

The Invariance of Domain for C^1 Fredholm maps of index zero

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Dedicated to Alfonso Vignoli on the occasion of his 60th anniversary

Summary: We give a version of the classical Invariance of Domain Theorem for nonlinear Fredholm maps of index zero between Banach spaces (and Banach manifolds). The proof is based on a finite dimensional reduction technique combined with a mod 2 degree argument for continuous maps between (finite dimensional) differentiable manifolds.

Keywords: Domain invariance, nonlinear Fredholm maps, mod 2 degree.

Classification: 58C25, 47H11.

1 Introduction

Is the one-to-one continuous image of an open set in \mathbf{R}^n still open? This problem is the so called *invariance of domain* and, as H. Freudenthal pointed out in his essay on the history of topology [6] (see also [12]), it has been considered of a great importance at the beginning of this (20th) century “for it was essential for the justification of the Poincaré continuation method in the theory of automorphic functions and the uniformization (of analytic functions).”

As it is well-known, a positive answer to the problem has been given by Brouwer in 1912 ([3]). This result is usually obtained as a consequence of the Odd Mapping Theorem, as it is shown, for instance, in [5]. A simpler proof can be done by using the Jordan Separation Theorem (see, e.g., [7] or [10]).

The extension of the Domain Invariance Theorem to Banach spaces is due to Schauder [9], who proved that if Ω is a bounded open subset of a Banach space E and $f : \overline{\Omega} \rightarrow E$ is a one-to-one map of the form $f = I - h$ with I the identity of E and $h : \overline{\Omega} \rightarrow E$ compact, then $f(\Omega)$ is open. Again this result can be deduced by the infinite dimensional version of the Odd Mapping

Theorem. A proof displaying this connection can be found, for instance, in [2], [5] or [7].

The result of Schauder is, clearly, a nonlinear counterpart of the well-known Fredholm alternative.

In this paper, we present a version of the Invariance of Domain Theorem for nonlinear Fredholm maps of index zero between Banach spaces (or, more generally, Banach manifolds). To prove our result, we use a finite dimensional reduction method for Fredholm maps that goes back to Caccioppoli (see [4]). In his paper, Caccioppoli defines a mod 2 degree for such type of maps, in order to give an answer to this question:

When does a point y_0 belong to the interior of the image of a Fredholm map of index zero?

His answer is the following:

Let E and F be Banach spaces, $f : \Omega \rightarrow F$ a Fredholm map of index zero defined on an open subset Ω of E , and $y_0 \in f(\Omega)$. If the mod 2 degree of f in Ω at y_0 is nonzero, then $f(\Omega)$ is a neighborhood of y_0 .

However, it should be observed that Caccioppoli does not provide sufficient conditions for the degree to be different from zero. It is a matter of definition that, if f is one-to-one and y is a regular value of f , belonging to $f(\Omega)$ and sufficiently close to y_0 , then the mod 2 degree of f in Ω at y_0 is equal to 1. On the other hand, it seems to be a nontrivial fact to prove the existence of a regular value belonging to the image of f , unless one preliminarily shows that such an image has an interior point. Actually, this is our aim here.

2 Preliminaries

Let E and F be two real Banach spaces. We recall that a bounded linear operator is said to be *Fredholm* if both $\text{Ker } L$ and $\text{coKer } L$ have finite dimension. In this case its *index* is the integer

$$\text{ind } L = \dim \text{Ker } L - \dim \text{coKer } L.$$

A map $f : M \rightarrow N$ between real Banach manifolds is Fredholm of index zero (see [11]) if it is C^1 and its Fréchet derivative $f'(x)$, from the tangent space $T_x M$ of M at x to the tangent space $T_{f(x)} N$ of N at $f(x)$, is Fredholm of index zero for any $x \in M$.

A map $f : M \rightarrow N$ between manifolds is said to be *proper* if $f^{-1}(K)$ is compact for any compact subset K of N . In particular, let us recall that Fredholm maps are locally proper (see [11]).

The proof of the Invariance of Domain Theorem we present here is based on a notion of mod 2 degree for continuous maps between (finite dimensional) differentiable manifolds.

A classical version of such a degree can be found, for example, in the textbook of Milnor [8], where the author takes into account continuous maps between compact manifolds.

The notion of degree we use in this paper is a straightforward extension of the mod 2 degree in [8] and it is defined in the following context.

Let $f : M \rightarrow N$ be a continuous map between two differentiable boundaryless manifolds of the same dimension. Given an open subset U of M and a point $y \in N$, the triple (f, U, y) is called *admissible for the mod 2 degree*, if $f^{-1}(y) \cap U$ is compact.

The mod 2 degree, denoted by \deg_2 , is a function defined in the class of all admissible triples, taking values in \mathbf{Z}_2 , and verifying the following properties:

i) (*Normalization*) If f is a homeomorphism of M onto N , then

$$\deg_2(f, M, y) = 1,$$

for all $y \in N$.

ii) (*Additivity*) If (f, M, y) is an admissible triple and U_1, U_2 are two open disjoint subsets of M such that $f^{-1}(y) \subset U_1 \cup U_2$, then

$$\deg_2(f, M, y) = \deg_2(f, U_1, y) + \deg_2(f, U_2, y).$$

iii) (*Homotopy invariance*) Let $H : M \times [0, 1] \rightarrow N$ be a continuous homotopy. Then, given any (continuous) path $y : [0, 1] \rightarrow N$, such that the set $\{(x, t) \in M \times [0, 1] : H(x, t) = y(t)\}$ is compact, $\deg_2(H(\cdot, t), M, y(t))$ does not depend on t .

iv) (*Topological invariance*) Let (f, M, y) be admissible. Given two differentiable manifolds W and Z with two homeomorphisms $\phi : M \rightarrow W$ and $\psi : N \rightarrow Z$, one has

$$\deg_2(f, M, y) = \deg_2(\psi \circ f \circ \phi^{-1}, W, \psi(y)).$$

The following consequences of the additivity property will be used in the proof of our main result.

v) (*Excision*) If (f, M, y) is admissible and U is an open neighborhood of $f^{-1}(y)$, then

$$\deg_2(f, M, y) = \deg_2(f, U, y).$$

vi) (*Existence*) If (f, M, y) is admissible and $\deg_2(f, M, y) = 1$, then the equation $f(x) = y$ admits a solution in M .

