Brauer tree algebras and Blocks of profinite groups

Ricardo J. Franquiz Flores,
Work together with John W. MacQuarrie

UNIVERSIDADE FEDERAL
DE MINAS GERAIS

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Framework!

- G is a finite group,
- k is a field of characteristic p with $p \mid |G|$.

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- k is a field of characteristic p with $p \mid |G|$.

$k[G]$ is not semisimple.

Blocks Decomposition- Finite case

Consider the set $\{e_i \in k[G] : 1 \leq i \leq |G|\}$ of primitive central idempotents of $k[G]$.

Then $k[G]$ has a decomposition into indecomposable direct algebra factors,

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where each factor is called *block* and each e_i is called block idempotent.

Blocks Decomposition- Finite case

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$$\begin{aligned} BU &= U \\ B'U &= 0, \forall\ B \neq B'. \end{aligned}$$

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$U = U_1 \oplus \cdots \oplus U_n$, where U_i lies in the block B_i.

Defect Groups for Blocks of Finite Groups

Let G be a finite group and A be a finite dimensional associative k-algebra.

A is a G-algebra

$$\begin{array}{ccc} G \times A & \longrightarrow & A \\ (g, a) & \longmapsto & {}^{g}a \end{array}$$

for any $a \in A$ and $g \in G$,

For every subgroup H of G the subalgebra of all H-fixed points in A is

$$A^H = \{a \in A \mid {}^{h}a = a \ \forall \ h \in H\}.$$

Defect Groups

Let H, L be subgroups of G with $H \leq L$. The **trace map** is the linear map

$$\begin{aligned} Tr_H^L : A^H &\longrightarrow A^L \\ Tr_H^L(a) &= \sum_{g \in L/H} {}^g a, \end{aligned}$$

where L/H denotes a set of coset representatives of H in L.

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Definition

Let B be a block of a finite group G with block idempotent e. A **defect group** of B is a minimal subgroup D of G such that $e \in Tr_D^G(k[G]^D)$.

Characterization of Defect Group

Brauer homomorphism: $Br_D : k[G]^D \longrightarrow k[G]^D / \sum_{Q \lneqq D} Tr_Q^D(k[G]^Q)$.

Theorem

Let D be a p-subgroup of a finite group G and B a block of G with block idempotent e. The following are equivalent:

1. *B has defect group D.*
2. *D is a maximal p-subgroup such that $Br_D(e) \neq 0$.*
3. *$e \in Tr_D^G(k[G]^D)$ and $Br_D(e) \neq 0$.*

Example:

$G = S_3 =< a, b : a^3 = b^2 = 1,\ bab^{-1} = a^1 >$ and let k be a field of characteristic 2.

$$k[G] = B_1 \times B_2 = k[G]e_1 \times k[G]e_2,$$

$$e_1 = 1 + a + a^2 \text{ and } e_2 = a + a^2$$

$Tr_{C_2}^G(e_1) = e_1$, where $C_2 =< b >$.
$Br_{C_2}(e_1) = 1 \neq 0$ and $Br_{C_2}(e_2) = 0$.

$Tr_1^G(a) = e_2$ and $Br_1(e_2) = e_2$.

Then, C_2 is defect group of B_1 and 1 is the defect group of B_2.

Blocks with cyclic defect group and Brauer tree algebras

Definition

A *Brauer tree* Γ is defined as a finite, connected, undirected graph without loops or cycles and with a cyclic ordering of the edges emanating from each vertex.

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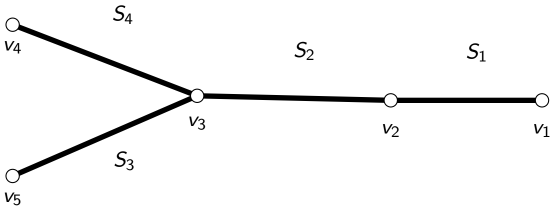
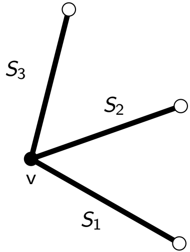


Brauer Trees

Definition

A Brauer tree Γ with an *exceptional* vertex and *multiplicity* m is a Brauer tree with a special vertex, in which the cyclic ordering will be repeated m times.

Brauer Trees



$$S_1, S_2, S_3, S_1, S_2, S_3, \ldots, S_1, S_2, S_3.$$

Let k be an algebraically closed field of characteristic p and let A be a finite dimensional k-algebra.

