REALIZATIONS OF AFFINE LIE ALGEBRAS

1. VERMA TYPE MODULES

Let \( \mathfrak{a} \) be a Lie algebra with a Cartan subalgebra \( \mathfrak{h} \) and root system \( \Delta \). A closed subset \( P \subset \Delta \) is called a partition if \( P \cap (-P) = \emptyset \) and \( P \cup (-P) = \Delta \). If \( \mathfrak{a} \) is finite-dimensional then every partition corresponds to a choice of positive roots in \( \Delta \) and all partitions are conjugate by the Weyl group. The situation is different in the infinite-dimensional case. If \( \mathfrak{a} \) is an affine Lie algebra then partitions are divided into a finite number of Weyl group orbits (cf. [JK], [F2]).

Given a partition \( P \) of \( \Delta \) we define a Borel subalgebra \( \mathfrak{b}_P \subset \mathfrak{a} \) generated by \( \mathfrak{h} \) and the root spaces \( \mathfrak{a}_\alpha \) with \( \alpha \in P \). All Borel subalgebras are conjugate in the finite-dimensional case. A parabolic subalgebra is a subalgebra that contains a Borel subalgebra. If \( \mathfrak{p} \) is a parabolic subalgebra of a finite-dimensional \( \mathfrak{a} \) then \( \mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_+ \) where \( \mathfrak{p}_0 \) is a reductive Levi factor and \( \mathfrak{p}_+ \) is a nilpotent subalgebra. Parabolic subalgebras correspond to a choice of a basis \( \pi \) of the root system \( \Delta \) and a subset \( S \subset \pi \). A classification of all Borel subalgebras in the affine case was obtained in [F2]. In this case not all of them are conjugate but there exists a finite number of conjugacy classes. These conjugacy classes are parametrized by parabolic subalgebras of the underlined finite-dimensional Lie algebra. Namely, let \( \mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_+ \) a parabolic subalgebra of \( \mathfrak{g} \) containing a fixed Borel subalgebra \( \mathfrak{b} \) of \( \mathfrak{g} \). Define

\[
B_p = \mathfrak{p}_+ \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{p}_0 \otimes t\mathbb{C}[t] \oplus \mathfrak{b} \oplus \mathbb{C}c \oplus \mathbb{C}d.
\]

For any Borel subalgebra \( \mathfrak{b} \) of \( \tilde{\mathfrak{g}} \) there exists a parabolic subalgebra \( \mathfrak{p} \) of \( \mathfrak{g} \) such that \( \mathfrak{b} \) is conjugate to \( B_p \).

When \( \mathfrak{p} \) coincides with \( \mathfrak{g} \), i.e. \( \mathfrak{p}_+ = 0 \), the corresponding Borel subalgebra \( B_\mathfrak{g} \) is the standard Borel subalgebra defined by the choice of positive roots in \( \mathfrak{g} \). Another extreme case is when \( \mathfrak{p}_0 = \mathfrak{h} \). This corresponds to the natural Borel subalgebra \( B_{\text{nat}} \) of \( \tilde{\mathfrak{g}} \) considered in [JK].

Given a parabolic subalgebra \( \mathfrak{p} \) of \( \mathfrak{g} \) let \( \lambda : B_\mathfrak{p} \to \mathbb{C} \) be a 1-dimensional representation of \( B_\mathfrak{p} \). Then one defines an induced Verma type \( \tilde{\mathfrak{g}} \)-module

\[
M_\mathfrak{p}(\lambda) = U(\tilde{\mathfrak{g}}) \otimes_{U(B_\mathfrak{p})} \mathbb{C}.
\]

The module \( M_\mathfrak{g}(\lambda) \) is the classical Verma module with highest weight \( \lambda \) [K]. In the case of natural Borel subalgebra we obtain imaginary Verma modules studied in [F1]. Note that the module \( M_\mathfrak{p}(\lambda) \) is \( U(\mathfrak{p}_-)-\)free, where \( \mathfrak{p}_- \) is the opposite subalgebra to \( \mathfrak{p}_+ \). The theory of Verma type modules was developed in [F2]. It follows immediately from the definition that, unless it is a classical Verma module, Verma type module with highest weight \( \lambda \) has a unique maximal submodule, it has both finite and infinite-dimensional weight spaces and it can be obtained using the parabolic induction from a standard Verma module \( M \) with highest weight \( \lambda \) over a certain affine Lie subalgebra. Moreover, if the central element \( c \) acts non-trivially on such Verma type module then the structure of this module is completely determined by the structure of module \( M \), which is well-known ([F2], [C1]).
2. Boson type realizations of Verma modules

