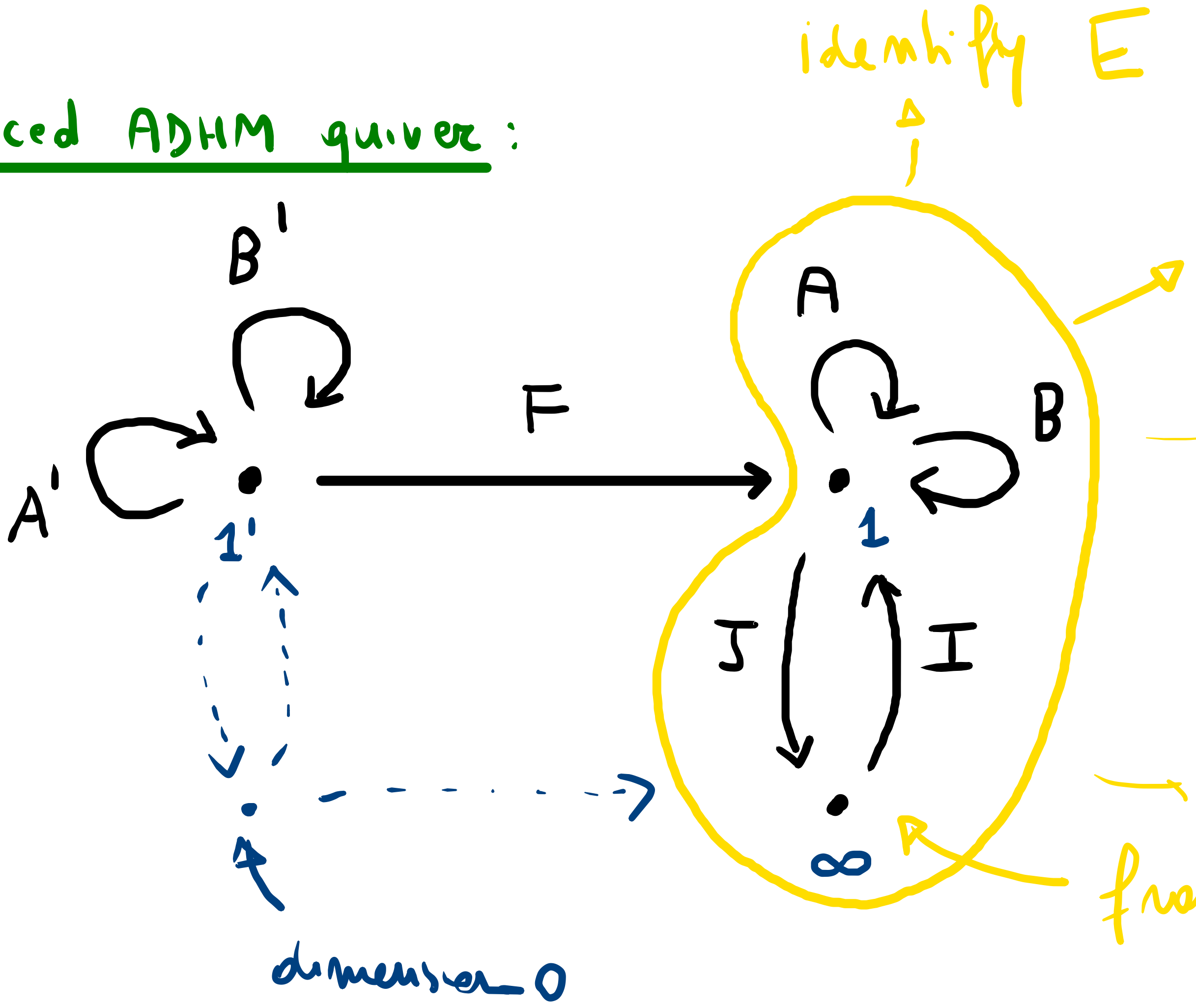
A quiver is finite oriented graph.

def A representation of a quiver Q is given by the choice of a vector space for every vertex and a linear map for every arrow.

