REPRESENTATIONS OF POSETS IN THE CATEGORY OF UNITARY SPACES

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Abstract. Representations of partially ordered sets (posets) in the category of linear spaces were introduced in the late sixties of XX century. Firstly we will recall basic definitions and results. We investigate a similar theory in the category of unitary spaces. There exists not many posets for which it is possible to classify all unitary representations up to the unitary equivalence, and the classification in that cases is rather simple (we will discuss these results). Therefore we impose the additional condition on representations (so-called orthoscalarity conditions). It turned out that orthoscalar unitary representations are strictly connected with stable linear representations. We will discuss some results for orthoscalar unitary representations which follow studying the stable linear representations of posets. I finish this note with some open questions.

0. Representations of posets in linear spaces

A poset $\mathcal{P}$ is a finite set $= \{1, \dots, n\}$ with a *partial order* $\prec$ which of course has not to coincide with the usual order on $\{1, \dots, n\}$.

To a poset $\mathcal{P}$ we associate a poset $\mathcal{P}^0 = \mathcal{P} \cup \{0\}$, with maximal element 0, i.e. $i \prec 0$, $i \in \mathcal{P}$. Then *a quiver $Q_{\mathcal{P}}$ associated to* $\mathcal{P}$ is the Hasse quiver of $\mathcal{P}^0$.

Example 1. If $\mathcal{P} = \{1, 2, 3, 4\}$ with no-relations between the points then its quiver $Q_{\mathcal{P}}$ has the following form

$$\begin{tikzcd}
 & & & 0 & & & \\
1 \arrow[urrr] & 2 \arrow[urr] & & & & 3 \arrow[ull] & 4 \arrow[ulll]
\end{tikzcd}$$

The notion of poset representations was introduced by Nazarova and Roiter (see for example [NR]). A *matrix-representations of $\mathcal{P}$ over a field k* is a block matrix

$$\mathcal{A} = [A_1| \dots |A_n]$$

over k. Two representations $\mathcal{A} = [A_1| \dots |A_n]$ and $\tilde{\mathcal{A}} = [\tilde{A}_1| \dots |\tilde{A}_n]$ are *equivalent* if $\mathcal{A}$ can be reduced to $\tilde{\mathcal{A}}$ by

- *elementary row transformations;*
- *elementary column transformations within blocks A_i;*
- *additions of linear combinations of columns of A_i to columns of A_j if $i \prec j$.*

Almost *equivalent definition* (up to a finite number of indecomposables) of representations is the following: *a representation* is a collection of k-vector spaces $(V_0; V_i)_{i\in\mathcal{P}}$ in which $V_i \subset V_0$ and $V_i \subset V_j$ if $i \prec j$. Two representations $(V_0; V_i)_{i\in\mathcal{P}}$ and $(\tilde{V}_0; \tilde{V}_i)_{i\in\mathcal{P}}$

are *equivalent* if there exist an invertible map $g : V_0 \to \tilde{V}_0$ such that $g(V_i) = \tilde{V}_i$ for all $i \in \mathcal{P}$. Yet another way to define the representations is to consider the representations of the quiver $Q_{\mathcal{P}}$ bounded by all commutative relation generated by non-oriented cycles in $Q_{\mathcal{P}}$, see [Simson, Chapter 14].

To a poset we associate a quadratic form (called *Tits form*)

$$q_{\mathcal{P}}(x_0; x_1, \ldots, x_n) = x_0^2 + \sum_{i=1}^{n} x_i^2 + \sum_{i \prec j} x_i x_j - \sum_{i=1}^{n} x_0 x_i;$$

A quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$ is said to be *weakly positive* (resp. *non-negative*) if $q(z) > 0$ (resp. $q(z) \geq 0$) for all non-zero $z = (z_1, \ldots, z_n) \in \mathbb{Z}^n$ with $z_i \geq 0$.

Recall two classification theorems. The first one is is due to Kleiner (see, for example, [Simson, Chapter 10]).

