Algebras, quivers and adjoint functors

Lecture 3 "Quiver $\rightleftarrows$ Algebra adjunction" (joint with John MacQuarrie)

Recall:

Lecture 1. "Adjoint" functors.

A pair
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$
is called adjoint pair if $\xi_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \simeq \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ for any $X \in Ob(\mathcal{C})$ and $Y \in Ob(\mathcal{D})$ which is "natural" in both variables.

Lecture 2. Given finite, acyclic quiver Q we get finite-dim algebra kQ — (path algebra).

Given an algebra A we associate a $GQ(A)$ its Gabriel quiver:

Receipt: vertices: the complete set of primitive idempotents $\{e_1, \dots, e_n\}$

arrows: between e_i and e_j are elements of basis of $e_i\, J(A)/J^2(A)\, e_j$

Let $\underline{\text{Quiv}}$ be a category of finite ~~acyclic~~ quivers ~~quivers~~

$\underline{\text{objects}}$: finite quivers

$\underline{\text{morfisms}}$: inclusion maps of quivers

$\underline{\text{SBAlg}}$: Category of basic algebras (Recall that $\underline{\text{basic}}$ means $A/J(A) \cong \prod k$)

$\underline{\text{objects}}$. Basic algebras.

$\underline{\text{Morfisms}}$: Algebra surjective homs.

The correspondence $Q \longmapsto k[Q]$ defines a contravariant functor $k[-]: \underline{\text{Quiv}}^{ae} \to \underline{\text{SBAlg}}$ as any $\beta: Q \hookrightarrow R$ defines

$$k[\beta]: k[R] \twoheadrightarrow k[Q]$$

$$p \longmapsto \begin{cases} \text{preimage}, & p \text{ in } p(Q) \\ 0, & \text{otherwise.} \end{cases}$$

But the Gabriel quiver construction, does not (!) define a functor $\text{SBAlg} \to \text{Quiv}$ as not unique (!)

- choice of idempotents
- choice of basis in $e_i \, J(A)/J^2(A) \, e_j$ not uniq

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① $\underline{\text{Factor}}$ category $\underline{\text{SBAlg}}_{\sim}$

Let $\mathcal{C}$ be an arbitrary category.

let $\sim$ is given on morfisms $\text{Mor}(\mathcal{C})$

That is $\text{Mor}(X, Y)$ is a union of equivalence classes which satisfy $[\alpha]=[\alpha'] \Rightarrow [\beta\alpha]=[\beta\alpha']$ and $[\alpha\beta]=[\alpha'\beta]$

when the composition makes sense.

Therefore we form a category $\mathcal{C}/\sim$ which will be called quotient-category.

Objects of $\mathcal{C}/\sim$ are the same as in $\mathcal{C}$

Morfisms: equivalence classes of morfisms.

$$\mathrm{Mor}_{\mathcal{C}/\sim}(X,Y) = \mathrm{Mor}(X,Y)/\sim.$$

with composition $[\beta][\alpha] = [\beta\alpha]$.

Def. Let $A,B \in \underline{\mathrm{SBAlg}}$, $\alpha_1, \alpha_2 \in \mathrm{Mor}(A,B)$

Define α_1, α_2 to be n-depth, denoting that by $\alpha_1 \sim_n \alpha_2$ if

$$(\alpha_1 - \alpha_2)(J^i(A)) \subseteq J^{i+1}(B), \quad 0 \leq i \leq n.$$

with $J^0(A) = A$.

Exer $\sim_n$ defines equivalence relation on $\mathrm{Mor}(\underline{\mathrm{SBAlg}})$.

$$\underline{\mathrm{SBAlg}}_n = \underline{\mathrm{SBAlg}}/\sim_n.$$

Let $\Pi_n : \underline{\mathrm{SBAlg}} \twoheadrightarrow \underline{\mathrm{SBAlg}}_n$ be corresponding quotient functor.