Definition

Given a Brauer tree Γ, we say that A is the *Brauer tree algebra* associated to Γ if

- There is a one-to-one correspondence between the edges of the tree and the isomorphism classes of simple A-modules,
-

Let k be an algebraically closed field of characteristic p and let A be a finite dimensional k-algebra.

Definition

Given a Brauer tree Γ, we say that A is the *Brauer tree algebra* associated to Γ if

-
- the top $P/rad(P)$ of the indecomposable projective A-module P is isomorphic to the socle of P,

- the projective cover P corresponding to the edge S is such that

$$rad(P)/soc(P) \cong U^v(S) \oplus U^w(S)$$

for two (possibly zero) uniserial A-modules $U^v(S)$ and $U^w(S)$, where v, w are the vertices adjacent to the edge S,

Brauer Tree Algebras

- if v is not the exceptional vertex and if v is adjacent to the edge S then $U^v(S)$ has $s(v) - 1$ composition factors, where $s(v)$ is the number of edges adjacent to v,
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Brauer Tree Algebras

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- if v is the exceptional vertex with multiplicity m, and if v is adjacent to S, then $U^v(S)$ has $m \cdot s(v) - 1$ composition factors,
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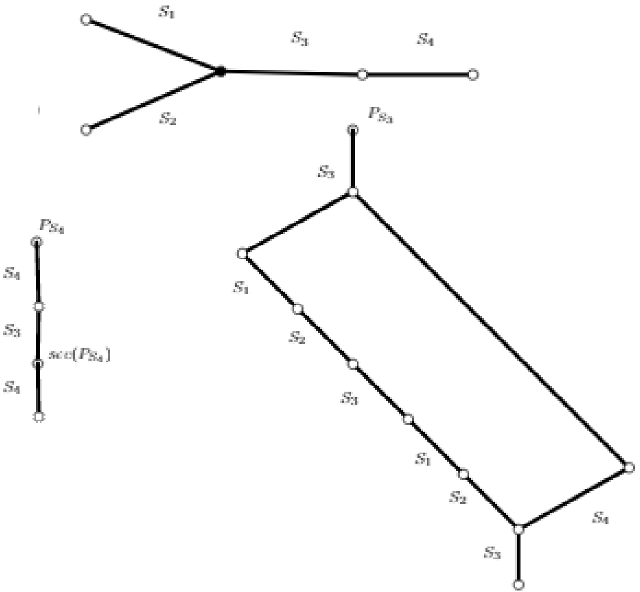
- if v is adjacent to S then the composition factors of $U^v(S)$ are described as

$$rad^j(U^v(S))/rad^{j+1}(U^v(S)) \cong \gamma_v^{j+1}(S),$$

for all j as long as j is smaller than the number of composition factors of $U^v(S)$.

Brauer Tree Algebras



-$\mathcal{S}$: A set of representatives of the isomorphism classes of simple modules in B.

-$|\mathcal{S}|$: number of elements of $\mathcal{S}$.

Theorem

Suppose k algebraically closed and B is a block of G with non-trivial cyclic defect group D. Then B is a Brauer tree algebra for a tree with $|\mathcal{S}|$ edges and multiplicity $\frac{|D|-1}{|\mathcal{S}|}$.

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What we did?

$G \rightarrow$ **Finite Group**
Block Theory of G:

- Block decomposition of $k[G]$ and finite dimensional modules,
- Defect groups,
- Blocks with cyclic defect groups, Brauer tree algebras.

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$G \to$ Profinite Group
Block Theory for G:

- Block decomposition of $k[[G]]$ and pseudocompact modules,
- Defect groups,
- Blocks with cyclic defect groups, Brauer tree algebras.

Blocks of Profinite Groups

Definition (Profinite Group)

A **Profinite Group** G is a topological group that can be expressed as inverse limit of inverse system of finite topological groups with discrete topology.

Example

- Every finite group is a profinite group.
- If G is a profinite group and I is a directed set of open normal subgroups of G, ordering by reverse inclusion, such that $\bigcap\{N : N \in I\} = 1_G$, then

$$G = \varprojlim_{N \in I} G/N.$$

Profinite Groups and Algebras

Example

- The p-adic integers $\mathbb{Z}_p$

$$\mathbb{Z}_p = \varprojlim_{i \in \mathbb{N}} \mathbb{Z}/p^i\mathbb{Z}$$

$$\begin{array}{c} \vdots \\ \downarrow \\ \mathbb{Z}/p^3\mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p^2\mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

Definition

- A profinite group is a **pro-p group** if it is the inverse limit of finite p-groups.
- A profinite group is **cyclic** if it has a dense cyclic abstract subgroup.

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$$

Definition

A pseudocompact k-algebra can be defined as an inverse limit of discrete finite dimensional algebras in the category of topological algebras.