Theorem 1. *(Kleiner) The following conditions are equivalent*

- *A poset $\mathcal{P}$ is representation-finite;*
- *Tits form $q_{\mathcal{P}}$ is weakly positive;*
- *A quiver $Q_{\mathcal{P}}$ of a poset $\mathcal{P}$ does not contain any of the following critical quivers*

$$\begin{tikzcd}
& \circ \arrow[d] & \\
\circ \arrow[r] & \circ & \circ \arrow[l] \\
& \circ \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \circ \arrow[d] & & \\
& & \circ \arrow[d] & & \\
\circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
& & & \circ \arrow[d] & & & \\
\circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l]
\end{tikzcd}$$

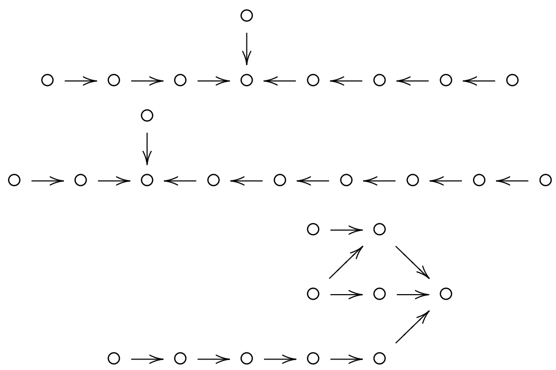
$$\begin{tikzcd}
& & \circ \arrow[d] & & & & & \\
\circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \circ \arrow[r] & \circ \arrow[dr] & \\
& & \circ \arrow[r] \arrow[ur] & \circ \arrow[r] & \circ \\
\circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[ur] &
\end{tikzcd}$$

The second one is due to L.Nazarova (see, for example, [Simson, Chapter 15]).

Theorem 2. *(Nazarova) The folowing conditions are equivalent*

- *A poset $\mathcal{P}$ is representation-tame;*
- *Tits form $q_{\mathcal{P}}$ is weakly non-negative;*
- *A quiver $Q_{\mathcal{P}}$ of the poset $\mathcal{P}$ does not contain any of the following critical quivers*

$$\begin{tikzcd}
\circ \arrow[dr] & \circ \arrow[d] & \\
\circ \arrow[r] & \circ & \circ \arrow[l] \\
& \circ \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \circ \arrow[d] & & \\
\circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] \\
& \circ \arrow[u] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \circ \arrow[d] & & & \\
& & \circ \arrow[d] & & & \\
\circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l]
\end{tikzcd}$$



1. Representations of posets in unitary spaces

From now on, all representations that we consider are over the field of complex numbers. We give the definitions similarly as in linear case. A *unitary representation* is a collection of complex vector spaces $(U_0; U_i)_{i\in\mathcal{P}}$ in which U_0 is a unitary space, $U_i \subset U_0$ and $U_i \subset U_j$ if $i \prec j$. Two representations $(U_0; U_i)_{i\in\mathcal{P}}$ and $(\tilde{U}_0; \tilde{U}_i)_{i\in\mathcal{P}}$ are *unitarily equivalent* if there exists a unitary map $g : U_0 \to \tilde{U}_0$ such that $g(U_i) = \tilde{U}_i$ for all $i \in \mathcal{P}$. A representations is *unitary indecomposable* if it is not equivalent to an orthogonal direct sum of two non-zero representations. Note also, that one can also give an analogue of matrix definition of representations.

Define a unitary Tits form $w_{\mathcal{P}}$ by

$$w_{\mathcal{P}}(x_0; x_1, \ldots, x_n) = x_0^2 + 2\sum_{i=1}^{n} x_i^2 + 2\sum_{i\prec j} x_i x_j - 2\sum_{i=1}^{n} x_0 x_i.$$

A *chain* is a linearly ordered poset. A *semi-chain* is a poset of the form $\mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i$, in which each $\mathcal{P}_i$ consists of either one point or two incomparable points, and $p^{(1)} \prec p^{(2)} \prec \cdots \prec p^{(k)}$ for all $p^{(i)} \in \mathcal{P}_i$.

The following theorems give the classification of posets by their unitary representations types ([BFSY-12]).

Theorem 3. *(Bondarenko, Futorny, Sergeichuk, Yusenko) The folowing conditions are equivalent*

- *A poset $\mathcal{P}$ is unitarily representation-finite;*
- *A poset $\mathcal{P} + \mathcal{P}$ is representation-finite;*
- *A poset $\mathcal{P}$ is linearly ordered (chain);*
- *Unitary Tits form $w_{\mathcal{P}}$ is weakly positive.*

Each indecomposable unitary representation of the chain has the following form

$$0 \to 0 \to \cdots \to \mathbb{C} \to \mathbb{C} \to \cdots \to \mathbb{C}$$

Theorem 4. *(Bondarenko, Futorny, Sergeichuk, Yusenko) The folowing conditions are equivalent*

