Paradoxes of Preferential Voting

What can go wrong with sophisticated voting systems designed to remedy problems of simpler systems.

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Preferential voting, developed by Thomas Hare [**12**] in the 1860's, is still used for major elections in Australia, Ireland and South Africa, as well as for local elections in many countries. From its inception, it has been touted as a way to promote full expression of electors' preferences and to ensure maximum and equitable consideration of each elector's vote. When used to fill several seats in a legislature, preferential voting provides representation for viable minorities and tends to distribute seats in proportion to the numbers of voters who favor the different parties.

It seeks to do all this on the basis of a single preferential (ranked) ballot by transferring votes, in part or in whole, from the most and least popular candidates to candidates with intermediate support. The most popular, elected first, don't need their "surpluses," and the least popular can never overcome their "deficits," so the transfers of both surpluses and deficits to the intermediate candidates determines which of these win. When there are n voters and c seats are to be filled, transfers are made sequentially until c candidates attain the vote quota needed for election. The quota is usually defined as

$$q = \left[\frac{n}{c+1}\right] + 1,$$

where brackets signify the integer part of the argument. We shall use this concept later.

Despite its tendencies to promote individual interests and fair representation, preferential voting has several surprising and potentially damning defects. We shall begin by illustrating four of these through an apocryphal story of an election among three contenders for mayor of a small town. In this deliberately simple case, a candidate ranked first on more than 50 percent of the ballots is elected; if there is no such candidate, the one with the fewest first-place votes is scratched, then the one of the remaining two who ranks higher on more ballots is elected. Since this procedure is tantamount to plurality (vote-for-one) voting followed by a two-candidate runoff election, the defects or paradoxes developed in our story apply also to the common plurality-runoff scheme.

The story's four paradoxes are summarized here for reference and for readers who may wish to test them on their own.

NO-SHOW PARADOX: *The addition of identical ballots with candidate x ranked last may change the winner from another candidate to x.*

THWARTED-MAJORITIES PARADOX: *A candidate who can defeat every other candidate in direct-comparison majority votes may lose the election.*

MULTIPLE-DISTRICTS PARADOX: *A candidate can win in each district separately, yet lose the general election in the combined districts.*

MORE-IS-LESS PARADOX: *If the winner were ranked higher by some voters, all else unchanged, then another candidate might have won.*

Following the story, we shall discuss general problems confronting voting schemes and mention interesting mathematical work on the subject. We then return to preferential voting to illustrate two other paradoxes that arise only in more complex situations. We conclude with a note on paradox probabilities.

A funny thing happened on the way to the polls

Mr. and Mrs. Smith's car broke down on the way to the polls just before closing time. The Smiths were intensely interested in a tight race for mayor of their town among Mrs. Bitt, Mr. Huff and Dr. Wogg.

The ballot for mayor asked each voter to rank the three candidates from first choice to third choice. The townspeople knew that the election would be decided by the simple preferential voting method, which had been instituted by local referendum some years earlier. Everyone in town was pleased that they used such a sensible procedure for electing the head of their local government.

The Smiths were of one mind about the candidates. They favored Bitt to Huff to Wogg, and therefore both would have voted BHW. Although they liked Mrs. Bitt best, they were almost as fond of Mr. Huff but disliked and mistrusted Dr. Wogg. Much to their regret, the Smith's car problem prevented them from making it to the polls before closing time.

Many of their fellow townspeople did. When Mrs. Smith opened her newspaper the next morning, her eye was caught by a headline proclaiming "Huff Elected as 1,608 Go to Polls." She and her husband were delighted that Dr. Wogg had not won. They did feel a twinge of regret that their friend, Mrs. Bitt, was beaten. Perhaps their votes would have made a difference.

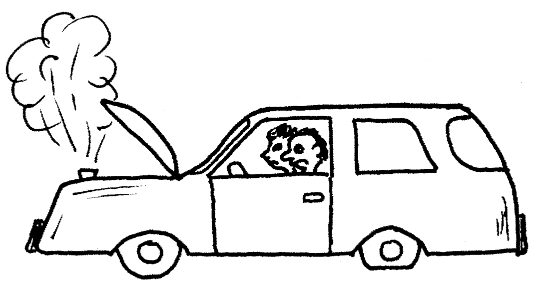
As Mrs. Smith read on, she noted that no candidate had gotten enough first-place votes to win outright. Mrs. Bitt had been scratched because she had the fewest first-place votes, and Mr. Huff went on to beat Dr. Wogg by a plurality of 917 to 691.