2.1. Finite-dimensional case. Consider first a finite-dimensional case. Let \( \mathfrak{g} = \text{Lie} G \) be a simple finite-dimensional Lie algebra with a Cartan decomposition \( \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \). Take a Borel subalgebra \( \mathfrak{b} = \mathfrak{n}^- \oplus \mathfrak{h} \). Let \( \mathfrak{b} = \text{Lie} B, \mathfrak{n}_\pm = \text{Lie} N_\pm \).

Consider the flag variety \( X = G/B \). Then \( X \) has a decomposition into open Schubert cells: \( X = \cup_{w \in W} C(w) \), where \( C(w) = B_+ w B_- \). The Weyl group and \( T = B_+ / N_+ \). The codimension of \( C(w) \) equals the length of \( w \). The subgroup \( N_+ \) acts on \( X \), and the largest orbit \( \mathcal{U} \) of this action can be identified with proper \( N_+ \). From Section 3 we know that the Lie algebra \( \mathfrak{g} \) can be mapped into vector fields on \( X \) and hence on \( \mathcal{U} \). Thus \( \mathfrak{g} \) can be embedded into the differential operators on \( \mathcal{U} \) of degree \( \leq 1 \). Since \( N_+ \cong \mathfrak{n}_+ \) via the exponential map, we conclude that \( N_+ \) can be identified with an affine space \( \Lambda^{\pm} \). Therefore, the ring of regular functions \( \mathcal{O}_\mathcal{U} \) on \( \mathcal{U} \) is just a polynomial ring in \( m = |\Delta_+| \) variables and \( \mathfrak{g} \) has an embedding into the Weyl algebra \( \mathcal{A}_m \). In particular, \( \mathcal{O}_\mathcal{U} \) is a \( \mathfrak{g} \)-module. In fact, a \( \mathfrak{g} \)-module \( \mathcal{O}_\mathcal{U} \) is isomorphic to a contragradient module \( M^+(0) \) with trivial highest weight.

Example 2.1. Let \( \mathfrak{g} = \mathfrak{sl}(2) \) with a standard basis \( e, h, [e, f] = h, [h, e] = 2e, [h, f] = -2f \). Let \( \mathfrak{b}_- = \text{span}(f, h) \). Then \( G = SL_2(\mathbb{C}) \) and the variety \( X = G/B_- \) can be identified with the projective line \( \mathbb{P}^1 \) which has a big cell \( \mathcal{U} = \mathbb{A}^1 \). Denote \( \mathcal{O}_\mathcal{U} = \mathbb{C}[x] \). Then one computes

\[
\begin{align*}
    e & \mapsto d/dx, h \mapsto -2xd/dx, f \mapsto -x^2d/dx.
\end{align*}
\]

Hence, \( M^+(0) \cong \mathbb{C}[x] \) as a \( \mathfrak{g} \)-module (see Example 10.2.1 in [FZ]).

In order to obtain a geometrical realization of Verma modules one needs to consider the minimal 1-point orbit of \( N_+ \) on \( X \).

Choosing another orbit of \( N_+ \) will give us a twisted Verma module. Twisted Verma modules are parametrized by the elements of the Weyl group and have the same character as corresponding Verma modules.

Remark 2.2. Consider again example of \( \mathfrak{sl}(2) \). Another way to get a realization of Verma module \( M(0) \) on Fock space \( \mathbb{C}[x] \) is the following. Apply to \( \mathfrak{sl}(2) \) an automorphism which is a composition of two anti-involutions: \( e \leftrightarrow f \), \( h \) is fixed, and \( x \leftrightarrow d/dx \). Then it gives the following realization in second order differential operators: \( f \mapsto x, h \mapsto -2xd/dx, e \mapsto -x(d/dx)^2 \).