Profinite Groups and Algebras

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Example

The *complete group algebra* of the profinite group G, is

$$k[[G]] = \varprojlim_{N \trianglelefteq_O G} k[G/N].$$

Let A be a pseudocompact k-algebra.

Definition

A **pseudocompact A-module** is a topological A-module U possessing a basis of 0 consisting of open submodules V of finite codimension that intersect in 0 and such that

$$U = \varprojlim_{V} U/V.$$

Block Decomposition

Let k be a field of characteristic p and A a pseudocompact k-algebra. Consider $E = \{e_i : i \in I\}$ be the complete set of orthogonal centrally primitive central idempotents of A.

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$$A = \prod_{i \in I} B_i,$$

where the $B_i = Ae_i$ are the blocks of A, and e_i is called block idempotent.

Block Decomposition

Let U be an A-module.

Definition

U **lies** in a block B of A if $BU = U$ and $B'U = 0$ for all $B' \neq B$.

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Proposition (F, MacQuarrie- 2021)

Let U be a pseudocompact A-module. Then

$$U = \prod_{i \in I} U_i,$$

where U_i lies in the block B_i.

Defect Groups

Relative Trace Map and Set

Let G be a profinite group and A a pseudocompact k-algebra.

A is a pseudocompact G-algebra

$$\begin{array}{ccc} G \times A & \longrightarrow & A \text{ (continuous!)} \\ (g, a) & \longmapsto & {}^{g}a. \end{array}$$

Relative Trace Map and Set

Definition

If H is a closed subgroup of G, the **subalgebra of fixed elements** of A by H is defined by $A^H = \{a : {}^h a = a, \ \forall \ h \in H\}$.

Example

$k[[G]]$ can be considered as a G-algebra with action of G given by conjugation, that is, ${}^g x = gxg^{-1}$, for each $g \in G$ and $x \in k[[G]]$.

Relative Trace Map and Set

Let A be a pseudocompact G-algebra

- If $H \leq_{O} G$

$$\begin{aligned} Tr_H^G : A^H &\longrightarrow A^G \\ a &\longmapsto \sum_{g \in G/H} {}^g a, \end{aligned}$$

where G/H denote a set of left coset representatives of H in G.

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  where G/H denote a set of left coset representatives of H in G.

- If $H \leq_{C} G$, define the **trace of** H as

$$Tr_H^G(A^H) = \bigcap_{N \trianglelefteq_O G} Tr_{HN}^G(A^{HN}) \subseteq A^G.$$

Defect groups

Recall that we treat $k[[G]]$ as a G-algebra with action given by conjugation. Let B be a block of a profinite group G with block idempotent e.

Definition

A **defect group** of B is a closed subgroup D of G such that $e \in Tr_D^G(k[[G]]^D)$ and minimal with this property.

D is a minimal element of the set

$$\{H \leq_C G : e \in Tr_H^G(k[[G]]^H)\}.$$

Characterization of Defect Groups

Brauer homomorphism: $Br_D : A^D \longrightarrow A^D / \overline{\sum_{Q \lneqq_O D} Tr_Q^D(A^Q)}$.

Theorem (F, MacQuarrie- 2021)

Let G be a profinite group, B a block of G with block idempotent e. The following are equivalent for a closed subgroup D of G:

1. *B has a defect group D.*
2. *$e \in Tr_D^G(k[[G]]^D)$ and $Br_D(e) \neq 0$.*
3. *D is a maximal pro-p subgroup of G such that $Br_D(e) \neq 0$.*

Blocks with Cyclic Defect Group and Brauer Tree Algebras

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- k will be an algebraically closed field of characteristic p.
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- $\mathcal{S}$ a set of representatives of the isomorphism classes of the simple modules in B.
- $\mathcal{P} = \{P_S : s \in \mathcal{S}\}$ a set of representatives of the isomorphism classes of indecomposable projective modules in B.

Block with Cyclic Defect Group

Observe that $\mathcal{S}$ and $\mathcal{P}$ are finite sets!. So we can take a set $\mathcal{N}$ of open normal subgroups of G acting trivially on $\mathcal{S}$.

Block with Cyclic Defect Group

Proposition (F, MacQuarrie- 2021)

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Definition

Let A be a pseudocompact k-algebra.

1. A pseudocompact A-module U is *pro-uniserial* if it can be expressed as the inverse limit of finite dimensional uniserial A-modules.

Pro-uniserial Modules

Definition

Let A be a pseudocompact k-algebra.

