

# On a formula for the spectral flow and its applications

Pierluigi Benevieri <sup>\*1</sup> and Paolo Piccione<sup>2</sup>

<sup>1</sup> Dipartimento di Matematica Applicata, Università degli Studi di Firenze,  
Via S. Marta 3, 50139 Firenze, Italy

<sup>2</sup> Departamento de Matemática, Universidade de São Paulo,  
Rua do Matão 1010, CEP 05508-900, São Paulo, SP, Brazil,  
E-mail: piccione@ime.usp.br  
URL: <http://www.ime.usp.br/~piccione>

**Key words** Fredholm operators, Spectral flow, periodic geodesic, semi-Riemannian manifolds, Maslov index

**MSC (2000)** 47A11, 47A53, 53D12, 58E10

We consider a continuous path of bounded symmetric Fredholm bilinear forms with arbitrary endpoints on a real Hilbert space, and we prove a formula that gives the spectral flow of the path in terms of the spectral flow of the restriction to a finite codimensional closed subspace. We also discuss the case of restrictions to a continuous path of finite codimensional closed subspaces. As an application of the formula, we introduce the notion of spectral flow for a periodic semi-Riemannian geodesic, and we compute its value in terms of the Maslov index.

Copyright line will be provided by the publisher

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Preliminaries. Fredholm bilinear forms.</b>	<b>3</b>
<b>3</b>	<b>Fredholm and commensurable pairs of closed subspaces</b>	<b>8</b>
3.1	Relative dimension and Fredholm pairs . . . . .	8
3.2	Commensurable subspaces . . . . .	11
3.3	Relative dimension of negative eigenspaces . . . . .	12
<b>4</b>	<b>On the spectral flow</b>	<b>16</b>
4.1	Generalities on the notion of spectral flow . . . . .	16
4.2	Restriction to a fixed subspace . . . . .	17
4.3	Continuous and smooth families of closed subspaces . . . . .	18
4.4	Spectral flow and restrictions to a continuous family of subspaces . . . . .	19
<b>5</b>	<b>Spectral flow along periodic semi-Riemannian geodesics</b>	<b>21</b>
5.1	Periodic geodesics . . . . .	21
5.2	Periodic frames and trivializations . . . . .	21
5.3	Spectral flow of a periodic geodesic . . . . .	23
5.4	Computation of the spectral flow . . . . .	23
<b>A</b>	<b>Group actions and fibrations over the infinite dimensional Grassmannian</b>	<b>25</b>
	<b>References</b>	<b>27</b>

---

P. P. is partially sponsored by CNPq and Fapesp.

\* Corresponding author E-mail: [pierluigi.benevieri@unifi.it](mailto:pierluigi.benevieri@unifi.it), Phone: +39 055 4796251, Fax: +39 055 471787

## 1 Introduction

The notion of *spectral flow* plays a central role in several areas of Calculus of Variations, including Morse theory and bifurcation theory; this is a fixed endpoint homotopy invariant integer associated to continuous paths of Fredholm symmetric bilinear forms on Hilbert spaces. In the modern formulations of Morse theory, it is now well understood that this notion is the natural substitute for the notion of Morse index for critical points of strongly indefinite variational problems. For instance, under suitable assumptions, the dimension of the intersection of stable and unstable manifolds of critical points of a smooth functional  $f$  defined on a Hilbert manifold is given by the spectral flow of the Hessian of  $f$  along the flow lines of  $\nabla f$  joining the two critical points (see [4]). In bifurcation theory, jumps of the spectral flow detect bifurcation from some given branch of critical points of a smooth curve of strongly indefinite smooth functionals (see [16]). Starting from the celebrated work of T. Yoshida [25], a series of results have been proven in the literature relating the spectral flow of a path of Dirac operators on partitioned manifolds to the geometry of the Cauchy data spaces (see [8, 9, 10, 11, 12, 13, 14, 22]); low dimensional topological invariants can be computed in terms of spectral flow (see [14, 25]).

A natural question in the above problems is to compute the spectral flow of *restrictions* to a given closed subspace, or more generally to a continuous path of closed subspaces, of a continuous path of Fredholm bilinear forms. In Calculus of Variations, restriction of the Hessian of smooth functionals corresponds to studying constrained variational problems. For instance, the typical Fredholm forms arising from geometrical variational problems are obtained from self-adjoint differential operators acting on sections of vector bundles over (compact) manifolds with boundary satisfying suitable boundary conditions. A formula for the spectral flow of restrictions in this case would allow to reduce the study of a general boundary condition to the usually easier case of Dirichlet conditions.

The aim of this paper is to prove formulas (Theorem 4.4 and Proposition 4.14) relating the spectral flow of a continuous path of Fredholm symmetric bilinear forms to the spectral flow of their restriction to a continuous path of finite codimensional closed subspaces, which is still Fredholm (Lemma 2.8).

Let us recall that the spectral flow of a path of symmetric bilinear forms is given by an algebraic count of eigenvalues passing through 0 in the spectrum of the path of self-adjoint operators that represent the bilinear forms relatively to some choice of inner products. However, a spectral theoretical approach to the restriction problem would not be successful, due to the fact that restrictions of bilinear forms correspond to left multiplication by a projection, and this operation in general perturbs the spectrum of a self-adjoint operator in a quite complicated way. In order to prove the desired result, we will use a different characterization of the spectral flow, which is given in terms of relative dimension of Fredholm pairs in the Grassmannian of all closed subspaces of a Hilbert space. The spectral flow of a path of Fredholm self-adjoint operators of the form symmetry plus compact is given by the relative dimension of the negative spectral subspaces at the endpoints. One proves that a finite codimensional reduction does not destroy the symmetry plus compact form of a Fredholm operator (Lemma 4.2); moreover, the relative dimension of the negative eigenspaces behaves well with respect to compact perturbations (Proposition 3.18).

As pointed out to the authors by one of the referees, it should be emphasized that one of the central results of the present paper, formula (4.2), was obtained before, though in a different formulation, by Chafeng Zhu see [26, Equation (44)], by a different proof, using results from Booss and Wojciechowski [12, Formula 15.20] or Dai and Zhang [15].

The case of restrictions to a varying family of closed finite codimensional subspaces (Proposition 4.14) is reduced to the case of a fixed subspace by means of a special class of trivialization of the family. We observe that one does not lose generality in considering only the case of paths of the form symmetry plus compact. Namely, let us recall that the spectral flow is invariant by the cogredient action of the general linear group of the Hilbert space on the space of self-adjoint Fredholm operators, and that all the orbits of this action meet the affine space of compact perturbations of a fixed symmetry. By an elementary principal fiber bundle argument, every path of class  $C^k$ ,  $k = 0, \dots, \infty, \omega$ , in the space of self-adjoint Fredholm operators is cogredient to a  $C^k$  path of compact perturbations of a symmetry.

The paper is finalized with the discussion of an application of our reduction formula in the context of semi-Riemannian geometry (Section 5). We will consider an orientation preserving periodic geodesic  $\gamma$  in a semi-Riemannian manifold  $(M, g)$ , and we will define its *spectral flow*, as a suitable generalization of the Morse index of the geodesic action functional, defined on the free loop space of  $M$ , at the critical point  $\gamma$ . Observe that, unless

the metric tensor  $g$  is positive definite, the standard Morse index of every nontrivial closed geodesic is infinite. Unlike the fixed endpoint case, in the periodic case the definition of spectral flow depends heavily on the choice of a periodic frame along the geodesic. Two distinct choices of a periodic frame along a given closed geodesic produce two paths of self-adjoint Fredholm operators that are in general neither fixed endpoint homotopic nor cogredient. Recall in analogy that periodic solutions of Hamiltonian systems on general symplectic manifolds do *not* have a well defined Conley–Zehnder index (i.e., independent of the choice of a periodic symplectic frame along the solution), unless one poses strong restrictions on the topology of the underlying manifold.

An application of Theorem 4.4 gives us a formula for the spectral flow of a periodic geodesic (Theorem 5.6), given in terms of the *Maslov index* and the so-called *concavity index* of the geodesic, plus a certain degeneracy term. The Maslov index is a symplectic invariant which is associated to the underlying fixed endpoint geodesic, while the concavity index is an integer invariant of periodic solutions of Hamiltonian systems, which was introduced by M. Morse in the context of Riemannian closed geodesics. A first, and somewhat surprising, consequence of the formula, is that the spectral flow is well defined regardless of the choice of a periodic frame. This fact is probably more interesting *in se* than the formula itself. Further developments of the theory are to be expected in the realm of Morse theory for semi-Riemannian periodic geodesics, which at the present stage is a largely unexplored field (see [7] for the stationary Lorentzian case, or [5] for the fixed endpoints Lorentzian case). A natural conjecture would be that, under suitable nondegeneracy assumptions, the difference of spectral flows at two distinct geodesics is equal to the dimension of the intersection between the stable and the unstable manifolds of the gradient flow at the two critical points in the free loop space.

An effort has been made in order to make the paper essentially self-contained. In Section 2 we recall a few preliminary basic facts on Fredholm operators and bilinear forms; the central result is Proposition 2.14, that gives an upper bound for the dimension of an isotropic subspace. Section 3 contains most of the basic facts in the theory of Fredholm pairs and commensurable pairs of closed subspaces and relative dimension, with complete proofs. The main result (Proposition 3.18) is a formula giving the relative dimension of the negative eigenspaces of a self-adjoint Fredholm operator and its restriction to any closed finite codimensional subspace of a Hilbert space. Section 4 contains material on the spectral flow, dealing mostly with the case of paths of Fredholm operators that are compact perturbations of a fixed symmetry. Theorem 4.4 gives a formula for the computation of the spectral flow of a path of Fredholm symmetric bilinear forms (with arbitrary endpoints) in terms of the spectral flow of its restriction to a finite codimensional closed subspaces, and some boundary terms. Observe that both the path and/or its restriction is allowed to admit degeneracies at the endpoints. In Proposition 4.14 we show how the same result can be employed to study the case of restrictions to a continuous path of closed finite codimensional subspaces. A discussion of the notion of continuity, or smoothness, for a path of closed subspaces of a Hilbert space is presented in subsection 4.3. Smoothness for a path is defined in terms of the smoothness of local trivializations for the path (Definition 4.5); we show that this is equivalent to the smoothness of the corresponding path of orthogonal projections in the Banach algebra of all bounded operators on the Hilbert spaces (Proposition 4.9). This characterization of continuity yields several interesting facts. First, as it is shown in Appendix A, one can find *global* trivializations, second, the trivialization can be chosen by a path of isometries of the Hilbert space. Such trivialization will be called an orthogonal trivialization; orthogonal trivializations are special cases of the so-called *splitting* trivializations, that are employed in the definition of spectral flow in the case of restriction to varying domains. Section 5 contains the geometrical application of the theory.

## 2 Preliminaries. Fredholm bilinear forms.

In this section we will recall some basic facts about the geometry of closed subspaces of a Hilbert space, and, in addition, some properties of bounded symmetric bilinear forms on Hilbert spaces. Basic references for this part are [19, Chapter 2], [12, Section 2], [22, Section 1] and [2, Chapter 4, § 4]. Virtually, most of the material discussed is well known to specialists; the authors' intention is merely to fix notations and to state the results in a way which is best suited for the purposes of the paper.

Throughout this paper we will denote by  $\mathcal{H}$  a real separable Hilbert space, endowed with inner product  $\langle \cdot, \cdot \rangle$ ; by  $\| \cdot \|$  we will indicate the relative norm. Many of the results presented here will *not* indeed depend on the choice of a specific Hilbert space inner product. Complex extensions of the theory are also very likely to exist, but we will not be concerned with the complex case here.

Given a closed subspace  $\mathcal{V}$  of  $\mathcal{H}$ ,  $P_{\mathcal{V}}$  will stand for the orthogonal projection onto  $\mathcal{V}$ , and  $\mathcal{V}^{\perp}$  will denote the orthogonal complement of  $\mathcal{V}$  in  $\mathcal{H}$ . Depending on the context we will use the same symbol  $P_{\mathcal{V}}$  for the projection with target space  $\mathcal{H}$  or  $\mathcal{V}$ . Given two closed subspaces  $\mathcal{V}$  and  $\mathcal{W}$  of  $\mathcal{H}$ ,  $P_{\mathcal{V}}^{\mathcal{W}}$  will represent the restriction to  $\mathcal{W}$  of  $P_{\mathcal{V}}$ ; an immediate calculation shows that the adjoint of  $P_{\mathcal{V}}^{\mathcal{W}}$  is  $P_{\mathcal{W}}^{\mathcal{V}}$ .

Let us warm up by singling out a few basic facts concerning closed subspaces, orthogonal projections and compact operators, that will be used explicitly or implicitly in our proofs.

**Lemma 2.1** *Let  $\mathcal{V}$  and  $\mathcal{W}$  be closed subspaces of  $\mathcal{H}$ ; the following statements hold true:*

1.  $\text{Ker}(P_{\mathcal{V}} + P_{\mathcal{W}}) = \mathcal{V}^{\perp} \cap \mathcal{W}^{\perp}$ , and  $\overline{\text{Im}(P_{\mathcal{V}} + P_{\mathcal{W}})} = (\text{Ker}(P_{\mathcal{V}} + P_{\mathcal{W}}))^{\perp} = \overline{\mathcal{V} + \mathcal{W}}$ ;
2. if  $\text{codim}(\mathcal{V} + \mathcal{W}) < +\infty$ , then  $\mathcal{V} + \mathcal{W}$  is closed;
3. if  $K : \mathcal{H} \rightarrow \mathcal{H}$  is a compact linear operator, then  $(I + K)\mathcal{V}$  is closed;
4. if  $\mathcal{V} \supseteq \mathcal{W}^{\perp}$ , then  $\text{codim}_{\mathcal{V}} = \text{codim}_{\mathcal{W}}(\mathcal{V} \cap \mathcal{W})$ ;
5. if  $\text{codim}_{\mathcal{V}} < +\infty$ , then any subspace of  $\mathcal{H}$  containing  $\mathcal{V}$  is closed;
6. If  $\dim \mathcal{V} < +\infty$ , then  $\dim((\mathcal{V} + \mathcal{W}^{\perp}) \cap \mathcal{W}) < +\infty$ .

*Proof.* To prove (1) observe in first place that  $\text{Ker}(P_{\mathcal{V}} + P_{\mathcal{W}}) \supseteq \mathcal{V}^{\perp} \cap \mathcal{W}^{\perp}$ . If  $x \in \text{Ker}(P_{\mathcal{V}} + P_{\mathcal{W}})$ , then

$$\|P_{\mathcal{V}}x\|^2 = \langle P_{\mathcal{V}}x, x \rangle = -\langle P_{\mathcal{W}}x, x \rangle = -\|P_{\mathcal{W}}x\|^2,$$

hence  $\|P_{\mathcal{V}}x\| = \|P_{\mathcal{W}}x\| = 0$ , and  $x \in \mathcal{V}^{\perp} \cap \mathcal{W}^{\perp}$ . The second equality in (1) follows immediately.

Statement (2) follows from the general fact that, given a bounded linear operator between Banach spaces  $T : \mathcal{F} \rightarrow \mathcal{G}$ , having image of finite codimension, then  $\text{Im} T$  is closed. This is an easy application of the Open Mapping Theorem. In the case,  $\mathcal{V} + \mathcal{W}$  is the image of the bounded operator from  $\mathcal{V} \times \mathcal{W}$  to  $\mathcal{H}$ , given by  $(x, y) \mapsto x + y$ .

The proof of (5) goes as follows. Let  $\mathcal{U}$  be any subspace of  $\mathcal{H}$  containing  $\mathcal{V}$ , and consider the quotient map  $\pi : \mathcal{H} \rightarrow \mathcal{H}/\mathcal{V}$ . Since this quotient is finite dimensional, then  $\pi(\mathcal{U})$  is closed, and, since  $\mathcal{U} \supseteq \mathcal{V} = \text{Ker} \pi$ , then  $\mathcal{U}$  is saturated, i.e.,  $\mathcal{U} = \pi^{-1}(\pi(\mathcal{U}))$ , which implies that  $\mathcal{U}$  is closed.

To prove (6) consider  $P_{\mathcal{W}}^{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{W}$ , which clearly has finite dimensional, hence closed, image. Then

$$\begin{aligned} \text{Im} P_{\mathcal{W}}^{\mathcal{V}} &= \overline{\text{Im} P_{\mathcal{W}}^{\mathcal{V}}} = (\text{Ker}(P_{\mathcal{W}}^{\mathcal{V}})^*)^{\perp} = (\text{Ker} P_{\mathcal{V}}^{\mathcal{W}})^{\perp} = \\ &(\mathcal{V}^{\perp} \cap \mathcal{W})^{\perp} \cap \mathcal{W} = (\mathcal{V} + \mathcal{W}^{\perp}) \cap \mathcal{W}. \end{aligned}$$

This concludes the proof. □

Moreover, an application of [19, Ch. 4, § 4, Theorem 4.2] yields the following Lemma.

**Lemma 2.2** *Let  $\mathcal{V}, \mathcal{W}$  be closed subspaces of  $\mathcal{H}$ . Then  $\mathcal{V} + \mathcal{W}$  is closed if and only if the operator  $P_{\mathcal{V}^{\perp}}^{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{V}^{\perp}$  has closed image.*

*Proof.* For  $w \in \mathcal{W}$ , set  $w_o = P_{\mathcal{V} \cap \mathcal{W}}w$  and  $w_{\perp} = w - w_o$ ; the result of [19, Ch. 4, § 4, Theorem 4.2] tells us that  $\mathcal{V} + \mathcal{W}$  is closed if and only if there exists  $c > 0$  such that  $\|P_{\mathcal{V}^{\perp}}w_{\perp}\| \geq c\|w_{\perp}\|$  for all  $w \in \mathcal{W}$ . In turn, this latter condition is equivalent to the fact that  $P_{\mathcal{V}^{\perp}}|_{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{V}^{\perp}$  has closed image. □

**Remark 2.3** Given closed subspaces  $\mathcal{V}, \mathcal{W}$  of a Banach space, let us recall Kato's definition of the constant  $\gamma(\mathcal{V}, \mathcal{W}) \in [0, 1]$ :

$$\gamma(\mathcal{V}, \mathcal{W}) = \inf_{u \in \mathcal{V} \setminus \mathcal{W}} \frac{\text{dist}(u, \mathcal{W})}{\text{dist}(u, \mathcal{V} \cap \mathcal{W})}.$$

It is proven in [19, Ch. 4, § 4, Theorem 4.2] that  $\mathcal{V} + \mathcal{W}$  is closed if and only if  $\gamma(\mathcal{V}, \mathcal{W}) > 0$ . Similarly, if  $L$  is a bounded linear operator between Banach spaces, then the image of  $L$  is closed if and only if the constant

$$\gamma(L) = \inf_{u \notin \text{Ker} L} \frac{\|Lu\|}{\text{dist}(u, \text{Ker} L)}$$

is positive. An immediate calculation shows that if  $\mathcal{V}, \mathcal{W}$  are closed subspaces of a Hilbert space  $\mathcal{H}$ , then  $\gamma(\mathcal{V}, \mathcal{W}) = \gamma(P_{\mathcal{V}^\perp} \mathcal{W})$ ; from this fact it follows immediately a proof of Lemma 2.2.

Let now  $B$  be a continuous bilinear form on  $\mathcal{H}$  and  $T : \mathcal{H} \rightarrow \mathcal{H}$  the continuous linear operator uniquely associated with  $B$ , that is,

$$B(x, y) = \langle Tx, y \rangle, \quad \forall x, y \in \mathcal{H}.$$

We define

$$\text{Ker } B = \{x \in \mathcal{H} : B(x, y) = 0, \forall y \in \mathcal{H}\}.$$

It is immediate to see that  $\text{Ker } B = \text{Ker } T$ . If  $\text{Ker } B = \{0\}$ , then  $B$  is said to be *nondegenerate*.

If a continuous bilinear form  $B$  is symmetric, then  $T$  is *self-adjoint*, that is,  $\langle Tx, y \rangle = \langle x, Ty \rangle$ , for all  $x, y \in \mathcal{H}$ .