2.2. Affine case. Consider now the loop algebra \( \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \). Sometimes it is more convenient to consider a completion of \( \hat{\mathfrak{g}} \) substituting the Laurent polynomials by the Laurent power series (for a geometric interpretation, but it is irrelevant to us. So jushy ignore the series). We will denote this Lie algebra by \( \hat{\mathfrak{g}}((t)) \). The corresponding loop group will be denoted by \( \hat{G}((t)) \). Fix a Cartan decomposition \( \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \) and consider a Borel subalgebra \( \mathfrak{b}_\pm = \mathfrak{n}_\pm \oplus \mathfrak{h} \). Denote

\[
\hat{\mathfrak{n}}_\pm = (\mathfrak{n}_\pm \oplus 1) \oplus (\mathfrak{g} \otimes t^\pm \mathbb{C}[t^\pm]),
\]

\[
\hat{\mathfrak{b}}_\pm = \hat{\mathfrak{n}}_\pm \oplus \mathfrak{h} \otimes \mathbb{C}[t].
\]

Let \( \hat{N}_\pm \) and \( \hat{B}_\pm \) be Lie groups corresponding to \( \hat{\mathfrak{n}}_\pm \) and \( \hat{\mathfrak{b}}_\pm \) respectively. Consider a flag variety \( X = \hat{G}((t))/\hat{B}_- \) which has a structure of a scheme of infinite type. As in the finite-dimensional case \( X \) splits into \( \hat{N}_+ \)-orbits of finite codimension, parametrized by the affine Weyl group. There is an analogue of a big cell \( \mathcal{U} \) in \( X \) which is a projective limit of affine spaces, and
hence, the ring of regular functions $O_U$ on $U$ is a polynomial ring in infinitely many variables. Thus $\mathcal{G}((t))$ acts on it by differential operators providing a realization for the contragradient Verma module with zero highest weight. Global sections of more general $\hat{N}_+$-equivariant sheaves on $X$ will produce an arbitrary highest weight. Other $\hat{N}_+$-orbits in $X$ correspond to twisted contragradient Verma modules. A striking difference with the finite-dimensional case is that we can not obtain standard Verma modules this way. They can be obtained considering $\hat{N}_+$-orbits on $G((t))/B_+$.

3. First free field realization

In the previous section we considered the case of classical Verma modules for affine Lie algebras. Consider the completion $b_{nat}$ in $\mathcal{G}((t))$ of the natural Borel subalgebra $n_- \otimes \mathbb{C}[t, t^{-1}] \oplus \mathcal{H} \otimes \mathbb{C}[t^{-1}]$. If $N_-$ is the Lie group corresponding to $n_-$ then $\mathcal{B}_{nat} = N_-((t))\mathcal{H}[[t^{-1}]]$ is the Borel subgroup corresponding to $b_{nat}$. Let $X = G((t))/\mathcal{B}_{nat}$. The difference with the classical case is that $X$ is not a scheme. This structure is called the semi-infinite manifold $[FZ]$, [V1]. It can be viewed as the space of maps from $Spec \mathbb{C}((t))$ to the finite-dimensional flag variety $G/B_-$. We can consider the $\hat{N}_+$-orbits on the semi-infinite manifold and, in particular, $\hat{N}_+((t))$ can be viewed as an analogue of the big cell $U$ in $G/B_-$. 

**Example 3.1.** ([FZ], 10.3.6). For simplicity we will only consider the case $\mathfrak{g} = \mathfrak{sl}(2)$. The corresponding semi-infinite manifold can be thought as $\mathbb{P}^1((t))$. The big cell $\mathfrak{h} \subset \mathbb{P}^1$ can be lifted to a big cell $\mathfrak{h}^+((t)) = \left\{ \left( x(t) - 1 \right)^t \right\}$, which coincides with the space of functions $F \simeq \mathbb{C}((t))$ on the punctured disc with the chosen coordinate $t$ on the disc. Denote

$$e_n = e \otimes t^n, \quad h_n = h \otimes t^n, \quad f_n = f \otimes t^n, \quad n \in \mathbb{Z}.$$ 

Then the corresponding representation by vector fields on $F$ is the following

$$e_n \mapsto \partial x_n, \quad h_n \mapsto -2 \sum_{m \in \mathbb{Z}} x_m \partial x_{n+m}, \quad f_n \mapsto - \sum_{m,k \in \mathbb{Z}} x_m x_k \partial x_{n+m+k}.$$ 