($\Leftrightarrow$ representation of the path algebra $\mathbb{C}Q$)

Enhanced ADHM quiver:

Q^{enh}_{ADHM}



ADHM quiver: used by Nakajima (1999) to describe framed sheaves on $\mathbb{P}^2$

$\rightsquigarrow$ moduli space $\mathcal{N}_\theta(r, c, c')$

$r := \dim W \longleftrightarrow \infty$

$c := \dim V \longleftrightarrow 1$

$c' := \dim V' \longleftrightarrow 1'$

We need to follow King's recipe (1994)

ADHM equation

+ Relations: $[A,B] + IJ = 0$; $[A',B'] = 0$;

$$AF - FA' = BF - FB' = JF = 0$$

$\left(\mathbb{C}Q/\mathcal{J}\right)$

+ Stability : $\underline{\theta} \in \mathbb{R}^{\overset{\text{set of vertices}}{I}} = \mathbb{R}^3$, $\underline{\theta} = \left(\overset{\overset{\infty}{\updownarrow}}{\theta_\infty}, \overset{\overset{1}{\updownarrow}}{\theta}, \overset{\overset{1'}{\updownarrow}}{\theta'} \right)$

+ normalization : $\boxed{r\theta_\infty + c\theta + c'\theta' = 0}$

def A subrepresentation of numerical type $(\tilde{r}, \tilde{c}, \tilde{c}')$ is a collection of invariant subspaces $\tilde{W}, \tilde{V}, \tilde{V}'$ of W, V, V', resp., of dimension $\tilde{r}, \tilde{c}, \tilde{c}'$, respectively.

def A representation X of Q is $\underline{\theta}$-(semi)stable if

•) $\underline{\theta} \cdot (0, \tilde{c}, \tilde{c}') \underset{(=)}{<} 0 \quad \forall$ nontrivial subrepresentation of numerical type $(0, \tilde{c}, \tilde{c}')$

••) $\underline{\theta} \cdot (r, \tilde{c}, \tilde{c}') \underset{(=)}{<} 0 \quad \forall$ proper subrepresentation of numerical type $(r, \tilde{c}, \tilde{c}')$

+ Action of the group

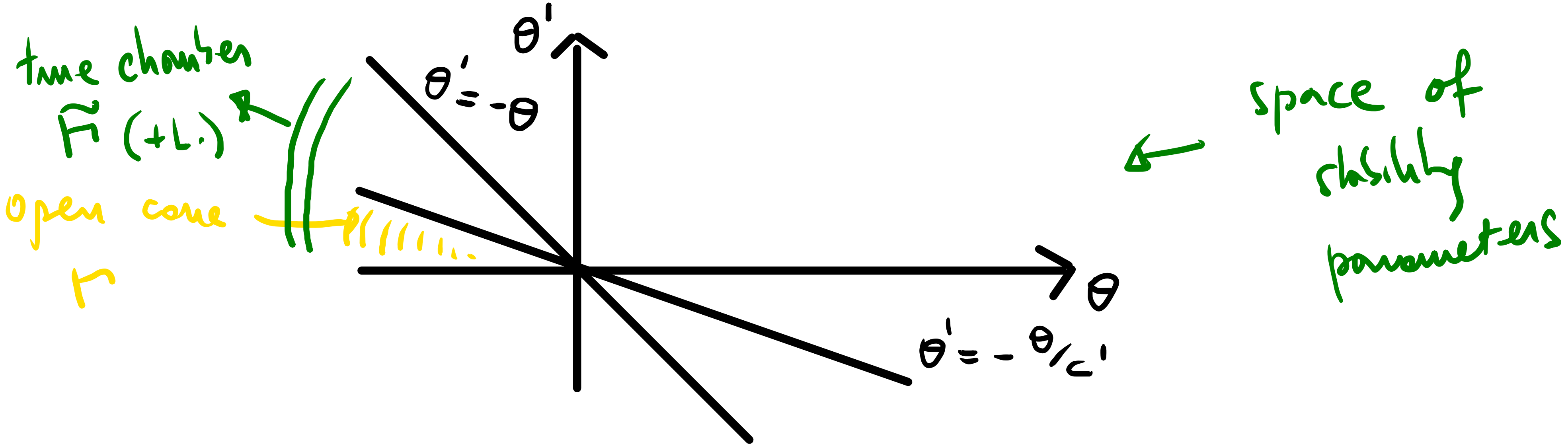
$$GL(V) \times GL(V')$$

Thm (von Flach, Jardim; 2017)

$\exists$ an isomorphism of schemes between

$$\mathcal{N}_{\underline{\theta}}(r, n+l, l) \xrightarrow{\sim} \mathcal{J}(r, n, l)$$

whenever $\underline{\theta}$ is chosen in here:



MOTIVATIONS:

(·) Bruzzo, Chuang, Diaconescu, Jardim, Pan, Zhang, "D-branes, surface operators, and ADHM quiver representations" (2011)

(··) Bonelli, Fasola, Tanzini, "Defects, nested instantons and comet-shaped quivers" (2021)

Thm (Von Flach, Jardim, L.)

