Formal deformations (continue)

0) Remembering

Given a k-algebra (associative) A, we consider two structures associated with A:

$\boxed{\text{Structure 1}}$ Formal deformations:

Associative $k[[t]]$-bilinear maps:

$$m : A[[t]] \times A[[t]] \longrightarrow A[[t]],$$

such that for any $a, b \in A$

$$m(a,b) = a \cdot b + \sum_{i=1}^{\infty} m_i(a,b)\, t^i$$

$$m(m(a,b),c) - m(a, m(b,c)) = 0$$

$\boxed{\text{Structure 2}}$ Poisson structure:

Bracket $\{\cdot,\cdot\} : A \times A \to A$ such that

a) $\{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\} = 0$ — Jacobi

b) $\{a, b\cdot c\} = \{a,b\}\cdot c + b\cdot\{a,c\}$ — on product as !

Formal deformations $\xrightarrow[\text{rise}]{\text{gives}}$ Poisson bracket

$$m : A[[t]]^2 \to A[[t]] \longmapsto \{a,b\} = \frac{1}{2}\left(m_1(a,b) - m_1(b,a)\right)$$

Deformation Quantization Problem:

Poisson bracket on A $\{\cdot,\cdot\}: A^2 \to A$ $\xrightarrow{?}$ Formal deformation s.t. $\{a,b\} = \frac{1}{2}(m_1(a,b) - m_1(b,a))$

Remark. Does not work in general, i.e. there are Poisson algebras with bracket which cannot come from f.d.

Theorem (Kontsevich, 97). If $A = C^\infty(M)$, algebra of smooth functions on differentiable manifold then

Every Poisson bracket $\xrightarrow{\text{gives rises}}$ associative formal deformation of A.

(1) Kontsevich's explicit formula

Lemma Given a Poisson bracket $\{\cdot,\cdot\}: C^\infty(M)^2 \to C^\infty(M)$ there are unique smooth functions α^{ij}, $1 \le i < j \le n$, such that
$$\{f,g\} = \sum_{i<j} \alpha^{ij}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\right)$$

We find a formal deformation of $C^\infty(M)$ in a form

$$f * g = f \cdot g + \sum_{n=1}^{\infty} B_n(f,g) \cdot \frac{t^n}{n!}$$

$B_n(f,g)$ — diferentional operator of f, g, α^{ij}

which is given in terms of special quivers:

We define G_n to be the set of quivers Γ, s.t.:

1) $\Gamma_0 = \{1,\dots,n\} \sqcup \{L,R\}$ — vertices in Γ

2) $\Gamma_1 = \{a_1,\dots,a_n, b_1,\dots,b_n\}$ — arrows in Γ.

3) $s(a_i)=s(b_i)=i \longrightarrow$ starting vertices of a_i, b_i are i.

4) Γ has neither loops nor doble arrows.

For instance

$$\begin{tikzcd}
 & 3 \arrow[dl, "b_3"'] \arrow[dr, "a_3"] & 4 \arrow[l, "b_4"'] \arrow[d, "a_4"] \\
1 \arrow[d, "a_1"'] \arrow[urr, bend left, "b_1"] & & 2 \arrow[ll, "a_2"] \arrow[d, "b_2"] \\
L & & R
\end{tikzcd} \in G_4$$

Exercicio Show that $|G_n| = (n\cdot(n+1))^n$.

Given $\Gamma \in G_n$ we define

$$B_{\Gamma,\alpha}(f,g) = \sum \prod_{i=1}^{n} \left(\prod_{a\in\Gamma(?,i)} \partial_{I(a)}\right)\alpha^{I(a_i),I(b_i)} \prod_{a\in\Gamma(?,L)} \partial_{I(a)} f \cdot \prod_{a\in\Gamma(?,R)} \partial_{I(a)} g .$$

$I:\{a_1,\dots,b_n\} \to \{1,\dots,d\}$

$$\sum \left(\partial_{i_2}\partial_{j_3}\alpha^{i_1,j_1}\right)\left(\partial_{i_3}\partial_{i_4}\alpha^{i_2,j_2}\right)\left(\partial_{j_3}\alpha^{i_3,j_3}\right)\left(\partial_{j_1}\alpha^{i_4,j_4}\right)\left(\partial_{i_1} f\right)\left(\partial_{j_2} g\right).$$

$I:\{a_1,a_2,a_3,\dots,b_4\} \to \{1,\dots,d\}.$

For example: $M=\mathbb{R}^2$. Γ — $L \xleftarrow{a_1} 1 \xrightarrow{b_1} R$.

$$B_{\Gamma,\alpha} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}$$

We define $\mathcal{B}_n = \sum_{\Gamma \in G_n} \omega_\Gamma B_{\Gamma,\alpha}$

Now we will construct ω_Γ.