Let $V = \mathbb{C}[x_m, m \in \mathbb{Z}]$. It is clear that the differential operators corresponding to $f_n$ are not well-defined on $V$ (they take values in some formal completion of $V$). One way to deal with this problem is to apply the anti-involutions:

$$e_n \leftrightarrow f_n, \quad h_n \leftrightarrow h_n; \quad x_n \leftrightarrow \partial x_n, \quad n \in \mathbb{Z}$$

which gives the following formulas:

$$f_n \mapsto \partial x_n, \quad h_n \mapsto -2 \sum_{m \in \mathbb{Z}} x_{n+m} \partial x_m, \quad e_n \mapsto - \sum_{m,k \in \mathbb{Z}} x_{n+m+k} \partial x_m \partial x_k.$$ 

These formulas define the first free field realization of $\hat{sl}(2)$ in the polynomial ring $\mathbb{C}[x_m, m \in \mathbb{Z}]$. This module is, in fact, a quotient $M(0)$ of the imaginary Verma module with trivial highest weight by a submodule generated by the elements $h_n \otimes 1, \quad n < 0$.

A boson type realization of the imaginary Verma module for $\hat{sl}(2)$ with a non-trivial central action was obtained by Bernard and Felder in the Fock space $\mathbb{C}[x_m, m \in \mathbb{Z}] \otimes \mathbb{C}[y_n, n > 0]$: 

**3. First free field realization**
leads to the construction of Wakimoto modules $[W]$. 

Lie algebras in $[C2]$ providing a realization of imaginary Verma modules.

\[ J_\delta \]

\[ y \]

\[ J \]

\[ a \]

\[ y \]

\[ x \]

\[ 4 \]

\[ f_{n} \mapsto x_{n}, \ h_{n} \mapsto -2 \sum_{m \in \mathbb{Z}} x_{m+n}\partial x_{m} + \delta_{n<0}y_{-n} + \delta_{n>0}2nK\partial y_{n} + \delta_{n,0}J, \]

\[ e_{n} \mapsto - \sum_{m,k \in \mathbb{Z}} x_{k+m+n}\partial x_{k}\partial x_{m} + \sum_{k>0} y_{k}\partial x_{-k-n} + 2K \sum_{m>0} m\partial y_{n}\partial x_{m-n} + (K\partial J)\partial x_{-n}. \]

This module is irreducible if and only if $K \neq 0$. If we let $K = 0$ and quotient out the submodule generated by $y_{m}, m > 0$ then the factormodule is irreducible if and only if $J \neq 0$ (cf. [F1]). This construction has been generalized for all affine Lie algebras in [C2] providing a realization of imaginary Verma modules.

4. Second free field realization

There is another way to correct the formulas obtained in Example 6.1, which leads to the construction of Wakimoto modules $[W]$.

Denote $a_{n} = \partial x_{n}, a_{n}^{*} = x_{-n}$ and consider formal power series

\[ a(z) = \sum_{n \in \mathbb{Z}} a_{n}z^{-n-1}, \quad a^{*}(z) = \sum_{n \in \mathbb{Z}} a_{n}^{*}z^{-n}. \]

Series $a(z)$ and $a^{*}(z)$ are called formal distributions. It is easy to see that $[a_{n}, a_{m}^{*}] = \delta_{n+m,0}$ and all other products are zero. The formulas in Example 6.1 can be rewritten as follows:

\[ e(z) \mapsto a(z), \quad h(z) \mapsto -2a^{*}(z)a(z), \quad f(z) \mapsto -a^{*}(z)^{2}a(z), \]

where $g(z) = \sum_{n \in \mathbb{Z}} g_{n}z^{-n-1}$ for $g \in \{c, f, h\}$. This realization is not well-defined since the annihilation and creation operators are in a wrong order. It becomes well-defined after the application of two anti-involutions described above. Then the formulas read:

\[ f(z) \mapsto a(z), \quad h(z) \mapsto 2a(z)a^{*}(z), \quad f(z) \mapsto -a(z)a^{*}(z)^{2}, \]

where $a_{n}$ and $a_{n}^{*}$ have the following meaning now $a_{n} = x_{n}, a_{n}^{*} = -\partial x_{-n}$. This is our quotient of the imaginary Verma module.