Toward the end of the article, on an inside page, Mrs. Smith read the tabulation of how the 1,608 voters cast their ballots shown in FIGURE 1.

Totals	Rankings	H over W	W over H
417	BHW	417	0
82	BWH	0	82
143	HBW	143	0
357	HWB	357	0
285	WBH	0	285
324	WHB	0	324
1608		917	691

FIGURE 1

It made her feel good that she and her husband would have voted with the largest of the six groups. As the article had noted earlier, Mrs. Bitt barely lost out on the initial count since Bitt, Huff and Wogg had first-place tallies of 499 (417 + 82), 500 (143 + 357), and 609 (285 + 324) respectively.



Mrs. Smith realized when she read this that Mr. Huff rather than Mrs. Bitt would have been scratched if she and her husband had voted. At least their friend, Mrs. Bitt, would have made the "runoff" if their car had not broken down.

Before leaving for her job as an actuary with an insurance company headquartered in the next town, Mrs. Smith decided to see what would have happened if she and her husband had voted. Her tabulation is shown in FIGURE 2.

Totals	Rankings	B over W	W over B
419	BHW	419	0
82	BWH	82	0
143	HBW	143	0
357	HWB	0	357
285	WBH	0	285
324	WHB	0	324
1610		644	966

FIGURE 2

To her chagrin, she saw that their votes would have made Dr. Wogg the winner even though he was ranked last on their ballots! This so shocked her that she checked her figures three times. When they refused to change, it hit her: *the whole thing depended on who was scratched after the initial count*. With Bitt out, Huff wins; with Huff out, Wogg wins. Even if 300 more people had voted BHW, Dr. Wogg would still have won.

Mrs. Smith was beginning to wonder if the town's procedure for electing a mayor was that sensible after all.

That evening, while reviewing her figures again, Mrs. Smith became aware of another curious fact. She realized that the winner, Mr. Huff, would have beaten either Mrs. Bitt (824 to 784) or Dr. Wogg (917 to 691) in a direct vote between the two. The "majority candidate"—that is, the candidate who could have beaten each of the others in direct pairwise votes—had indeed won. However, if the Smiths had voted, then not only would their last choice have won but the winner, Dr. Wogg, would not have been the candidate favored in separate pairwise contests to each of the other candidates.

At this point, Mrs. Smith suspected that their election procedure might be more than a little flawed and wondered if further probing might uncover other unusual possibilities. She vowed to make time for this over the weekend.

The Smith's town had two voting districts, called East and West. When the weekend came round, Mrs. Smith decided to compare the outcome with what might have happened in the separate districts. She suspected that the winner, Mr. Huff, might have lost in one if not both districts. The paper had reported that 588 people voted in the East and 1,020 had voted in the West. Moreover, it gave the East-West splits shown in FIGURE 3.

Totals	Rankings	East	West
417	BHW	160	257
82	BWH	0	82
143	HBW	143	0
357	HWB	0	357
285	WBH	0	285
324	WHB	285	39
1608		588	1020

FIGURE 3

Applying precisely the same election rule used for the general election to the East and the West separately, Mrs. Smith found that Mrs. Bitt would have won in both districts! She felt this was truly amazing since both Huff and Wogg had sizable majorities over Bitt in the overall electorate, so there was no way that Mrs. Bitt could have won in the combined districts.

Moreover, as Mrs. Smith noted, a multiple-district winner like Mrs. Bitt could be a "minority candidate" in the sense that this candidate would be defeated by every other candidate in direct-comparison votes. She also realized that this anomaly could arise only when different candidates were scratched on the first rounds in the several "district elections." In fact, Huff was scratched in the East, whereas Wogg was scratched in the West.

Mrs. Smith was now convinced that she had a very strong case against the supposedly sensible system used to elect the mayor of their town. At her request, the chairman of the local Election Board called a special meeting of the board to review her findings.

On the night before the board meeting, as she was going over her figures, Mrs. Smith discovered another irregularity. While pondering what would have occurred if she and her husband had voted (see FIGURE 2), she realized that if two or more of the 82 voters with ranking BWH had moved Wogg into first place (WBH), then Bitt rather than Huff would have been scratched and Huff rather than Wogg would have won. In other words, an increase in support for Dr. Wogg would have changed him from a winner to a loser! Extraordinary, thought Mrs. Smith, as she prepared her flip charts for her presentation to the Election Board.

