① **Duflo isomorfism**

Let $\mathfrak{g}$ — Lie algebra,
$U(\mathfrak{g})$ — its universal enveloping,
$S(\mathfrak{g})$ — its symmetric algebra.
Using PBW theorem we have that

$$I_{PBW}: S(\mathfrak{g}) \to U(\mathfrak{g})$$
$$X_1 \odot \ldots \odot X_n \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

is an isomorfism of vector spaces.

$\mathfrak{g}$ acts on $U(\mathfrak{g})$ by ad-action.

$$ad_X(u) = Xu - uX, \quad X \in \mathfrak{g},\ u \in U(\mathfrak{g}).$$

$\mathfrak{g}$ —"— on $S(\mathfrak{g})$ by ext. of ad-action on itself.

$$ad_X(Y^n) = n \cdot [X, Y] \cdot Y^{n-1}, \quad X, Y \in \mathfrak{g}.$$

or given a monomial $Y_1 \odot \ldots \odot Y_n \in S(\mathfrak{g})$

$$ad_X(Y_1 \odot \ldots \odot Y_n) = \sum_i Y_1 \odot \ldots \odot [X, Y_i] \odot \ldots \odot Y_n.$$

Exercise Show that

$$\begin{tikzcd} S(\mathfrak{g}) \arrow[r, "I_{PBW}"] \arrow[d, "ad_X"'] & U(\mathfrak{g}) \arrow[d, "ad_X"] \\ S(\mathfrak{g}) \arrow[r, "I_{PBW}"'] & U(\mathfrak{g}) \end{tikzcd}$$

is commutative for any $X \in \mathfrak{g}$, i.e.

$$ad_X \circ I_{PBW} = I_{PBW} \circ ad_X .$$

It follows that,

$$I_{PBW}: S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} U(\mathfrak{g})^{\mathfrak{g}}$$

is a vector space isomorfism.

Exercise 2 Show that ~~Z(U)~~

$$Z(U(\mathfrak{g})) \cong U(\mathfrak{g})^{\mathfrak{g}}.$$

center

Remark Unfortunatelly I_{PBW} is not (!) an algebra isomorfism. (!).

Theorem (Duflo)

$$S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{J} S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{I_{PBW}} U(\mathfrak{g})^{\mathfrak{g}}$$ — is an isomorfism

$J \in Aut(S(\mathfrak{g}))$

Duflo element

Remark.

- Duflo theorem works for any fin.-dim. Lie algebra.
- For a semi-simple Lie algebra it is just a Harish-Chandra theorem.

① **Harish-Chandra map**

Let $\mathfrak{g}$ be semi-simple Lie algebra.

$$\mathfrak{g} = \underset{\bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha}{\underset{\|}{\mathfrak{n}^-}} \oplus \mathfrak{h} \oplus \underset{\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha}{\underset{\|}{\mathfrak{n}^+}} \text{ — Cartan decomp.}$$

PBW implies that

not direct sum (!)

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus \left(U(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^- U(\mathfrak{g})\right)$$

Exercise 3 Show that if $z \in Z(U(\mathfrak{g}))$

$$\Rightarrow z \in U(\mathfrak{h}) \oplus \left(U(\mathfrak{g})\mathfrak{n}^+ \cap \mathfrak{n}^- U(\mathfrak{g})\right).$$

Let $\gamma: U(\mathfrak{g}) \to \underset{S(\mathfrak{h})}{\underset{\cong}{U(\mathfrak{h})}}$ — be a projection on first factor

Let $\rho = \frac{1}{2} \sum_{\alpha_i \in \Delta^+} \alpha_i$, consider

$$\tau: \mathfrak{h} \to S(\mathfrak{h})$$
$$H \mapsto H - \rho(H) \cdot 1,$$

τ extends to an automorfism $\tau: S(\mathfrak{h}) \to S(\mathfrak{h})$.

Theorem (Harish-Chandra) $S(\mathfrak{g})^{\mathfrak{g}} \supseteq S(\mathfrak{h})^{W}$

The map

$$\tilde{\gamma} = \gamma \circ \tau : \mathcal{U}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{h})^{W}$$

is an ~~endo~~ isomorfism of algebras in which $S(\mathfrak{h})^{W}$ – is a set of invariant elements w.r.t. Weyl group W.