**Definition 2.4** Given a continuous bilinear form  $B$ , the *Morse index* of  $B$  is the (possibly infinite) integer number

$$n_-(B) = \sup \{ \dim \mathcal{V} : B|_{\mathcal{V} \times \mathcal{V}} \text{ is negative definite} \}.$$

We denote by  $n_+(B)$  the Morse index of  $-B$ , also called the *Morse coindex* of  $B$ . Of course one has

$$n_+(B) = \sup \{ \dim \mathcal{V} : B|_{\mathcal{V} \times \mathcal{V}} \text{ is positive definite} \}.$$

**Definition 2.5** A symmetric continuous bilinear form  $B$  on  $\mathcal{H}$ , associated with a (self-adjoint) Fredholm operator, is called a *symmetric Fredholm form* on  $\mathcal{H}$ .

A self-adjoint Fredholm operator has null index.

**Standing assumption.** From now on  $B$  will denote a symmetric Fredholm form on  $\mathcal{H}$  and  $T$  will be the self-adjoint Fredholm operator  $T$  associated with  $B$ .

By the spectral theory of the self-adjoint Fredholm operators, there exists a unique orthogonal splitting of  $\mathcal{H}$  induced by  $B$ ,

$$H = V^-(T) \oplus V^+(T) \oplus \text{Ker } T, \tag{2.1}$$

such that  $V^-(T)$  and  $V^+(T)$  are both  $T$ -invariant,  $B|_{V^-(T) \times V^-(T)}$  is negative definite and  $B|_{V^+(T) \times V^+(T)}$  is positive definite.

In addition, since  $V^-(T)$  and  $V^+(T)$  are  $T$ -invariant and orthogonal, they are also  $B$ -orthogonal, that is,  $B(x, y) = 0$  for any  $x \in V^-(T)$  and any  $y \in V^+(T)$ .

With a slight abuse of notation, we will refer to  $V^-(T)$  and  $V^+(T)$  respectively as the *negative* and the *positive eigenspaces* of  $B$ .

**Remark 2.6** Observe that the Morse index of a symmetric Fredholm form  $B$  coincides with the (possibly infinite) dimension of the negative eigenspace  $V^-(T)$ .

Given a subspace  $\mathcal{V}$  of  $\mathcal{H}$ , we define the *B-orthogonal complement* of  $\mathcal{V}$  as the subspace of  $\mathcal{H}$

$$\mathcal{V}^{\perp B} = \{x \in \mathcal{H} : B(x, y) = 0, \forall y \in \mathcal{V}\}.$$

**Remark 2.7** Given a closed subspace  $\mathcal{V}$  of  $\mathcal{H}$ , we have the following properties.

- i)  $\mathcal{V}^{\perp B}$  is closed and  $\text{Ker } T \subseteq \mathcal{V}^{\perp B}$ , the proof is immediate.
- ii) If  $\mathcal{V}$  has finite codimension, then  $\mathcal{V}^{\perp B}$  is finite dimensional. Indeed,

$$\mathcal{V}^{\perp B} = \{x \in \mathcal{H} : \langle Tx, y \rangle = 0, \forall y \in \mathcal{V}\} = \{x \in \mathcal{H} : \langle x, Ty \rangle = 0, \forall y \in \mathcal{V}\}.$$

That is,  $\mathcal{V}^{\perp B}$  is orthogonal to  $T(\mathcal{V})$ , which has finite codimension since  $T$  is Fredholm and  $\mathcal{V}$  has finite codimension. More precisely,

$$\dim \mathcal{V}^{\perp B} = \text{codim } \mathcal{V} + \dim (\text{Ker } T \cap \mathcal{V}).$$

- iii) Similarly, if  $\mathcal{V}$  has finite dimension, then  $\mathcal{V}^{\perp B}$  has finite codimension coinciding with  $\dim \mathcal{V} - \dim \text{Ker } T|_{\mathcal{V}}$ .
- iv) In general,  $\mathcal{V} + \mathcal{V}^{\perp B} \neq \mathcal{H}$ , even when  $B$  is nondegenerate.

**Lemma 2.8** *If  $\mathcal{V}$  is a closed subspace of  $\mathcal{H}$ , having finite codimension, then the restriction  $B|_{\mathcal{V} \times \mathcal{V}}$  is Fredholm.*

*Proof.* The kernel of  $B|_{\mathcal{V} \times \mathcal{V}}$  is given by  $\mathcal{V} \cap \mathcal{V}^{\perp B}$ , which is finite dimensional. If  $T$  is the Fredholm self-adjoint operator that represents  $B$ , then  $B|_{\mathcal{V} \times \mathcal{V}}$  is represented by  $P_{\mathcal{V}} \circ T|_{\mathcal{V}}$ , whose image contains  $T(\mathcal{V}) \cap \mathcal{V}$ , which has finite codimension.  $\square$

Let  $\mathcal{V}$  be a closed subspace of  $\mathcal{H}$ . Denote by  $\tilde{T} : \mathcal{V} \rightarrow \mathcal{V}$  the operator associated with  $B|_{\mathcal{V} \times \mathcal{V}}$  and by  $T_2 : \mathcal{V}^{\perp B} \rightarrow \mathcal{V}^{\perp B}$  the operator associated with  $B|_{\mathcal{V}^{\perp B} \times \mathcal{V}^{\perp B}}$ . Notice that  $T_2 = P_{\mathcal{V}^{\perp B}} \circ T|_{\mathcal{V}^{\perp B}}$ .

**Lemma 2.9** *In the above notation we have the following results.*

1. *If  $\mathcal{V} \cap \mathcal{V}^{\perp B} = \{0\}$ , then  $B|_{\mathcal{V} \times \mathcal{V}}$  is nondegenerate.*
2. *If  $\mathcal{V}$  is finite dimensional or finite codimensional and  $B|_{\mathcal{V} \times \mathcal{V}}$  is nondegenerate, then  $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^{\perp B}$ .*
3.  *$\text{Ker } \tilde{T}$  and  $\text{Ker } T$  are contained in  $\text{Ker } T_2$ . If in particular  $\mathcal{H} = \mathcal{V} + \mathcal{V}^{\perp B}$  (not necessarily direct sum), then  $\text{Ker } T = \text{Ker } T_2$ .*
4. *If  $B$  is nondegenerate and  $\mathcal{V} + \mathcal{V}^{\perp B} = \mathcal{H}$ , then  $\mathcal{V} \cap \mathcal{V}^{\perp B} = \{0\}$ .*
5. *If  $\mathcal{V}$  is finite dimensional or finite codimensional, then*

$$(\mathcal{V} \cap \mathcal{V}^{\perp B})^{\perp B} = \mathcal{V} + \mathcal{V}^{\perp B}.$$

*Proof.* (1) If  $x \in \text{Ker } \tilde{T}$ , then  $Tx$  is orthogonal to  $\mathcal{V}$ , that is,  $x \in \mathcal{V}^{\perp B}$ . Hence  $x = 0$  and  $B|_{\mathcal{V} \times \mathcal{V}}$  is nondegenerate.

(2) Let  $v \in \mathcal{V} \cap \mathcal{V}^{\perp B}$  be given. As  $v \in \mathcal{V}^{\perp B}$ , it is orthogonal to  $T(\mathcal{V})$ , that is,  $0 = \langle Tv', v \rangle = \langle v', Tv \rangle$  for any  $v' \in \mathcal{V}$ . This implies that  $Tv$  is orthogonal to  $\mathcal{V}$  and so  $\tilde{T}v = 0$  ( $v$  belongs to  $\mathcal{V}$ , hence  $\tilde{T}v$  is well defined). Thus  $v = 0$  since  $B|_{\mathcal{V} \times \mathcal{V}}$  is nondegenerate. Notice that the proof that  $\mathcal{V} \cap \mathcal{V}^{\perp B} = \{0\}$  does not require any information about the dimension of  $\mathcal{V}$ .

Now, if  $\mathcal{V}$  has finite codimension, then  $\mathcal{V}^{\perp B}$  has finite dimension. Hence, if  $\mathcal{V}$  is finite dimensional or finite codimensional, then  $\mathcal{V} + \mathcal{V}^{\perp B}$  is closed being the sum of two closed subspaces of  $\mathcal{H}$  such that one of them has finite dimension.

To show that  $\mathcal{V} + \mathcal{V}^{\perp B} = \mathcal{H}$  consider an element  $v$  of the orthogonal complement of  $\mathcal{V} + \mathcal{V}^{\perp B}$  in  $\mathcal{H}$ . We have that  $v \in T(\mathcal{V})$  since this latter coincides with  $(\mathcal{V}^{\perp B})^{\perp}$ . Let  $x \in \mathcal{V}$  be such that  $Tx = v$ . As  $Tx$  is orthogonal to  $\mathcal{V}$ , then  $\tilde{T}x = 0$  and this implies that  $x = 0$  since  $\tilde{T}$  is injective. Therefore,  $v = 0$  and we have finally  $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^{\perp B}$ .

(3) If  $x \in \text{Ker } T$ , then  $\langle Tx, y \rangle = 0$  for each  $y \in \mathcal{V}$ , that is,  $x \in \mathcal{V}^{\perp B}$  and  $T_2x$  is well defined. As  $Tx = 0$ , trivially  $T_2x = 0$ , that is,  $\text{Ker } T \subseteq \text{Ker } T_2$ . Given  $x \in \text{Ker } \tilde{T}$ , in the decomposition  $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^{\perp B}$  write  $Tx = \tilde{T}x + P_{\mathcal{V}^{\perp B}}Tx = P_{\mathcal{V}^{\perp B}}Tx$ . Hence  $\langle Tx, y \rangle = 0$  for each  $y \in \mathcal{V}$ , that is,  $x \in \mathcal{V}^{\perp B}$  and  $T_2x$  is well defined. In the decomposition  $\mathcal{H} = (\mathcal{V}^{\perp B})^{\perp} \oplus \mathcal{V}^{\perp B}$  denote by  $Q$  the orthogonal projection onto  $(\mathcal{V}^{\perp B})^{\perp}$ . Then,

$$0 = \langle Tx, y \rangle = \langle QTx + T_2x, y \rangle = \langle T_2x, y \rangle, \quad \forall y \in \mathcal{V}^{\perp B}.$$

Then  $T_2x = 0$ , that is,  $\text{Ker } \tilde{T} \subseteq \text{Ker } T_2$ .

In the particular case when  $\mathcal{H} = \mathcal{V} + \mathcal{V}^{\perp B}$ , let  $x \in \text{Ker } T_2$  be given. Given any  $z \in \mathcal{H}$ , let us write  $z = z_1 + z_2$ , where  $z_1 \in \mathcal{V}$  and  $z_2 \in \mathcal{V}^{\perp B}$ . We have

$$\langle Tx, z \rangle = \langle Tx, z_1 \rangle + \langle Tx, z_2 \rangle.$$

The product  $\langle Tx, z_1 \rangle$  vanishes since  $x \in \mathcal{V}^{\perp B}$  and  $z_1 \in \mathcal{V}$ , and the term  $\langle Tx, z_2 \rangle$  is zero since  $T_2x = 0$ , that is  $Tx \in (\mathcal{V}^{\perp B})^{\perp}$ . Hence  $\langle Tx, z \rangle = 0$  for any  $z \in \mathcal{H}$ , that means  $Tx = 0$ .

(4) Let  $x \in \mathcal{V} \cap \mathcal{V}^{\perp B}$  be given. Given any  $z \in \mathcal{H}$ , write  $z = z_1 + z_2$ , where  $z_1 \in \mathcal{V}$  and  $z_2 \in \mathcal{V}^{\perp B}$ . Then

$$\langle Tx, z \rangle = \langle Tx, z_1 \rangle + \langle Tx, z_2 \rangle = 0.$$

In fact,  $\langle Tx, z_1 \rangle = 0$  since  $x \in \mathcal{V}^{\perp B}$  and  $z_1 \in \mathcal{V}$ , while  $\langle Tx, z_2 \rangle = 0$  since  $x \in \mathcal{V}$  and  $z_2 \in \mathcal{V}^{\perp B}$ . Hence  $\langle Tx, z \rangle = 0$  for any  $z \in \mathcal{H}$  and this implies that  $x = 0$  since  $B$  is nondegenerate.

(5) It is a consequence of the following properties shown (in a more general setting) in [7]: *given two closed subspaces  $S_1$  and  $S_2$  of  $\mathcal{H}$ , then*

$$\text{i) } (S_1 + S_2)^{\perp B} = S_1^{\perp B} \cap S_2^{\perp B},$$

$$\text{ii) } (S_1^{\perp B})^{\perp B} = S_1 + \text{Ker } T.$$

First of all one can show that

$$(\mathcal{V} \cap \mathcal{V}^{\perp B})^{\perp B} = ((\mathcal{V} + \text{Ker } T) \cap \mathcal{V}^{\perp B})^{\perp B}$$

(even if  $\mathcal{V} \cap \mathcal{V}^{\perp B}$  could be strictly contained in  $(\mathcal{V} + \text{Ker } T) \cap \mathcal{V}^{\perp B}$ ; this is the case when  $\mathcal{V}$  does not contain  $\text{Ker } T$ ). Indeed, fix an element  $x \in (\mathcal{V} \cap \mathcal{V}^{\perp B})^{\perp B}$  and let  $w \in (\mathcal{V} + \text{Ker } T) \cap \mathcal{V}^{\perp B}$  be given. One can write  $w = v + k$ , where  $v \in \mathcal{V}$  and  $k \in \text{Ker } T$ . Since  $\text{Ker } T \subseteq \mathcal{V}^{\perp B}$ , then  $k$  belongs to  $\mathcal{V}^{\perp B}$ , as  $w$ , and thus  $v \in \mathcal{V}^{\perp B}$  as well. That is,  $v \in \mathcal{V} \cap \mathcal{V}^{\perp B}$  and this implies

$$\langle Tx, w \rangle = \langle Tx, v \rangle + \langle Tx, k \rangle = 0,$$

since  $\langle Tx, v \rangle$  and  $\langle Tx, k \rangle$  both vanish. Therefore,  $x \in ((\mathcal{V} + \text{Ker } T) \cap \mathcal{V}^{\perp B})^{\perp B}$ , that is,  $(\mathcal{V} \cap \mathcal{V}^{\perp B})^{\perp B} \subseteq ((\mathcal{V} + \text{Ker } T) \cap \mathcal{V}^{\perp B})^{\perp B}$ .

The inclusion  $((\mathcal{V} + \text{Ker } T) \cap \mathcal{V}^{\perp B})^{\perp B} \subseteq (\mathcal{V} \cap \mathcal{V}^{\perp B})^{\perp B}$  follows immediately from the inclusion  $\mathcal{V} \cap \mathcal{V}^{\perp B} \subseteq (\mathcal{V} + \text{Ker } T) \cap \mathcal{V}^{\perp B}$ .

Let us now conclude the proof of the statement (5). By the previous item ii) we have  $\mathcal{V} + \text{Ker } T = (\mathcal{V}^{\perp B})^{\perp B}$ . Hence, by i),  $(\mathcal{V} + \text{Ker } T) \cap \mathcal{V}^{\perp B} = (\mathcal{V}^{\perp B} + \mathcal{V})^{\perp B}$ . By ii),  $((\mathcal{V}^{\perp B} + \mathcal{V})^{\perp B})^{\perp B} = \mathcal{V}^{\perp B} + \mathcal{V}$ , recalling that  $\text{Ker } T \subseteq \mathcal{V}^{\perp B}$ .

Summarizing the arguments,

$$(\mathcal{V} \cap \mathcal{V}^{\perp B})^{\perp B} = ((\mathcal{V} + \text{Ker } T) \cap \mathcal{V}^{\perp B})^{\perp B} = \mathcal{V}^{\perp B} + \mathcal{V}$$

and the proof is complete.  $\square$

**Remark 2.10** If  $\mathcal{V} + \mathcal{V}^{\perp B}$  is strictly contained in  $\mathcal{H}$ ,  $\text{Ker } T$  does not necessarily coincides with  $\text{Ker } T_2$  and  $\mathcal{V} \cap \mathcal{V}^{\perp B}$  is not necessarily empty. Examples, even in finite dimension, could be easily provided and left to the reader.

**Definition 2.11** A subspace  $\mathcal{Z}$  of  $\mathcal{H}$  is said to be *isotropic* for  $B$  if  $B(z, z) = 0$  for any  $z \in \mathcal{Z}$ .

Any subspace of  $\text{Ker } T$  is clearly isotropic, but one can easily find examples of symmetric Fredholm forms having isotropic subspaces not contained in the kernel of the associated operator.

**Lemma 2.12** *Suppose that  $B$  admits an isotropic subspace  $\mathcal{Z}$  which is not contained in  $\text{Ker } T$ . Then  $B$  is indefinite, that is, there exist  $x, y \in \mathcal{H}$  such that  $\langle Tx, x \rangle > 0$  and  $\langle Ty, y \rangle < 0$ .*

*Proof.* Let  $v \in \mathcal{H}$  such that  $\langle Tv, v \rangle = 0$  and  $w := Tv \neq 0$ . For any  $\alpha \in \mathbb{R}$  we have

$$\langle T(\alpha v + w), \alpha v + w \rangle = 2\alpha \|w\|^2 + \langle Tw, w \rangle.$$

The claim follows choosing  $x = \alpha_1 v + w$  and  $y = \alpha_2 v + w$ , with any  $\alpha_1 > -\frac{|\langle Tw, w \rangle|}{2\|w\|^2}$  and any  $\alpha_2 < -\frac{|\langle Tw, w \rangle|}{2\|w\|^2}$ .  $\square$

**Corollary 2.13** *If  $B$  is positive (resp. negative) semidefinite, then  $B|_{(\text{Ker } T)^\perp \times (\text{Ker } T)^\perp}$  is positive (resp. negative) definite.*

Lemma 2.12 above allows us to prove the next result connecting the Morse index of  $B$  and a given isotropic space  $\mathcal{Z}$ .

**Proposition 2.14** *If  $\mathcal{Z}$  is an isotropic subspace of  $\mathcal{H}$ , then*

$$\dim \mathcal{Z} \leq n_-(B) + \dim(\mathcal{Z} \cap \text{Ker } T) \quad \text{and} \quad \dim \mathcal{Z} \leq n_+(B) + \dim(\mathcal{Z} \cap \text{Ker } T).$$

*Proof.* Let us prove just the first inequality, the proof of the second one is analogous. If  $\mathcal{Z}$  is infinite dimensional (this is the case when, for instance, it is not closed), one has  $n_-(B) = +\infty$  and this could be easily verified using the proof of the above Lemma 2.12. In this case the inequality  $\dim \mathcal{Z} \leq n_-(B) + \dim(\mathcal{Z} \cap \text{Ker } T)$  immediately follows.

Suppose now that  $\dim \mathcal{Z} < +\infty$ . If  $\mathcal{Z}$  is contained in  $\text{Ker } T$ , the result trivially holds. If  $\mathcal{Z}$  is not contained in  $\text{Ker } T$ , then  $B$  is indefinite and  $n_-(B)$  is strictly positive (or  $+\infty$ ). Call  $\mathcal{V}$  the orthogonal complement of  $\mathcal{Z} \cap \text{Ker } T$  in  $\mathcal{Z}$  and recall the spectral decomposition (2.1) of  $\mathcal{H}$ , induced by  $B$ ,  $\mathcal{H} = V^-(T) \oplus V^+(T) \oplus \text{Ker } T$ .

Given  $z \in \mathcal{V}$ , if  $P_{V^-(T)}z = 0$ , then  $z \in V^+(T)$  and thus

$$\langle Tz, z \rangle \geq 0.$$

On the other hand  $\langle Tz, z \rangle = 0$  as  $z$  belongs to  $\mathcal{Z}$ . Then  $z = 0$  and  $P_{V^-(T)}^\mathcal{V}$  is injective. Consequently

$$\dim \mathcal{Z} - \dim(\mathcal{Z} \cap \text{Ker } T) = \dim \mathcal{V} = \dim(\text{Im } P_{V^-(T)}^\mathcal{V}) \leq n_-(B)$$

and the proposition is proven.  $\square$

### 3 Fredholm and commensurable pairs of closed subspaces

#### 3.1 Relative dimension and Fredholm pairs

The following notion of Fredholm pair of closed subspaces of  $\mathcal{H}$  has been introduced by Kato (see [19]). See also [6] for the notion of index of a pair of orthogonal projections.

