USP talk

(May, 18, 2017)

"Invariant polynomials on truncated multicurrent algebras" (joint with A. Savage)

§1 Notation.

* $\mathbb{F}$: (algebraically closed) field of characteristic 0;
* $\mathfrak{g}$: finite-dimensional (semisimple) Lie algebra over $\mathbb{F}$;
* $A = \dfrac{\mathbb{F}[t_1, \dots, t_n]}{I}$, onde $n>0$ e I é um ideal de codim. finita gerado por monom.;
* $S(\mathfrak{g}\otimes A) = \bigoplus_{k\geq 0} S^k(\mathfrak{g}\otimes A)$: (polynomial functions) symmetric algebra on the f-d. vector space $(\mathfrak{g}\otimes A)$ endowed with the $(\mathfrak{g}\otimes A)$-action induced from the adjoint representation of $(\mathfrak{g}\otimes A)$;
* $S(\mathfrak{g}\otimes A)^{\mathfrak{g}\otimes A}$ = the set of invariant (under this $(\mathfrak{g}\otimes A)$-action) polynomial functions on $(\mathfrak{g}\otimes A)$.

Example: $\mathfrak{g} = \mathfrak{sl}_2$, $\mathbb{F} = \mathbb{C}$, $A = \dfrac{\mathbb{C}[x,y]}{\langle x^2, y^2\rangle}$ or $\dfrac{\mathbb{C}[t]}{\langle t^{n+1}\rangle}$.

§2. Goals

* Construct elements in $S(\mathfrak{g}\otimes A)^{\mathfrak{g}\otimes A}$;
* Find a set of algebraically independent generators for $S(\mathfrak{g}\otimes A)^{\mathfrak{g}\otimes A}$;
* Application: describe how $Z(\mathfrak{g}\otimes A)$, the center of $U(\mathfrak{g}\otimes A)$ acts on finite-dimensional irreps of $(\mathfrak{g}\otimes A)$.

§3. Motivation

* If $\mathbb{F}$ is alg. closed, $\mathfrak{g}$ is semisimple, B is an assoc., comm., finitely generated algebra, and Γ is a finite abelian group acting on $\mathfrak{g}$ and B (freely on maxspec B), then: irreducible finite-dimensional $(\mathfrak{g}\otimes B)^{\Gamma}$-modules are isomorphic to tensor products of (irred) evaluation modules for Lie algebras of the form $(\mathfrak{g}\otimes A)$, A as above.

* If $\mathbb{F}$ is alg. closed and $\mathfrak{g}$ is semisimple, then: $\mathfrak{g}$-modules in the same block in Cat. $\mathcal{O}$ have the same central character.
  - Block decomposition: given an abelian category $\mathcal{A}$, a block is a full subcategory of $\mathcal{A}$ consisting of objects in an equivalence class of the smallest equivalence relation on $Obj(\mathcal{A})$ satisfying:

$$L_1 \sim L_2 \quad \text{iff} \quad Ext^1(L_1, L_2) \neq 0. \qquad (L_1, L_2 : \text{irrep}).$$

If $\mathcal{A}$ is Artinian, then: $M = \bigoplus_{b:\text{blocks}} M_b$ and

$$Ext^i_{\mathcal{A}}(M,N) \cong \prod_{b:\text{blocks}} Ext^i_b(M_b, N_b) \quad \forall i \geq 0.$$

  - Central character: if Z is a comm. algebra, then any algebra homomorphism $\chi : Z \to \mathbb{F}$ is a central character. Denote $M_\chi = \{ m \in M \mid (z - \chi(z))^k m = 0 \text{ for some } k>0 \}$. If $Hom(M_{\chi_1}, M_{\chi_2}) \neq 0$, then $\chi_1 = \chi_2$.

* The centr of $U(\mathfrak{g}\otimes A)$ is isomorphic to $S(\mathfrak{g}\otimes A)^{\mathfrak{g}\otimes A}$:
  - as a vector space via the symmetrization map,
  - as an algebra via Duflo's isomorphism.

§4. Results

§§4.1. Construction.

* Denote $\Omega = \mathbb{N}^\ell$, $\ell > 0$. (So that $\mathbb{F}[t_1, \dots, t_\ell] \cong \mathbb{F}[\Omega]$.)
  Let $I = \langle \Omega_0 \rangle$, where $\Omega_0 \subseteq \Omega$ is a subset such that
  $$\omega \cdot \omega_0 \in \Omega_0 \quad \forall \omega \in \Omega \text{ and } \omega_0 \in \Omega_0,$$
  and $\Omega_1 = \Omega \setminus \Omega_0$ is a finite set. (So that $\mathbb{F}[\Omega]/I$ is finite dimensional and generated by monomials.)
  Let $A = \mathbb{F}[\Omega]/I$, fix a basis $\{t^\omega \mid \omega \in \Omega_1\}$, and denote $a = \sum_{\omega \in \Omega_1} t^\omega \in A$.