$XY = [X,Y] + YX = H + YX$

$\Rightarrow \Omega = \frac{1}{2}H^2 + H + 2YX, \quad \gamma(\Omega) = \frac{1}{2}H^2 + H.$

$\Delta^+(sl_2) = \{\alpha\}, \quad W_{sl_2} = \{id, H \to -H\}.$

$\Rightarrow \tau : H \to H - \rho(H)\cdot 1 = H - \frac{1}{2}\alpha(H)\cdot 1 = \underline{H-1}$ ($\alpha(H) = 2$)

$\Rightarrow \tilde{\gamma} = \frac{1}{2}(H-1)^2 + (H-1) = \frac{1}{2}H^2 - \frac{1}{2} = \frac{1}{2}(H^2-1)$

(W-invariant)!

(a) $\tilde{\gamma} = \gamma \circ \sigma$. Let $z_1, z_2 \in Z(\mathcal{U}(\mathfrak{g}))$

$z_1 - \gamma(z_1) \in \mathcal{U}(\mathfrak{g})n_+$

$z_2 - \gamma(z_2) \in \mathcal{U}(\mathfrak{g})n_+$

$z_1 \cdot z_2 - \gamma(z_1)\gamma(z_2) = z_1(\underbrace{z_2 - \gamma(z_2)}_{\in\, \mathcal{U}(\mathfrak{g})n_+}) + \gamma(z_2)\cdot(\underbrace{z_1 - \gamma(z_1)}_{\in\, \mathcal{U}(\mathfrak{g})n^+})$

But $\mathcal{U}(\mathfrak{g})n^+$ is an ideal $\Rightarrow$

$z_1 \cdot z_2 - \gamma(z_1)\cdot\gamma(z_2) \in \mathcal{U}(\mathfrak{g})n^+$

(b) Let $\lambda \in \mathfrak{h}^*$ be a weight. Consider a Verma module

$$Verm(\lambda - \rho) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}(\lambda - \rho)$$

Ex. Any $z \in Z(\mathcal{U}(\mathfrak{g}))$ acts trivially on $Verm(\lambda - \rho)$, i.e. if v_+ is highest weight vector then $z \cdot v_+ = \underbrace{\varphi_\lambda(z)}_{\in\, \mathbb{C}} v_+$

② sl_2 – example

sl_2 – is a 3-dim Lie algebras with gen. X, Y, H with relations

$[H,X] = 2X, \quad [H,Y] = -2Y, \quad [X,Y] = H.$

Q. $Z(\mathcal{U}(sl_2))$ – ?.

Exercise Show that $\Omega = \frac{1}{2}H^2 + XY + YX$ generates $Z(\mathcal{U}(sl_2))$

(Casimir element)

③ Proof of Harish-Chandra thm.

Strategy:

a) $\tilde{\gamma}$ – is algebra homomorfism;

b) $\tilde{\gamma} : \mathcal{U}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{h})^{W}$;

c) $\tilde{\gamma}$ – is a bijection.

But also we have that

$z_1 z_2 - \gamma(z_1 z_2) \in \mathcal{U}(\mathfrak{g})n_+$

hence

$\underbrace{\gamma(z_1 z_2) - \gamma(z_1)\gamma(z_2)}_{0} \in \mathcal{U}(\mathfrak{g})n^+$

($\gamma(z_1 z_2) \in \mathcal{U}(\mathfrak{h})$, $\gamma(z_1)\gamma(z_2) \in \mathcal{U}(\mathfrak{h})$)

$\Rightarrow \gamma(z_1 z_2) = \gamma(z_1)\cdot\gamma(z_2)$

$\Rightarrow \gamma$ – is homomorfism.

σ – obviously, $\Rightarrow \tilde{\gamma} = \gamma \circ \sigma$ – is homomorf.

And also we have that

$$\varphi_\lambda(z) = \lambda(\tilde{\gamma}(z)) = \tilde{\gamma}(z)(\lambda).$$

Rem. This holds for any cyclic, highest weigh module of $\mathfrak{g}$.

Let α_i be a simple root, and

$$m = \langle \lambda, \alpha_i \rangle.$$

Hence $Y_i^m v_+ \in \mathrm{Verm}(\lambda-\rho)_\mu$ with

$$\mu = \lambda - \rho - m\alpha_i = s_i\lambda - \rho$$

For $sl(2)$ algebra generated by X_i, Y_i the vector v_+ is highest weight vector with weight $m - 1 = \langle \lambda - \rho, \alpha_i \rangle$.