**Definition 3.1** Given two closed subspaces  $\mathcal{V}$  and  $\mathcal{W}$  of  $\mathcal{H}$ , we will say that  $(\mathcal{V}, \mathcal{W})$  is a *Fredholm pair* if  $\dim(\mathcal{V} \cap \mathcal{W}) < +\infty$  and  $\text{codim}(\mathcal{V} + \mathcal{W}) < +\infty$ . We will denote by  $\mathcal{FP}(\mathcal{H})$  the set of all Fredholm pairs of closed subspaces in  $\mathcal{H}$ ; for  $(\mathcal{V}, \mathcal{W}) \in \mathcal{FP}(\mathcal{H})$  we set

$$\text{ind}(\mathcal{V}, \mathcal{W}) = \dim(\mathcal{V} \cap \mathcal{W}) - \text{codim}(\mathcal{V} + \mathcal{W}).$$

We observe that, by part (2) of Lemma 2.1, if  $(\mathcal{V}, \mathcal{W}) \in \mathcal{FP}(\mathcal{H})$  then  $\mathcal{V} + \mathcal{W}$  is closed, and so

$$\text{ind}(\mathcal{V}, \mathcal{W}) = \dim(\mathcal{V} \cap \mathcal{W}) - \dim((\mathcal{V} + \mathcal{W})^\perp) = \dim(\mathcal{V} \cap \mathcal{W}) - \dim(\mathcal{V}^\perp \cap \mathcal{W}^\perp).$$

Establishing if a given pair of closed subspaces is a Fredholm pair is not always easy; usually, the nontrivial part of the proof is to show that the sum of the spaces is closed. Once this is done, the finite codimensionality is obtained using orthogonality arguments. For this reason, it will be essential to determine criteria for establishing the Fredholmness of pairs; most of such criteria are given in terms of orthogonal projections.

**Proposition 3.2** *Given two closed subspaces  $\mathcal{V}$  and  $\mathcal{W}$  of  $\mathcal{H}$ ,  $(\mathcal{V}, \mathcal{W}) \in \mathcal{FP}(\mathcal{H})$  if and only if  $P_{\mathcal{V}^\perp}^\mathcal{W} : \mathcal{W} \rightarrow \mathcal{V}^\perp$  is a Fredholm operator. In this case,  $\text{ind}(\mathcal{V}, \mathcal{W})$  equals the Fredholm index of  $P_{\mathcal{V}^\perp}^\mathcal{W}$ .*

*Proof.* In first place,

$$\text{Ker } P_{\mathcal{V}^\perp}^\mathcal{W} = \mathcal{V} \cap \mathcal{W}. \tag{3.1}$$

If  $(\mathcal{V}, \mathcal{W}) \in \mathcal{FP}(\mathcal{H})$ , then  $\mathcal{V} + \mathcal{W}$  is closed, and, by Lemma 2.2,  $P_{\mathcal{V}^\perp}^\mathcal{W}$  has closed image. Moreover,

$$\begin{aligned} \text{Im } P_{\mathcal{V}^\perp}^\mathcal{W} &= \overline{\text{Im } P_{\mathcal{V}^\perp}^\mathcal{W}} = (\text{Ker } (P_{\mathcal{V}^\perp}^\mathcal{W})^*)^\perp = (\text{Ker } P_{\mathcal{V}^\perp}^{\mathcal{W}^\perp})^\perp \\ &= (\mathcal{V}^\perp \cap \mathcal{W}^\perp)^\perp \cap \mathcal{V}^\perp = \overline{\mathcal{V} + \mathcal{W}} \cap \mathcal{V}^\perp = (\mathcal{V} + \mathcal{W}) \cap \mathcal{V}^\perp, \end{aligned} \tag{3.2}$$

(see also the proof of part (6) in Lemma 2.1). By part (4) of Lemma 2.1,

$$\text{codim}_{\mathcal{V}^\perp}((\mathcal{V} + \mathcal{W}) \cap \mathcal{V}^\perp) = \text{codim}(\mathcal{V} + \mathcal{W}) < +\infty, \quad (3.3)$$

hence  $P_{\mathcal{V}^\perp}^{\mathcal{W}}$  is Fredholm. Conversely, if  $P_{\mathcal{V}^\perp}^{\mathcal{W}}$  is Fredholm, then, by Lemma 2.2,  $\mathcal{V} + \mathcal{W}$  is closed; moreover, the adjoint  $(P_{\mathcal{V}^\perp}^{\mathcal{W}})^* = P_{\mathcal{W}}^{\mathcal{V}^\perp}$  is also Fredholm, and thus

$$\text{codim}(\mathcal{V} + \mathcal{W}) = \dim((\mathcal{V} + \mathcal{W})^\perp) = \dim(\mathcal{V}^\perp \cap \mathcal{W}^\perp) = \dim(\text{Ker } P_{\mathcal{W}}^{\mathcal{V}^\perp}) < +\infty.$$

The last statement in the thesis follows readily from (3.1), (3.2) and (3.3).  $\square$

**Corollary 3.3** *If  $(\mathcal{V}, \mathcal{W}) \in \mathcal{FP}(\mathcal{H})$ , then  $(\mathcal{W}, \mathcal{V})$  and  $(\mathcal{V}^\perp, \mathcal{W}^\perp)$  are in  $\mathcal{FP}(\mathcal{H})$ , and  $\text{ind}(\mathcal{V}, \mathcal{W}) = \text{ind}(\mathcal{W}, \mathcal{V}) = -\text{ind}(\mathcal{V}^\perp, \mathcal{W}^\perp)$ .*

**Proof.** The fact that  $(\mathcal{W}, \mathcal{V}) \in \mathcal{FP}(\mathcal{H})$  follows directly from the definition of Fredholm pairs, as well as the equality  $\text{ind}(\mathcal{V}, \mathcal{W}) = \text{ind}(\mathcal{W}, \mathcal{V})$ . Moreover, since the adjoint of  $P_{\mathcal{V}^\perp}^{\mathcal{W}}$  is  $P_{\mathcal{W}}^{\mathcal{V}^\perp}$ , it follows from Proposition 3.2 that  $(\mathcal{V}^\perp, \mathcal{W}^\perp) \in \mathcal{FP}(\mathcal{H})$ , and that  $\text{ind}(\mathcal{V}^\perp, \mathcal{W}^\perp) = \text{ind}(P_{\mathcal{W}}^{\mathcal{V}^\perp}) = -\text{ind}(P_{\mathcal{V}^\perp}^{\mathcal{W}}) = -\text{ind}(\mathcal{V}, \mathcal{W})$ .  $\square$

Here is yet another characterization of Fredholm pairs.

**Corollary 3.4**  *$(\mathcal{V}, \mathcal{W}) \in \mathcal{FP}(\mathcal{H})$  if and only if the difference  $P_{\mathcal{V}} - P_{\mathcal{W}} : \mathcal{H} \rightarrow \mathcal{H}$  is Fredholm.*

**Proof.** Consider the operators

$$\begin{aligned} \tilde{T} : \mathcal{H} &\rightarrow \mathcal{W} \oplus \mathcal{W}^\perp, & \tilde{T}(x) &= (P_{\mathcal{W}}x, P_{\mathcal{W}^\perp}x), \\ T_2 : \mathcal{W} \oplus \mathcal{W}^\perp &\rightarrow \mathcal{V}^\perp \oplus \mathcal{V}, & T_2(w, w_\perp) &= (P_{\mathcal{V}^\perp}w, -P_{\mathcal{V}}w_\perp), \\ T_3 : \mathcal{V}^\perp \oplus \mathcal{V} &\rightarrow \mathcal{H}, & T_3(v, v_\perp) &= v + v_\perp. \end{aligned}$$

Clearly,  $\tilde{T}$  and  $T_3$  are isomorphisms, and the composition  $T_3 \circ T_2 \circ \tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$  is Fredholm if and only if  $T_2$  is Fredholm. We have

$$\begin{aligned} T_3(T_2(\tilde{T}z)) &= T_3(T_2(P_{\mathcal{W}}z, P_{\mathcal{W}^\perp}z)) = P_{\mathcal{V}^\perp}(P_{\mathcal{W}}z) - P_{\mathcal{V}}(P_{\mathcal{W}^\perp}z) \\ &= P_{\mathcal{V}^\perp}(P_{\mathcal{W}}z) + P_{\mathcal{V}}(P_{\mathcal{W}}z) - P_{\mathcal{V}}(P_{\mathcal{W}}z) - P_{\mathcal{V}}(P_{\mathcal{W}^\perp}z) = P_{\mathcal{W}}z - P_{\mathcal{V}}z. \end{aligned}$$

Now,  $T_2 = P_{\mathcal{V}^\perp}^{\mathcal{W}} \oplus (-P_{\mathcal{V}}^{\mathcal{W}^\perp})$ , and this is Fredholm if and only if both  $P_{\mathcal{V}^\perp}^{\mathcal{W}}$  and  $P_{\mathcal{V}}^{\mathcal{W}^\perp}$  are; the conclusion follows now from Proposition 3.2 and Corollary 3.3.  $\square$

As to the *sum* of orthogonal projections onto Fredholm pairs, we have the following result.

**Lemma 3.5** *Let  $\mathcal{V}, \mathcal{W}$  be closed subspaces of  $\mathcal{H}$  such that  $\mathcal{V} \cap \mathcal{W} = \{0\}$  and such that  $\mathcal{V} + \mathcal{W}$  is closed. Then, the image of  $P_{\mathcal{V}} + P_{\mathcal{W}} : \mathcal{H} \rightarrow \mathcal{H}$  is  $\mathcal{V} + \mathcal{W}$ . In particular, if  $\mathcal{V} + \mathcal{W} = \mathcal{H}$ , then  $P_{\mathcal{V}} + P_{\mathcal{W}}$  is surjective.*

**Proof.** Obviously,  $\text{Im}(P_{\mathcal{V}} + P_{\mathcal{W}}) \subseteq \mathcal{V} + \mathcal{W}$ . Since  $\mathcal{V} + \mathcal{W}$  is closed and  $P_{\mathcal{V}}P_{\mathcal{V}+\mathcal{W}} = P_{\mathcal{V}}$ ,  $P_{\mathcal{W}}P_{\mathcal{V}+\mathcal{W}} = P_{\mathcal{W}}$ , we can replace  $\mathcal{H}$  by  $\mathcal{V} + \mathcal{W}$  and assume that  $\mathcal{V} + \mathcal{W} = \mathcal{H}$ .

Since  $\mathcal{V} \cap \mathcal{W} = \{0\}$ , then there exists a (unique) linear operator  $A : \mathcal{V}^\perp \rightarrow \mathcal{V}$  whose graph

$$\text{Graph}(A) = \{z + Az : z \in \mathcal{V}^\perp\} \subseteq \mathcal{H}$$

is  $\mathcal{W}$ . Then,  $\mathcal{H} = \mathcal{V} + \text{Graph}(A)$ .

Clearly,  $A$  is bounded because its graph is closed (Closed Graph Theorem). It is easy to show that the graph of the negative adjoint map  $-A^* : \mathcal{V} \rightarrow \mathcal{V}^\perp$  is equal to  $\mathcal{W}^\perp$ ; namely, if  $y \in \mathcal{W}$ , then  $y = z + Az$  for some  $z \in \mathcal{V}^\perp$ . Now, if  $x \in \mathcal{V}$ , we have

$$\langle x - A^*x, y \rangle = \langle x - A^*x, z + Az \rangle = -\langle A^*x, z \rangle + \langle x, Az \rangle = 0,$$

i.e.,  $\text{Graph}(-A^*) \subseteq \mathcal{W}^\perp$ . On the other hand, choose  $t \in \mathcal{W}^\perp$  and write  $t = t_{\mathcal{V}} + t_{\mathcal{V}^\perp}$ , where  $t_{\mathcal{V}} \in \mathcal{V}$  and  $t_{\mathcal{V}^\perp} \in \mathcal{V}^\perp$ . Since  $\mathcal{W} = \text{Graph}(A)$ , we have:

$$\langle t, z + Az \rangle = 0, \quad \forall z \in \mathcal{V}^\perp,$$

i.e.,

$$0 = \langle t_{\mathcal{V}} + t_{\mathcal{V}^\perp}, z + Az \rangle = \langle t_{\mathcal{V}}, Az \rangle + \langle t_{\mathcal{V}^\perp}, z \rangle = \langle A^* t_{\mathcal{V}} + t_{\mathcal{V}^\perp}, z \rangle$$

for all  $z \in \mathcal{V}^\perp$ , which implies  $A^* t_{\mathcal{V}} + t_{\mathcal{V}^\perp} \in \mathcal{V}$ . But  $A^* t_{\mathcal{V}} + t_{\mathcal{V}^\perp} \in \mathcal{V}^\perp$ , hence  $A^* t_{\mathcal{V}} + t_{\mathcal{V}^\perp} = 0$ , and  $t_{\mathcal{V}^\perp} = -A^* t_{\mathcal{V}}$ ,  $t = t_{\mathcal{V}} - A^* t_{\mathcal{V}} \in \text{Graph}(-A^*)$ , that is  $\text{Graph}(-A^*) \supseteq \mathcal{W}^\perp$ . That is,  $\text{Graph}(-A^*) = \mathcal{W}^\perp$ .

Let us now determine the image of  $P_{\mathcal{V}} + P_{\mathcal{W}}$ ; let  $r \in \mathcal{H}$  be fixed, we search  $s \in \mathcal{H}$  with  $P_{\mathcal{V}}s + P_{\mathcal{W}}s = r$ . Write  $r = z + Az + t$ , with  $z \in \mathcal{V}^\perp$  and  $t \in \mathcal{V}$ , and set

$$s = (z + Az) + (c - A^*c),$$

where  $c = t - Az \in \mathcal{V}$ . Observe that  $z + Az \in \mathcal{W}$  and  $c - A^*c \in \mathcal{W}^\perp$ , i.e.,  $P_{\mathcal{W}}s = z + Az$ . Writing  $s = (Az + c) + (z - A^*c)$ , we have  $Az + c \in \mathcal{V}$  and  $z - A^*c \in \mathcal{V}^\perp$ . Hence  $P_{\mathcal{V}}s = Az + c = t$ . In conclusion,  $P_{\mathcal{V}}s + P_{\mathcal{W}}s = z + Az + t = r$  and the proof is concluded.  $\square$

As to the image of  $P_{\mathcal{V}} + P_{\mathcal{W}}$  for a general Fredholm pair  $(\mathcal{V}, \mathcal{W})$ , we have the following lemma.

**Lemma 3.6** *Given a Fredholm pair  $(\mathcal{V}, \mathcal{W})$ , the image of  $P_{\mathcal{V}} + P_{\mathcal{W}}$  has finite codimension in  $\mathcal{H}$ .*

*Proof.* By Proposition 3.2,  $P_{\mathcal{V}^\perp}^{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{V}^\perp$  and  $P_{\mathcal{W}^\perp}^{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{W}^\perp$  are Fredholm, and thus their adjoints  $P_{\mathcal{W}^\perp}^{\mathcal{V}^\perp} : \mathcal{V}^\perp \rightarrow \mathcal{W}$  and  $P_{\mathcal{V}^\perp}^{\mathcal{W}^\perp} : \mathcal{W}^\perp \rightarrow \mathcal{V}$  are Fredholm. It follows that  $\mathcal{W}' = P_{\mathcal{W}}(\mathcal{V}^\perp)$  has finite codimension in  $\mathcal{W}$ , and that  $\mathcal{V}' = P_{\mathcal{V}}(\mathcal{W}^\perp)$  has finite codimension in  $\mathcal{V}$ . But  $(P_{\mathcal{V}} + P_{\mathcal{W}})(\mathcal{W}^\perp + \mathcal{V}^\perp) = P_{\mathcal{V}}(\mathcal{W}^\perp) + P_{\mathcal{W}}(\mathcal{V}^\perp) = \mathcal{V}' + \mathcal{W}'$ , hence the image of  $P_{\mathcal{V}} + P_{\mathcal{W}}$  has finite codimension in  $\mathcal{V} + \mathcal{W}$ . Since  $\mathcal{V} + \mathcal{W}$  has finite codimension in  $\mathcal{H}$ , it follows that  $P_{\mathcal{V}} + P_{\mathcal{W}}$  has image of finite codimension in  $\mathcal{H}$ .  $\square$

We can now extend the result of Lemma 3.5 to pairs of closed subspaces  $\mathcal{V}$  and  $\mathcal{W}$  whose intersection is not zero.

**Proposition 3.7** *Let  $\mathcal{V}, \mathcal{W} \subseteq \mathcal{H}$  be closed subspaces with  $\dim(\mathcal{V} \cap \mathcal{W}) < +\infty$ . Then,  $P_{\mathcal{V}} + P_{\mathcal{W}}$  is Fredholm if and only if  $(\mathcal{V}, \mathcal{W})$  is a Fredholm pair.*

*Proof.* If  $P_{\mathcal{V}} + P_{\mathcal{W}}$  is Fredholm, then  $\mathcal{V} + \mathcal{W}$  is a closed and finite codimensional subspace of  $\mathcal{H}$  because it contains the image of  $P_{\mathcal{V}} + P_{\mathcal{W}}$ . Conversely, if  $(\mathcal{V}, \mathcal{W})$  is a Fredholm pair, by part (1) of Lemma 2.1 one has  $\text{Ker}(P_{\mathcal{V}} + P_{\mathcal{W}}) = \mathcal{V}^\perp \cap \mathcal{W}^\perp = (\mathcal{V} + \mathcal{W})^\perp$ , hence  $\dim(\text{Ker}(P_{\mathcal{V}} + P_{\mathcal{W}})) < +\infty$ . By Lemma 3.6, the image of  $P_{\mathcal{V}} + P_{\mathcal{W}}$  has finite codimension in  $\mathcal{H}$ , which concludes the proof.  $\square$

Set:

$$\mathcal{E}(\mathcal{H}) = \left\{ (\mathcal{V}, \mathcal{W}) : (\mathcal{V}, \mathcal{W}^\perp) \in \mathcal{FP}(\mathcal{H}) \right\}.$$

It follows immediately from Proposition 3.2 that  $(\mathcal{V}, \mathcal{W}) \in \mathcal{E}(\mathcal{H})$  if and only if  $P_{\mathcal{V}}^{\mathcal{W}}$  is Fredholm.

**Corollary 3.8**  *$\mathcal{E}(\mathcal{H})$  is an equivalence relation in the set of all closed subspaces of  $\mathcal{H}$ . If  $(\mathcal{V}, \mathcal{W}), (\mathcal{W}, \mathcal{Z}) \in \mathcal{E}(\mathcal{H})$ , then  $\text{ind}(\mathcal{V}, \mathcal{Z}^\perp) = \text{ind}(\mathcal{V}, \mathcal{W}^\perp) + \text{ind}(\mathcal{W}, \mathcal{Z}^\perp)$ .*

*Proof.* The reflexivity and the symmetry of  $\mathcal{E}(\mathcal{H})$  follow easily from Corollary 3.3. The transitivity and equality on the index will follow by proving that  $P_{\mathcal{V}^\perp}^{\mathcal{Z}^\perp}$  is a compact (in fact, a finite rank) perturbation of the composition  $P_{\mathcal{V}^\perp}^{\mathcal{W}^\perp} \circ P_{\mathcal{W}^\perp}^{\mathcal{Z}^\perp}$ , using the fact that the Fredholm index of operators is stable by compact perturbations, and additive by composition. Consider the difference  $P_{\mathcal{V}^\perp}^{\mathcal{Z}^\perp} - P_{\mathcal{V}^\perp}^{\mathcal{W}^\perp} P_{\mathcal{W}^\perp}^{\mathcal{Z}^\perp} = P_{\mathcal{V}^\perp}^{\mathcal{W}^\perp} P_{\mathcal{W}^\perp}^{\mathcal{Z}^\perp}$ ; we have

$$\text{Ker}(P_{\mathcal{V}^\perp}^{\mathcal{Z}^\perp} P_{\mathcal{W}^\perp}^{\mathcal{Z}^\perp}) = P_{\mathcal{W}^\perp}^{-1}(\mathcal{V}) = \overline{\mathcal{V} + \mathcal{W}^\perp} = \mathcal{V} + \mathcal{W}^\perp.$$

Hence

$$\text{Ker}(P_{\mathcal{V}^\perp}P_{\mathcal{W}}|_{\mathcal{Z}^\perp}) = (\mathcal{V} + \mathcal{W}^\perp) \cap \mathcal{Z}^\perp.$$

Such a space has finite codimension in  $\mathcal{Z}^\perp$ , because

$$((\mathcal{V} + \mathcal{W}^\perp) \cap \mathcal{Z}^\perp)^\perp \cap \mathcal{Z}^\perp = \overline{(\mathcal{V}^\perp \cap \mathcal{W}) + \mathcal{Z}} \cap \mathcal{Z}^\perp = ((\mathcal{V}^\perp \cap \mathcal{W}) + \mathcal{Z}) \cap \mathcal{Z}^\perp.$$

The last equality follows from the fact that  $\mathcal{V}^\perp \cap \mathcal{W}$  is finite dimensional, so that  $(\mathcal{V}^\perp \cap \mathcal{W}) + \mathcal{Z}$  is closed; moreover,  $((\mathcal{V}^\perp \cap \mathcal{W}) + \mathcal{Z}) \cap \mathcal{Z}^\perp$  has finite dimension (recall part (6) of Lemma 2.1). This shows that the restriction of  $P_{\mathcal{V}^\perp} - P_{\mathcal{V}^\perp}P_{\mathcal{W}^\perp}$  to  $\mathcal{Z}^\perp$  has finite rank, which concludes the proof.  $\square$

### 3.2 Commensurable subspaces

Let us now recall the notion of commensurable spaces and relative dimension, introduced in [1] (see also [2]).

