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ON THE UNIQUENESS OF THE DEGREE FOR NONLINEAR FREDHOLM MAPS OF INDEX ZERO BETWEEN BANACH MANIFOLDS

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ABSTRACT. In some previous papers we presented a fairly simple construction of a topological degree for C^1 Fredholm maps of index zero between Banach manifolds which verifies the three fundamental properties of the classical degree theory: normalization, additivity and homotopy invariance. We show here that this degree is unique. Precisely, by an axiomatic approach similar to the one due to Amann-Weiss, we prove that there exists at most one real function satisfying the above properties, and this function must be integer valued.

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1. Introduction

In [2] and [3] we developed a degree theory for a class of C^1 Fredholm maps of index zero between real Banach manifolds. By our construction we extended and simplified the Elworthy-Tromba approach to the degree theory avoiding the concept of Fredholm structure and any related notion of orientation on the source and target manifolds (see [6] and [7]).

To this purpose we introduced a concept of orientability for Fredholm maps of index zero between Banach manifolds. This notion does not coincide with that given by Fitzpatrick, Pejsachowicz and Rabier (see [8] and references therein), it is stable (in the sense that any map "sufficiently close" to an orientable or nonorientable map

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inherits the same property), and not based on the Leray-Schauder degree. Moreover, in the finite dimensional case, it turns out to be equivalent to the concept of orientability for maps between not necessarily orientable manifolds introduced, with completely different methods, by Dold in [5]. In particular, when $f : M \to N$ is a map acting between finite dimensional orientable manifolds of the same dimension, an orientation of f (in our sense) can be regarded as a pair of orientations of M and N, up to an inversion of both of them.

Our notion of orientability is based on an elementary, purely algebraic, definition of orientation for an algebraic Fredholm linear operator of index zero $L : E \to F$ acting between real vector spaces (no additional structure is needed). When the vector spaces E and F are actually Banach and the operator L is bounded, an orientation of L induces, by a sort of continuity, an orientation on any operator L' sufficiently close to L (in the operator norm). Thus, roughly speaking, an oriented map from an open subset Ω of E into F is a nonlinear Fredholm map of index zero $f : \Omega \to F$ together with a function α which assigns, in a continuous way, an orientation $\alpha(x)$ of the Fréchet derivative $Df(x)$ of f at any $x \in \Omega$. This notion of oriented map between real Banach spaces is easily extended to the context of real Banach manifolds.

Concerning our notion of degree, consider two Banach manifolds M and N and let $f : M \to N$ be an oriented Fredholm map of index zero. Given an open subset U of M and an element $y \in N$, the triple (f, U, y) is said to be *admissible* if $f^{-1}(y) \cap U$ is compact. Our degree is defined as a map from the class of all admissible triples into $\mathbb{Z}$ such that the classical properties of degree theory are verified.

The most significant properties of the degree (and the related concept of orientation) are proved in [2, 3]. The purpose of this paper is to investigate the problem of the uniqueness of the degree, that is, the problem to determine which properties, thought as axioms, ensure that there exists a unique map verifying those properties.

In their celebrated paper [1] of 1973 Amann and Weiss showed that both the Brouwer degree and the Leray-Schauder degree are uniquely determined by three properties, namely *Normalization*, *Additivity* and *Homotopy invariance*, which they considered as axioms. As regards the sole finite dimensional case, the uniqueness of the Brouwer degree has been previously established by Führer (see [9] and [10]).

In this paper we obtain an analogous result concerning our degree. Namely, we prove that there exists at most one real valued map, defined in the class of the admissible triples, which verifies a particular Normalization property (stated for oriented diffeomorphisms), with the more classical Additivity and Homotopy invariance properties.

2. The concept of determinant in infinite dimension

Consider a real vector space E and denote by $\Psi(E)$ the set of endomorphisms of E of the form $I - K$, where K has finite dimensional image. It is known (see e.g. [11]) that a notion of determinant is well defined for the operators of $\Psi(E)$. Precisely, let $T = I - K \in \Psi(E)$ be given. If E_0 is any nontrivial finite dimensional subspace of E containing the image of K, the determinant of the restriction of T to E_0 is well defined. It is easy to verify that this value does not depend on E_0. Thus, the *determinant* $\det T$ of T is defined as the determinant of the restriction of T to any nontrivial finite dimensional subspace of E containing the image of K.