② Category of Vquivers

Finite Vquiver is given by:

- finite set of vertices $VQ_0^* = \{*\} \cup VQ_0$

  $VQ_0 = \{e_1, \ldots, e_n\}$

- For any pair of verties $e, f \in VQ_0^*$, finite-dim vector space $VQ_{e,f}$, such that $VQ_{*,e} = VQ_{e,*} = 0$ for any e in VQ_0.

Denote by Σ_{VQ} free k-module generated by VQ_0, considered as semisimple algebra via
$$e_i \cdot e_j = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}, \quad e_i, e_j \in VQ_0.$$

By VQ_1 we denote direct sum $\bigoplus_{e,f \in VQ_0} VQ_{e,f}$ which is Σ_{VQ}-Σ_{VQ} bimodule.

Morfisms of finite Vquivers $\rho: VQ \to VR$ given by:

- pointed map $\rho_0: VQ_0^* \to VR_0^*$ (i.e. $\rho_0(*) = *$) which is bijection on elements non-going to $*$.
- linear maps $\rho_{e,f}: VQ_{e,f} \to VR_{\rho_0(e), \rho_0(f)}$ for any pair of verties $e, f \in VQ_0$

We say that ρ is surjective if $\rho_{e,f}$ surjective.

SVQuiv - category with:

Objects: finite Vquivers

Morfisms: surjective homs of Vquivers.

VQuiver is called acyclic if there exist $n>0$, such that
$$VQ_1 \otimes_{\Sigma} \Sigma \cdots \otimes_{\Sigma} \Sigma \, VQ_1 = 0.$$

$SVQuiv^{ac}$ be a full category of acyclic quivers.

We have a natural functor

$$V[-]: Quiv \longrightarrow SVQuiv$$
$$Q \longrightarrow (\{*\} \underset{*}{\cup} Q_0, VQ_1)$$

$VQ_{i,j} = k[\text{arrows betwen } i \to j]$.

$V[-]$ is a contravariant functor.

(3) Functor "path algebra".

let (Σ, V) be any pair, in which

- Σ - algebra
- V - Σ-Σ bimodule.

Connect $T(\Sigma, V) = \Sigma \oplus V \oplus V \underset{\Sigma}{\otimes}_{\Sigma} V \oplus \ldots \oplus V \otimes \ldots \otimes V \ldots$

calling tensor algebra.

Prop. (Universal property of tensor algebra) V - Σ-Σ bimodule.

let A, Σ be k-algebras.

let $\varphi_0 : \Sigma \to A$
$\varphi_1 : V \to A$, in which

φ_0 - is algebra hom
φ_1 - is bimodu hom (considering A as bimodule giving by φ_0)

$\Rightarrow \exists!$ algebra hom $T(\Sigma, V) \to A\}$, such that
$\varphi|_{\Sigma} = \varphi_0$, $\varphi|_V = \varphi_1$

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let $VQ = (VQ_0^{\vee}, VQ_{e,t})$ be a "finite acyclic Vquiver.

Define $k[VQ] = T(\Sigma_{VQ}, VQ_1)$ - basic, finite-dim

$\Rightarrow k[VQ] \in \underline{SBAlg}$

Let $p: VQ \to VR$ - surjective Vquiver map.
Therefore this generates two maps

$$\varphi_0 : \Sigma_{VQ} \twoheadrightarrow \Sigma_{VR} \subset k[VR]$$

$$\varphi_1 : VQ_1 \to VR_1 \subset k[VR]$$

$\Rightarrow \exists!$ homomorfism

$$k[p] : k[VQ] \twoheadrightarrow k[VR].$$

<u>Prop 2</u> Construction above gives rise to covariant functors

$$k[-] : \underline{S\,VQuiv}^{ac} \longrightarrow \underline{SBAlg}$$

$$k_n[-] = \Pi_n \circ k[-] : \underline{S\,VQuiv}^{ac} \longrightarrow \underline{SBAlg}$$

(4) Functor "Gabriel Quiver"
Let A be finite-dim algebra.

<u>Theorem</u> (Wedderburn-Maltcev).