$\mathcal{N}(r,c,c')$ has a perfect obstruction theory

$\forall$ repr. X one has $H^0(\mathcal{E}(X)) = H^3(\mathcal{E}(X)) = 0$.

$$\rightsquigarrow \text{ expected dimension } \begin{cases} r(2c-c') & \text{when } c'>1 \\ r(2c-1)+1 & \text{when } c'=1 \end{cases}$$

Deformation complex $\mathcal{C}(X)$:

$$
\begin{matrix} \mathrm{End}(V) \\ \oplus \\ \mathrm{End}(V') \end{matrix}
\xrightarrow{d_0}
\begin{matrix} \mathrm{End}(V)^{\oplus 2} \\ \oplus \\ \mathrm{Hom}(W,V) \\ \oplus \\ \mathrm{Hom}(V,W) \\ \oplus \\ \mathrm{End}(V')^{\oplus 2} \\ \oplus \\ \mathrm{Hom}(V',V) \end{matrix}
\xrightarrow{d_1}
\begin{matrix} \mathrm{End}(V) \\ \oplus \\ \mathrm{Hom}(V',V)^{\oplus 2} \\ \oplus \\ \mathrm{Hom}(V',W) \\ \oplus \\ \mathrm{End}(V') \end{matrix}
\xrightarrow{d_2}
\mathrm{Hom}(V',V)
$$

where :

$$d_0\begin{pmatrix} h \\ h' \end{pmatrix} = \begin{pmatrix} [h, A] \\ [h, B] \\ hI \\ -Jh \\ [h', A'] \\ [h', B'] \\ hF - Fh' \end{pmatrix} ; \quad d_1\begin{pmatrix} a \\ b \\ i \\ j \\ a' \\ b' \\ f \end{pmatrix} = \begin{pmatrix} [a, B] + [A, b] + Ij + iJ \\ Af + aF - Fa' - fA' \\ Bf + bF - Fb' - fB' \\ jF + Jf \\ [a', B'] + [A', b'] \end{pmatrix} ;$$

$$d_2(c_1, c_2, c_3, c_4, c_5) = c_1 F + B c_2 - c_2 B' - A c_3 + c_3 A' - I c_4 - F c_5 .$$

Open problems:

$$\left(\begin{array}{ccccc} \mathcal{E} & \hookrightarrow & \mathcal{E}^{\vee\vee} & \to & \mathcal{E}^{\vee\vee}/\mathcal{E} \\ & & & & \vdots \\ & & & & \mathrm{supp}\ \mathcal{E}^{\vee\vee}/\mathcal{E} \end{array} \right)$$

(1) Is $\mathcal{N}(r, c, 1)$ smooth?

Why should it be? $\quad r = 1 \longleftrightarrow$ nested Hilbert schemes of $\mathbb{C}^2$

It is well-known that $\mathcal{N}(1, c, c')$ is smooth if and only if $c' = 1$ (Cheah, 1998).

(**Rmk**: $\mathcal{N}(r, 3, 2)$ are indeed singular for any $r \geq 1$)

Proposition (von Flach, Jardim, L.)

Inside $\mathcal{N}(r,c,1)$ one has a family of unobstructed points corresponding to parameters $\lambda_1,\dots,\lambda_c \in \mathbb{C}$, with $\lambda_i \neq \lambda_j$ for $i \neq j$:

$$A = B = \operatorname{diag}(\lambda_1,\dots,\lambda_c)\,; \quad I = \begin{pmatrix} 1 & & \\ \vdots & & 0_{c\times(r-1)} \\ 1 & & \end{pmatrix}; \quad J = 0\,;$$

$$F = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \qquad A' = B' = (\lambda_1).$$

(2) Is $\mathcal{N}(r,c,c')$ connected? Is it irreducible?

Idea: to study the fibers of the projection

$$\begin{array}{ccc} (E,F,\varphi) & & \mathcal{N}(r,c,c') \\ \big\downarrow & & \big\downarrow q \\ (F,\varphi) & & \mathcal{M}(r,c-c') \end{array}$$

$\leftarrow$ Nakajima's moduli space (smooth + irreducible)

THANK YOU!