A different approach was suggested by Wakimoto ([W]) who introduced the normal ordering. Denote

\[ a(z)_{-} = \sum_{n<0} a_{n}z^{-n-1}, \quad a(z)_{+} = \sum_{n\geq0} a_{n}z^{-n-1} \]

and define the normal ordering as follows

\[ :a(z)b(z): = a(z)_{-}b(z) + b(z)a_{+}(z). \]

Let now

\[ a_{n} = \{ x_{n}, \quad n < 0 \}, \quad a_{n}^{*} = \{ x_{-n}, \quad n \leq 0 \}, \quad b_{m} = \{ m\partial y_{m}, \quad m > 0 \}, \quad y_{-m} = \{ \partial x_{-n}, \quad n > 0 \}. \]

Here $[a_{n}, a_{m}^{*}] = [b_{n}, b_{m}] = \delta_{n+m,0}$.

**Theorem 4.1.** ([W]). The formulas

\[ c \mapsto K, \quad e(z) \mapsto a(z), \quad h(z) \mapsto -2 : a^{*}(z)a(z) : + b(z), \]

\[ f(z) \mapsto - : a^{*}(z)^{2}a(z) : + K\partial z a^{*}(z) + a^{*}(z)b(z) \]
define the second free field realization of the affine \( sl(2) \) acting on the space \( \mathbb{C}[x_n, n \in \mathbb{Z}] \otimes \mathbb{C}[y_m, m > 0] \).

These modules are celebrated Wakimoto modules. They were defined for an arbitrary affine Lie algebra by Feigin and Frenkel [FF1], [FF2]. Generically Wkimoto modules are isomorphic to Verma modules.

5. Intermediate Wakimoto Modules

So far we considered two extreme cases of Borel subalgebras in the affine Lie algebras: standard and natural. But if the rank of the second free field realizations. For affine Lie algebras of type \( A \) rearranging the annihilation and creation operators as it was done in the first and the second free field realizations. Modules \( \text{sl}(n+1) \) this has been accomplished in [CF], where a series of boson type realizations was constructed depending on the parameter \( 0 \leq r \leq n \). If \( r = n \) this construction coincides with the construction of Wakimoto modules. On the other hand when \( r = 0 \) the obtained representation gives a Fock space realization described in [C2].

Let \( 0 \leq r \leq n, \gamma \in \mathbb{C}^*, k = \gamma^2 - (r + 1) \). Let \( H_i, E_i, F_i, i = 1, \ldots, n \) be the standard basis for \( \mathfrak{g} = \text{sl}(n+1) \). Denote \( X_m = r^m \otimes X \) for \( X,Y \in \mathfrak{g} \) and \( m \in \mathbb{Z} \). Let \( \{\alpha_1, \ldots, \alpha_n\} \) be a basis for \( \Delta^+ \), the positive set of roots for \( \mathfrak{g} \), such that \( H_i = \delta_i \) and let \( \Delta_r \) be the root system with basis \( \{\alpha_1, \ldots, \alpha_r\} \) (\( \Delta_r = \emptyset \), if \( r = 0 \)) of the Lie subalgebra \( \mathfrak{g}_r = \text{sl}(r+1) \). Denote by \( \mathcal{H}_r \) a Cartan subalgebra of \( \mathfrak{g} \), spanned by \( H_i, i = 1, \ldots, r \). Set \( \mathcal{H}_0 = 0, \mathcal{H}_r = \mathcal{H}_r \oplus \mathbb{C}c \oplus \mathbb{C}d \).

Denote by \( E_{im}, F_{im}, H_{im}, i = 1, \ldots, n, m \in \mathbb{Z} \), the generators of the loop algebra corresponding to \( \mathfrak{g} \).

Let \( \mathfrak{a} \) be the infinite dimensional Heisenberg algebra with generators \( a_{ij,m}, a^*_{ij,m}, \) and \( 1, 1 \leq i \leq j \leq n \) and \( m \in \mathbb{Z} \), subject to the relations

\[
[a_{ij,m}, a_{kl,n}] = [a^*_{ij,m}, a^*_{kl,n}] = 0, \\
[a_{ij,m}, a^*_{kl,n}] = \delta_{kl}\delta_{ij}\delta_{m+n,0}1, \\
[a_{ij,1}, 1] = [a^*_{ij,1}, 1] = 0.
\]