1. A pseudocompact A-module U is *pro-uniserial* if it can be expressed as the inverse limit of finite dimensional uniserial A-modules.
2. The simple module S is a *composition factor* of the pseudocompact module U if it a composition factor of some finite dimensional quotient of U.

Radical and Socle

Let $P = P_S \in \mathcal{P}$. Then

- The **socle** of P ($soc(P)$) is the maximal closed semisimple submodule of P.
- The **radical** of P ($rad(P)$) is the intersection of the maximal open submodules of P.

Lemma (F, MacQuarrie- 2021)

If $P = \varprojlim_N P_N$, then, for each $i \geq 1$, $rad^i(P) = \varprojlim_N rad^i(P_N)$ and $soc(P) = \varprojlim_N soc(P_N)$

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Observe that it is possible that $soc(P) = 0$.

Indecomposable Projective Modules

Proposition (F, MacQuarrie- 2021)

Fix $P \in \mathcal{P}$. There are unique pro-uniserial submodules X, Y of P satisfying the following properties:

1. $X \cap Y = soc(P)$.
2. $X + Y = rad(P)$.
3. $\frac{rad(P)}{soc(P)} \cong \frac{X}{soc(P)} \oplus \frac{Y}{soc(P)}$, *and the modules* $\frac{X}{soc(P)}, \frac{Y}{soc(P)}$ *have no composition factors in common.*

If $X = \varprojlim_N X_N$ and $Y = \varprojlim_N Y_N$, denote by

-$Fac(X) \subseteq \mathcal{S}$ the set of distinct representatives of the isomorphism classes of composition factors of X,

-$Fac(Y) \subseteq \mathcal{S}$ the set of distinct representatives of the isomorphism classes of composition factors of Y.

Now consider the set $\mathcal{N}$ of open normal subgroups of G acting trivially on each $S \in \mathcal{S}$ such that $Fac(X) = Fac(X_N)$ and $Fac(Y) = Fac(Y_N)$ for $N \in \mathcal{N}$.

$$B = \varprojlim_N B_N, \;\; D = \varprojlim_N DN/N, \;\; P = \varprojlim_N P_N$$

For each N, B_N is the Brauer tree algebra of the Brauer tree $\Gamma(B_N)$.

Brauer Trees for Blocks of the Profinite Group G

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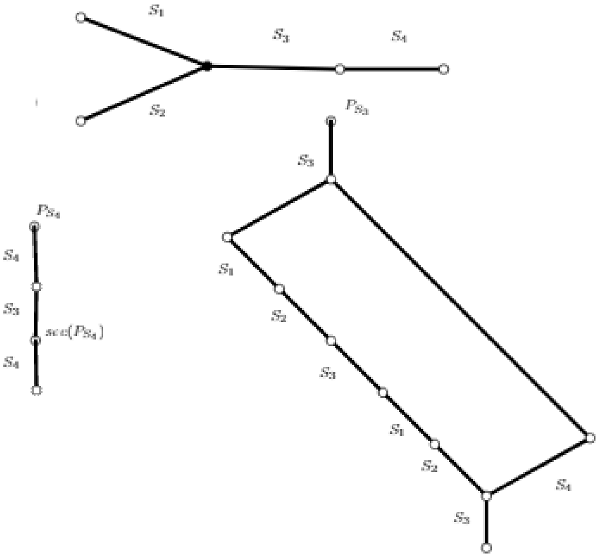

Brauer Trees for Blocks of the Profinite Group G

Lemma (F, MacQuarrie- 2021)

Via the canonical identification $\mathcal{S} = \mathcal{S}_N$*, the Brauer trees* $\Gamma(B_N)$ *are equal for each* $N \in \mathcal{N}$*, except for the multiplicity* m_N*.*

Lemma (F, MacQuarrie- 2021)

Via the canonical identification $\mathcal{S} = \mathcal{S}_N$, the Brauer trees $\Gamma(B_N)$ are equal for each $N \in \mathcal{N}$, except for the multiplicity m_N.

Define the **Brauer tree of** B to be $\Gamma(B) := \Gamma(B_N)$, for any $N \in \mathcal{N}$, except for the multiplicity m of the exceptional vertex, which is $\frac{|D|-1}{|\mathcal{S}|}$ if D is finite, or ∞ if D is infinite.

Theorem (F, MacQuarrie- 2021)

Let B be a block of a profinite group G with cyclic defect group D. Then B is the Brauer tree algebra of the Brauer tree $\Gamma(B)$ in the following sense:

1. *There is a one-to-one correspondence between the edges of $\Gamma(B)$ and the elements of $\mathcal{S}$.*
2.
3.
4.

