AN INTRODUCTION TO REPRESENTATION THEORY.

2. Lecture 2. Irreducible and indecomposable representations. Density theorem.

2.1. Indecomposable representations and density theorem.

Definition 1. A *direct sum* of two representations V_1 and V_2 of an algebra A is a representation $V_1 \oplus V_2$ with the action $\rho(x, y) = \rho_1(x) \oplus \rho_2(y)$.

Definition 2. A nonzero representation V of an algebra A is said to be *indecomposable* if it is not isomorphic to a direct sum of two nonzero representations.

If a representation is irreducible representation then it is indecomposable. The converse is false in general (see in examples).

Definition 3. A *semisimple* representation of A is a direct sum of of simple (irreducible) representations.

Example 1. Some examples:

(1) Assume that $V_1 = k$ is one-dimensional representation of k. Then $V_1 \oplus V_1$ is $k \oplus k$ with
$$\rho : x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad x \in k$$
(2) Let $V = k^2$ be a representation of $k[x]$ given by
$$\rho : x \mapsto \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, \quad x \in k.$$
This representation is indecomposable but not irreducible (the subspace $\{(x, 0) \mid x \in k\}$ is invariant). In particular it is not semi-simple.

(3) Let V be an irreducible representation of A of dimension n. Then $Y = \mathrm{End}(V)$, with action of A by left multiplication, is a semisimple representation of A, isomorphic to nV (the direct sum of n copies of V). Indeed, any basis $v_1, \ldots, v_n$ of V gives rise to an isomorphism of representations $\mathrm{End}(V) \to nV$, given by $x \mapsto (xv_1, \ldots, xv_n)$.

Let us discuss the case $A = k[x]$. Since this algebra is commutative, the irreducible representations of A are always 1-dimensional representations $\rho(x) = \lambda \in k$.

The classification of indecomposable representations of $k[x]$ is more interesting. Recall that any linear operator on a finite dimensional vector space V can be reduced to Jordan normal form. More specifically, recall that the Jordan block $J_{\lambda,n}$ is the operator on k^n which in the standard basis is given by the formulas $J_{\lambda,n} e_i = \lambda e_i + e_{i-1}$ for $i > 1$, and

$J_{\lambda,n}e_1 = \lambda e_1$. Then for any linear operator $B : V \to V$ there exists a basis of V such that the matrix of B in this basis is a direct sum of Jordan blocks. This implies that all the indecomposable representations of A are $V_{\lambda,n} = k^n, \lambda \in k$, with $\rho(x) = J_{\lambda,n}$. The fact that these representations are indecomposable and pairwise non-isomorphic follows from the Jordan normal form theorem (which in particular says that the Jordan normal form of an operator is unique up to permutation of blocks).

Proposition 1. Let $V_i, \ldots, V_m$ be irreducible finite dimensional pairwise nonisomorphic representations of A, and W be a subrepresentation of $V = \oplus_{i=1}^m n_i V_i$. Then W is isomorphic to $\oplus_{i=1}^m r_i V_i$, $r_i \le n_i$, and the inclusion $\varphi : W \to V$ is a direct sum of inclusions $\varphi_i : r_i V_i \to n_i V_i$ given by multiplication of a row vector of elements of V_i (of length r_i) by a certain r_i-by-n_i matrix X_i with linearly independent rows: $\varphi(v_1, \ldots, v_{r_i}) = (v_1, \ldots, v_{r_i})X_i$.

Proof. The proof is by induction in $n := \sum_{i=1}^m n_i$. The base of induction ($n = 1$) is clear. To perform the induction step, let us assume that W is nonzero, and fix an irreducible subrepresentation $P \subset W$. Recall that such P exists. By Schurfs lemma, P is isomorphic to V_i for some i, and the inclusion $\varphi : P \to V$ factors through $n_i V_i$, and after the identification of P with V_i is given by the formula $v \mapsto (vq_1, \ldots, vq_{n_i})$, where $q_l \in k$ are not all zero.

Now note that the group $G_i = GL_{n_i}(k)$ of invertible n_i-by-n_i matrices over k acts on $n_i V_i$ by $(v_1, \ldots, v_{n_i}) \mapsto (v_1, \ldots, v_{n_i})g_i$ (and by the identity on $n_j V_j$, $j \neq i$), and therefore acts on the set of subrepresentations of V, preserving the property we need to establish: namely, under the action of g_i, the matrix X_i goes to $X_i g_i$, while $X_j, j \neq i$ don't change. Take $g_i \in G_i$ such that $(q_1, \ldots, q_{n_i})g_i = (1, 0, \ldots, 0)$. Then W_{g_i} contains the first summand V_i of $n_i V_i$ (namely, it is Pg_i), hence $Wg_i = V_i \oplus W'$, where $W' \subset n_1 V_1 \oplus \ldots n_m V_m$ is the kernel of the projection of Wg_i to the first summand V_i along the other summands. Thus the required statement follows from the induction assumption. □