This notion of determinant verifies the following property, which generalized the analogous well known result in finite dimension. For the details see [4, Proposition 3.1].

Lemma 2.1. *Let E and F be two real vector spaces. If $S : F \to E$ is an isomorphism and $T \in \Psi(E)$, then $S^{-1}TS \in \Psi(F)$ and $\det(S^{-1}TS) = \det(T)$.*

3. Orientability for Fredholm maps

The section is devoted to a summary of the notion of orientability for nonlinear Fredholm maps of index zero between Banach manifolds, introduced in [2, 3].

As a first step we consider two real vector spaces E and F and we give a definition of orientation for a linear Fredholm operator $L : E \to F$ (at this level no topological structure is needed). Let us recall that L is said to be (*algebraic*) *Fredholm* if $\operatorname{Ker} L$ and $\operatorname{coKer} L = F/\operatorname{Im} L$ are finite dimensional. The *index* of L is the integer

$$\operatorname{ind} L = \dim \operatorname{Ker} L - \dim \operatorname{coKer} L.$$

Given a Fredholm operator of index zero L, we call *corrector* of L a linear operator $A : E \to F$ such that

i) $\operatorname{Im} A$ has finite dimension,

ii) $L + A$ is an isomorphism.

It is easy to check that the set $\mathcal{C}(L)$ of correctors of L is nonempty. We define in $\mathcal{C}(L)$ the following equivalence relation. Given $A, B \in \mathcal{C}(L)$, consider the automorphism

$$T = (L + B)^{-1}(L + A) = I - (L + B)^{-1}(B - A)$$

of E. Clearly, $K := (L + B)^{-1}(B - A)$ has finite dimensional image. Therefore, the determinant of T is well defined and nonzero. We say that A is *equivalent* to B or, more precisely, A is *L-equivalent* to B, if

$$\det\big((L + B)^{-1}(L + A)\big) > 0.$$

In [2] it is shown that this is actually an equivalence relation on $\mathcal{C}(L)$ with two equivalence classes. This relation provides a concept of orientation of L.

Definition 3.1. *Let L be a linear Fredholm operator of index zero between real vector spaces. Each one of the two classes of $\mathcal{C}(L)$ is an* orientation *of L, and L is* oriented *when one of them is chosen. Any of the two orientations of L is called* opposite *to the other. If L is oriented, the elements of its orientation are called the* positive correctors *of L.*

The following notions of natural and unnatural orientations of an isomorphism will be often mentioned throughout the paper.

Definition 3.2. *An oriented isomorphism L is said to be* naturally oriented *if the trivial operator is a positive corrector, and we will refer to this orientation as the* natural orientation *of L. Conversely, L is* unnaturally oriented *if the trivial operator is not a positive corrector; in this case L assumes the* unnatural orientation. *The* sign *of an oriented isomorphism L is defined as* $\operatorname{sign} L = 1$ *if L is naturally oriented and* $\operatorname{sign} L = -1$ *otherwise.*

From now on, the real vector spaces E and F will have the additional structure of Banach spaces. Any Fredholm operator between Banach spaces will be assumed to be bounded. Moreover, $L(E,F)$ will denote the Banach space of bounded linear operators from E into F and $\Phi_0(E,F)$ will be the open subset of $L(E,F)$ of the Fredholm operators of index zero. Given $L \in \Phi_0(E,F)$, the symbol $\mathcal{C}(L)$ now denotes, with a slight abuse of notation, the set of bounded correctors of L, which is still nonempty. Of course, the definition of orientation of $L \in \Phi_0(E,F)$ can be given as the choice of one of the two equivalence classes of bounded correctors of L, according to the above equivalence relation.

In the context of Banach spaces an orientation of a bounded linear Fredholm operator of index zero induces an orientation to any sufficiently close operator. Precisely, consider $L \in \Phi_0(E,F)$ and a corrector A of L. Suppose that L is oriented with A positive corrector. Since the set of the isomorphisms of E into F is open in $L(E,F)$, then A is a corrector of every T in a suitable neighborhood W of L in $\Phi_0(E,F)$. Thus, any $T \in W$ can be oriented by taking A as a positive corrector. This fact leads us to the following definition.