**Definition 3.9** Two closed subspaces  $\mathcal{V}$  and  $\mathcal{W}$  of  $\mathcal{H}$  are called *commensurable* if  $P_{\mathcal{V}} - P_{\mathcal{W}}$  is a compact operator. The *relative dimension* of  $\mathcal{V}$  with respect to  $\mathcal{W}$  is defined as

$$\dim(\mathcal{V}, \mathcal{W}) = \dim \mathcal{V} \cap \mathcal{W}^\perp - \dim \mathcal{W} \cap \mathcal{V}^\perp.$$

An easy computation shows that  $P_{\mathcal{V}} - P_{\mathcal{W}}$  is compact if and only if so are both  $P_{\mathcal{V}^\perp}P_{\mathcal{W}}$  and  $P_{\mathcal{W}^\perp}P_{\mathcal{V}}$ . Indeed:

$$P_{\mathcal{V}} - P_{\mathcal{W}} = P_{\mathcal{V}}(P_{\mathcal{W}} + P_{\mathcal{W}^\perp}) - (P_{\mathcal{V}} + P_{\mathcal{V}^\perp})P_{\mathcal{W}} = P_{\mathcal{V}}P_{\mathcal{W}^\perp} - P_{\mathcal{V}^\perp}P_{\mathcal{W}},$$

and

$$P_{\mathcal{V}}P_{\mathcal{W}^\perp} = (P_{\mathcal{V}} - P_{\mathcal{W}})P_{\mathcal{W}^\perp}, \quad P_{\mathcal{V}^\perp}P_{\mathcal{W}} = P_{\mathcal{V}^\perp}(P_{\mathcal{W}} - P_{\mathcal{V}}).$$

As a consequence, if  $\mathcal{V}$  and  $\mathcal{W}$  are commensurable, then  $I - P_{\mathcal{V}^\perp}P_{\mathcal{W}}$  and  $I - P_{\mathcal{W}^\perp}P_{\mathcal{V}}$  are Fredholm operators of index zero being compact perturbations of Fredholm operators of index zero ( $I$  denotes the identity on  $\mathcal{H}$ ). Therefore,

$$\mathcal{W} \cap \mathcal{V}^\perp = \text{Ker}(I - P_{\mathcal{V}^\perp}P_{\mathcal{W}}) \quad \text{and} \quad \mathcal{V} \cap \mathcal{W}^\perp = \text{Ker}(I - P_{\mathcal{W}^\perp}P_{\mathcal{V}})$$

are finite dimensional and then the above definition of relative dimension is well posed.

It follows directly from the definition that commensurability is an equivalence relation in the set of closed subspaces of  $\mathcal{H}$ ; we will set

$$\mathcal{C}(\mathcal{H}) = \{(\mathcal{V}, \mathcal{W}) : \mathcal{V} \text{ is commensurable with } \mathcal{W}\}.$$

Let us see the following property (see [2]).

**Lemma 3.10** *If  $\mathcal{V}$ ,  $\mathcal{W}$  and  $\mathcal{Z}$  are closed commensurable subspaces of  $\mathcal{H}$ , then*

$$\dim(\mathcal{V}, \mathcal{Z}) = \dim(\mathcal{V}, \mathcal{W}) + \dim(\mathcal{W}, \mathcal{Z}).$$

**Remark 3.11** Two subspaces  $\mathcal{V}$  and  $\mathcal{W}$  of  $\mathcal{H}$  of finite codimension are commensurable since  $P_{\mathcal{W}^\perp}P_{\mathcal{V}}$  and  $P_{\mathcal{V}^\perp}P_{\mathcal{W}}$  are compact having finite dimensional image. If  $\text{codim } \mathcal{V} = n$  and  $\text{codim } \mathcal{W} = m$ , by the above lemma it follows

$$\dim(\mathcal{V}, \mathcal{W}) = \dim(\mathcal{V}, \mathcal{H}) + \dim(\mathcal{H}, \mathcal{W}) = m - n.$$

In particular, if  $L : \mathcal{H} \rightarrow \mathcal{H}$  is a Fredholm operator of index zero, then  $(\text{Ker } L)^\perp$  e  $\text{Im } L$  are commensurable and their relative dimension is zero.

This property clearly fails if  $\mathcal{V}$  or  $\mathcal{W}$  has infinite codimension. Consider also the particular case when  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ , direct sum of infinite dimensional subspaces, and

$$L = \begin{pmatrix} 0 & L_{12} \\ L_{21} & 0 \end{pmatrix}$$

where  $L_{12}$  e  $L_{21}$  are isomorphisms. Then  $\mathcal{H}_1$  e  $\mathcal{H}_2$  are isomorphic, but not commensurable.

**Proposition 3.12**  $\mathcal{C}(\mathcal{H}) \subsetneq \mathcal{E}(\mathcal{H})$ . If  $(\mathcal{V}, \mathcal{W}) \in \mathcal{C}(\mathcal{H})$ , then

$$\dim(\mathcal{V}, \mathcal{W}) = \text{ind}(\mathcal{V}, \mathcal{W}^\perp). \quad (3.4)$$

*Proof.* If  $(\mathcal{V}, \mathcal{W}) \in \mathcal{C}(\mathcal{H})$ , then the difference  $P_{\mathcal{V}} - P_{\mathcal{W}}$  is compact, and so the kernel of the Fredholm operator  $I + P_{\mathcal{V}} - P_{\mathcal{W}}$  is finite dimensional:

$$\text{Ker}(I + P_{\mathcal{V}} - P_{\mathcal{W}}) = \text{Ker}(P_{\mathcal{V}} + P_{\mathcal{W}^\perp}) = \mathcal{V}^\perp \cap \mathcal{W}.$$

On the other hand,

$$\text{codim}(\mathcal{V}^\perp + \mathcal{W}) \leq \text{codim}(\text{Im}(P_{\mathcal{V}^\perp} + P_{\mathcal{W}})) = \text{codim}(\text{Im}(I + P_{\mathcal{W}} - P_{\mathcal{V}})) < +\infty.$$

This proves that  $(\mathcal{V}^\perp, \mathcal{W}) \in \mathcal{FP}(\mathcal{H})$ , i.e.,  $\mathcal{C}(\mathcal{H}) \subseteq \mathcal{E}(\mathcal{H})$ . The proof of formula (3.4) is straightforward.

To see that  $\mathcal{C}(\mathcal{H})$  actually does not coincide with  $\mathcal{E}(\mathcal{H})$  consider the following example. Let  $H$  be a real infinite dimensional separable Hilbert space, and set

$$\mathcal{H} = H \times H, \quad \mathcal{V} = H \times \{0\} \quad \text{and} \quad \mathcal{W} = \{(x, x) : x \in H\}.$$

Obviously,  $\mathcal{V} \cap \mathcal{W} = \{0\}$  and  $\mathcal{V} + \mathcal{W} = \mathcal{H}$ , so that  $(\mathcal{V}, \mathcal{W}) \in \mathcal{FP}(\mathcal{H})$  and  $(\mathcal{V}, \mathcal{W}^\perp) \in \mathcal{E}(\mathcal{H})$ . An immediate calculation shows that  $P_{\mathcal{V}}P_{\mathcal{W}} : \mathcal{H} \rightarrow \mathcal{H}$  is given by  $P_{\mathcal{V}}P_{\mathcal{W}}(a, b) = \frac{1}{2}(a + b)$ , which is clearly not a compact operator on  $\mathcal{H}$ , so  $(\mathcal{V}, \mathcal{W}^\perp) \notin \mathcal{C}(\mathcal{H})$ .  $\square$

The following results will be useful in the sequel.

**Proposition 3.13** [2, Prop. 2.3.2]. Given two self-adjoint Fredholm operators  $L$  and  $L'$  such that  $L - L'$  is compact, the negative (resp. the positive) eigenspaces are commensurable.

**Proposition 3.14** [2, Prop. 2.3.6]. Let  $B$  be a symmetric Fredholm form on  $\mathcal{H}$  and  $T$  the self-adjoint Fredholm operator associated with  $B$ . Let  $\mathcal{V}$  be a closed subspace of  $\mathcal{H}$ . Suppose that  $B$  is negative definite on  $\mathcal{V}$  and positive semidefinite on  $\mathcal{V}^{\perp B}$ . Then  $(\mathcal{V}, V^+(T) \oplus \text{Ker } T)$  is a Fredholm pair of index zero.

### 3.3 Relative dimension of negative eigenspaces

Let us recall that  $B$  denotes a symmetric Fredholm form on the Hilbert space  $\mathcal{H}$  and  $T$  is the self-adjoint Fredholm operator associated with  $B$ .

Let  $\mathcal{V}$  be a closed subspace of  $\mathcal{H}$  of finite codimension. Call  $\tilde{T} = P_{\mathcal{V}} \circ T|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}$  the linear operator associated with  $B|_{\mathcal{V} \times \mathcal{V}}$ , which is clearly a self-adjoint Fredholm operator (since  $B|_{\mathcal{V} \times \mathcal{V}}$  is symmetric).

Recalling the spectral decomposition (2.1) of  $\mathcal{H}$ , induced by  $B$ , in this subsection we prove that  $V^-(T)$  and  $V^-(\tilde{T})$  are commensurable and we give some results concerning the relative dimension  $\dim(V^-(T), V^-(\tilde{T}))$  in different particular cases. The most general case, when  $\mathcal{V}$  is any finite codimensional subspace of  $\mathcal{H}$ , will be tackled in Proposition 3.18 below.

**Proposition 3.15** Given  $T$  and  $\tilde{T}$  as above,  $V^-(T)$  and  $V^-(\tilde{T})$  are commensurable.

*Proof.* Define  $\hat{T} : \mathcal{H} \rightarrow \mathcal{H}$  as  $\hat{T} := i \circ \tilde{T} \circ P_{\mathcal{V}}$  where  $i : \mathcal{V} \hookrightarrow \mathcal{H}$  is the inclusion. It is immediate to see that  $\hat{T}$  is a Fredholm operator of index zero. In fact, the index of  $P_{\mathcal{V}}$  coincides with the codimension of  $\mathcal{V}$  in  $\mathcal{H}$ , while  $\text{ind } i = -\text{codim } \mathcal{V}$ . It is known that the composition of Fredholm operators is a Fredholm operator whose index is the sum of the indices of the components.

In the decomposition  $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^\perp$  we can represent  $\hat{T}$  in the block-matrix form as

$$\hat{T} = \begin{pmatrix} \tilde{T} & 0 \\ 0 & 0 \end{pmatrix}.$$

As  $\tilde{T}$  is self-adjoint, so is  $\hat{T}$ .

Since  $\mathcal{V}$  has finite codimension in  $\mathcal{H}$ , it follows that  $T - \widehat{T}$  is compact. Indeed, consider the block-matrix representation of  $T$  in the splitting  $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^\perp$ :

$$T = \begin{pmatrix} \widetilde{T} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where  $T_{12} = P_{\mathcal{V}} \circ T|_{\mathcal{V}^\perp}$ ,  $T_{21} = P_{\mathcal{V}^\perp} \circ T|_{\mathcal{V}}$  and  $T_{22} = P_{\mathcal{V}^\perp} \circ T|_{\mathcal{V}^\perp}$ . These three operators have finite dimensional image. Therefore,

$$T - \widehat{T} = \begin{pmatrix} 0 & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

turns out to have finite dimensional image, and then it is compact. We obtain, by Proposition 3.13, that  $V^-(T)$  and  $V^-(\widehat{T})$  are commensurable.

Consider now the spectral decompositions of  $\mathcal{H}$  induced by  $\widehat{T}$  and of  $\mathcal{V}$  induced by  $\widetilde{T}$ :

$$\mathcal{H} = V^-(\widehat{T}) \oplus V^+(\widehat{T}) \oplus \text{Ker } \widehat{T}, \quad \mathcal{V} = V^-(\widetilde{T}) \oplus V^+(\widetilde{T}) \oplus \text{Ker } \widetilde{T}.$$

Since  $\text{Ker } \widehat{T} = \text{Ker } \widetilde{T} \oplus \mathcal{V}^\perp$ , we have

$$\mathcal{H} = V^-(\widetilde{T}) \oplus V^+(\widetilde{T}) \oplus \text{Ker } \widehat{T}.$$

The Fredholm form associated with  $\widehat{T}$  is negative definite on  $V^-(\widetilde{T})$  and positive on  $V^+(\widetilde{T})$ , as the definition of  $\widehat{T}$  immediately shows. In addition both the spaces are invariant with respect to  $\widehat{T}$ . Therefore, by the uniqueness of the spectral decomposition, the above formula is actually the spectral decomposition of  $\mathcal{H}$  by  $\widehat{T}$ , that is,

$$V^-(\widehat{T}) = V^-(\widetilde{T}) \quad \text{and} \quad V^+(\widehat{T}) = V^+(\widetilde{T}).$$

We have seen that  $V^-(T)$  and  $V^-(\widehat{T})$  are commensurable. Of course, so are  $V^-(T)$  and  $V^-(\widetilde{T})$  and the proof is complete.  $\square$

Lemmas 3.16 and 3.17 below give an answer to the question concerning the relative dimension of  $(V^-(T), V^-(\widetilde{T}))$  in two particular cases. These results are interesting in themselves and propaedeutic to Proposition 3.18.

**Lemma 3.16** *Suppose  $\mathcal{H} = \mathcal{V} + \mathcal{V}^{\perp B}$ . Let  $T_2 := P_{\mathcal{V}^{\perp B}} \circ T|_{\mathcal{V}^{\perp B}} : \mathcal{V}^{\perp B} \rightarrow \mathcal{V}^{\perp B}$  be the linear operator associated with  $B|_{\mathcal{V}^{\perp B} \times \mathcal{V}^{\perp B}}$ . One has*

$$\dim(V^-(T), V^-(\widetilde{T})) = \dim V^-(T_2).$$

*Proof.* It is immediate to see that  $\mathcal{V} \cap \mathcal{V}^{\perp B}$  is an isotropic space for  $B$ . Hence

$$V^\pm(\widetilde{T}) \cap V^\pm(T_2) = V^\pm(\widetilde{T}) \cap \text{Ker } T_2 = \text{Ker } \widetilde{T} \cap V^\pm(T_2) = \{0\}.$$

Thus, given

$$\widehat{V}^- := V^-(\widetilde{T}) \oplus V^-(T_2) \quad \text{and} \quad \widehat{V}^+ := V^+(\widetilde{T}) \oplus V^+(T_2), \tag{3.5}$$

and recalling that  $\text{Ker } \widetilde{T} \subseteq \text{Ker } T_2 = \text{Ker } T$  (Lemma 2.9), we have

$$H = \widehat{V}^- \oplus \widehat{V}^+ \oplus \text{Ker } T.$$

Let us show that:

- a)  $B$  is negative definite on  $\widehat{V}^-$  and positive on  $\widehat{V}^+$ ;
- b)  $\widehat{V}^-$  and  $\widehat{V}^+$  are  $B$ -orthogonal.

a) Let  $x \in \widehat{V}^-$  be given and write  $x = x_1 + x_2$  in the splitting  $\widehat{V}^- = V^-(\widetilde{T}) \oplus V^-(T_2)$ . We have

$$\langle Tx, x \rangle = \langle Tx_1 + Tx_2, x_1 + x_2 \rangle = \langle Tx_1, x_1 \rangle + \langle Tx_2, x_2 \rangle = \langle \widetilde{T}x_1, x_1 \rangle + \langle T_2x_2, x_2 \rangle$$

(notice that  $\langle Tx_1, x_2 \rangle = 0 = \langle Tx_2, x_1 \rangle$  since  $x_1 \in \mathcal{V}$  and  $x_2 \in \mathcal{V}^{\perp B}$ ). The last two summands are, by definition of  $\widehat{V}^-$ , less or equal to zero, and not both zero if  $x \neq 0$ . Then  $B$  is negative definite on  $\widehat{V}^-$ . The proof of the analogous result for  $\widehat{V}^+$  is identical and omitted.

b) Let  $x \in \widehat{V}^-$  and  $y \in \widehat{V}^+$  be given. By the decompositions (3.5), write  $x = x_1 + x_2$  and  $y = y_1 + y_2$ . Hence

$$\langle Tx, y \rangle = \langle Tx_1 + Tx_2, y_1 + y_2 \rangle = \langle Tx_1, y_1 \rangle + \langle Tx_2, y_2 \rangle = \langle \widetilde{T}x_1, y_1 \rangle + \langle T_2x_2, y_2 \rangle = 0.$$

The last equality is due to the fact that  $V^-(\widetilde{T})$  and  $V^+(\widetilde{T})$  are  $(B|_{\mathcal{V} \times \mathcal{V}})$ -orthogonal, while  $V^-(T_2)$  and  $V^+(T_2)$  are  $(B|_{\mathcal{V}^{\perp B} \times \mathcal{V}^{\perp B}})$ -orthogonal.

We are now in the position to apply Proposition 3.14 to the pair  $(\widehat{V}^-, V^+(T) \oplus \text{Ker } T)$  obtaining that it is a Fredholm pair of index zero.