This algebra acts on \( \mathbb{C}[x_{ij,m}|i,j,m \in \mathbb{Z}, 1 \leq i \leq j \leq n] \) by

\[
a_{ij,m} \mapsto \begin{cases} \\
\partial/\partial x_{ij,m} & \text{if } m \geq 0, \text{ and } j \leq r \\
x_{ij,m} & \text{otherwise,}
\end{cases}
\]

\[a^*_{ij,m} \mapsto \begin{cases} \\
x_{ij,-m} & \text{if } m \leq 0, \text{ and } j \leq r \\
-\partial/\partial x_{ij,-m} & \text{otherwise.}
\end{cases}
\]

and \( 1 \) acts as an identity. Hence we have an \( \mathfrak{a} \)-module generated by \( v \) such that

\[
a_{ij,m} v = 0, \quad m \geq 0 \text{ and } j \leq r, \quad a^*_{ij,m} v = 0, \quad m > 0 \text{ or } j > r.
\]

Let \( \mathfrak{a}_r \) denote the subalgebra generated by \( a_{ij,m} \) and \( a^*_{ij,m} \) and \( 1 \), where \( 1 \leq i \leq j \leq r \) and \( m \in \mathbb{Z} \). If \( r = 0 \), we set \( \mathfrak{a}_r = 0 \).

Let \( ((\alpha_i|\alpha_j)) \) be the Cartan matrix for \( \text{sl}(n+1) \) and let

\[
\mathfrak{B}_{ij} := (\alpha_1|\alpha_j)(\gamma^2 - \delta_{i>r}\delta_{j>r}(r+1) + r^2\delta_{i,r+1}\delta_{j,r+1}).
\]
Let $\mathfrak{b}$ be the Heisenberg Lie algebra with generators $b_{jm}, 1 \leq i \leq n, m \in \mathbb{Z}$, $1$, and relations $[b_{jm}, b_{jp}] = m \mathfrak{b}_{ij} \delta_{m+p,0} 1$ and $[b_{jm}, 1] = 0$.

For each $1 \leq i \leq n$ fix $\lambda_i \in \mathbb{C}$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then the algebra $\hat{\mathfrak{b}}$ acts on the space $\mathbb{C}[y, m]$, $m \in \mathbb{N}^+$, $1 \leq i \leq n$ by

$$
b_{j0} \mapsto \lambda_i, \quad b_{j-m} \mapsto e_i \cdot y_m, \quad b_{jm} \mapsto m e_i \cdot \frac{\partial}{\partial y_m} \quad \text{for } m > 0$$

and $1 \mapsto 1$. Here

$$y_m = (y_{1m}, \ldots, y_{nm}), \quad \frac{\partial}{\partial y_m} = \left( \frac{\partial}{\partial y_{1m}}, \ldots, \frac{\partial}{\partial y_{nm}} \right)$$

and $e_i$ are vectors in $\mathbb{C}^n$ such that $e_i \cdot e_j = \mathfrak{b}_{ij}$ where $\cdot$ means the usual dot product.

For any $1 \leq i \leq j \leq n$, we define

$$a_{ij}^*(z) = \sum_{n \in \mathbb{Z}} a_{ij,n}^* z^{-n}, \quad a_{ij}(z) = \sum_{n \in \mathbb{Z}} a_{ij,n} z^{-n-1}$$

and

$$b_i(z) = \sum_{n \in \mathbb{Z}} b_{in} z^{-n-1}.$$

Then

$$[b_i(z), b_j(w)] = \mathfrak{b}_{ij} \partial_z \delta(z - w), \quad [a_{ij}(z), a_{kl}^*(w)] = \delta_{ik} \delta_{jl} 1 \delta(z - w),$$

where

$$\delta(z - w) = z^{-1} \sum_{n \in \mathbb{Z}} \left( \frac{z}{w} \right)^n.$$

is the formal delta function.

Set

$$a_{ij}(z)_+ = a_{ij}(z), \quad a_{ij}(z)_- = 0$$

$$a_{ij}^*(z)_+ = 0, \quad a_{ij}^*(z)_- = a_{ij}^*(z),$$

if $j > r$.

Denote $\mathbb{C}[x] = \mathbb{C}[x_{ij,m}|i,j,m \in \mathbb{Z}, 1 \leq i \leq j \leq n]$ and $\mathbb{C}[y] = \mathbb{C}[y_{i,m}|i,m \in \mathbb{N}^*, 1 \leq i \leq n]$.

**Remark 5.1.** Note that

$$a_{ij}(z) a_{kl}^*(z) := \sum_{m \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} a_{ij,n} a_{kl,m-n}^* \right) z^{-m-1}$$

is well defined on $\mathbb{C}[x] \otimes \mathbb{C}[y][[z, z^{-1}]]$ for all $l > r$ or if $l \leq r$ and $j \leq r$.