Definition 3.3. *Let X be a topological space and $h : X \to \Phi_0(E,F)$ a continuous map. An* orientation *of h is a continuous choice of an orientation $\alpha(x)$ of $h(x)$ for each $x \in X$, where 'continuous' means that for any $x \in X$ there exists $A \in \alpha(x)$ which is a positive corrector of $h(x')$ for any x' in a neighborhood of x. A map is* orientable *when it admits an orientation and* oriented *when an orientation is chosen.*

The properties of this notion of orientation are discussed in [2, 3]. Here we recall just those results which will be used in the sequel.

Proposition 3.4. *An orientable map $h\colon X \to \Phi_0(E,F)$ admits at least two orientations. If, in particular, X is connected, then h admits exactly two orientations. In addition, if X is simply connected and locally path connected, then h is orientable.*

Remark 3.5. *Referring to the above proposition, let α be an orientation of h and, for any $x \in X$, assign to $h(x)$ the opposite orientation $\overline{\alpha}(x)$ of $\alpha(x)$. We obtain in this way an orientation of h, which we call the* opposite orientation *of α. This explains why h admits at least two orientations.*

Notice that the set in which two orientations α and β of h coincide is open. For this reason the set where β equals the opposite orientation $\overline{\alpha}$ of α is open. Thus it turns out to be open the set in which α and β do not coincide. Therefore, if X is connected, h admits exactly two orientations, one the opposite of the other.

In the case when $h(x)$ is an isomorphism for all $x \in X$, with X not necessarily connected, it is easy to see, by Definition 3.3, that h is orientable and admits (at least) the two orientations, say α and β, such that, for every $x \in X$, $\alpha(x)$ is the natural orientation of $h(x)$ and $\beta(x)$ is the unnatural one. We will call these orientations of h the natural *and* unnatural orientation*, respectively.*

Definition 3.3 allows us to give a notion of orientability for Fredholm maps of index zero between Banach spaces. Recall that, given an open subset Ω of E, a C^1 map $g : \Omega \to F$ is *Fredholm of index n* if its Fréchet derivative, $Dg(x)$, is a Fredholm operator of index n for all $x \in \Omega$.

Definition 3.6. *An* orientation *of a Fredholm map of index zero $g : \Omega \to F$ is an orientation of the derivative $Dg : \Omega \to \Phi_0(E,F)$, and g is* orientable*, or* oriented*, if so is Dg according to Definition 3.3.*

The above definition can be extended to the context of real Banach manifolds. Recall that, given two real Banach manifolds M and N, a C^1 map $f : M \to N$ is *Fredholm of index n* if its Fréchet derivative, $Df(x) : T_xM \to T_{f(x)}N$, is Fredholm of index n for any $x \in M$. Actually, if M and N are such that there exists a Fredholm map of index zero $f : M \to N$, then the two manifolds can be modeled on the same Banach space. Thus, we will proceed by assuming that M and N are two real Banach manifolds, modeled on a Banach space E.

Definition 3.7. *Let $f : M \to N$ be a Fredholm map of index zero. An* orientation *α of f is a continuous choice of an orientation $\alpha(x)$ of $Df(x)$ for any $x \in M$; where 'continuous' means that, given a selection of positive correctors $\{A_x \in \alpha(x)\}_{x\in M}$, and* *two charts, $\phi : U \to E$ of M and $\psi : V \to E$ of N, with $f(U) \subseteq V$, the family of linear operators*

$$\left\{ D\psi(f(\phi^{-1}(z)))\, A_{\phi^{-1}(z)}\, D\phi^{-1}(z) \right\}_{z \in \phi(U)}$$

defines an orientation of the composite map $\psi f \phi^{-1} : \phi(U) \to E$. The map f is orientable *if admits an orientation and* oriented *when an orientation is chosen.*

The next property will play a role in the proof of Lemma 5.5.

Remark 3.8. *Let $f : M \to N$ be an oriented map and call α its orientation. Consider two charts, $\phi : U \to E$ of M and $\psi : V \to E$ of N, with $f(U) \subseteq V$ and U connected. Given $u_0 \in U$, let A_{u_0} belong to $\alpha(u_0)$. Without loss of generality suppose that, for any u in U, the operator A_u, defined as*

$$A_u := D\psi^{-1}(\psi(f(u))) \Big(D\psi(f(u_0))\, A_{u_0}\, D\phi^{-1}(\phi(u_0)) \Big) D\phi(u),$$

is a corrector of $Df(u)$. Then, by Definitions 3.3 and 3.7, it is easy to see that A_u is actually a positive corrector of $Df(u)$, that is, belongs to $\alpha(u)$. Roughly speaking, this property asserts that, since there is a sort of continuous dependence on u of A_u, then, if u is close to u_0, A_u is a positive corrector of $Df(u)$.