The next day Mrs. Smith so impressed the board that they decided to appoint a select panel—chaired by Mrs. Smith, of course—to recommend a better election procedure. In particular, the board charged the panel with devising a system that would avoid all the paradoxes uncovered by Mrs. Smith.

At the panel's first meeting, one member suggested that they retain ranked voting but simply use the ballots to determine which of the several candidates was the majority candidate. He explained that this would directly resolve the Thwarted-Majorities Paradox and, moreover, would also take care of Mrs. Smith's other three paradoxes.

Mrs. Smith responded that this was a very good idea up to a point, but that it would not solve all their problems. She had been reading up on the subject and proceeded to tell the panel about the most famous paradox of them all, variously known as "Condorcet's phenomenon" [**4**], the "paradox of voting," and the "paradox of cyclical majorities."

Condorcet's phenomenon occurs when every candidate is beaten by some other candidate under direct-comparison voting. Mrs. Smith pointed out that this was not the case in their election, but it was certainly possible. For example, if 1,600 total ballots had been cast, with

400	for	BHW
500	for	WBH
700	for	HWB,

then Bitt beats Huff 900 to 700, Huff beats Wogg 1,100 to 500, and Wogg beats Bitt 1,200 to 400.

At this point, another panel member suggested that perhaps their problems would vanish if they used the method that his lodge used to choose its president. This method awards 2 points to a first-place vote, 1 point to a second-place vote, and 0 points to a third-place vote. The winner is the candidate with the most points. He noted that it could be extended in a straightforward way when there are more than three candidates.

The panel determined that this point-scoring system—sometimes referred to as Borda's "method of marks" [**2**], [**5**]—would resolve all of Mrs. Smith's paradoxes, with the possible exception of the Thwarted-Majorities Paradox. A quick review of the election data showed that the majority candidate, Mr. Huff, would have won under the point-scoring system. However, the panel also noticed that if 50 or so BHW voters had preferred Wogg to Huff and voted BWH, then despite the fact that Mr. Huff would remain the majority candidate, Dr. Wogg would win under the point-scoring system (see FIGURE 4).

Totals	Rankings	B Points	H Points	W Points
367	BHW	734	367	0
132	BWH	264	0	132
143	HBW	143	286	0
357	HWB	0	714	357
285	WBH	285	0	570
324	WHB	0	324	648
1608		1426	1691	1707

FIGURE 4

Confused and tired, the panel agreed that they had done enough for one meeting. Their next meeting was set for the following Wednesday.

Problems of voting systems

We end our story at this point because, in a sense, it has no end. The panel could meet forever without being able to fulfill its charge from the Election Board to avoid all four paradoxes. This is because there is a metaparadox lurking in the background which, in simplified form, says that *no* election procedure can simultaneously resolve Mrs. Smith's second and third paradoxes.

Let us elaborate. We assume, as before, that voters rank the candidates from most preferred to least preferred. With a fixed number of candidates, but any potential number of voters, Young [**21**] (see also [**22**]) showed in one of the most mathematically elegant papers on the subject that in order to avoid the Multiple-Districts Paradox as well as to satisfy fundamental equity conditions for voters and candidates, one must use a type of point-scoring system. In an attempt to avoid the Thwarted-Majorities Paradox, it is necessary to assign more points to a first-place vote than to a second-place vote, and so forth, which of course takes care of the No-Show and More-Is-Less Paradoxes.

However, given any set of decreasing point values for the various places, it is always possible to construct an example with a majority candidate who is not elected by the point-scoring system. In fact, nearly two hundred years ago Condorcet recognized that it is possible to construct examples with a majority candidate who is not elected by *any* point-scoring system with decreasing point values [**4**],[**9**]. For example, if there are seven voters such that

3	have	BHW
2	have	HWB
1	has	HBW
1	has	WBH,

then B has a 4-to-3 majority over each of H and W, but H beats B under *every* point-scoring system that assigns more points to a second-place vote than to a third-place vote.

Many other problems and paradoxes that plague preferential voting and other election systems seem to have surfaced only recently. The More-Is-Less Paradox, better known in the literature as the monotonicity paradox, was shown by Smith [**20**] to affect virtually all successive-elimination procedures based on point scoring. Further results on the monotonicity paradox appear in [**10**]. Within the specific context of preferential voting, the More-Is-Less Paradox and the Multiple-Districts Paradox are discussed in [**6**], [**7**].