Let $\mathfrak{b}_r$ be the Borel subalgebra of $\mathfrak{G}$ corresponding to a parabolic subalgebra of $\mathfrak{G}$ whose semisimple part of the Levi factor is $\mathfrak{G}_r$.

Fix $\lambda \in \mathcal{H}^*$ and let $M_r(\lambda)$ be the Verma type module associated with $\mathfrak{b}_r$ and $\tilde{\lambda}$. When $r = n$ this module is a standard Verma module while in the case $r = 0$ we get an imaginary Verma module. Denote by $v_\lambda$ the generator of $M_r(\lambda)$.

Let $\lambda_r = \lambda|_{\mathfrak{h}^*_r}$. The module $M_r(\lambda)$ contains a $\mathfrak{G}_r$-submodule $M(\lambda_r) = U(\mathfrak{g})/(v_\lambda)$ which is isomorphic to the standard Verma module for $\mathfrak{G}_r$. If $\lambda(\mathfrak{c}) \neq 0$ then the submodule structure of $M_r(\lambda)$ is completely determined by the submodule structure of $M(\lambda_r)$ [C1], [FS].
Define
\[ E_i(z) = \sum_{n \in \mathbb{Z}} E_{in} z^{-n-1}, \quad F_i(z) = \sum_{n \in \mathbb{Z}} F_{in} z^{-n-1}, \quad H_i(z) = \sum_{n \in \mathbb{Z}} H_{in} z^{-n-1}, \]
for \(1 \leq i \leq n\).

**Theorem 5.2.** ([CF]). Let \( \lambda \in \mathfrak{h}^* \) and set \( \lambda_i = \lambda(H_i) \). The generating functions are
\[ F_i(z) \mapsto a_{ii} + \sum_{j=i+1}^{n} a_{ij} a_{i+1,j}, \]
\[ H_i(z) \mapsto 2 : a_{ii} a_{ii}^* : + \sum_{j=i+1}^{n-1} \left( : a_{jj} a_{jj}^* : - : a_{j,i-1} a_{j,i-1}^* : \right) + \sum_{j=i+1}^{n} \left( : a_{ij} a_{ij}^* : - : a_{i+1,j} a_{i+1,j}^* : \right) + b_i, \]
\[ E_i(z) \mapsto a_{ii}^* \left( \sum_{k=1}^{i-1} a_{k,i-1} a_{k,i-1}^* - \sum_{k=1}^{i} a_{k} a_{k}^* \right) + \sum_{k=1}^{i} a_{i+1,k} a_{i+1,k}^* - \sum_{k=1}^{i-1} a_{k,i-1} a_{k,i-1}^* - a_{ii}^* b_i - \left( \delta_{i \geq r} (r + 1) + \delta_{i < r} (i + 1) - \gamma^2 \right) \partial a_{ii}^*, \]
where \( c \mapsto \gamma^2 - (r + 1) \)
define a representation on the Fock space \( \mathbb{C}[x] \otimes \mathbb{C}[y] \). In the above \( a_{jj}, a_{ij}^* \) and \( b_i \) denotes \( a_{ij}(z), a_{ij}^*(z) \) and \( b_i(z) \) respectively.

This boson type realization of \( \tilde{\mathfrak{s}\mathfrak{l}}(n+1) \) depends on the parameter \( 0 \leq r \leq n \) and defines a module structure on the Fock space \( \mathbb{C}[x] \otimes \mathbb{C}[y] \) which is called an intermediate Wakimoto module. Denote it by \( W_{n,r}(\lambda, \gamma) \) and consider a \( \tilde{\mathfrak{g}} \)-submodule \( W = U(\tilde{\mathfrak{g}})[(v_\lambda, \partial)] \cong W_{n,r}(\lambda, \gamma) \) of \( W_{n,r}(\lambda, \gamma) \). Then \( W \) is isomorphic to the Wakimoto module \( W_{\lambda, \gamma} \), \( \gamma \in \mathfrak{h}^* \), \( [\mathfrak{f}\mathfrak{f}^2] \), where \( \lambda(r) = \lambda|_{\mathfrak{h}^*}, \gamma = \gamma^2 - (r + 1) \). Since generically Wakimoto modules are isomorphic to Verma modules, intermediate Wakimoto modules provide a realization for generic Verma type modules.

**References**


8 REALIZATIONS OF AFFINE LIE ALGEBRAS