The following result is the analogue for Fredholm maps of Proposition 3.4 (see [2, 3]).

Proposition 3.9. *An orientable map $f\colon M \to N$ admits at least two orientations. If, in particular, M is connected, then f admits exactly two orientations. In addition, if M is simply connected, then f is orientable.*

Remark 3.10. *Given $f\colon M \to N$, if $Df(x)$ is invertible for every $x \in M$, then f can be oriented in such a way that $Df(x)$ is naturally oriented for every $x \in M$. We call this orientation the* natural orientation *of f. Analogously, if $Df(x)$ is unnaturally oriented for every $x \in M$, we say that f has the* unnatural orientation *(see also Remark 3.5).*

Definition 3.11. *Let $H : M \times [0,1] \to N$ be a C^1 map. We call H a* Fredholm homotopy *if it is Fredholm of index 1 or, equivalently, if any partial map $H_t : x \mapsto H(x,t)$ (defined on M) is Fredholm of index zero. An* orientation *α of H is a continuous choice of an orientation $\alpha(x,t)$ of $DH_t(x)$ for any $(x,t) \in M \times [0,1]$; where 'continuous' means that given any two charts, $\phi : U \to E$ of M and $\psi : V \to E$ of N, with $H(U \times [0,1]) \subseteq V$, the following map, from $\phi(U) \times [0,1]$ into $\Phi_0(E)$, defined by*

$$(z,t) \mapsto D\psi(H(\phi^{-1}(z),t))\, DH_t(\phi^{-1}(z))\, D\phi^{-1}(z),$$

can be oriented (according to Definition 3.3) choosing as positive correctors the operators

$$D\psi(H(\phi^{-1}(z),t))\, A\, D\phi^{-1}(z),$$

with $A \in \alpha(\phi^{-1}(z), t)$ and $(z,t) \in \phi(U) \times [0,1]$. The map H is orientable *if it admits an orientation and* oriented *when an orientation is chosen.*

Given an oriented Fredholm homotopy $H : M \times [0,1] \to N$, it is immediate to observe that any partial map $H_t : M \to N$ is oriented. Conversely, we have the following result (see [2, 3]).

Proposition 3.12. *Given a Fredholm homotopy $H : M \times [0,1] \to N$, suppose that $H_{\bar{t}}$ is orientable for a given $\bar{t} \in [0,1]$. Then H is orientable. In addition, assume that $H_{\bar{t}}$ is oriented and call α its orientation. Then there exists a unique orientation of H, say β, such that $\beta(x,\bar{t}) = \alpha(x)$ for any $x \in M$.*

4. Degree for oriented maps

In this section we give a summary of the construction, given in [2, 3], of the degree for oriented Fredholm maps of index zero between Banach manifolds.

Definition 4.1. *Let $f : M \to N$ be an oriented Fredholm map of index zero. Given an open subset U of M and an element $y \in N$, the triple (f, U, y) is said to be* admissible *if $f^{-1}(y) \cap U$ is compact.*

The degree is defined as a map which to every admissible triple (f, U, y) assigns an integer, $\deg(f, U, y)$, in such a way that the following three *fundamental properties* hold:

i) (*Normalization*) Let $f : M \to N$ be a diffeomorphism onto an open subset of N. If f is naturally oriented, then

$$\deg(f, M, y) = 1, \quad \forall y \in f(M).$$

ii) (*Additivity*) Given an admissible triple (f, U, y) and two disjoint open subsets U_1, U_2 of U such that $f^{-1}(y) \cap U \subseteq U_1 \cup U_2$, then,

$$\deg(f, U, y) = \deg(f|_{U_1}, U_1, y) + \deg(f|_{U_2}, U_2, y).$$

iii) (*Homotopy invariance*) Let $H : M \times [0,1] \to N$ be an oriented Fredholm homotopy. Let $y : [0,1] \to N$ be a continuous path. If the set

$$\{(x,t) \in M \times [0,1] : H(x,t) = y(t)\}$$

is compact, then $\deg(H_t, M, y(t))$ does not depend on $t \in [0,1]$.