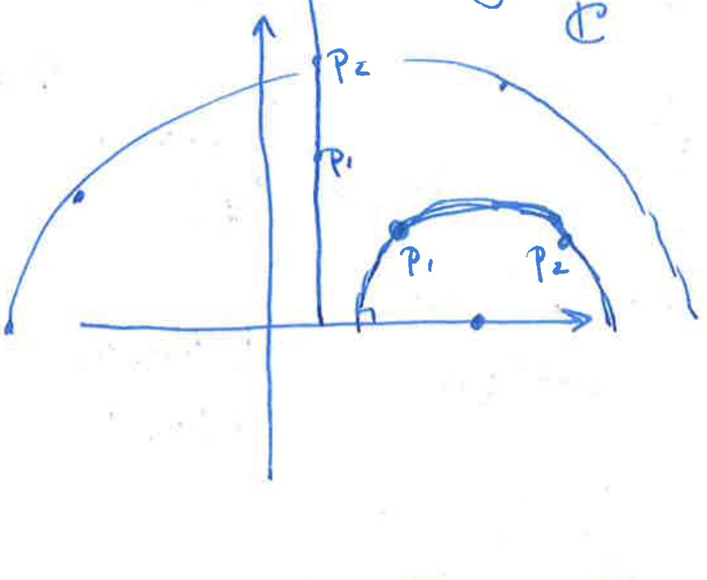
Let $\mathcal{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$ – upper half-plane in $\mathbb{C}$.

We endow $\mathcal{H}$ with hyperbolic metric, i.e. length element has a form

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}, \text{ and the distance}$$

$$d(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle) = \operatorname{arccosh}\left(1 + \frac{(x_2-x_1)^2 + (y_2-y_1)^2}{2 y_1 y_2}\right).$$

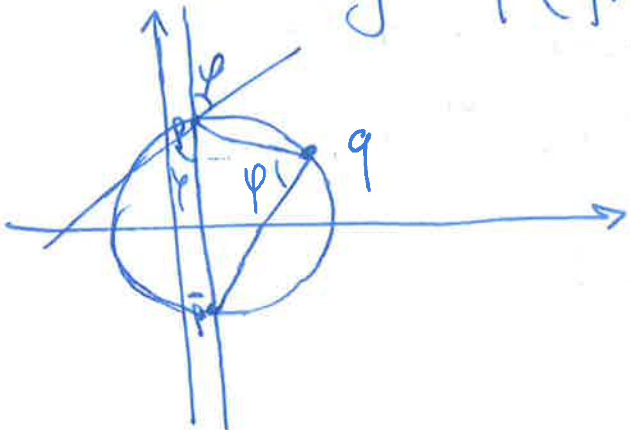
"straight lines" = geodesics has a form



$\ell(p,q)$ – geodesic between $p, q \in \mathcal{H}$ (either vertical line or half circle whose centre is on real axis).

$\ell(p,\infty)$ – vertical line going from p to ∞.

We denote by $\varphi(p,q)$ the angle from $\ell(p,\infty), \ell(p,q)$.



Hence we have that

$$\varphi(p,q) = \arg(q-p) - \arg(q-\bar{p})$$

$$= \arg\left(\frac{q-p}{q-\bar{p}}\right)$$

Exercise Show that

$$\varphi(p,q) = \arg\left(\frac{q-p}{q-\bar{p}}\right) = \frac{1}{2i}\log\left(\frac{q-p}{q-\bar{p}}\cdot\frac{\overline{q-p}}{\overline{q-\bar{p}}}\right).$$

Hence $\varphi : (p,q) \longmapsto \varphi(p,q)$ - is analytic.

Hence it admits a continuous extension to

$$\bar{\mathcal{H}}\times\bar{\mathcal{H}} = \{ z_1 \times z_2 \in \mathbb{C}^2 \mid \operatorname{Im} z_1 \geq 0,\ \operatorname{Im} z_2 \geq 0,\ z_1 \neq z_2 \}.$$

Now for $n \geq 0$ let $\mathcal{H}_n = \{ (p_1,\dots,p_n) \in \mathcal{H}^n \mid p_i \neq p_j \}$

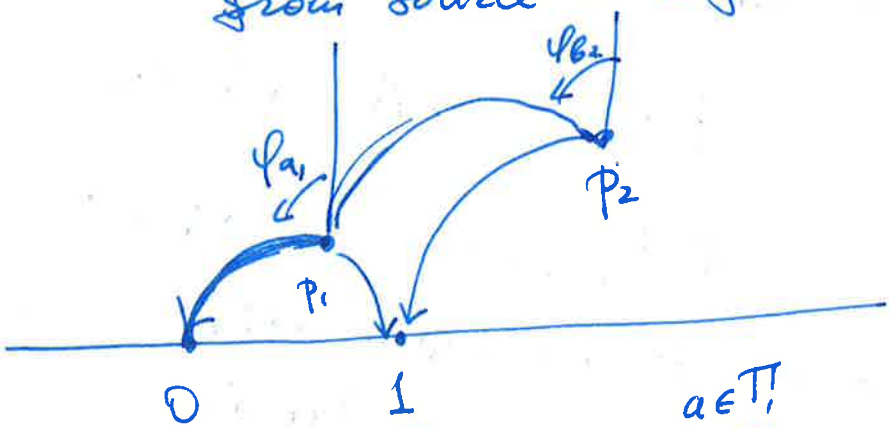
Given $\Gamma \in G_n$ we think of $\mathcal{H}_n$ as the set of all "geodesic drawings" of Γ in the closure of $\mathcal{H}$.