Observe that  $\widehat{V}^-$  and  $V^-(T)$  are commensurable. Indeed  $V^-(\widetilde{T})$  and  $V^-(T)$  are commensurable by Proposition 3.15; in addition  $\widehat{V}^-$  and  $V^-(\widetilde{T})$  are of course commensurable since  $V^-(T_2)$  has finite dimension. Now, recalling that  $V^-(T)$  is the orthogonal complement of  $V^+(T) \oplus \text{Ker } T$ , by formula (3.4) it follows

$$\dim(\widehat{V}^-, V^-(T)) = 0.$$

In addition, it is immediate to see that

$$\dim(\widehat{V}^-, V^-(\widetilde{T})) = \dim V^-(T_2).$$

By Lemma 3.10 we have

$$\dim(V^-(T), V^-(\widetilde{T})) = \dim V^-(T_2)$$

and the proof is complete.  $\square$

**Lemma 3.17** *Let  $\mathcal{Z}$  be a finite dimensional subspace of  $\mathcal{H}$ , isotropic with respect to  $B$ , and call  $L : \mathcal{Z}^{\perp B} \rightarrow \mathcal{Z}^{\perp B}$  the operator associated with  $B|_{\mathcal{Z}^{\perp B} \times \mathcal{Z}^{\perp B}}$ . Then  $V^-(T)$  is commensurable with  $V^-(L)$  and*

$$\dim(V^-(T), V^-(L)) = \dim \mathcal{Z} - \dim(\mathcal{Z} \cap \text{Ker } T).$$

*Proof.* Since  $\mathcal{Z}$  is isotropic, we have  $\mathcal{Z} \subseteq \mathcal{Z}^{\perp B}$ . Observe that  $\mathcal{Z}^{\perp B}$  is the orthogonal complement of  $T(\mathcal{Z})$  in  $\mathcal{H}$ . Therefore the codimension of  $\mathcal{Z}^{\perp B}$  in  $\mathcal{H}$  is finite and

$$\text{codim } \mathcal{Z}^{\perp B} = \dim \mathcal{Z} - \dim(\text{Ker } T \cap \mathcal{Z}). \quad (3.6)$$

The kernel of  $B|_{\mathcal{Z}^{\perp B} \times \mathcal{Z}^{\perp B}}$  is

$$\begin{aligned} \text{Ker } B|_{\mathcal{Z}^{\perp B} \times \mathcal{Z}^{\perp B}} &= \{x \in \mathcal{Z}^{\perp B} : \langle Tx, y \rangle = 0, \forall y \in \mathcal{Z}^{\perp B}\} \\ &= \{x \in \mathcal{Z}^{\perp B} : \langle x, Ty \rangle = 0, \forall y \in \mathcal{Z}^{\perp B}\}. \end{aligned}$$

That is,  $\text{Ker } B|_{\mathcal{Z}^{\perp B} \times \mathcal{Z}^{\perp B}}$  is a subspace of the orthogonal complement of  $T(\mathcal{Z}^{\perp B})$  in  $\mathcal{H}$ . Hence, taking into account (3.6), one has

$$\dim \text{Ker } B|_{\mathcal{Z}^{\perp B} \times \mathcal{Z}^{\perp B}} \leq \dim \mathcal{Z} - \dim(\text{Ker } T \cap \mathcal{Z}) + \dim \text{Ker } T.$$

Since  $\mathcal{Z}$  is isotropic, we have that  $\mathcal{Z} \subseteq \text{Ker } B|_{\mathcal{Z}^{\perp B} \times \mathcal{Z}^{\perp B}}$ . Of course  $\text{Ker } T \subseteq \mathcal{Z}^{\perp B}$ . Since

$$\dim(\mathcal{Z} + \text{Ker } T) = \dim \mathcal{Z} - \dim(\text{Ker } T \cap \mathcal{Z}) + \dim \text{Ker } T,$$

it follows

$$\text{Ker } B|_{\mathcal{Z}^{\perp B} \times \mathcal{Z}^{\perp B}} = \mathcal{Z} + \text{Ker } T.$$

The spectral decomposition of  $\mathcal{Z}^{\perp B}$  with respect to  $B|_{\mathcal{Z}^{\perp B} \times \mathcal{Z}^{\perp B}}$  is

$$\mathcal{Z}^{\perp B} = V^-(L) \oplus V^+(L) \oplus (\mathcal{Z} + \text{Ker } T),$$

and  $\mathcal{V} := V^-(L) \oplus V^+(L)$  is the orthogonal complement of  $\mathcal{Z} + \text{Ker } T$  in  $\mathcal{Z}^{\perp B}$ . Observe that  $B|_{\mathcal{V} \times \mathcal{V}}$  is nondegenerate and then, by (2) in Lemma 2.9,

$$\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^{\perp B}.$$

Since  $\mathcal{Z} + \text{Ker } T$  is  $B$ -orthogonal to  $\mathcal{V}$  it turns out to be contained in  $\mathcal{V}^{\perp B}$ . An immediate computation says that

$$\dim \mathcal{V}^{\perp B} = 2(\dim \mathcal{Z} - \dim(\mathcal{Z} \cap \text{Ker } T)) + \dim \text{Ker } T.$$

Call  $\tilde{T}$  the operator associated with  $B|_{\mathcal{V} \times \mathcal{V}}$  and  $T_2$  that associated with  $B|_{\mathcal{V}^{\perp B} \times \mathcal{V}^{\perp B}}$ . By Proposition 3.16, we have that  $V^-(T)$  and  $V^-(\tilde{T})$  are commensurable and

$$\dim(V^-(T), V^-(\tilde{T})) = \dim V^-(T_2).$$

On the other hand  $V^-(L) = V^-(\tilde{T})$ . Therefore, the proof is complete if we show that  $\dim V^-(T_2) = \dim \mathcal{Z} - \dim(\mathcal{Z} \cap \text{Ker } T)$ .

It is crucial now to notice that  $\mathcal{Z} \subseteq \mathcal{V}^{\perp B}$ ; this immediately follows from the inclusion  $\mathcal{Z} + \text{Ker } T \subseteq \mathcal{V}^{\perp B}$ . By Proposition 2.14 we have, since  $\mathcal{Z}$  is isotropic,

$$\dim V^+(T_2) \geq \dim \mathcal{Z} - \dim(\mathcal{Z} \cap \text{Ker } T), \quad \dim V^-(T_2) \geq \dim \mathcal{Z} - \dim(\mathcal{Z} \cap \text{Ker } T).$$

Then

$$\dim V^+(T_2) = \dim \mathcal{Z} - \dim(\mathcal{Z} \cap \text{Ker } T) = \dim V^-(T_2)$$

and the proof is complete.  $\square$

We are now in the position to present the main result of this section, concerning the relative dimension of the negative eigenspaces of a self-adjoint Fredholm operator and its restriction to any closed finite codimensional subspace of  $\mathcal{H}$ .

**Proposition 3.18** *Let  $B$  be a Fredholm symmetric bilinear form on  $\mathcal{H}$  and let  $\mathcal{V}$  be a closed finite codimensional subspace of  $\mathcal{H}$ . Denote by  $T : \mathcal{H} \rightarrow \mathcal{H}$  and  $\tilde{T} = P_{\mathcal{V}} \circ T|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}$  the self-adjoint Fredholm operators associated with  $B$  and  $B|_{\mathcal{V} \times \mathcal{V}}$  respectively. Then:*

$$\dim(V^-(T), V^-(\tilde{T})) = n_-(B|_{\mathcal{V}^{\perp B} \times \mathcal{V}^{\perp B}}) + \dim(\mathcal{V} \cap \mathcal{V}^{\perp B}) - \dim(\mathcal{V} \cap \text{Ker } T).$$

*Proof.* Clearly  $\mathcal{Z} := \mathcal{V} \cap \mathcal{V}^{\perp B}$  is an isotropic space. In addition it is finite dimensional since so is  $\mathcal{V}^{\perp B}$  (Remark 2.7, statement ii). Let  $R : \mathcal{Z}^{\perp B} \rightarrow \mathcal{Z}^{\perp B}$  be the linear operator associated with  $B|_{\mathcal{Z}^{\perp B} \times \mathcal{Z}^{\perp B}}$ . Then, by Lemma 3.17,

$$\dim(V^-(T), V^-(R)) = \dim \mathcal{Z} - \dim(\mathcal{Z} \cap \text{Ker } T).$$

Now, as  $\mathcal{Z}^{\perp B} = \mathcal{V} + \mathcal{V}^{\perp B}$  by statement (5) in Lemma 2.9, we can apply to  $\mathcal{Z}^{\perp B}$  Lemma 3.16 and we obtain

$$\dim(V^-(R), V^-(\tilde{T})) = n_-(B|_{\mathcal{V}^{\perp B} \times \mathcal{V}^{\perp B}}).$$

By Lemma 3.10 the claim follows.  $\square$

## 4 On the spectral flow

### 4.1 Generalities on the notion of spectral flow

Let us denote by  $\mathcal{F}_{\text{sa}}(\mathcal{H})$  the set of self-adjoint Fredholm operators in  $\mathcal{H}$ . Given a continuous path  $T : [a, b] \rightarrow \mathcal{F}_{\text{sa}}(\mathcal{H})$ , we will denote by  $\text{sf}(T, [a, b])$  the *spectral flow* of  $T$  on the interval  $[a, b]$ , which is an integer number that gives, roughly speaking, the net number of eigenvalues of  $T$  that pass through the value 0.

There exist several equivalent definitions of the spectral flow in the literature, although the reader should note that there exist different conventions on the contribution of the endpoints in the case when  $T_a$  and/or  $T_b$  are not invertible.

A possible definition of spectral flow using functional calculus is given in [23] as follows. Let  $t_0 = a < t_1 < \dots < t_N = b$  be a partition of  $[a, b]$ , and  $a_1, \dots, a_N$  be positive numbers with the property that, denoting by  $\chi_I$  the characteristic function of the interval  $I$ , for  $i = 1, \dots, N$  the following hold:

- (a) the map  $[t_{i-1}, t_i] \ni t \mapsto \chi_{[-a_i, a_i]}(T_t)$  is continuous,
- (b)  $\chi_{[-a_i, a_i]}(T_t)$  is a projection onto a finite dimensional subspace of  $\mathcal{H}$ .

Then,  $\text{sf}(T, [a, b])$  is defined by the sum

$$\text{sf}(T, [a, b]) = \sum_{i=1}^N = [\text{rk}(\chi_{[0, a_i]}(T_{t_i}) - \text{rk}(\chi_{[0, a_i]}(T_{t_{i-1}}))],$$

where  $\text{rk}(P)$  denotes the rank of a projection  $P$ . With this definition, in the particular case when  $T$  is a path of essentially positive operators, that is, the negative spectrum of each operator  $T_t$  has only isolated eigenvalues of finite multiplicity, then the spectral flow of  $T$  is given by

$$\text{sf}(T, [a, b]) = n_-(T_b) + \dim(\text{Ker } T_b) - n_-(T_a) - \dim(\text{Ker } T_a).$$

The spectral flow is additive by concatenation of paths, and invariant by fixed-endpoints homotopies, and it therefore defines a  $\mathbb{Z}$ -valued homomorphism on the fundamental groupoid of  $\mathcal{F}_{\text{sa}}(\mathcal{H})$ . In fact, one shows easily that the spectral flow is invariant by the larger class of homotopies that leave constant the dimension of the kernel at the endpoints. Moreover, the spectral flow is invariant by cogredience, i.e., given Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2$ , a continuous path  $T : [a, b] \rightarrow \mathcal{F}_{\text{sa}}(\mathcal{H}_2)$  and a continuous path of isomorphisms  $S : [a, b] \rightarrow \text{Iso}(\mathcal{H}_1, \mathcal{H}_2)$ , then the spectral flow of the path  $[a, b] \ni t \mapsto S_t^* T_t S_t \in \mathcal{F}_{\text{sa}}(\mathcal{H}_1)$  equals the spectral flow of  $T$ .

We are interested in computing the spectral flow of paths of self-adjoint Fredholm operators that are compact perturbations of a fixed symmetry of the Hilbert space  $\mathcal{H}$ . By a *symmetry* of  $\mathcal{H}$  we mean a bounded operator  $\mathfrak{J}$  on  $\mathcal{H}$  of the form  $\mathfrak{J} = P_{\mathcal{W}} - P_{\mathcal{W}^\perp} = 2P_{\mathcal{W}} - I$ , where  $\mathcal{W}$  is a given closed subspace of  $\mathcal{H}$ . Equivalently,  $\mathfrak{J}$  is a symmetry if it is self-adjoint and it satisfies  $\mathfrak{J}^2 = I$ , the identity map of  $\mathcal{H}$ .

A symmetry  $\mathfrak{J}$  can be represented, with respect to the decomposition  $\mathcal{H} = \mathcal{W} \oplus \mathcal{W}^\perp$ , by the matrix

$$\begin{pmatrix} I_{\mathcal{W}} & 0 \\ 0 & -I_{\mathcal{W}^\perp} \end{pmatrix}$$

where  $I_{\mathcal{W}}$  and  $I_{\mathcal{W}^\perp}$  are the identity maps of  $\mathcal{W}$  and  $\mathcal{W}^\perp$ , respectively.

A compact perturbation of  $\mathfrak{J}$  is essentially positive, essentially negative or strongly indefinite according to whether  $\mathcal{W}^\perp$  is finite dimensional,  $\mathcal{W}$  is finite dimensional, or both  $\mathcal{W}$  and  $\mathcal{W}^\perp$  are infinite dimensional, respectively. Of course the last case could happen only if  $\mathcal{H}$  is infinite dimensional.

Given a continuous curve  $T : [a, b] \rightarrow \mathcal{F}_{\text{sa}}(\mathcal{H})$  of the form  $T_t = \mathfrak{J} + K_t$ , where  $\mathfrak{J}$  is a symmetry of  $\mathcal{H}$  and  $K_t$  is a self-adjoint compact operator on  $\mathcal{H}$ , then the spectral flow of  $T$  can be computed in terms of the notion of relative dimension, recalled in the above section, as follows: by Proposition 3.13 the spaces  $V^-(T_a)$  and  $V^-(T_b)$  are commensurable, and

$$\text{sf}(T, [a, b]) = \dim(V^-(T_a), V^-(T_b)). \quad (4.1)$$

Here comes an immediate observation, that will be useful ahead.

**Proposition 4.1** *For a continuous path  $T : [a, b] \rightarrow \mathcal{F}_{\text{sa}}(\mathcal{H})$  of the form  $T_t = \mathfrak{J} + K_t$ , where  $\mathfrak{J}$  is a symmetry of  $\mathcal{H}$  and  $K_t$  is a self-adjoint compact operator on  $\mathcal{H}$ , the spectral flow  $\text{sf}(T, [a, b])$  depends only on the endpoints  $T_a$  and  $T_b$ .  $\square$*

## 4.2 Restriction to a fixed subspace

An important property, stated in the following lemma, says that if  $\mathcal{V}$  is a closed subspace of  $\mathcal{H}$  of finite codimension, then  $P_{\mathcal{V}} \circ T_t|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}$  is a path of self-adjoint compact perturbations of a fixed symmetry of  $\mathcal{V}$ .

**Lemma 4.2** *Let  $T : [a, b] \rightarrow \mathcal{F}_{\text{sa}}(\mathcal{H})$  be a continuous curve of the form  $T_t = \mathfrak{J} + K_t$ , where  $\mathfrak{J}$  is a symmetry of  $\mathcal{H}$  and  $K_t$  is a self-adjoint compact operator on  $\mathcal{H}$ , and consider a closed subspace  $\mathcal{V}$  of  $\mathcal{H}$  of finite codimension. Call  $\tilde{T} : [a, b] \rightarrow \mathcal{F}_{\text{sa}}(\mathcal{V})$  the continuous curve of self-adjoint operators defined as  $\tilde{T}_t = P_{\mathcal{V}} \circ T_t|_{\mathcal{V}}$ . Then, there exist a symmetry  $\mathfrak{J}_{\mathcal{V}}$  of  $\mathcal{V}$  and a continuous path of self-adjoint compact operators  $C_t$  on  $\mathcal{V}$  such that*

$$\tilde{T}_t = \mathfrak{J}_{\mathcal{V}} + C_t, \quad t \in [a, b].$$

**Proof.** The operator  $\tilde{\mathfrak{J}}_{\mathcal{V}} = P_{\mathcal{V}} \circ \mathfrak{J}|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}$  is self-adjoint, and its square  $(\tilde{\mathfrak{J}}_{\mathcal{V}})^2$  is easily computed as the sum of the identity of  $\mathcal{V}$  and a finite rank operator. Namely, the space  $\mathcal{W} = \mathfrak{J}^{-1}(\mathcal{V}) \cap \mathcal{V} = \mathfrak{J}(\mathcal{V}) \cap \mathcal{V}$  has finite codimension in  $\mathcal{V}$ , it is invariant by  $\mathfrak{J}$ , and  $(\mathfrak{J}|_{\mathcal{W}})^2 = I_{\mathcal{W}}$ . The symmetry  $\mathfrak{J}_{\mathcal{V}}$  is obtained applying next Lemma to the operator  $S = \tilde{\mathfrak{J}}_{\mathcal{V}}$  on the Hilbert space  $\mathcal{V}$ .  $\square$

**Lemma 4.3** *Let  $S$  be a self-adjoint operator on a Hilbert space  $\mathcal{G}$  such that  $S^2 - I$  has finite rank. Then,  $S$  is a finite-rank perturbation of a symmetry  $\mathfrak{L}$  of  $\mathcal{G}$ .*

**Proof.**  $S^2 - I$  is self-adjoint and it has closed image (finite dimensional), thus  $\mathcal{G}$  is given by the orthogonal sum of closed subspaces, that is,  $\mathcal{G} = \text{Ker}(S^2 - I) + \text{Im}(S^2 - I)$ . The symmetry  $\mathfrak{L}$  is given by

$$\mathfrak{L} = \begin{cases} S & \text{on Ker}(S^2 - I) \\ I & \text{on Im}(S^2 - I). \end{cases} \quad \square$$

We are now in the position to present the following result, which concerns the difference between the spectral flow of a path of symmetric Fredholm forms on  $\mathcal{H}$  and the spectral flow of its restriction to a finite codimensional closed subspace of  $\mathcal{H}$ .

In the theorem  $\text{B}_{\text{sym}}(\mathcal{H})$  will denote the set of symmetric Fredholm forms on  $\mathcal{H}$ , while  $\mathcal{F}_{\text{sa}}(\mathcal{H})$ , as said before, will stand for the set of self-adjoint Fredholm operators in  $\mathcal{H}$ .

**Theorem 4.4** *Consider a continuous path  $B : [a, b] \rightarrow \text{B}_{\text{sym}}(\mathcal{H})$  of symmetric Fredholm forms on  $\mathcal{H}$ . Let  $\mathcal{V}$  be a finite codimensional closed subspace of  $\mathcal{H}$  and denote by  $T : [a, b] \rightarrow \mathcal{F}_{\text{sa}}(\mathcal{H})$  and  $\tilde{T} : [a, b] \rightarrow \mathcal{F}_{\text{sa}}(\mathcal{V})$  the continuous paths of self-adjoint Fredholm operators associated with  $B$  and to the restriction  $B|_{\mathcal{V} \times \mathcal{V}}$ , respectively. Assume that  $T_t = \mathfrak{J} + K_t$  for all  $t \in [a, b]$ , where  $\mathfrak{J}$  is a symmetry of  $\mathcal{H}$  and  $K_t$  is compact for all  $t$ . Then,*

$$\begin{aligned} \text{sf}(T, [a, b]) - \text{sf}(\tilde{T}, [a, b]) &= \dim(V^-(T_a), V^-(\tilde{T}_a)) - \dim(V^-(T_b), V^-(\tilde{T}_b)) \\ &= n_-(B_a|_{\mathcal{V}^{\perp B_a} \times \mathcal{V}^{\perp B_a}}) + \dim(\mathcal{V} \cap \mathcal{V}^{\perp B_a}) - \dim(\mathcal{V} \cap \text{Ker } B_a) \\ &\quad - n_-(B_b|_{\mathcal{V}^{\perp B_b} \times \mathcal{V}^{\perp B_b}}) - \dim(\mathcal{V} \cap \mathcal{V}^{\perp B_b}) + \dim(\mathcal{V} \cap \text{Ker } B_b). \end{aligned} \quad (4.2)$$

**Proof.** By Lemma 4.2,  $\tilde{T}$  is a path of compact perturbations of a symmetry of  $\mathcal{V}$ . Therefore, using formula (4.1) we obtain readily

$$\text{sf}(T, [a, b]) - \text{sf}(\tilde{T}, [a, b]) = \dim(V^-(T_a), V^-(T_b)) - \dim(V^-(\tilde{T}_a), V^-(\tilde{T}_b)).$$

Recalling that the commensurability of subspaces is an equivalence relation and applying Lemma 3.10, it follows that

$$\begin{aligned} \dim(V^-(T_a), V^-(T_b)) - \dim(V^-(\tilde{T}_a), V^-(\tilde{T}_b)) &= \\ &= \dim(V^-(T_a), V^-(\tilde{T}_a)) - \dim(V^-(T_b), V^-(\tilde{T}_b)). \end{aligned}$$

The conclusion of the proof is an immediate consequence of Proposition 3.18.  $\square$

Note that  $\mathcal{V} \cap \mathcal{V}^{\perp B} = \text{Ker}(B|_{\mathcal{V} \times \mathcal{V}})$ .

### 4.3 Continuous and smooth families of closed subspaces

In Subsection 4.4 below we will extend formula 4.2 to the case when the subspace  $\mathcal{V}$  in Theorem 4.4 is not constant but depends on  $t$ . To this end we devote this subsection to a summary of the concept of smooth family (or smooth path) of closed subspaces of  $\mathcal{H}$ , recalling also some crucial properties, important for our construction. The goal is to determine the existence of a special class of *trivializations* for smooth, or continuous, curves of closed subspaces. Most of the material discussed in this subsection is known to specialists, nevertheless it will be useful to give a formal proof of the essential results, for the reader's convenience.

In the following definition, being  $L(\mathcal{H})$  the space of bounded linear operators of  $\mathcal{H}$  into itself,  $GL(\mathcal{H})$  is the open subset of  $L(\mathcal{H})$  of the automorphisms. The space of bounded linear operators between two Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  is denoted by  $L(\mathcal{H}_1, \mathcal{H}_2)$ .

**Definition 4.5** Let  $I \subseteq \mathbb{R}$  be an interval and  $\mathcal{D} = \{\mathcal{V}_t\}_{t \in I}$  be a family of closed subspaces of  $\mathcal{H}$ . We say that  $\mathcal{D}$  is a  $C^k$  family of closed subspaces of  $\mathcal{H}$ ,  $k = 0, \dots, \infty, \omega^1$  if for all  $t_0 \in I$  there exist  $\varepsilon > 0$ , a  $C^k$  map  $\Psi : I \cap ]t_0 - \varepsilon, t_0 + \varepsilon[ \rightarrow GL(\mathcal{H})$  and a closed subspace  $\mathcal{V}_* \subseteq \mathcal{H}$  such that  $\Psi_t(\mathcal{V}_t) = \mathcal{V}_*$  for all  $t \in I \cap ]t_0 - \varepsilon, t_0 + \varepsilon[$ .