The degree is first defined in the special case when (f, U, y) is a *regular triple*, that is, when (f, U, y) is admissible and y is a regular value for f in U. This implies that $f^{-1}(y) \cap U$ is a finite set. In this case we define

$$\deg(f, U, y) = \sum_{x \in f^{-1}(y) \cap U} \operatorname{sign} Df(x), \tag{4.1}$$

where, recalling Definition 3.2, $\operatorname{sign} Df(x) = 1$ if $Df(x) : T_xM \to T_yN$ is naturally oriented, and $\operatorname{sign} Df(x) = -1$ otherwise.

Now, given any admissible triple (f, U, y), let us recall that, as a byproduct of Sard–Smale Lemma [12], we know that the set of regular values of f is dense in N. Thus, using also the fact that Fredholm maps are locally proper, we prove (see [2, Lemma 3.2]) that, given any admissible triple (f, U, y), if U_1 and U_2 are sufficiently small open neighborhoods of $f^{-1}(y) \cap U$, and $y_1, y_2 \in N$ are two regular values for f, sufficiently close to y, then

$$\deg(f, U_1, y_1) = \deg(f, U_2, y_2).$$

This property implies that the following definition of degree for general admissible triples is well posed.

Definition 4.2. *Let (f, U, y) be admissible and let W be any open neighborhood of $f^{-1}(y) \cap U$ such that $\overline{W} \subseteq U$ and f is proper on $\overline{W}$. Let V be an open connected neighborhood of y in N which is disjoint from $f(\partial W)$. Define*

$$\deg(f, U, y) = \deg(f, W, z),$$

where z is any regular value for $f|_W$ belonging to V.

5. Uniqueness of the degree

In this section we prove the main result of the paper, i.e. that there exists at most one real map, defined in the class of all admissible triples, which verifies the three fundamental properties: Normalization, Additivity and Homotopy invariance. Thus, such a map turns out to be integer valued and necessarily coincides with the degree for oriented Fredholm maps between real Banach manifolds, whose definition has been recalled in the above section.

We proceed as follows. Let $\mathbb{T}$ be the family of all admissible triples and call $d : \mathbb{T} \to \mathbb{R}$ a map which verifies the three fundamental properties. We prove first that, if $f : M \to N$ is an unnaturally oriented diffeomorphism into N, then

$$d(f, M, y) = -1, \quad \forall y \in f(M). \tag{5.1}$$

Therefore, as a consequence of the above equality and the first two fundamental properties, we show that, for every regular triple (f, U, y), one has

$$d(f, U, y) = \sum_{x \in f^{-1}(y) \cap U} \operatorname{sign} Df(x). \tag{5.2}$$

The above formula ensures the uniqueness of d on the subfamily of $\mathcal{T}$ of regular triples. Moreover, by the Homotopy invariance property and by using, as a crucial tool, the local properness of nonlinear Fredholm maps, we prove the uniqueness of

d. Finally, since the function deg verifies the three fundamental properties, one has $d = \deg$.

To help the reader we divide the section in four steps.

Step 1. This is a preliminary part in which we show some properties of d following from the Additivity.

Given any oriented Fredholm map $f : M \to N$, the triple $(f, \emptyset, y)$ is admissible for all $y \in N$, being the empty set compact. Therefore, by the Additivity property, we get

$$d(f, \emptyset, y) = d(f|_\emptyset, \emptyset, y) + d(f|_\emptyset, \emptyset, y),$$

and

$$d(f|_\emptyset, \emptyset, y) = d(f|_\emptyset, \emptyset, y) + d(f|_\emptyset, \emptyset, y)$$

Hence, one has

$$d(f, \emptyset, y) = d(f|_\emptyset, \emptyset, y) = 0.$$

By the above equality and the Additivity we obtain the following (often neglected) Localization property.

Proposition 5.1 (*Localization*)**.** *Let (f, U, y) be an admissible triple. Then,*

$$d(f, U, y) = d(f|_U, U, y).$$

Proof. By the Additivity one has

$$d(f, U, y) = d(f|_U, U, y) + d(f|_\emptyset, \emptyset, y).$$

Then, the assertion follows being $d(f|_\emptyset, \emptyset, y) = 0$. □

Another consequence of the Additivity (and of the Localization) is the Excision property, which basically assert that $d(f, U, y)$ depends only on the behavior of f in any neighborhood of $f^{-1}(y) \cap U$.