$\{1,\dots,n\}$ vertices $\longrightarrow$ $\{p_1,\dots,p_n\}$ points

L, R vertices $\longrightarrow$ $0, 1$ on real axis

arrows $\longrightarrow$ geodesics segment from source to target.

$$\begin{tikzcd}
1 \arrow[d, "a_1"'] \arrow[dr, "b_1"] & 2 \arrow[l, "b_2"'] \arrow[d, "a_2"] \\
L & R
\end{tikzcd}$$



$a \in \Gamma$

We define the functions $\varphi_a : \mathcal{H}_n \longrightarrow \mathbb{R}$, by

$$\varphi_a(p_1,\dots,p_n) = \varphi(p_{s(a)}, p_{t(a)}). \quad p_L = 0,\ p_R = 1$$

Finally
$$\omega_\Gamma = \frac{1}{(2\pi)^n} \int_{H_n} \bigwedge_{i=1}^{n} \left(d\varphi_{a_i} \wedge d\varphi_{b_i}\right).$$

Lemma (Kontsevich) Integral converges.

Theorem (—"—) the formula
$$f * g = f \cdot g + \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\Gamma \in G_n} \omega_\Gamma B_{\Gamma,\alpha}(f,g)$$
defines a formal quantization of the given Poisson bracket.

② More precise version o Kontsevich's theorem.

Let M be differentiable manifold, $A = C^\infty(M)$.

A multidifferentiable operator on M is a map
$$P : A^m \longrightarrow A, \text{ such that in local system } x_1, \dots, x_d$$
$$P(f_1, \dots, f_m) = \sum a_{\nu_1, \dots, \nu_m} \left(\frac{\partial^{|\nu_1|}}{\partial x^{\nu_1}} f_1\right) \dots \left(\frac{\partial^{\nu_m}}{\partial x^{\nu_m}} f_m\right)$$
where ν_i are multi-indices and $a_{\nu_1, \dots, \nu_m}$ smooth functions which vanish for almost all $(\nu_1, \dots, \nu_m)$

A <u>star product</u> on M is an associative formal deformation

$$* = \sum_{n \geq 0} B_n t^n \quad \text{such that}$$

B_n are bidifferentiable operators

(that is, $B_n : A \times A \to A$, B_n is differentiable).

There is an action of <u>gauge group</u> on star product.

Namely, let $\mathcal{J}_d$ denote the group of $\mathbb{R}[[t]]$-module automorfisms $g = \sum_{n \geq 0} g_n t^n$ of $A[[t]]$, s.t.

g_0 — identity

g_n - differential operators.

Namely $\quad g(f) = f + g_1(f) \cdot t + \dots + g_n(f) \cdot t^n + \dots \quad f \in A.$

$$g\Big(\sum_{n \geq 0} f_n t^n\Big) = \sum_{n \geq 0} f_n t^n + \sum_{\substack{n \geq 0 \\ m \geq 1}} g_m(f_n) \cdot t^{n+m}$$

g_i - are differentiable operators.

Two star product $*$ and $*'$ are <u>equivalent</u> if

$$g(f_1 * f_2) = g(f_1) *' g(f_2), \quad f_1, f_2 \in A[[t]].$$

<u>Theorem</u> (Kontsevich, 97)

The set of equivalence classes of star products on diff. manifold M, can be identified with the set of equivalence classes of

Poisson structures depending formally on $\hbar$

$$\alpha = \alpha_1 t + \alpha_2 t^2 + \dots + \in \Gamma(M, \Lambda^2 T_M)[[t]]$$

$[\alpha, \alpha] = 0$ — Schouten - Nijenhuis bracket on polyvector fields.

③ Some unprecise statements and Duflo's isomorfism

Given Poisson manifold M, its poisson bracket is represented as certain element $\alpha \in T^1_{poly}(M)$ which satisfy Maurer - Cartan equation. (= Poisson)

For other point of view any star product of ~~po~~ multidifferentiable is an element of $\gamma \in D_{poly}(M)$ - of operator such that satisfy Maurer-Cartan (= associativity)

Theorem (Kontsevich) There is a canonical quasi-isomorfism

$$\varphi : T_{poly}(M) \longrightarrow D_{poly}(M) \text{ which}$$

induces an algebra isomorfism between Homologies algebras.

Corolario In the case $M = g^*$ — dual to fin. dim Lie algebra this leads to $H^*(g, S(g)) \Rightarrow HH^*(\mathcal{U}(g), \mathcal{U}(g))$ which in 0-degree gives $S(g)^g \simeq \mathcal{U}(g)^g$

Theorem (Kontsevich) Such isomorfism coincides with Duflo.