The pair  $(\mathcal{V}_*, \Psi)$  as above will be called a  $C^k$ -local trivialization of the family  $\mathcal{D}$  around  $t_0$ . The following criterion of smoothness holds.

**Proposition 4.6** Let  $I \subseteq \mathbb{R}$  be an interval,  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces and  $F : I \mapsto L(\mathcal{H}_1, \mathcal{H}_2)$  be a  $C^k$  map,  $k = 0, 1, \dots, \infty, \omega$ , such that each  $F_t$  is surjective. Then, the family of  $\mathcal{V}_t = \text{Ker } F_t$  is a  $C^k$ -family of closed subspaces of  $\mathcal{H}_1$ .

*Proof.* See for instance [18, Lemma 2.9]. □

Let  $\mathcal{D} = \{\mathcal{V}_t\}_{t \in I}$  be a family of closed subspaces of  $\mathcal{H}$ . Proposition 4.9 below relates the smoothness of  $\mathcal{D}$  with the smoothness of the path  $t \mapsto P_{\mathcal{V}_t}$  of the orthogonal projections onto  $\mathcal{V}_t$ , for  $t \in I$ . Any  $P_{\mathcal{V}_t}$  is considered having  $\mathcal{H}$  as target space. We need first two preliminary lemmas.

**Lemma 4.7** Let  $P, Q$  be two projections such that  $\|P - Q\| < 1$ . Then, the restriction  $P_{\text{Im } P}^{\text{Im } Q} : \text{Im } Q \rightarrow \text{Im } P$  is an isomorphism.

*Proof.* Assume  $x \in \text{Im } Q \setminus \{0\}$  and  $Px = 0$ ; then  $\|Px - Qx\| = \|Qx\| = \|x\|$ , which implies  $\|P - Q\| \geq 1$ . Thus,  $P_{\text{Im } P}^{\text{Im } Q}$  is injective. We now need to show that  $\text{Im}(P_{\text{Im } P}^{\text{Im } Q})$  is equal to  $\text{Im } P$ ; to this aim, it suffices to show that  $\text{Im}(PQ) = \text{Im } P$ . This follows easily from the equality

$$PQ = P(Q + I - P),$$

observing that, since  $\|P - Q\| < 1$ , then  $I + Q - P$  is an isomorphism of  $\mathcal{H}$ . □

**Lemma 4.8** Let  $\mathcal{H}_0$  and  $\mathcal{H}_1$  be Hilbert spaces, and let  $L : \mathcal{H}_0 \rightarrow \mathcal{H}_1$  be a bounded linear operator. Set  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ ; then, the orthogonal projection  $P_{\text{Graph}(L)}$  onto the graph of  $L$  is given by

$$\begin{aligned} P_{\text{Graph}(L)}(x, y) = & \\ & (x + L^*(I + LL^*)^{-1}(y - Lx), L(x + L^*(I + LL^*)^{-1}(y - Lx))) = \\ & (x + L^*(I + LL^*)^{-1}(y - Lx), y - (I + LL^*)^{-1}(y - Lx)). \end{aligned} \quad (4.3)$$

*Proof.* It follows by a straightforward calculation, keeping in mind that the orthogonal complement of  $\text{Graph}(L)$  in  $\mathcal{H}$  is  $\{(-L^*b, b) : b \in \mathcal{H}_1\}$ . □

Formula (4.3) shows that the orthogonal projection onto the graph of  $L$  is written as a smooth function of  $L$ . We are now ready for the main result of the subsection.

**Proposition 4.9** Let  $J \subseteq \mathbb{R}$  be an interval, and let  $\mathcal{D} = \{\mathcal{V}_t\}_{t \in J}$  be a family of closed subspaces of  $\mathcal{H}$ . Then, for all  $k = 0, 1, \dots, \infty, \omega$ ,  $\mathcal{D}$  is a  $C^k$ -family of subspaces of  $\mathcal{H}$  if and only if the map  $t \mapsto P_{\mathcal{V}_t}$ , from  $J$  into  $L(\mathcal{H})$ , is of class  $C^k$ .

<sup>1</sup> The symbol  $C^\omega$  means analytic.

**Proof.** Assume that  $t \mapsto P_{\mathcal{V}_t}$  is of class  $C^k$ ; set  $Q_t = I - P_{\mathcal{V}_t}$ , so that  $\mathcal{V}_t = \text{Ker } Q_t$  for all  $t$ . Fix  $t_0 \in J$ , for  $t \in J$  near  $t_0$ , by continuity we can assume  $\|Q_t - Q_{t_0}\| < 1$ . We claim that, for  $t \in J$  near  $t_0$ , the map  $F_t = Q_{t_0}Q_t : \mathcal{H} \rightarrow \text{Im } Q_{t_0}$  is surjective; namely,  $\text{Im } F_t = \text{Im}(Q_{t_0}|_{\text{Im } Q_t})$ , and the claim follows from Lemma 4.7. Moreover,  $\text{Ker } F_t = \text{Ker } Q_t$  because, by Lemma 4.7,  $Q_{t_0}|_{\text{Im } Q_t}$  is injective. Since  $t \mapsto F_t$  is of class  $C^k$ ,  $\mathcal{D}$  is a  $C^k$ -family of closed subspaces of  $\mathcal{H}$  by Proposition 4.6.

For the converse, we will show that the projections  $P_{\mathcal{V}_t}$  can be written as smooth functions of a local trivialization. Assume  $\mathcal{D}$  of class  $C^k$ ; choose  $t_0 \in J$ , and let  $(\mathcal{V}_*, \Psi)$  be a local trivialization of  $\mathcal{D}$  around  $t_0$ ; set  $\phi_t = \Psi_t^{-1}$ . Up to replacing  $\Psi_t$  with  $\Psi_{t_0}^{-1}\Psi_t$ , we can assume  $\mathcal{V}_* = \mathcal{V}_{t_0}$  and  $\mathcal{V}_t = \phi_t(\mathcal{V}_{t_0})$  for all  $t$  near  $t_0$ . Write  $\mathcal{H} = \mathcal{V}_{t_0} \oplus \mathcal{V}_{t_0}^\perp$  and write  $\phi_t$  in blocks relatively to this decomposition of  $\mathcal{H}$  as:

$$\phi_t = \begin{pmatrix} \phi_t^{11} & \phi_t^{12} \\ \phi_t^{21} & \phi_t^{22} \end{pmatrix};$$

observe that the smoothness of  $\Psi_t$  is equivalent to the smoothness of the blocks  $\phi_t^{ij}$ . Since  $\phi_{t_0}|_{\mathcal{V}_{t_0}}$  is the identity on  $\mathcal{V}_{t_0}$ ,  $\phi_{t_0}^{11}$  is the identity, and by continuity,  $\phi_t^{11}$  is invertible for  $t$  near  $t_0$ . An immediate computation shows that, setting  $L_t : \mathcal{V}_{t_0} \rightarrow \mathcal{V}_{t_0}^\perp$ ,

$$L_t = \phi_t^{21} \circ (\phi_t^{11})^{-1},$$

then  $\mathcal{V}_t = \text{Graph}(L_t)$ . Using Lemma 4.8, the projection  $P_{\mathcal{V}_t}$  onto  $\mathcal{V}_t$  can be written as a smooth function of  $\phi_t$ , which proves that  $t \mapsto P_{\mathcal{V}_t}$  is of class  $C^k$ .  $\square$

**Remark 4.10** The above proposition tells us that, given a  $C^k$  family  $\mathcal{D} = \{\mathcal{V}_t\}_{t \in [a,b]}$  of closed subspaces of  $\mathcal{H}$ , there exists, for any  $t_0 \in [a, b]$ , a local trivialization  $(\mathcal{V}_*, \Psi)$  of  $\mathcal{D}$  around  $t_0$  such that  $\Psi_t(\mathcal{V}_t^\perp) = \mathcal{V}_*^\perp$  for all  $t$  in a neighborhood  $I$  of  $t_0$ .

**Definition 4.11** A local trivialization  $(\mathcal{V}_*, \Psi)$  of  $\mathcal{D}$  around  $t_0$  is called a *local splitting trivialization* if  $\Psi_t(\mathcal{V}_t^\perp) = \mathcal{V}_*^\perp$ .

Actually, as an immediate consequence of Corollary A.3, we obtain the following *global* result, that is, the existence of a global splitting trivialization of isometries.

**Proposition 4.12** *Given a  $C^k$  family  $\mathcal{D} = \{\mathcal{V}_t\}_{t \in [a,b]}$  of closed subspaces of  $\mathcal{H}$ , there exists a global trivialization  $(\mathcal{V}_*, \Psi)$  of  $\mathcal{D}$  such that  $\Psi_t \in \text{O}(\mathcal{H})$  for all  $t \in [a, b]$ .*

#### 4.4 Spectral flow and restrictions to a continuous family of subspaces

The additivity by concatenation of paths and invariance by cogredience properties, recalled in Subsection 4.1, allow us to extend the definition of spectral flow to the case of paths of Fredholm operators with varying domains.

Assume that  $[a, b] \ni t \mapsto T_t$  is a continuous map of bounded operators on  $\mathcal{H}$  and  $\mathcal{D} = \{\mathcal{V}_t\}_{t \in [a,b]}$  is a continuous family of closed subspaces such that, taking the orthogonal projection  $P_{\mathcal{V}_t}$  as a map with target space  $\mathcal{V}_t$  for every  $t \in [a, b]$ , the operator  $P_{\mathcal{V}_t} \circ T_t|_{\mathcal{V}_t} : \mathcal{V}_t \rightarrow \mathcal{V}_t$  is Fredholm and self-adjoint. Let  $(\mathcal{V}_*, \Psi)$  be a trivialization of  $\mathcal{D}$ , and denote by  $P_*$  the orthogonal projection onto  $\mathcal{V}_*$ . Then, we have a continuous family  $[a, b] \ni t \mapsto \tilde{T}_t \in \mathcal{F}_{\text{sa}}(\mathcal{V}_*)$  of self-adjoint Fredholm operators on  $\mathcal{V}_*$ , obtained by setting<sup>2</sup>

$$\tilde{T}_t = P_* \circ (\Psi_t|_{\mathcal{V}_t}) \circ P_{\mathcal{V}_t} \circ T_t \circ (\Psi_t|_{\mathcal{V}_t})^* : \mathcal{V}_* \longrightarrow \mathcal{V}_*. \quad (4.4)$$

We define the spectral flow  $\text{sf}(T, \mathcal{D}; [a, b])$  of the path  $T = (T_t)_{t \in [a,b]}$  restricted to the varying domains  $\mathcal{D} = (\mathcal{V}_t)_{t \in [a,b]}$  by

$$\text{sf}(T, \mathcal{D}; [a, b]) = \text{sf}(\tilde{T}, [a, b]). \quad (4.5)$$

In order to prove that this is a valid definition, one needs the following lemma.

**Lemma 4.13** *The right hand side of equality (4.5) does not depend on the choice of a trivialization of the family  $\mathcal{D}$ .*

<sup>2</sup>  $(\Psi_t|_{\mathcal{V}_t})^* = P_{\mathcal{V}_t} \Psi_t^*|_{\mathcal{V}_*}$ .

**Proof.** Assume that  $(\widehat{\mathcal{V}}_\star, \widehat{\Psi})$  is another trivialization of  $\mathcal{D}$ . Denoting by  $\widehat{P}_\star$  the orthogonal projection onto  $\widehat{\mathcal{V}}_\star$ , set

$$\widehat{T}_t = \widehat{P}_\star \circ (\widehat{\Psi}_t|_{\mathcal{V}_t}) \circ P_{\mathcal{V}_t} \circ T_t \circ (\widehat{\Psi}_t|_{\mathcal{V}_t})^* : \widehat{\mathcal{V}}_\star \longrightarrow \widehat{\mathcal{V}}_\star$$

and denote by  $\Phi_t : \mathcal{V}_\star \rightarrow \widehat{\mathcal{V}}_\star$  the isomorphism  $(P_{\mathcal{V}_t} \circ (\widehat{\Psi}_t^*|_{\widehat{\mathcal{V}}_\star}))^{-1} \circ P_{\mathcal{V}_t} \circ (\Psi_t^*|_{\mathcal{V}_\star})$ . If  $\widetilde{T}_t$  is as in formulas (4.4), then

$$\widetilde{T}_t = \Phi_t^* \circ \widehat{T}_t \circ \Phi_t$$

for all  $t$ , hence  $\text{sf}(\widetilde{T}, [a, b]) = \text{sf}(\widehat{T}, [a, b])$ , by the cogredient invariance of the spectral flow.  $\square$

Our aim is to show how the result of Theorem 4.4 can be employed in the computation of the spectral flow in the case of varying domains. Towards this goal, we observe preliminarily that if  $(\mathcal{V}_\star, \Psi)$  is an orthogonal trivialization of  $\mathcal{D}$ , then  $\Psi_t^*(\mathcal{V}_\star) = \mathcal{V}_t$  for<sup>3</sup> all  $t$ ; this simplifies formula (4.4), in that  $(\Psi_t|_{\mathcal{V}_t})^* = \Psi_t^*|_{\mathcal{V}_\star} = \Psi_t^{-1}|_{\mathcal{V}_\star}$ . Moreover, it is easy to show that, given a continuous path  $[a, b] \ni t \mapsto U_t$  with values in  $O(\mathcal{H})$ , the set of the orthogonal automorphisms of  $\mathcal{H}$ , then the spectral flow of the path  $[a, b] \ni t \mapsto T_t$  restricted to a continuous family of subspaces of  $\mathcal{H}$ ,  $\mathcal{D} = \{\mathcal{V}_t\}_{t \in [a, b]}$  is equal to the spectral flow of the path  $[a, b] \ni t \mapsto U_t T_t U_t^*$  restricted to the family  $\{U_t(\mathcal{V}_t)\}_{t \in [a, b]}$ .

We are now ready for the following result.

**Proposition 4.14** *Let  $T : [a, b] \rightarrow \mathcal{F}_{\text{sa}}(\mathcal{H})$  be a continuous path of the form  $T_t = \mathfrak{J} + K_t$ , where  $\mathfrak{J}$  is a symmetry of  $\mathcal{H}$  and  $K_t$  is, for any  $t \in [a, b]$ , a self-adjoint compact operator on  $\mathcal{H}$ . Consider a continuous family  $\mathcal{D} = \{\mathcal{V}_t\}_{t \in [a, b]}$  of (finite codimensional) closed subspaces of  $\mathcal{H}$ . Then, there exists an orthogonal trivialization  $(\mathcal{V}_\star, \Psi)$  of  $\mathcal{D}$  (with  $\mathcal{V}_\star$  finite codimensional) and a symmetry  $\widetilde{\mathfrak{J}} : \mathcal{H} \rightarrow \mathcal{H}$  such that  $\Psi_t T_t \Psi_t^* - \widetilde{\mathfrak{J}}$  is compact for all  $t \in [a, b]$ .*

**Proof.** Choose any orthogonal trivialization  $(\mathcal{V}_\star, \Phi)$  of  $\mathcal{D}$ , so that by what has been just observed, the spectral flow of  $T$  restricted to  $\mathcal{D}$  equals the spectral flow of  $t \mapsto \Phi_t T_t \Phi_t^* = \Phi_t \mathfrak{J} \Phi_t^* + \Phi_t K_t \Phi_t^*$  restricted to the fixed subspace  $\Phi_t(\mathcal{V}_t) = \mathcal{V}_\star$ .

Since  $\Phi_t$  is orthogonal, then, for all  $t$ ,  $\widehat{\mathfrak{J}}_t = \Phi_t \mathfrak{J} \Phi_t^*$  is a symmetry of  $\mathcal{H}$ ; the operator  $\Phi_t K_t \Phi_t^*$  is clearly compact. By Lemma 4.2, if  $P_\star$  is the orthogonal projection onto  $\mathcal{V}_\star$ , the operator  $P_\star \Phi_t T_t \Phi_t^*|_{\mathcal{V}_\star} \in \mathcal{F}_{\text{sa}}(\mathcal{V}_\star)$  is of the form  $\mathfrak{J}_t^* + C_t$ , where  $t \mapsto \mathfrak{J}_t^*$  is a continuous path of symmetries of the Hilbert space  $\mathcal{V}_\star$ . Now, by Corollary A.4, there exists a continuous path  $t \mapsto U_t \in O(\mathcal{V}_\star)$  and a fixed symmetry  $\mathfrak{J}_\star$  of  $\mathcal{V}_\star$  with the property that  $U_t \mathfrak{J}_t^* U_t^* = \mathfrak{J}_\star$  for all  $t$ . Extend  $\mathfrak{J}_\star$  to a symmetry  $\widetilde{\mathfrak{J}}$  of  $\mathcal{H}$  by setting  $\widetilde{\mathfrak{J}}|_{\mathcal{V}_\star^\perp}$  equal to the identity, and each  $U_t$  to an orthogonal operator  $W_t \in O(\mathcal{H})$  by setting  $W_t|_{\mathcal{V}_\star^\perp}$  equal to the identity. Observe that  $W_t$  commutes with  $P_\star$  for all  $t$ , since  $\mathcal{V}_\star$  is  $W_t$ -invariant. Then, the required trivialization of  $\mathcal{D}$  is obtained by setting  $\Psi_t = W_t \Phi_t$  for all  $t$ .  $\square$

Using an orthogonal trivialization as in Proposition 4.14, Theorem 4.4 can now be employed in the computation of the spectral flow of restrictions to a varying family of finite codimensional subspaces.

**Theorem 4.15** *Consider a continuous path  $B : [a, b] \rightarrow \text{B}_{\text{sym}}(\mathcal{H})$  of symmetric Fredholm forms on  $\mathcal{H}$  and denote by  $T : [a, b] \rightarrow \mathcal{F}_{\text{sa}}(\mathcal{H})$  the continuous paths of self-adjoint Fredholm operators associated with  $B$ . Consider a continuous family  $\mathcal{D} = \{\mathcal{V}_t\}_{t \in [a, b]}$  of (finite codimensional) closed subspaces of  $\mathcal{H}$  and let  $(\mathcal{V}_\star, \Psi)$  be an orthogonal trivialization of  $\mathcal{D}$  and  $\widetilde{\mathfrak{J}} : \mathcal{H} \rightarrow \mathcal{H}$  be a symmetry such that  $\Psi_t T_t \Psi_t^* - \widetilde{\mathfrak{J}}$  is compact for all  $t \in [a, b]$ . Denote by  $\widetilde{T} : [a, b] \rightarrow \mathcal{F}_{\text{sa}}(\mathcal{V}_\star)$  the path  $\widetilde{T}_t = P_\star \Psi_t T_t \Psi_t^*|_{\mathcal{V}_\star}$ , where  $P_\star$  is the projection onto  $\mathcal{V}_\star$ .*

*Then, we have*

$$\text{sf}(T, [a, b]) - \text{sf}(T, \mathcal{D}; [a, b]) = \dim(V^-(T_a), V^-(\widetilde{T}_a)) - \dim(V^-(T_b), V^-(\widetilde{T}_b)). \quad (4.6)$$

<sup>3</sup> This holds more generally for splitting trivializations.

**Proof.** Denote  $\widehat{T}_t = \Psi_t T_t \Psi_t^* : \mathcal{H} \rightarrow \mathcal{H}$ , for any  $t \in [a, b]$ . Since  $T$  and  $\widehat{T}$  are cogredient, their spectral flows coincide, and, since  $\widehat{T}$  is a path of compact perturbations of a symmetry, we have

$$\mathfrak{sf}(T, [a, b]) = \dim(V^-(T_a), V^-(T_b)) = \mathfrak{sf}(\widehat{T}, [a, b]) = \dim(V^-(\widehat{T}_a), V^-(\widehat{T}_b)).$$

Applying Theorem 4.4, we have

$$\mathfrak{sf}(\widehat{T}, [a, b]) - \mathfrak{sf}(\widetilde{T}, [a, b]) = \dim(V^-(\widehat{T}_a), V^-(\widetilde{T}_a)) - \dim(V^-(\widehat{T}_b), V^-(\widetilde{T}_b)),$$

and, finally, by Lemma 3.10 the claim follows.  $\square$

## 5 Spectral flow along periodic semi-Riemannian geodesics

In this section we will discuss an application to semi-Riemannian geometry of our spectral flow formula. We will define the *spectral flow* of the index form along a periodic geodesic in a semi-Riemannian manifold, and we will compute its value in terms of the Maslov index of the geodesic. In the Riemannian (i.e., positive definite) case, the spectral flow is equal to the Morse index of the geodesic action functional at the closed geodesic, and the Maslov index is given by the number of conjugate points along a geodesic. In the general semi-Riemannian case, it is well known that the Morse index of the geodesic action functional is infinite.