As far as we know, the No-Show Paradox, which is closely related to the More-Is-Less Paradox, is not discussed elsewhere. However, another no-show paradox seems to have been discovered many years ago [**14**], [**17**]. This other paradox says that one of the candidates elected by preferential voting could have ended up a loser if additional people who ranked him in first place had actually voted. An example of this paradox appears in the next section.

The paradoxes discussed here and elsewhere [**9**], [**15**] reveal only the surface effects of deeper aspects of aggregation structures, such as those developed by Young [**21**]. Recent work on these structures was stimulated in large measure by Kenneth Arrow's classic impossibility theorem [**1**]. This theorem shows that a few simple and appealing conditions for aggregating diverse rankings into a consensus ranking are incompatible. Numerous variants of Arrow's theorem now exist [**8**], [**13**], [**19**], and these have been joined by related results [**11**], [**13**], [**16**], [**18**] which show that virtually every sensible election procedure for multicandidate elections is vulnerable to strategic manipulation by voters. In other words, there will be situations in which some voters can benefit by voting contrary to their true, or sincere, preferences.

An example of the latter phenomenon occurs in our story of the Smiths. If they had voted their true preference order, BHW, then Dr. Wogg would have won under preferential voting. However, if they had voted HBW, or any other order that did not have Mrs. Bitt in first place, then Mr. Huff would have won. Hence, by voting strategically (i.e., falsely), the Smiths would have helped to elect their second choice (H) rather than their last choice (W).

More paradoxes of preferential voting

Additional flaws in preferential voting can arise only when there are more than three contenders. We shall illustrate two of these after describing a general, and widely used, procedure for preferential voting.

In the general case, voters rank the candidates from most preferred to least preferred on their ballots. To be elected, a candidate must receive a quota q of weighted votes.

Each voter begins with voting weight 1. First-place votes are tallied for each candidate; those with q or more are elected. If c' are elected on this first round and $0 < c' < c$, then the weight of each voter whose first choice was elected is decreased from 1 to a nonnegative number (0 if there is no "surplus" over quota) so that the sum of all weights becomes $n - qc'$. Elected candidates are removed from the ballots, and new rounds follow until c candidates are elected, as described below.

After removal of the elected candidates, unelected candidates move up in the ballot rankings to fill in top places, and the process is repeated with a new, weighted tally of unelected candidates now in first place. Again, q is used as the quota for election. The process continues until either all c seats have been filled, or no unelected candidate gets at least q weighted votes in the latest tally. In the latter case, the candidate with the *smallest* weighted first-place tally is scratched, ballot rankings (but not voter weights) are revised accordingly, and the process continues until c seats are filled.

Instead of our earlier story, suppose now that Bitt, Foxx, Huff and Wogg are vying for two seats on the town council, and that 100 people vote as follows:

34	BHFW
25	FBHW
26	HWBF
9	WBFH
6	WHFB

The quota is 34, so Bitt is elected first. Since exactly 34 people voted for Bitt in first place, the weights of these voters are reduced to 0, leaving

25	FHW
26	HWF
9	WFH
6	WHF

Since none of the others reaches the quota, Wogg is scratched. Then Foxx, who has 34 (25 + 9) votes to 32 (26 + 6) for Huff, wins the second seat.

Now suppose five more Foxx supporters (FBHW) had voted, giving

34	BHFW
30	FBHW
26	HWBF
9	WBFH
6	WHFB

The new quota is $(105/3) + 1 = 36$. Since no candidate reaches the quota, Wogg is scratched:

34	BHF
9	BFH
30	FBH
26	HBF
6	HFB

At this point Bitt passes the quota with 43 (34 + 9) votes and is elected as before. Since Bitt exceeded the quota by 7 first-place votes, each of her 43 supporters retains 7/43 of a vote, giving aggregates of

5.5	HF	from Bitt's surplus
1.5	FH	from Bitt's surplus
30	FH	
26	HF	
6	HF	

Since Huff now surpasses the quota with 37.5 (26 + 6 + 5.5), he becomes the second candidate elected. Thus Foxx, a winner in the first case, becomes a loser when five more voters show up with him in first place.

Our final paradox was suggested by a statement on a recent ballot of a professional society that listed eight candidates for four seats on the society's Nominating Committee [**3**]. The election was conducted by preferential voting. Society members were advised to mark candidates in order of preference until they were ignorant or indifferent concerning candidates whom they did not rank. The preferential voting system described earlier is easily modified to account for partial rankings: if a voter's marked candidates are removed or scratched before all seats are filled, that voter is then treated as if he never voted in the first place.