- *A poset $\mathcal{P}$ is unitarily representation-tame;*

- *A poset $\mathcal{P}+\mathcal{P}$ is representation-tame;*
- *A poset $\mathcal{P}$ is a semi-chain;*
- *Unitary Tits form $w_{\mathcal{P}}$ is weakly non-negative.*

Let us show the description of indecomposable representations of the semi-chain in the case when it consists of two incomparable points. Apart from obvious one-dimensional representation there exists an infinite number of non-equivalent two-dimensional representations:

$$\begin{tikzcd} & \mathbb{C}\langle e_1, e_2\rangle & \\ \mathbb{C}\langle e_1\rangle \arrow[ur] & & \mathbb{C}\langle e_1+\lambda e_2\rangle \arrow[ul] \end{tikzcd}$$

Such representations are non-equivalent for different values of λ and there are no other indecomposable representations. There are no indecomposable representations in higher dimensions. Note that the description of all representations of each semi-chain looks similarly to this example.

If the poset is not a semi-chain then it is unitarily wild. Roughtly speaking that means that it contains a problem of classification of the pair of self-adjoint matrices under unitary equivalence, which is unsolved (see [OS] for more details).

So we study the representations which satisfy some additional conditions

2. Orthoscalar representations of posets

Let $\chi = (\chi_0; \chi_i)_{i\in\mathcal{P}} \in \mathbb{R}_+^{|\mathcal{P}|+1}$. We say that a unitary representation $(U_0; U_i)_{i\in\mathcal{P}}$ is χ-orthoscalar if the following condition holds

$$\chi_1 P_{U_1} + \cdots + \chi_n P_{U_n} = \chi_0 I,$$

where P_{U_i} is the orthogonal projection on subspace U_i. Representations of this kind in similar context as well as a terminology were introduced in the series of papers by Kruglyak, Nazarova and Roiter (see [KNR], references therein and other papers in ArXiv). Such relations appear in many areas of mathematics, see for example [Klyachko, Totaro, Wun] and references therein.

My aim is to make a survey on results concerning χ-orthoscalar unitary representations which follows studying the linear representation of special type.

If we take a trace from two side of last equality we get

$$\chi_1 \dim U_1 + \ldots + \chi_n \dim U_n = \chi_0 \dim U_0. \tag{2.1}$$

Let us take a subspace $M \subset U_0$ and let P_M be corresponding orthoprojection on M. It is not hard to prove that we have

$$\chi_1 \dim U_1 \cap M + \ldots + \chi_n \dim U_n \cap M < \chi_0 \dim M, \tag{2.2}$$

if representations is indecomposable. If a given system $(V_0; V_i)_{i\mathcal{P}}$ of subspaces satisfies (2.1) and (2.2) we say that it is χ-*stable.*

So, *if* indecomposable unitary representation is $(U_0; U_i)_{i\in\mathcal{P}}$ *is χ-orthoscalar then* the corresponding linear representations $(U_0; U_i)_{i\in\mathcal{P}}$ *is χ-stable.* In fact it is also vice versa, in the sense that given a χ-stable linear representations $(V_0; V_i)_{i\in\mathcal{P}}$ it is possible to choose a scalar product in V_0 in such a way that this representation becomes χ-orthoscalar. Note that this stability coincides with the stability on corresponding bound quiver $Q_{\mathcal{P}}$ introduced by A.King in [King], see [WY11] for the details.

Moreover, we have the following

Proposition 5. *If two χ-orthoscalar representations are equivalent as linear representations then they are unitarily equivalent.*

Using this observation and studying stability for linear representations the following result was obtained (see [SY11])

Theorem 6. *(Samoilenko, Yusenko) The following conditions are equivalent*

- *A poset $\mathcal{P}$ is representation-finite;*
- *For any weight χ, a poset $\mathcal{P}$ has just finite number of non-equivalent χ-orthoscalar representations.*

In one direction this theorem follows from previous Proposition. In other direction one should show the existence of infinite number of χ-stable linear representations for critical posets.

Also we have the following result (see [WY11])

Theorem 7. *(Weist, Yusenko) Assume that $\mathcal{P}$ is representation-wild than there exists a weight χ such that a poset $\mathcal{P}$ has an infinite number of non-equivalent χ-orthoscalar representations which depends on an arbitrary number of complex parameters.*

The last theorem says that if $\mathcal{P}$ is representation-wild then the classification of unitarily non-equivalent χ-orthoscalar can be a hard problem. But the following conjecture is still remains unsolved

Conjecture 1. *Assume that $\mathcal{P}$ is representation-wild. There exists a weight χ such that the classification of χ-orthoscalar representations of $\mathcal{P}$ is unitarily representation-wild problem.*

As it was mentioned to build a χ-orthoscalar representation one can find χ-stable linear representation. So we naturally ask:

Question 1. *Under which condition a given linear representation is stable with some weight?*

One of the necessary conditions is that it has to be schurian (i.e., with trivial endomorphism ring), but it is not sufficient as next statements show.