3 The invariance of domain

The following is our version of the invariance of domain for Fredholm maps.

Theorem 1. *Let M and N be two real Banach manifolds and $f : M \rightarrow N$ be an injective Fredholm map of index zero. Then $f(M)$ is open in N .*

Proof. The above statement is clearly equivalent to the following. If E and F are real Banach spaces, U is an open subset of E and $f : U \rightarrow F$ is an injective Fredholm map of index zero, then $f(U)$ is open. Thus, we will prove our result directly in the context of Banach spaces.

Take $y_0 \in f(U)$ and denote $x_0 = f^{-1}(y_0)$. There exists an open neighborhood V of x_0 in U such that $\bar{V} \subset U$ and f is proper on \bar{V} (recall that Fredholm maps are locally proper). Let F_0 be any finite dimensional subspace of F through y_0 such that $F_0 + \text{Range} f'(x_0) = F$, i.e. f is transverse to F_0 at x_0 . Without loss of generality, we may assume that f is transverse to F_0 at any $x \in V$ (see e.g. [1]). Since $f(\partial V)$ is closed in F and does not contain y_0 , there exists an open ball B centered at y_0 such that $\bar{B} \cap f(\partial V) = \emptyset$. We will show that B is contained in $f(U)$. Let $y \in B$ and denote $Y = \text{span}(F_0 \cup \{y\})$. The map f is still transverse to the subspace Y at any $x \in V$ and, consequently, the set $f^{-1}(Y) \cap V$ is a submanifold of U (containing x_0) of the same dimension as Y (again see e.g. [1]). Denote $W = f^{-1}(Y) \cap V \cap f^{-1}(B)$ and observe that the restriction $f|_W : W \rightarrow Y$ is a continuous (actually C^1) map between two manifolds of the same dimension with $(f|_W)^{-1}(y_0)$ compact, since it reduces to $\{x_0\}$. Therefore, as pointed out in the preliminaries, the mod 2 degree, $\deg_2(f|_W, W, y_0)$, is well defined. Let us show that $\deg_2(f|_W, W, y_0) = 1$. By excision, to compute this degree it suffices to restrict f to a coordinate neighborhood W_1 of x_0 in W . Hence, using local charts, by the topological invariance, one can replace the triple

$(f|_{W_1}, W_1, y_0)$ with a triple (f_1, Ω, η) , where Ω is a bounded open subset of \mathbf{R}^n , $f_1 : \Omega \rightarrow \mathbf{R}^n$ is injective, and $\eta \in f_1(\Omega)$. In this situation, as it is well-known (see e.g. [7]), the oriented Brouwer degree is ± 1 . Consequently, the mod 2 degree, $\deg_2(f_1, \Omega, \eta)$, is equal to 1 and, thus, $\deg_2(f|_W, W, y_0) = 1$ as well, as claimed.

Now, since B is a ball containing y_0 and y , the line segment Γ joining y_0 and y is contained in $B \cap Y$. Moreover, $(f|_W)^{-1}(\Gamma)$ is a compact subset of W (observe, in fact, that $\Gamma \cap f(\partial W) = \emptyset$). Thus, by the homotopy invariance of the degree, $\deg_2(f|_W, W, (1-t)y_0 + ty)$ is well defined and independent of t . Consequently, $\deg_2(f|_W, W, y) = 1$, which implies, by the existence property, that $(f|_W)^{-1}(y)$ is nonempty. \square

Obviously, as a consequence of Theorem 1, we get the following nonlinear Fredholm alternative:

If $f : M \rightarrow N$ is Fredholm of index zero, $x_0 \in M$, and the equation $f(x) = y$ has at most one solution (close to x_0) for any y in a convenient neighborhood of $y_0 = f(x_0)$, then $f(x) = y$ is (uniquely) solvable for y sufficiently close to y_0 .

An easy improvement of our previous result is the following.

Theorem 2. *Let M be a real Banach manifold, F a real Banach space, $f : M \rightarrow F$ the sum of two maps, g and h , where g is Fredholm of index zero and h is continuous with $h(M)$ in a finite dimensional space. Assume that f is injective. Then $f(M)$ is open in F .*

Sketch of the proof. The proof is in the outline of that of Theorem 1, with only minor differences. More precisely, in the same notation as above, choose F_0 containing $\{y_0\} \cup h(M)$ and such that $F_0 + \text{Range } g'(x_0) = F$ (observe that, F_0 being a vector space, $g(x_0) \in F_0$). In addition, consider the manifold $g^{-1}(Y) \cap V$ and take $W = g^{-1}(Y) \cap V \cap (f)^{-1}(B)$. \square

We close the paper with a question whose answer we believe is affirmative, but, as far as we know, unknown.

Does the above domain invariance result hold if the perturbation h of g , instead of having finite dimensional image, is assumed to be only locally compact?

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