Hence $X_i \cdot Y_i^m \cdot v_+ = 0$, and using $[X_j, Y_i] = 0$, if $i \neq j$ we have

$X_j \cdot Y_i^m \cdot v_+ = 0$. $\Rightarrow$ n^+ anihilates $Y_i^m v_+$

$\Rightarrow$ $Y_i^m \cdot v_+$ – is highest weight vector with the weight $s_i\lambda - \rho = \mu$

Let M – be highest weight vector it generates.

M_1 – highest weight, cyclic

$\Rightarrow$ Z acts trivially and

$$\varphi_{s_i\lambda}(z) = \varphi_\lambda(z)$$

or

$$(\tilde{\gamma}(z))(w\lambda) = (\tilde{\gamma}(z))(\lambda), \; w \in W.$$

$$\Rightarrow \tilde{\gamma} : \underset{\mathcal{U}(g)^g}{Z(\mathcal{U}(g))} \to S(h)^W.$$

ⓒ <u>Outline</u> <u>of</u> <u>bijectivity</u>

Supose that we prove

$f : S(g)^g \xrightarrow{\sim} S(h)^W$ – prove this

We know that

$$s : \mathcal{U}(g)^g \xrightarrow{\sim} S(g)^g$$

Let $S_k(g) = S^0(g) \oplus S^1(g) \oplus \ldots \oplus S^k(g)$ be a filtration on g.

Hence it induces a filtration on $S(h)$

<u>Ex.</u> Show that for any $z \in \mathcal{U}_k(g) \cap Z(g)$ we have

$$(f \circ s)(z) \equiv \tilde{\gamma}(z) \mod S_{k-1}(g)$$

$\Rightarrow$ proves that γ is an isom. inductively

<u>Aim</u> is to

Construct an ~~map~~ isomorfism

$$f : S(g)^g \xrightarrow{\sim} S(h)^W$$

<u>Obs</u> We can construct an isom. f^* between $S(g^*)^g$ and $S(h^*)^W$ and then use <u>Killing form argument</u>

$S(g^*)^g$ – is a polinomial algebra on g.

given $P = X_{i_1}^* \ldots X_{i_k}^* \in S(g^*)^g$, $x_{i_j} \in g^*$

it acts as

$$(x_{i_1}^* \ldots x_{i_k}^*)(X) = X_{i_1}^*(X) \ldots X_{i_k}^*(X), \; \forall X \in g$$

Let $\pi : g \to GL(V)$ be a fin. dim. representation of g. Consider a symmetric function

$$(X_1, \ldots X_k) \mapsto \sum_{\sigma \in S_k} \mathrm{tr}(\pi(X_{\sigma(1)}) \ldots \pi$$

g acts on $S(g^*)$ via contragredient

$$Y F(X_1 \ldots X_k) = F([Y, X_1], \ldots X_n) + \ldots F(X_1, \ldots [Y, X_k])$$

aplying to $F(X_1 \ldots X_n) = \mathrm{tr}\,\pi(X_1) \ldots \mathrm{tr}\,\pi(X_n)$ easy to see that $YF = 0$

or $X \mapsto \mathrm{tr}\,\pi(X)^n \in S(g^*)^g$.

If $F \in S(g^*)^g$ is invariant then in particular it is invariant under automorfism

$$\tau_i = (\exp \mathrm{ad}\, e_i)(\exp \mathrm{ad} - f_i)(\exp \mathrm{ad}\, e_i)$$

hence restriction map

$$f^*: S(\mathfrak{g}^*) \to S(\mathfrak{h}^*)$$

is in fact

$$f^*: S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{h}^*)^W.$$

Easy to show that:

- f^* is an injection
- $S(\mathfrak{h}^*)^W$ is spanned by

all functions of the form $x \to \mathrm{tr}\,\pi(x)^k$

$\pi \in \mathrm{Rep}(\mathfrak{g})$, $k \in \mathbb{Z}^+$.

$\Rightarrow$ f^* – is isomorfism.

$\Rightarrow$ gives rise to a isomorfism $f: S(\mathfrak{g})^{\mathfrak{g}} \to S(\mathfrak{h})^W$.

This proves the theorem.