Theorem (F, MacQuarrie- 2021)

Let B be a block of a profinite group G with cyclic defect group D. Then B is the Brauer tree algebra of the Brauer tree $\Gamma(B)$ in the following sense:

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2. *the projective cover P of the simple module corresponding to the edge S is such that*

$$rad(P)/soc(P) \cong U^v(S) \oplus U^w(S)$$

   for two (possibly zero) pro-uniserial modules $U^v(S)$ and $U^w(S)$, where v, w are the vertices adjacent to the edge S,
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3. *if v is not the exceptional vertex and if v is adjacent to the edge S then $U^v(S)$ has $s(v) - 1$ composition factors, where $s(v)$ is the number of edges adjacent to v,*
4.

Blocks as Brauer Tree Algebras

Theorem (F, MacQuarrie- 2021)

Let B be a block of a profinite group G with cyclic defect group D. Then B is the Brauer tree algebra of the Brauer tree $\Gamma(B)$ in the following sense:

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4. *if v is the exceptional vertex with multiplicity m, and if v is adjacent to S, then $U^v(S)$ has $m \cdot s(v) - 1$ composition factors if m is finite, or infinitely many if $m = \infty$,*

5. if v is adjacent to S then the composition factors of $U^v(S)$ are described as

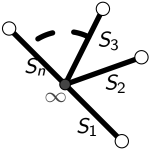
$$rad^j(U^v(S))/rad^{j+1}(U^v(S)) \cong \gamma_v^{j+1}(S),$$

for all j as long as j is smaller than the number of composition factors of $U^v(S)$,

6.

5.
6. The socle of P is zero if, and only if, S is adjacent to a vertex of infinite multiplicity. Otherwise, $soc(P) \cong S$.

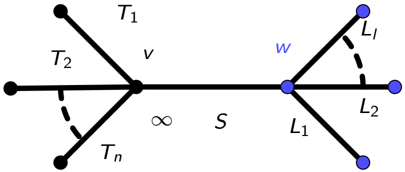
Brauer Trees Star Type



Theorem (F, MacQuarrie- 2021)

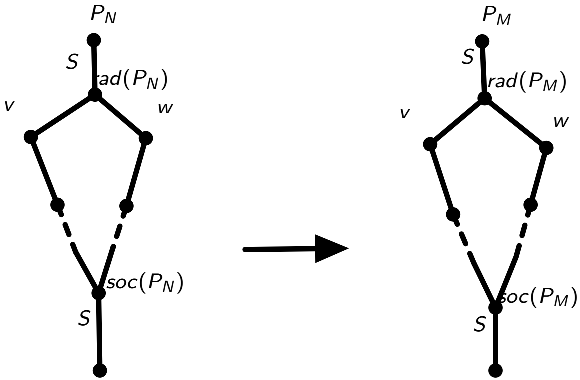
Let B be a block of a profinite group G with infinite cyclic defect group D. Then $\Gamma(B)$ is of star type.

Proof:



Proof:

$$P = \varprojlim_N P_N,\ \varphi_{MN} : P_N \to P_M \text{ if } N \leq M$$



Corollary (F, MacQuarrie- 2021)

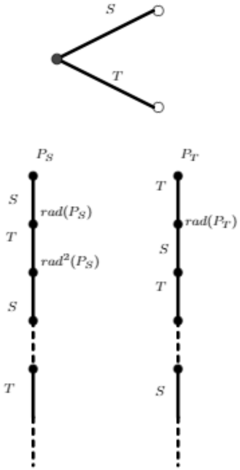
Let B be a block of a profinite group, having defect group $\mathbb{Z}_p$. Then

1. *The indecomposable projective B-modules are pro-uniserial.*
2. *B has global dimension 1.*
3. *If B has n simple modules, then B is Morita equivalent to the completed path algebra $k[[Q]]$, where Q is an oriented cycle of length n.*

Example:

- k be a field of characteristic 5,
- $G_n = < a, b \mid a^{5^n} = b^2 = 1, bab = a^{-1} >$ (dihedral group of order $2 \cdot 5^n$), $C_{5^n} = < a >$ (cyclic subgroup of order 5^n)
- $G = \varprojlim_n G_n$,
- $C = \varprojlim_n C_{5^n} \cong \mathbb{Z}_5$
- $k[[G]]$ indecomposable algebra

Example:



Next Steps!!

1. “Pseudocompact" Brauer graph algebra and properties.
2. Indecomposable projective modules of “pseudocompact" graph algebras.
   - Almost split sequence for pseudocompact modules.
   - What happen with $rad(P)/soc(P)$?

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