Corollary 2. *Let V be an irreducible finite dimensional representation of A, and $v_1, \ldots, v_n \in V$ be any linearly independent vectors. Then for any $w_1, \ldots, w_n \in V$ there exists an element $a \in A$ such that $av_i = w_i$.*

Proof. Assume the contrary. Then the image of the map $A \to nV$ given by $a \mapsto (av_1, \ldots, av_n)$ is a proper subrepresentation, it corresponds to an r-by-n matrix X, $r < n$. Thus, taking $a = 1$, we see that there exist vectors $u_1, \ldots, u_r \in V$ such that $(u_1, \ldots, u_r)X = (v_1, \ldots, v_n)$. Let $(q_1, \ldots, q_n)$ be a nonzero vector such that $X(q_1, \ldots, q_n)T = 0$ (it exists because $r < n$). Then $\sum q_i v_i = (u_1, \ldots, u_r)X(q_1, \ldots, q_n)T = 0$, i.e. $Pq_i v_i = 0$ - a contradiction with the linear independence of v_i. □

Theorem 3. *(the Density Theorem).*

(i) *Let V be an irreducible finite dimensional representation of A. Then the map $\rho : A \to \mathrm{End}V$ is surjective.*

(ii) *Let $V = V_1 \oplus \cdots \oplus V_r$, where V_i are irreducible pairwise nonisomorphic finite dimensional representations of A. Then the map $\oplus_{i=1}^r \rho_i : A \to \oplus_{i=1}^r \mathrm{End}(V_i)$ is surjective.*

Proof. (i) Let B be the image of A in $\mathrm{End}(V)$. Our aim is to show that $B = End(V)$. Let $c \in \mathrm{End}(V)$, $v_1, \ldots, v_n$ be a basis of V, and $w_i = cv_i$. By Corollary 2, there exists $a \in A$ such that $av_i = w_i$. Then $\rho(a) = c$, so $c \in B$, and we are done.

(ii) Let B_i be the image of A in $\mathrm{End}(V)$, and B be the image of A in $\oplus_{i=1}^r \mathrm{End}(V_i)$. Recall that as a representation of A, $\oplus_{i=1}^r \mathrm{End}(V_i)$ is semisimple: it is isomorphic to $\oplus_{i=1}^r d_i V_i$, where $d_i = \dim V_i$. Then by Proposition 2.2, $B = \oplus_i B_i$. On the other hand, (i) implies that $B_i = \mathrm{End}(V_i)$. Thus (ii) follows. □

2.2. **Direct sum of matrix algebras.**

Definition 4. Let A be an algebra, then it is dual $A^{op} = a \in A$ is an algebra with multiplication $a \cdot b = ba$.

Definition 5. (Dual representation) Let V be a representation of any algebra A. Then the dual representation V is the representation of the opposite algebra A^{op} with the action

$$\rho : a \mapsto \phi_a \in \mathrm{End}(V^*), \quad \phi_a(f(v)) = f(av).$$

Direct sum of matrix algebras is an algebra $A = \oplus_{i=1}^r \mathrm{Mat}_{d_i}(k)$.

Theorem 4. *Let $A = \oplus_{i=1}^r \mathrm{Mat}_{d_i}(k)$. Then the irreducible representations of A are $V_1 = k^{d_1}, \ldots, V_r = k^{d_r}$ and any finite dimensional representation of A is a direct sum of copies of $V_1, \ldots, V_r$.*

Proof. First, the given representations are clearly irreducible, as for any $v \neq 0$, $w \in V_i$, there exists $a \in A$ such that $av = w$. Next, let X be an n-dimensional representation of A. Then, X^* is an n-dimensional representation of A^{op}. But $(\mathrm{Mat}_{d_i}(k))^{op} \cong \mathrm{Mat}_{d_i}(k)$ with isomorphism $\varphi(X) = X^T$, as $(BC)^T = C^T B^T$. Thus, $A \cong A^{op}$ and X^* may be viewed as an n-dimensional representation of A. Define

$$\varphi : \oplus_{i=1}^n A \mapsto X^*$$

by

$$\varphi(a_1, \ldots, a_n) = a_1 y_1 + \cdots + a_n y_n,$$

where $\{yi\}$ is a basis of X^*. φ is clearly surjective, as $k \subset A$. Thus, the dual map $\varphi^* : X \to A^{n*}$ is injective. But $A^{n*} \cong A^n$ as representations of A (check it!). Hence, $\mathrm{Im}\varphi^* \cong X$ is a subrepresentation of A^n. Also, $\mathrm{Mat}_{d_i}(k) = d_i V_i$, so $A = \oplus_{i=1}^r d_i V_i$, $A^n = \oplus_{i=1}^r n d_i V_i$, as a representation of A. Hence $X = \oplus_{i=1}^r m_i V_i$. □

Home work.

(1) Let $A = \mathbb{C}[G]$ be a group algebra of a finite group G. Show that a representation V of A is indecomposable if and only if it is irreducible.