### 5.1 Periodic geodesics

We will consider throughout an  $n$ -dimensional semi-Riemannian manifold  $(M, g)$ , denoting by  $\nabla$  the covariant derivative of its Levi-Civita connection, and by  $R$  its curvature tensor, chosen with the sign convention  $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ .

Let  $\gamma : [0, 1] \rightarrow M$  be a periodic geodesic in  $M$ , i.e.,  $\gamma(0) = \gamma(1)$  and  $\dot{\gamma}(0) = \dot{\gamma}(1)$ . We will assume that  $\gamma$  is *orientation preserving*, which means that the parallel transport along  $\gamma$  is orientation preserving. If  $M$  is orientable, then every closed geodesic is orientation preserving. Moreover, given any closed geodesic  $\gamma$ , its two-fold iteration  $\gamma^{(2)}$ , defined by  $\gamma^{(2)}(t) = \gamma(2t)$ , is always orientation preserving.

We will denote by  $\frac{D}{dt}$  the covariant differentiation of vector fields along  $\gamma$ ; recall that the *index form*  $I_\gamma$  is the bounded symmetric bilinear form defined on the Hilbert space of all periodic vector fields of Sobolev class  $H^1$  along  $\gamma$ , given by

$$I_\gamma(V, W) = \int_0^1 g\left(\frac{D}{dt}V, \frac{D}{dt}W\right) + g(RV, W) dt, \tag{5.1}$$

where we set  $R = R(\dot{\gamma}, \cdot)\dot{\gamma}$ . Closed geodesics in  $M$  are the critical points of the geodesic action functional  $f(\gamma) = \frac{1}{2} \int_0^1 g(\dot{\gamma}, \dot{\gamma}) dt$  defined in the *free loop space*  $\Omega M$  of  $M$ ;  $\Omega M$  is the Hilbert manifold of all closed curves in  $M$  of Sobolev class  $H^1$ . The index form  $I_\gamma$  is the second variation of  $f$  at the critical point  $\gamma$ ; unless  $g$  is positive definite, the Morse index of  $f$  at each nonconstant critical point is infinite. The notion of Morse index is replaced by the notion of spectral flow.

### 5.2 Periodic frames and trivializations

Consider a smooth periodic orthonormal frame  $\mathbf{T}$  along  $\gamma$ , i.e., a smooth family  $[0, 1] \ni t \mapsto T_t$  of isomorphisms:

$$T_t : \mathbb{R}^n \longrightarrow T_{\gamma(t)}M, \tag{5.2}$$

with  $T_0 = T_1$ , and

$$g(T_t e_i, T_t e_j) = \epsilon_i \delta_{ij}, \tag{5.3}$$

where  $\{e_i\}_{i=1, \dots, n}$  is the canonical basis of  $\mathbb{R}^n$ ,  $\epsilon_i \in \{-1, 1\}$  and  $\delta_{ij}$  is the Kronecker symbol. The existence of such a frame is guaranteed by the orientability assumption on the closed geodesic. The pull-back by  $T_t$  of the

metric  $g$  gives a symmetric nondegenerate bilinear form  $G$  on  $\mathbb{R}^n$ , whose index is the same as the index of  $g$ ; note that this pull-back does not depend on  $t$ , by the orthogonality assumption on the frame  $\mathbf{T}$ . In the sequel, we will also denote by  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  the symmetric linear operator defined by  $(Gv) \cdot w$ ; By (5.3),  $G$  satisfies

$$G^2 = I. \quad (5.4)$$

For all  $t \in ]0, 1]$ , define by  $\mathcal{H}_t^\gamma$  the Hilbert space of all  $H^1$ -vector fields  $V$  along  $\gamma|_{[0,t]}$  satisfying

$$T_0^{-1}V(0) = T_t^{-1}V(t).$$

Observe that the definition of  $\mathcal{H}_t^\gamma$  depends on the choice of the periodic frame  $\mathbf{T}$ , however,  $\mathcal{H}_1^\gamma$ , which is the space of all periodic vector fields along  $\gamma$ , does not depend on  $\mathbf{T}$ . Although in principle there is no necessity of fixing a specific Hilbert space inner product, it will be useful to have one at disposal, and this will be chosen as follows. For all  $t \in ]0, 1]$ , consider the Hilbert space

$$H_{\text{per}}^1([0, t], \mathbb{R}^n) = \left\{ \bar{V} \in H^1([0, t], \mathbb{R}^n) : \bar{V}(0) = \bar{V}(t) \right\}.$$

There is a natural Hilbert space inner product in  $H_{\text{per}}^1([0, t], \mathbb{R}^n)$  given by

$$\langle \bar{V}, \bar{W} \rangle = \bar{V}(0) \cdot \bar{W}(0) + \int_0^t \bar{V}'(s) \cdot \bar{W}'(s) \, ds, \quad (5.5)$$

where  $\cdot$  is the Euclidean inner product in  $\mathbb{R}^n$ . The map  $\Psi_t : \mathcal{H}_t^\gamma \rightarrow H_{\text{per}}^1([0, t], \mathbb{R}^n)$  defined by  $\Psi_t(V) = \bar{V}$ , where  $\bar{V}(s) = T_s^{-1}(V(s))$  is an isomorphism; the space  $\mathcal{H}_t^\gamma$  will be endowed with the pull-back of the inner product (5.5) by the isomorphism  $\Psi_t$ . Denote by  $\bar{R}_t \in L(\mathbb{R}^n)$  the pull-back by  $T_t$  of the endomorphism  $R_{\gamma(t)} = R(\dot{\gamma}, \cdot)\dot{\gamma}$  of  $T_{\gamma(t)}M$ :

$$\bar{R}_t = T_t^{-1} \circ R_{\gamma(t)} \circ T_t;$$

observe that  $t \mapsto \bar{R}_t$  is a smooth map of  $G$ -symmetric endomorphisms of  $\mathbb{R}^n$ . Finally, denote by  $\Gamma_t \in L(\mathbb{R}^n)$  the *Christoffel symbol* of the frame  $\mathbf{T}$ , defined by

$$\Gamma_t(v) = T_t^{-1} \left( \frac{D}{dt} V \right) - \frac{d}{dt} \bar{V}(t),$$

where  $\bar{V}$  is any vector field satisfying  $\bar{V}(t) = v$ , and  $V = \Psi_t^{-1}(\bar{V})$ . The push-forward by  $\Psi_t$  of the index form  $I_\gamma$  on  $\mathcal{H}_t^\gamma$  is given by the bounded symmetric bilinear form  $\bar{I}_t$  on  $H_{\text{per}}^1([0, t], \mathbb{R}^n)$  defined by

$$\begin{aligned} \bar{I}_t(\bar{V}, \bar{W}) = \int_0^t G(\bar{V}'(s), \bar{W}'(s)) + G(\Gamma_s \bar{V}(s), \bar{W}'(s)) + G(\Gamma_s \bar{W}(s), \bar{V}'(s)) \\ + G(\Gamma_s \bar{V}(s), \Gamma_s \bar{W}(s)) + G(\bar{R}_s \bar{V}(s), \bar{W}(s)) \, ds. \end{aligned} \quad (5.6)$$

Finally, for  $t \in ]0, 1]$ , we will consider the isomorphism

$$\Phi_t : H_{\text{per}}^1([0, t], \mathbb{R}^n) \rightarrow H_{\text{per}}^1([0, 1], \mathbb{R}^n),$$

defined by  $\bar{V} \mapsto \tilde{V}$ , where  $\tilde{V}(s) = \bar{V}(st)$ ,  $s \in [0, 1]$ . The push-forward by  $\Phi_t$  of the bilinear form  $\bar{I}_t$  is given by the bounded symmetric bilinear form  $\tilde{I}_t$  on  $H_{\text{per}}^1([0, 1], \mathbb{R}^n)$  defined by:

$$\begin{aligned} \tilde{I}_t(\tilde{V}, \tilde{W}) = \frac{1}{t^2} \int_0^1 G(\tilde{V}'(r), \tilde{W}'(r)) + tG(\Gamma_{tr} \tilde{V}(r), \tilde{W}'(r)) + tG(\Gamma_{tr} \tilde{W}(r), \tilde{V}'(r)) \\ + t^2G(\Gamma_{tr} \tilde{V}(r), \Gamma_{tr} \tilde{W}(r)) + t^2G(\tilde{R}_{tr} \tilde{V}(r), \tilde{W}(r)) \, dr. \end{aligned} \quad (5.7)$$

### 5.3 Spectral flow of a periodic geodesic

For  $t \in ]0, 1]$ , define the Fredholm bilinear form  $B_t$  on the Hilbert space  $H_{\text{per}}^1([0, 1], \mathbb{R}^n)$  by setting

$$B_t = t^2 \cdot \tilde{I}_t. \tag{5.8}$$

From (5.7) we obtain immediately the following result.

**Lemma 5.1** *The map  $]0, 1] \ni t \mapsto B_t$  can be extended continuously to  $t = 0$  by setting:*

$$B_0(\tilde{V}, \tilde{W}) = \int_0^1 G(\tilde{V}'(r), \tilde{W}'(r)) \, dr. \quad \square$$

Observe that  $\text{Ker } B_0$  is one-dimensional, and it consists of all constant vector fields.

**Proposition 5.2** *For all  $t \in [0, 1]$ , the bilinear form  $\tilde{I}_t$  on  $H_{\text{per}}^1([0, 1], \mathbb{R}^n)$  is represented with respect to the inner product (5.5) by a compact perturbation of the symmetry  $\mathfrak{J}$  of  $H_{\text{per}}^1([0, 1], \mathbb{R}^n)$  given by  $\tilde{V} \mapsto G\tilde{V}$ .*

**Proof.** First, observe that  $B_t$  is a compact perturbation of  $B_0$ . Namely, from (5.7) we get:

$$\begin{aligned} B_t(\tilde{V}, \tilde{W}) - B_0(\tilde{V}, \tilde{W}) &= \int_0^1 tG(\Gamma_{tr}\tilde{V}(r), \tilde{W}'(r)) + tG(\Gamma_{tr}\tilde{W}(r), \tilde{V}'(r)) \\ &\quad + t^2G(\Gamma_{tr}\tilde{V}(r), \Gamma_{tr}\tilde{W}(r)) + t^2G(\tilde{R}_{tr}\tilde{V}(r), \tilde{W}(r)) \, dr. \end{aligned}$$

The integral above defines a bilinear map which is continuous in the  $H^{\frac{1}{2}}$ -topology, and thus it is represented by a compact operator, since the inclusion  $H^1 \hookrightarrow H^{\frac{1}{2}}$  is compact. Next, observe that  $B_0$  is represented by a compact perturbation of the symmetry  $\mathfrak{J}$ . For,

$$\langle \mathfrak{J}\tilde{V}, \tilde{W} \rangle - B_0(\tilde{V}, \tilde{W}) = G(\tilde{V}(0), \tilde{W}(0)),$$

which is continuous in the  $C^0$ -topology, hence represented by a compact operator. Note that  $\mathfrak{J}$  is self-adjoint and, by (5.4),  $\mathfrak{J}^2 = I$ ; thus,  $\mathfrak{J}$  is a symmetry. This concludes the proof.  $\square$

**Definition 5.3** *The spectral flow  $\text{sf}(\gamma)$  of the closed geodesic  $\gamma$  is defined as the spectral flow of the continuous path of Fredholm bilinear forms  $[0, 1] \ni t \mapsto B_t$  on the Hilbert space  $H_{\text{per}}^1([0, 1], \mathbb{R}^n)$ .*

**Remark 5.4** *The fact that the definition of  $\text{sf}(\gamma)$  does not depend on the choice of a smooth periodic orthonormal frame along  $\gamma$  is a nontrivial fact, and it will be proven in next subsection by giving an explicit formula for its computation.*

We observe here that the paths of Fredholm bilinear forms  $B_t$  as above produced by two distinct periodic trivializations of the tangent bundle are in general neither fixed endpoint homotopic, nor cogredient. Namely, two distinct trivializations *differ* by a closed path in the (connected component of the identity of the) Lie group  $O(G)$  of all  $G$ -preserving linear isomorphisms of  $\mathbb{R}^n$ , which is not simply connected.

### 5.4 Computation of the spectral flow

There is an integer valued invariant associated to every (fixed endpoints) geodesic in a semi-Riemannian manifold  $(M, g)$ , called the *Maslov index*. This is a symplectic invariant, which is computed as an intersection number in the Lagrangian Grassmannian of a symplectic vector space. Details on the definition and the computation of the Maslov index for a given geodesic  $\gamma$ , that will be denoted by  $i_{\text{Maslov}}(\gamma)$  can be found in [17, 18, 24].

As for the definition of spectral flow, there are several conventions in the literature concerning the computation of the contribution to the Maslov index of the endpoints of the geodesic. In this section we will convention<sup>4</sup> that in the computation of the Maslov index  $i_{\text{Maslov}}(\gamma)$  it is also considered the contribution of the initial point of  $\gamma$ ; the value of this contribution is easily computed to be equal to  $n_-(g)$ , which is the index of the semi-Riemannian metric tensor  $g$ .

<sup>4</sup> This is not a standard choice in the literature.

Recall that a *Jacobi field* along  $\gamma$  is a smooth vector field  $J$  along  $\gamma$  that satisfies the second order linear equation

$$DDdtJ(t) = R(\dot{\gamma}(t), J(t)) \dot{\gamma}(t), \quad t \in [0, 1].$$

Let us denote by  $\mathcal{J}_\gamma$  the  $2n$ -dimensional real vector space of all Jacobi fields along  $\gamma$ . Let us introduce the following spaces:

$$\begin{aligned} \mathcal{J}_\gamma^{\text{per}} &= \left\{ J \in \mathcal{J}_\gamma : J(0) = J(1), \frac{D}{dt}J(0) = \frac{D}{dt}J(1) \right\}, \\ \mathcal{J}_\gamma^0 &= \left\{ J \in \mathcal{J}_\gamma : J(0) = J(1) = 0 \right\}, \quad \text{and} \\ \mathcal{J}_\gamma^* &= \left\{ J \in \mathcal{J}_\gamma : J(0) = J(1) \right\}. \end{aligned}$$

It is well known that  $\mathcal{J}_\gamma^{\text{per}}$  is the kernel of the index form  $I_\gamma$  defined in (5.1), while  $\mathcal{J}_\gamma^0$  is the kernel of the restriction of the index form to the space of vector fields along  $\gamma$  vanishing at the endpoints. We denote by  $n_\gamma^{\text{per}}$  and  $n_\gamma^0$  the dimensions of  $\mathcal{J}_\gamma^{\text{per}}$  and  $\mathcal{J}_\gamma^0$  respectively. The nonnegative integer  $n_\gamma^{\text{per}}$  is the nullity of  $\gamma$  as a periodic geodesic, i.e., the nullity of the Hessian of the geodesic action functional at  $\gamma$  in the space of closed curves. Observe that  $n_\gamma^{\text{per}} \geq 1$ , as  $\mathcal{J}_\gamma^{\text{per}}$  contains the one-dimensional space spanned by the tangent field  $J = \dot{\gamma}$ . Similarly,  $n_\gamma^0$  is the nullity of  $\gamma$  as a fixed endpoint geodesic, i.e., it is the nullity of the Hessian of the geodesic action functional at  $\gamma$  in the space of fixed endpoints curves in  $M$ . In this case,  $n_\gamma^0 > 0$  if and only if  $\gamma(1)$  is conjugate to  $\gamma(0)$  along  $\gamma$ .

Given a semi-Riemannian geodesic  $\gamma$ , the spectral flow of the path of symmetric Fredholm bilinear forms  $[0, 1] \ni t \mapsto B_t$  restricted to the space  $H_0^1([0, 1], \mathbb{R}^n)$  will be denoted by  $\mathfrak{sf}_0(\gamma)$ . A formula giving the value of this integer is proven in [17, Proposition 2]:

**Proposition 5.5** *Given any (closed) semi-Riemannian geodesic  $\gamma$ , the following equality holds:*

$$\mathfrak{sf}_0(\gamma) = n_\gamma^0 - n_-(g) - i_{\text{Maslov}}(\gamma). \quad \square \tag{5.9}$$

Finally, the last ingredient needed for the computation of the spectral flow of a closed geodesic is the so called *index of concavity* of  $\gamma$ , that will be denoted by  $i_{\text{conc}}(\gamma)$ . This is a nonnegative integer invariant associated to periodic solutions of Hamiltonian systems, first introduced by M. Morse [21] in the context of closed Riemannian geodesic. In our notations,  $i_{\text{conc}}(\gamma)$  is equal to the index of the symmetric bilinear form:

$$(J_1, J_2) \longmapsto g\left(\frac{D}{dt}J_1(1) - \frac{D}{dt}J_1(0), J_2(0)\right),$$

defined on the vector space  $\mathcal{J}_\gamma^*$ . It is not hard to show that this bilinear form is symmetric, in fact, it is given by the restriction of the index form  $I_\gamma$  to  $\mathcal{J}_\gamma^*$ .

It is now easy to apply Theorem 4.4 in order to obtain a formula for the spectral flow of an oriented closed geodesic.

**Theorem 5.6** *Let  $(M, g)$  be a semi-Riemannian manifold and let  $\gamma : [0, 1] \rightarrow M$  be a closed oriented geodesic in  $M$ . Then, the spectral flow  $\mathfrak{sf}(\gamma)$  is given by the following formula:*

$$\mathfrak{sf}(\gamma) = \dim(\mathcal{J}_\gamma^{\text{per}} \cap \mathcal{J}_\gamma^0) - i_{\text{Maslov}}(\gamma) - i_{\text{conc}}(\gamma) - n_-(g). \tag{5.10}$$

**Proof.** Set  $\mathcal{H} = H_{\text{per}}^1([0, 1], \mathbb{R}^n)$ ,  $\mathcal{V} = H_0^1([0, 1], \mathbb{R}^n)$  in Theorem 4.4; The difference  $\mathfrak{sf}(\gamma) - \mathfrak{sf}_0(\gamma)$  is thus given by the sum of six terms, that are computed easily as follows. The space  $\mathcal{V}^{\perp B_0}$  coincides with the kernel of  $B_0$ , and it is given by the one dimensional space of constant vector fields on  $[0, 1]$ ; the restriction of  $B_0$  to such space vanishes identically. Moreover,  $\mathcal{V} \cap \mathcal{V}^{\perp B_0} = \mathcal{V} \cap \text{Ker } B_0 = \{0\}$ . A straightforward partial integration argument shows that the space  $\mathcal{V}^{\perp B_1}$  is given by  $\mathcal{J}_\gamma^*$ . By definition, the index of the restriction of  $B_1$  to this space equals  $i_{\text{conc}}(\gamma)$ . The space  $\mathcal{V} \cap \mathcal{V}^{\perp B_1} = \text{Ker}(B_1|_{\mathcal{V} \times \mathcal{V}})$  is given by  $\mathcal{J}_\gamma^0$ . Finally,  $\text{Ker } B_1 = \mathcal{J}_\gamma^{\text{per}}$ , thus  $\text{Ker } B_1 \cap \mathcal{V} = \mathcal{J}_\gamma^{\text{per}} \cap \mathcal{J}_\gamma^0$ . Formula (5.10) follows now immediately from (5.9).  $\square$

Formula (5.10) proves in particular that the definition of spectral flow for a periodic geodesic  $\gamma$  does not depend on the choice of an orthonormal frame along  $\gamma$ .