* There exists a unique homomorphism of (comm.) algebras $\tau_a : S(g) \to S(g \otimes A)$ satisfying
  $$\tau_a(x) = x \otimes a \qquad \forall x \in g.$$

* Since A is $\mathbb{Z}^\ell$-graded, one induces a $\mathbb{Z}^\ell$-grading on $S(g \otimes A)$. Given $p \in S(g)$ and $r \in \mathbb{Z}^\ell$, denote by

  p_r : the **homogeneous** component of $\tau_a(p)$ on $dgr = r$.

* If $p \in S^k(g)$, then $p_r \neq 0$ only if $r \in \Omega_1^k$.

§§4.2. Invariance.

* If Ω_1 has a greatest element μ and $p \in S^k(g)^g$, then: $p_r \in S^k(g \otimes A)^{g \otimes A}$ for all $r \in (k\mu - \Omega_1)$, $k > 0$.
  (Usual order induced by addition on $\mathbb{N}^\ell$: $(m_1, \dots, m_\ell) \leq (n_1, \dots, n_\ell)$ iff $m_1 \leq n_1, \dots, m_\ell \leq n_\ell$.)

* If g is a f.d. simple Lie algebra, then $S(g)^g \cong S(\mathfrak{h})^W$ (Chevalley's Theorem) and thus $S(g)^g$ is isomorphic to a polynomial algebra on $rk(g) = \dim(\mathfrak{h})$ generators (and degrees given by the exponents of g).

§§4.3. Algebraic independence.

* Assume Ω_1 has a maximal element μ. (Because in this case, $A \cong \dfrac{\mathbb{F}[t_1, \dots, t_{\ell-1}]}{\langle t^\omega \mid \omega \geq (\mu_1, \dots, \mu_{\ell-1})\rangle} \otimes \dfrac{\mathbb{F}[t_\ell]}{\langle t_\ell^\omega \mid \omega \geq \mu_\ell\rangle}$.)

A set $\{p^{(i)} \in S^{k_i}(g) \mid i \in \{1, \dots, r\}\}$ is algebraically independent iff the set $\{p^{(i)}_r \in S^{k_i}(g \otimes A) \mid i \in \{1, \dots, r\},\ r \in (k_i\mu - \Omega_1)\}$ is algebraically independent.

(* If g is quadratic (that is, g admits a symmetric, non-degenerate, invariant bilinear form), then: a set $\{q^{(i)} \in S^{k_i}(g^*)^g \mid i \in \{1, \dots, r\}\}$ is algebraically independent if and only if the set $\{q^{(i)}_r \in S^{k_i}(g \otimes A)^{g \otimes A} \mid i \in \{1, \dots, r\},\ r \in \Omega_1\}$ is algebraically independent.)

§§4.4. Generators.

* If g is finite-dimensional semisimple, Ω_1 has a maximal element μ, and $\{p^{(i)} \in S^{k_i}(g)^g \mid i \in \{1, \dots, r\}\}$ is an algebraically independent set of generators for $S(g)^g$, then $\{p^{(i)}_r \in S^{k_i}(g \otimes A)^{g \otimes A} \mid i \in \{1, \dots, r\},\ r \in (k_i\mu - \Omega_1)\}$ is an algebraically independent set of generators for $S(g \otimes A)^{g \otimes A}$.

* Remarks:
  - The semisimple condition is used to move the problem from $S(a^*)$ to $S(a)$ and back. We need to move to $S((g \otimes A)^*)$ in order to use results of Kostant about transversal slices $(x_+ + g^{x_-})$, since g^* is not a ss. Lie alg.
  - The condition that Ω_1 has a maximal element is not very restrictive, because we can pull-back $(g \otimes A)$-mods

i) $\tilde{A} = \dfrac{\mathbb{F}[{}_L\Omega]}{\langle {}_L\Omega_0 \rangle}$;

ii) $({}_L\Omega \setminus {}_L\Omega_0)$ has a maximal element;

iii) $\tilde{A} \twoheadrightarrow A$ as an algebra.

For instance, $\dfrac{\mathbb{C}[x,y]}{\langle x^2, y^2 \rangle} \twoheadrightarrow \dfrac{\mathbb{C}[x,y]}{\langle x,y \rangle^2}$.

§5. Application.

* Let $\mathfrak{g}$ be semisimple and $A \neq \mathbb{F}$. Then the center of $U(\mathfrak{g}\otimes A)$ acts on any finite-dimensional, irreducible $(\mathfrak{g}\otimes A)$-module via the restriction of the augmentation map

$$\varepsilon: U(\mathfrak{g}\otimes A) \to \mathbb{F}.$$
$$(x\otimes a) \mapsto 0$$

In particular, all finite-dimensional irreducible $(\mathfrak{g}\otimes A)$-module have the same central character. That is, the center of $U((\mathfrak{g}\otimes A)^\Gamma)$ does not separate blocks of the category of f.-d. $(\mathfrak{g}\otimes A)^\Gamma$-modules.