- Every indecomposable representation of a poset $\mathcal{P}$ is stable with some weight iff $\mathcal{P}$ is representation-finite (Gruchevoy, Yusenko [GY10]);
- If a poset $\mathcal{P}$ is representation-tame and primitive than every schurian representation of $\mathcal{P}$ is stable with some weight (Hille, de la Peña [HdlP-02]);

- For some non-primitive posets $\mathcal{P}$ and some dimension vectors d every schurian representation of $\mathcal{P}$ in dimension d is stable with some weight (Yusenko, Weist [WY11]);
- If a poset $\mathcal{P}$ is representation-wild then there exist schurian representations which are not stable with any weight (see [WY11]).

The following conjecture reminds unsolved (which finishes the classification):

Conjecture 2. *Every Schurian representation of representation-tame poset is stable with some weight.*

3. Another open problems

In [Dr74] there were introduced Coxeter functors for representations of posets (see the original paper for the definitions).

Question 2. *Is it true that Drozd's Coxeter functors map stable representations into stable ones?*

Differentiation of a representation of a poset were introduced by Nazarova and Roiter for 'matrix language'. We recall the construction for 'subspace language' which was given originally by Gabriel (see [Gabriel72]).

Let c be a maximal element in $\mathcal{P}$. Denote by $c^{\triangledown} = \{i \in \mathcal{P} \mid i \preceq c\}$. We say that $\mathcal{P}$ is *differentiable* with respect to a maximal element c if the width of the poset $\mathcal{P} - c^{\triangledown}$ is less then 3. Define a poset $\mathcal{P}'_c$ as a union of the poset $\mathcal{P} - \{c\}$ and the set

$$\{r \vee s \mid r, s \in \mathcal{P},\ \text{and } r, s, c \text{ are incomparable}\}$$

On $\mathcal{P}'_c$ we have an induced partial order. For example if $\mathcal{P}$ has the following form

$$\begin{tikzcd}
4 & 5 & 6 \\
1 \arrow[u] & 2 \arrow[ul] \arrow[u] & 3 \arrow[u]
\end{tikzcd}$$

then $\mathcal{P}'_4$ has the form

$$\begin{tikzcd}
 & & 5 \vee 6 & \\
 & 5 \arrow[r] & 5 \vee 3 \arrow[u] & 6 \arrow[ul] \\
1 & 2 \arrow[u] & & 3 \arrow[ul] \arrow[u]
\end{tikzcd}$$

Define a functor (see also [Gabriel72])

$$\partial_c : \mathcal{P} - sp \to \mathcal{P}'_c - sp$$

by the formula $\partial_c((V_0; V_i)_{i \in \mathcal{P}}) = (V'_0; V'_i)_{i \in \mathcal{P}'_c}$, where

$$\begin{aligned}
V'_0 &= V_c, \\
V'_i &= V_i \cap V_c; \quad \text{if } i \in \mathcal{P}, \\
V'_{r \vee s} &= (V_r + V_s) \cap V_c.
\end{aligned}$$

For example if V has the form

$$\begin{tikzcd}
\langle e_{123}, e_1\rangle & \langle e_1, e_2\rangle & \langle e_2, e_3\rangle \\
\langle e_{123}\rangle \arrow[u] & \langle e_1\rangle \arrow[u] \arrow[ul] & \langle e_3\rangle \arrow[u]
\end{tikzcd}$$

then V' is of the form

$$\begin{tikzcd}
 & & \langle e_{123}, e_1\rangle & \\
 & \langle e_1\rangle \arrow[r] & \langle e_{123}, e_1\rangle \arrow[u] & \langle e_{23}\rangle \arrow[ul] \\
\langle e_{123}\rangle & \langle e_1\rangle \arrow[u] & & 0 \arrow[u] \arrow[ul]
\end{tikzcd}$$

where $e_{n_1\dots n_k}$ denotes $e_{n_1} + \dots + e_{n_k}$, and e_j is j-th coordinate vector.

See [Gabriel72] and [Simson] for the properties of this functor.

Question 3. *Is it true that Differentiation functor maps stable representations into stable ones?*

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