**Remark 5.7** Our definition of spectral flow along a closed geodesic has used a periodic orthonormal frame along the geodesic, which exists only if the geodesic is orientation preserving. We observe however that the right hand side of formula (5.10) is defined for every closed geodesic, regardless of its orientability, which suggests that (5.10) can be taken as the definition of spectral flow in the nonorientable case. Let us sketch briefly how the right-hand side of (5.10) can be obtained as a spectral flow of paths of Fredholm operators. Given a nonorientable closed geodesic  $\gamma$ , choose an arbitrary smooth frame  $\mathbf{T}$  along  $\gamma$  as in (5.2), which will *not* satisfy  $T_0 = T_1$ ; set  $S = T_1^{-1}T_0 \in \text{GL}(\mathbb{R}^n)$ . Then, the spectral flow  $\mathfrak{sf}(\gamma)$  is defined as the difference  $\mathfrak{sf}_S(\gamma) - \mathfrak{n}_S$ , where  $\mathfrak{sf}_S(\gamma)$  is the spectral flow of the path of Fredholm bilinear forms  $[0, 1] \ni t \mapsto B_t$  given in (5.8) on the space

$$H_S^1([0, 1], \mathbb{R}^n) = \left\{ \tilde{V} \in H^1([0, 1], \mathbb{R}^n) : \tilde{V}(1) = S\tilde{V}(0) \right\},$$

and  $\mathfrak{n}_S$  is the index of the restriction of the metric tensor  $g$  to the image of the operator  $S - I$  (compare with Definition 5.3). Note that  $S = I$  in the orientation preserving case. With such definition, formula (5.10) holds also in the non orientation-preserving case. This is proven easily using Theorem 4.4, as in the proof of Theorem 5.6. One sets  $\mathcal{H} = H_S^1([0, 1], \mathbb{R}^n)$ ,  $\mathcal{V} = H_0^1([0, 1], \mathbb{R}^n)$ , and observes that in this case the space  $\mathcal{V}^{\perp B_0}$  consists of all affine maps  $\tilde{V} : [0, 1] \rightarrow \mathbb{C}^n$  of the form  $\tilde{V}(t) = (S - I)B + B$ , where  $B$  is an arbitrary vector in  $\mathbb{C}^n$ . The restriction of the the Hermitian form  $B_0$  to such space equals the index of the restriction of  $g$  to the image of  $S - I$ , from which the desired conclusion follows.

## A Group actions and fibrations over the infinite dimensional Grassmannian

In this appendix we will study the fibrations over the Grassmannian of all closed subspaces of a Hilbert space  $\mathcal{H}$  determined by the actions of the general linear group  $\text{GL}(\mathcal{H})$  and of the orthogonal group  $\text{O}(\mathcal{H})$ .

Let  $\mathcal{H}$  be an infinite dimensional separable Hilbert space; denote, as in the previous sections, by  $\text{L}(\mathcal{H})$  the Banach algebra of all bounded linear operators on  $\mathcal{H}$ , by  $\text{L}_{\text{sa}}(\mathcal{H})$  (resp.,  $\text{L}_{\text{as}}(\mathcal{H})$ ) the subspace of  $\text{L}(\mathcal{H})$  of self-adjoint operators (resp., of anti-symmetric operators), by  $\text{GL}(\mathcal{H})$  the Banach Lie group of all bounded linear isomorphisms of  $\mathcal{H}$  and by  $\text{O}(\mathcal{H})$  the subset of  $\text{GL}(\mathcal{H})$  consisting of isometries of  $\mathcal{H}$ :

$$\text{O}(\mathcal{H}) = \{T \in \text{GL}(\mathcal{H}) : T^*T = TT^* = I\}.$$

By a well known result due to Kuiper [20],  $\text{O}(\mathcal{H})$  is contractible;  $\text{O}(\mathcal{H})$  is a smooth embedded submanifold of  $\text{GL}(\mathcal{H})$ , being the inverse image  $f^{-1}(I) \cap \text{GL}(\mathcal{H})$  of the submersion  $\text{L}(\mathcal{H}) \ni T \mapsto T^*T \in \text{L}_{\text{sa}}(\mathcal{H})$ . The tangent space  $T_1\text{GL}(\mathcal{H})$  is  $\text{L}(\mathcal{H})$ ; the tangent space  $T_1\text{O}(\mathcal{H})$  is the subspace  $\text{L}_{\text{as}}(\mathcal{H})$ . Denote by  $\text{Gr}(\mathcal{H})$  the Grassmannian of all closed subspaces of  $\mathcal{H}$ , which is a metric space endowed with the metric  $\text{dist}(\mathcal{V}, \mathcal{W}) = \|P_{\mathcal{V}} - P_{\mathcal{W}}\|$ . There is an action  $\text{GL}(\mathcal{H}) \times \text{Gr}(\mathcal{H}) \rightarrow \text{Gr}(\mathcal{H})$  given by  $(T, \mathcal{V}) \mapsto T(\mathcal{V})$ .

The set  $\text{Gr}(\mathcal{H})$  has a real analytic Banach manifold structure, the action of  $\text{GL}(\mathcal{H})$  is analytic, and so is its restriction to the orthogonal group (see for instance [3]). The connected components of  $\text{Gr}(\mathcal{H})$  are the sets

$$\text{Gr}_{k_1, k_2}(\mathcal{H}) = \{\mathcal{V} \in \text{Gr}(\mathcal{H}) : \dim(\mathcal{V}) = k_1, \dim(\mathcal{V}^\perp) = k_2\},$$

where  $k_1, k_2 \in \mathbb{N} \cup \{+\infty\}$  are not both finite numbers. The action of  $\text{O}(\mathcal{H})$  is transitive on each connected component of  $\text{Gr}(\mathcal{H})$ . For all  $\mathcal{W} \in \text{Gr}(\mathcal{H})$ , the tangent space  $T_{\mathcal{W}}\text{Gr}(\mathcal{H})$  is identified with the Banach space  $\text{L}(\mathcal{W}, \mathcal{W}^\perp)$  of all bounded linear operators  $X : \mathcal{W} \rightarrow \mathcal{W}^\perp$ .

Here comes a simple result on group actions, submersion and fibrations.

**Lemma A.1** *Let  $M$  be a Banach manifold and let  $G$  be a Banach Lie group acting smoothly and transitively on  $M$ :*

$$G \times M \ni (g, m) \mapsto g \cdot m \in M.$$

*Let  $m \in M$  be fixed, and denote by  $\beta_m : G \rightarrow M$  the map  $\beta_m(g) = g \cdot m$ .*

- (a) *If  $\beta_m$  is a submersion at  $g = 1$ , then  $\beta_m$  is a submersion.*
- (b) *If  $\beta_m$  is a submersion, then  $\beta_m$  is a smooth fibration with typical fiber the isotropy group  $G_m$ .*

**Proof.** Denote by  $L_g : G \rightarrow G$  the left translation by  $g$ :  $L_g(h) = gh$ , and by  $\gamma_g : M \rightarrow M$  the diffeomorphism  $\gamma_g(m) = g \cdot m$ . Then,  $\beta_m \circ L_g = \gamma_g \circ \beta_m$ ; differentiating at  $h = 1$  gives

$$d\beta_m(g) \circ dL_g(1) = d\gamma_g(m) \circ d\beta_m(1).$$

Note that  $dL_g(1)$  and  $d\gamma_g(m)$  are isomorphisms. Thus, if  $d\beta_m(1)$  is surjective, then so is  $d\beta_m(g)$ . Similarly, if  $\text{Ker}(d\beta_m(1))$  is complemented, then so is  $\text{Ker}(d\beta_m(g)) = dL_g(1)[\text{Ker}(d\beta_m(1))]$ . This proves part (a).

For part (b), it suffices to show the existence of local trivializations. Note that the stabilizer  $G_m$  of  $m$  is a Lie subgroup of  $G$ , being the inverse image of a value of a submersion:  $G_m = \beta_m^{-1}(m)$ . Let  $S : U \subseteq M \rightarrow G$  be a local section of  $\beta_m$ ; local sections exist by the assumption that  $\beta_m$  is a submersion. Then, a trivialization of  $\beta_m^{-1}(U)$  is given by

$$U \times G_m \ni (x, g) \longmapsto s(x)g \in \beta^{-1}(U).$$

Obviously, this map is smooth, and its inverse is given by

$$\beta_m^{-1}(U) \ni h \longmapsto (h \cdot m, s(h \cdot m)^{-1}h) \in U \times G_m,$$

which is also smooth. □

**Proposition A.2** *Let  $\mathcal{W} \in \text{Gr}(\mathcal{H})$  be fixed and let  $\text{Gr}_{k_1, k_2}(\mathcal{H})$  be its connected component in  $\text{Gr}(\mathcal{H})$ . The map  $\beta_{\mathcal{W}} : \text{GL}(\mathcal{H}) \rightarrow \text{Gr}_{k_1, k_2}(\mathcal{H})$ , defined by  $\beta_m(T) = T(\mathcal{W})$ , is a real analytic fibration. The same conclusion holds for the restriction of  $\beta_m$  to  $\text{O}(\mathcal{H})$ .*

**Proof.** By part (a) and (b) of Lemma A.1, it suffices to show that the linear map  $d\beta_{\mathcal{W}}(1) : \text{L}(\mathcal{H}) \rightarrow \text{L}(\mathcal{W}, \mathcal{W}^\perp)$  is surjective and that it has complemented kernel, as well as its restriction to  $\text{L}_{\text{as}}(\mathcal{H})$ . An explicit computation gives:

$$d\beta_{\mathcal{W}}(1)X = P_{\mathcal{W}^\perp} \circ X|_{\mathcal{W}}, \quad \forall X \in \text{L}(\mathcal{H}),$$

where  $P_{\mathcal{W}^\perp}$  is the orthogonal projection onto  $\mathcal{W}^\perp$ . Writing  $X : \mathcal{W} \oplus \mathcal{W}^\perp \rightarrow \mathcal{W} \oplus \mathcal{W}^\perp$  in block form:

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

then  $d\beta_{\mathcal{W}}(1)X = X_{21} : \mathcal{W} \rightarrow \mathcal{W}^\perp$ . Clearly, a complement in  $\text{L}(\mathcal{H})$  for the kernel of this map is the closed subspace of  $\text{L}(\mathcal{H})$  consisting of operators  $Y$  that are written in block form as  $Y = \begin{pmatrix} 0 & 0 \\ Y_{21} & 0 \end{pmatrix}$ , where  $Y_{21} \in \text{L}(\mathcal{W}, \mathcal{W}^\perp)$ .

Similarly, the kernel of  $d\beta_{\mathcal{W}}(1) : \text{L}_{\text{as}}(\mathcal{H}) \rightarrow \text{L}(\mathcal{W}, \mathcal{W}^\perp)$  consists of all anti-symmetric operators  $X$  that are written in block form as  $X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}$ , where  $X_{11} \in \text{L}_{\text{as}}(\mathcal{W})$  and  $X_{22} \in \text{L}_{\text{as}}(\mathcal{W}^\perp)$ . A complement for this space in  $\text{L}_{\text{as}}(\mathcal{H})$  is given by the closed subspace of  $\text{L}_{\text{as}}(\mathcal{H})$  consisting of all operators  $Y$  that have block form  $Y = \begin{pmatrix} 0 & Y_{12} \\ Y_{12}^* & 0 \end{pmatrix}$ , with  $Y_{12} \in \text{L}(\mathcal{W}^\perp, \mathcal{W})$ .

Moreover, it is easy to check that  $d\beta_{\mathcal{W}}(1) : \text{L}_{\text{as}}(\mathcal{H}) \rightarrow \text{L}(\mathcal{W}, \mathcal{W}^\perp)$  (and thus also  $d\beta_{\mathcal{W}}(1) : \text{L}(\mathcal{H}) \rightarrow \text{L}(\mathcal{W}, \mathcal{W}^\perp)$ ) is surjective. Namely, given any  $A \in \text{L}(\mathcal{W}, \mathcal{W}^\perp)$ , there exists  $X \in \text{L}_{\text{as}}(\mathcal{H})$  whose lower down block  $X_{21}$  relative to the decomposition  $\mathcal{H} = \mathcal{W} \oplus \mathcal{W}^\perp$  equals  $A$ , for instance,  $X = \begin{pmatrix} 0 & -A^* \\ A & 0 \end{pmatrix}$ . This concludes the proof. □

**Corollary A.3** *Given any curve  $\mathcal{V} : [a, b] \rightarrow \text{Gr}(\mathcal{H})$  of class  $C^k$ ,  $k = 0, \dots, \infty, \omega$ , given any  $\mathcal{W}$  in the connected component  $\text{Gr}_{k_1, k_2}(\mathcal{H})$  of  $\mathcal{V}_a$  in  $\text{Gr}(\mathcal{H})$  and any isometry  $\varphi : \mathcal{H} \rightarrow \mathcal{H}$  such that  $\varphi(\mathcal{W}) = \mathcal{V}_a$ , then there exists a curve  $\Phi : [a, b] \rightarrow \text{O}(\mathcal{H})$  of class  $C^k$  such that  $\Phi_t(\mathcal{W}) = \mathcal{V}_t$  for all  $t \in [a, b]$  and with  $\Phi_a = \varphi$ .*

Proof.  $\Phi$  is a lifting of the curve  $\mathcal{V}$  in the fibration  $\beta_{\mathcal{W}}$ :

$$\begin{array}{ccc}
 & & \text{O}(\mathcal{H}) \\
 & \nearrow \Phi & \downarrow \beta_{\mathcal{W}} \\
 [a, b] & \xrightarrow{\mathcal{V}} & \text{Gr}_{k_1, k_2}(\mathcal{H})
 \end{array}$$

□

It is interesting to restate the result above in terms of symmetries. Recall that by a symmetry of  $\mathcal{H}$  we mean a self-adjoint operator  $\mathfrak{J}$  on  $\mathcal{H}$  such that  $\mathfrak{J}^2 = I$ . Denote by  $\mathfrak{S}(\mathcal{H})$  the closed subset of  $\text{O}(\mathcal{H})$  consisting of all symmetries of  $\mathcal{H}$ ; the bijection  $\mathfrak{S}(\mathcal{H}) \ni \mathfrak{J} \mapsto \text{Ker}(\mathfrak{J} - I) \in \text{Gr}(\mathcal{H})$  is a homeomorphism, whose inverse is

$$\text{Gr}(\mathcal{H}) \ni \mathcal{V} \mapsto P_{\mathcal{V}} - P_{\mathcal{V}^\perp} \in \mathfrak{S}(\mathcal{H}).$$

This bijection carries the action  $\text{O}(\mathcal{H}) \times \text{Gr}(\mathcal{H}) \ni (U, \mathcal{V}) \mapsto U(\mathcal{V}) \in \text{Gr}(\mathcal{H})$  into the cogredient action:

$$\text{O}(\mathcal{H}) \times \mathfrak{S}(\mathcal{H}) \ni (U, \mathfrak{J}) \mapsto U\mathfrak{J}U^* \in \mathfrak{S}(\mathcal{H}),$$

i.e., if  $\mathfrak{J} = P_{\mathcal{V}} - P_{\mathcal{V}^\perp}$ , then  $U\mathfrak{J}U^* = P_{U(\mathcal{V})} - P_{U(\mathcal{V})^\perp}$ . Thus, Corollary A.3 can be translated as follows.

**Corollary A.4** *Let  $[a, b] \ni t \mapsto \mathfrak{J}_t \in \text{GL}(\mathcal{H})$  be a map of class  $C^k$ ,  $k = 0, \dots, \infty, \omega$ , where  $\mathfrak{J}_t \in \mathfrak{S}(\mathcal{H})$  for all  $t$ . Then, there exists a  $C^k$  map  $[a, b] \ni t \mapsto U_t \in \text{O}(\mathcal{H})$  and a fixed symmetry  $\mathfrak{J} \in \mathfrak{S}(\mathcal{H})$  such that  $U_t \mathfrak{J}_t U_t^* = \mathfrak{J}$  for all  $t$ .* □

## References

- [1] A. Abbondandolo, *A new cohomology for the Morse theory of strongly indefinite functionals on Hilbert spaces*, *Topol. Methods Nonlin. Anal.*, **9** (1997), 325–382.
- [2] A. Abbondandolo, *Morse Theory for Hamiltonian Systems*, Pitman Research Notes in Mathematics, vol. 425, Chapman & Hall, London, 2001.
- [3] A. Abbondandolo, P. Majer, *Infinite dimensional Grassmannians*, arXiv:math/0307192.
- [4] A. Abbondandolo, P. Majer, *A Morse complex for infinite dimensional manifolds. I*, *Adv. Math.* **197**, no. 2 (2005), 321–410.
- [5] A. Abbondandolo, P. Majer, *A Morse complex for Lorentzian geodesics*, preprint 2006, math.DG/0605261.
- [6] J. Avron, R. Seiler, B. Simon, *The index of a pair of projections*, *J. Funct. Anal.* **120**, no. 1 (1994), 220–237.
- [7] L. Biliotti, F. Mercuri, P. Piccione, *On a Gromoll-Meyer type Theorem in Compact Stationary Lorentzian Manifolds*, *Comm. Anal. Geom.* **16** no. 2 (2008), 333–393.
- [8] D. Bleecker, B. Booss-Bavnbek, *Spectral Invariants of Operators of Dirac Type on Partitioned Manifolds*, *Aspects of boundary problems in analysis and geometry*, *Oper. Theory Adv. Appl.* **151** (2004), 1–130.
- [9] B. Booss-Bavnbek, K. Furutani, *The Maslov Index: a Functional Analytical Definition and the Spectral Flow Formula*, *Tokyo J. Math.* **21** (1998), 1–34.
- [10] B. Booss-Bavnbek, K. Furutani, N. Otsuki, *Criss-cross reduction of the Maslov index and a proof of the Yoshida-Nicolaescu theorem*, *Tokyo J. Math.* **24**, no. 1 (2001), 113–128.
- [11] B. Booss-Bavnbek, M. Lesch, J. Phillips, *Spectral flow of paths of self-adjoint Fredholm operators*, *Quantum gravity and spectral geometry (Napoli, 2001)*. *Nuclear Phys. B Proc. Suppl.* **104** (2002), 177–180.
- [12] B. Booss-Bavnbek, K. Wojciechowski, *Desuspension of Splitting Elliptic Symbols II*, *Ann. Glob. Analysis and Geometry* **4**, No. 3 (1986), 349–400.
- [13] B. Booss-Bavnbek, K. P. Wojciechowski, *Elliptic Boundary Value Problems for Dirac Operators*, Birkhäuser, Boston, 1993.
- [14] S. Cappell, R. Lee, E. Miller, *Self-Adjoint Elliptic Operators and Manifold Decompositions Part III: Determinant Line Bundles and Lagrangian Intersection*, *Comm. Pure Appl. Math.* **52** (1999), 543–611.
- [15] X. Dai, W. Zhang, *Higher spectral flow*, *J. Funct. Anal.* **157** (1998), 432–469.
- [16] P. M. Fitzpatrick, J. Pejsachowicz, L. Recht, *Spectral Flow and Bifurcation of Strongly Indefinite Functionals Part I. General Theory*, *J. Funct. Anal.* **162** (1) (1999), 52–95.
- [17] R. Giambò, P. Piccione, A. Portaluri, *Computation of the Maslov index and the spectral flow via partial signatures*, *C. R. Math. Acad. Sci. Paris* **338**, No. 5 (2004), 397–402.
- [18] F. Giannoni, A. Masiello, P. Piccione, and D. V. Tausk, *A generalized index theorem for Morse-Sturm systems and applications to semi-Riemannian geometry*, *Asian J. Math.* **5** (2001), 441–472.

- [19] T. Kato, *Perturbation theory for linear operators*, Grundlehren der Mathematischen Wissenschaften, vol. 132, Springer-Verlag, Berlin 1980.
- [20] N. H. Kuiper, *The homotopy type of the unitary group of Hilbert space*, *Topology* **3** (1965), 19–30.
- [21] M. Morse, *Calculus of Variations in the Large*, Am. Math. Soc., *Coll. Publ.* 1934.
- [22] L. I. Nicolaescu, *The Maslov Index, the Spectral Flow, and Decomposition of Manifolds*, *Duke Math. J.* **80**, No. 2(1995), 485–533.
- [23] J. Phillips, *Self-adjoint Fredholm Operators and Spectral Flow*, *Canad. Math. Bull.* **39** (4) (1996), 460–467.
- [24] P. Piccione, D. V. Tausk, *The Morse Index Theorem in semi-Riemannian Geometry*, *Topology* **41**, no. 6 (2002), 1123–1159.
- [25] T. Yoshida, *Floer Homology and Splittings of Manifolds*, *Ann. Math.* **134** (1991), 227–323.
- [26] C. Zhu, *A generalized Morse index theorem*, in B. Booss–Bavnbek et al. (eds), *Analysis, geometry and topology of elliptic operators*, World Sci. Publ., Hackensack, NJ, 2006, 493–540.