- There exists a subalgebra Σ in A such that $A = \Sigma \oplus J(A)$ (as k-vector spaces) and $\Sigma \simeq A/J(A)$ (as algebras)
- For any two subalgebras Σ and Σ' such that $A = \Sigma \oplus J(A) = \Sigma' \oplus J(A)$, there exists $w \in J(A)$ such that

$$\Sigma' = (1+w)\,\Sigma\,(1+w)^{-1}.$$

let $A \in \underline{SBAlg}$ we define

$GQ(A) \in \underline{SVQuiv}$. The "key" idea is to define the vertices of $GQ(-)$ as the orbits of certain action of $J(A)$.

Any element $w \in J(A)$ defines an automorfism

$$a \longmapsto \underbrace{(1+w)\, a\, (1+w)^{-1}}_{{}^{(1+w)}a \, ''}, \quad a \in A.$$

Denote By

~~Denote~~ by $G(A) \triangleleft \operatorname{Inn Aut}(A)$, the group of all such automorfisms, And by

$$G_a = \{ {}^{1+w}a \mid w \in J(A) \} \text{ — orbit of } a \in A \text{ under } G(A).$$

let A be any basic finite dim algebra.
And $s: A/J \to A$ any split of $\pi: A \twoheadrightarrow A/J$.
$P(s) = \{ s(j_1), \dots, s(j_n) \}$ — primitive idemp of A corresponding to split s, $j_1, \dots, j_n$ is a basis in A/J.

Define $GQ(A)$ by

$$GQ(A)_0 := \{*\} \,\dot{\cup}\, \{ G_e \mid e \in P(s) \}.$$

$$GQ(A)_{G_e, G_f} := e \, {J(A)}/{J^2(A)} \, f, \text{ for fixed } e, f \in P(s).$$

<u>Prop.</u> $GQ(A)$ is well-defined, i.e. does not define on the choice of ~~splits~~ split s.

Let $A, B \in \underline{SBAlg}$, with $\mathcal{G}(A) = G$
$\mathcal{G}(B) = H$.

Given a morfism $\alpha: A \to B$ we define a morfism $GQ(\alpha): GQ(A) \to GQ(B)$ by

$$GQ(\alpha)({}^{G}e) = {}^{H}\alpha(e)$$

$$GQ(\alpha)_{{}^{G}e,\,{}^{G}f} : e^{J(A)}/_{J^2(A)}{}^{f} \to \alpha(e)^{J(B)}/_{J^2(B)}{}^{\alpha(f)}$$

$$e(j + J^2(A))f \mapsto \alpha(e)(\alpha(j) + J^2(B))\alpha(f).$$

<u>Proposition</u> Everything is well-defined. And we have a functor

$$GQ(-): \underline{SBAlg} \to \underline{SVquiv}$$

<u>Ex.</u>

$$\begin{tikzcd} \underline{SBAlg} \arrow[r, "GQ(-)"] \arrow[d, "\Pi_n"'] & \underline{SVQuiv} \\ \underline{SBAlg_n} \arrow[ur, "GQ_n(-)"'] \arrow[u, phantom, shift left=6ex, "\circlearrowleft"] & \end{tikzcd}$$

– there is unique functor such that diagram commutes

(5) "<u>Adjunction</u>" Let $\underline{SBAlg}^{ac}$ be a full subcategory of $\underline{SBAlg}$ such that $GQ(A)$ is acyclic.

we have

$$\begin{tikzcd} \underline{Quiv}^{ac} \arrow[r, "V[-]"] & \underline{SVQuiv}^{ac} \arrow[r, bend left, "k[-]"] \arrow[dr, bend right, "k_1[-]"'] & \underline{SBAlg}^{ac} \arrow[l, bend left, "GQ(-)"] \arrow[d, "\Pi_1"] \\ & & \underline{SBAlg_1} \arrow[ul, bend left=40, "GQ_1(-)"] \end{tikzcd}$$

<u>Theorem</u>

$k_1[-]$ is left adjoint to $GQ_1(-)$.