The ballot statement alluded to in the preceding paragraph claimed that "there is no tactical advantage to be gained by marking few candidates." FIGURE 2, suitably modified, shows that this is false. Suppose again that Foxx is in the race for two council seats along with Bitt, Huff, and Wogg, and that votes are precisely the same as shown in FIGURE 2, except that Foxx is the first choice of all 1,610 voters:

Totals	Rankings
419	FBHW
82	FBWH
143	FHBW
357	FHWB
285	FWBH
324	FWHB
1610	

Then Foxx wins a seat, and matters proceed as before when he is removed from the ballots, giving Dr. Wogg the other seat.

But suppose that Mr. and Mrs. Smith had voted just F instead of FBHW, i.e., had voted only for their first choice. Then, after Foxx is removed, we revert to FIGURE 1, where Mr. Huff wins the other seat. By voting only for their first choice, the Smiths prevent their last choice from winning the second seat.

This example provides a second instance of how some voters might induce preferred outcomes by misrepresenting their true preferences. In the present case, misrepresentation takes the form of a deliberate truncation of one's ranking rather than a false but complete ranking.

Paradox probabilities

Although virtually all voting systems for elections with three or more candidates can produce counterintuitive and disturbing outcomes, preferential voting is especially vulnerable because of its sequential elimination and vote-transfer provisions. Nevertheless, this system is still widely used in several countries.

Defenders of preferential voting—and there have been many over the past century—might argue that the paradoxes of preferential voting are not a problem because they occur so infrequently in practice. They would, we presume, claim that a few contrived examples should not deter us from using a carefully refined system that has proved its worth in countless elections.

Although probabilities of paradoxes have been estimated in other settings [**9**], we know of no attempts to assess the likelihoods of the paradoxes of preferential voting discussed above, and would propose this as an interesting possibility for investigation. Is it indeed true that serious flaws in preferential voting such as the No-Show Paradox and the More-Is-Less Paradox are sufficiently rare as to cause no practical concern?

References

[1] K. J. Arrow, Social Choice and Individual Values, 2nd ed., Wiley, New York, 1963.

[2] Jean-Charles de Borda, Mémoire sur les élections au scrutin, Histoire de l'Académie Royale des Sciences, Paris, 1781.

[3] S. J. Brams, The AMS nominating system is vulnerable to truncation of preferences, Notices Amer. Math. Soc., 29 (1982) 136–138.

[4] Marquis de Condorcet, Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix, Paris, 1785.

[5] A. de Grazia, Mathematical derivation of an election system, Isis, 44 (1953) 42–51.

[6] G. Doron, The Hare voting system is inconsistent, Political Studies, 27 (1979) 283–286.

[7] G. Doron and R. Kronick, Single transferable vote: an example of a perverse social choice function, American Journal of Political Science, 21 (1977) 303–311.

[8] P. C. Fishburn, The Theory of Social Choice, Princeton University Press, 1973.

[9] ______, Paradoxes of voting, American Political Science Review, 68 (1974) 537–546.

[10] ______, Monotonicity paradoxes in the theory of elections, Discrete Applied Mathematics, 4 (1982) 119–134.

[11] A. Gibbard, Manipulation of voting schemes: a general result, Econometrica, 41 (1973) 587–601.

[12] T. Hare, The Election of Representatives, Parliamentary and Municipal: A Treatise, Longman, Green, London, 1861.

[13] J. S. Kelly, Arrow Impossibility Theorems, Academic Press, New York, 1978.

[14] J. C. Meredith, Proportional Representation in Ireland, Dublin, 1913.

[15] R. G. Niemi and W. H. Riker, The choice of voting systems, Scientific American, 234 (1976) 21–27.

[16] P. K. Pattanaik, Strategy and Group Choice, North-Holland, New York, 1978.

[17] Report of the Royal Commission Appointed to Enquire into Electoral Systems, HMSO, London, 1910.

[18] M. A. Satterthwaite, Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions, Journal of Economic Theory, 10 (1975) 187–217.

[19] A. K. Sen, Collective Choice and Social Welfare, Holden-Day, San Francisco, 1970.

[20] J. H. Smith, Aggregation of preferences with variable electorate, Econometrica, 41 (1973) 1027–1041.

[21] H. P. Young, Social choice scoring functions, SIAM J. Appl. Math., 28 (1975) 824–838.

[22] H. P. Young and A. Levenglick, A consistent extension of Condorcet's election principle, SIAM J. Appl. Math., 35 (1978) 285–300.