Proposition 5.2 (*Excision*)**.** *If (f, U, y) is admissible and V is an open subset of U such that $f^{-1}(y) \cap U \subseteq V$, then (f, V, y) is admissible and*

$$d(f, U, y) = d(f, V, y).$$

Proof. The triple (f, V, y) is clearly admissible. From the Additivity and the fact that $d(f|_\emptyset, \emptyset, y) = 0$, it follows

$$d(f, U, y) = d(f|_V, V, y).$$

On the other hand, the Localization implies that

$$d(f, V, y) = d(f|_V, V, y).$$

Thus, the assertion follows. □

From the Excision we obtain the Existence property.

Proposition 5.3 (*Existence*)**.** *Let $d(f,U,y)$ be nonzero. Then, the equation $f(x)=y$ admits at least one solution in U.*

Proof. Assume that $f^{-1}(y)\cap U$ is empty. By the Excision property, taking $V=\emptyset$, we get
$$d(f,U,y)=d(f,\emptyset,y)=0,$$
which contradicts the assumption. □

Remark 5.4. *As an immediate consequence of the Additivity and the Localization properties it follows that, given an admissible triple (f,U,y) and two disjoint open subsets U_1, U_2 of U such that $f^{-1}(y)\cap U\subseteq U_1\cup U_2$, one has*
$$d(f,U,y)=d(f,U_1,y)+d(f,U_2,y). \tag{5.3}$$

The reader who is familiar with the degree theory probably observes that the above equality (5.3) *is the classical version of the Additivity property, which is usually mentioned in the literature. Actually, we believe not possible to prove the above Localization property by means of this classical version of the Additivity.*

Step 2. Let $f:M\to N$ be an unnaturally oriented diffeomorphism and $y\in f(M)$ be given. In this step we prove that $d(f,M,y)=-1$.

Let $\phi:U\to\widetilde{U}$ be a chart, where

i) U is an open subset of M containing $f^{-1}(y)$;
ii) $\widetilde{U}$ is an open subset of the Banach space E.

Up to an isomorphism, we can regard E as
$$E=\mathbb{R}\times E_2. \tag{5.4}$$

Without loss of generality, assume that
$$\widetilde{U}=(-1,1)\times\widetilde{V},$$
where $\widetilde{V}$ is an open ball in E_2 centered at zero. Up to a diffeomorphism between $\widetilde{U}$ and another open subset of E, we can assume, without loss of generality, that $\phi(f^{-1}(y))\in(1/2,1)\times\widetilde{V}$.

Let $\gamma:(-1,1)\to[0,1)$ be a map verifying the following assumptions:

i) γ is C^1 and surjective;
ii) $\gamma(t)=\gamma(-t)$, for any t;
iii) $\gamma(t)=|t|$, for any $t\in(-1,-1/2]\cup[1/2,1)$;
iv) $\gamma'(t)\neq 0$, for any $t\neq 0$.

Clearly, condition iv) implies that γ is injective on $[0,1)$ and on $(-1,0]$, and, by condition ii), $\gamma'(0) = 0$. Consider the C^1 map $\Gamma : \widetilde{U} \to \widetilde{U}$, defined as $\Gamma(t,x) = (\gamma(t), x)$. Given any fixed $(\bar{t}, \overline{x}) \in \widetilde{U}$, the Fréchet derivative of Γ at $(\bar{t}, \overline{x})$ can be represented, with respect to the splitting (5.4), by the matrix

$$D\Gamma(\bar{t}, \overline{x}) = \begin{pmatrix} \gamma'(\bar{t}) & 0 \\ 0 & I_2 \end{pmatrix},$$

where I_2 stands for the identity of E_2. It is immediate to observe that $D\Gamma(\bar{t}, \overline{x})$ is a Fredholm operator of index zero, as sum of a Fredholm operator of index zero and a finite dimensional operator. Hence, Γ is a Fredholm map of index zero.

Define $g : U \to N$ by

$$g(u) = \left(f\phi^{-1}\Gamma\phi\right)(u).$$

Since g is the composition of Fredholm maps of index zero, then it is Fredholm of index zero (recall that the composition of two Fredholm maps of indices m and n is Fredholm of index $m + n$).

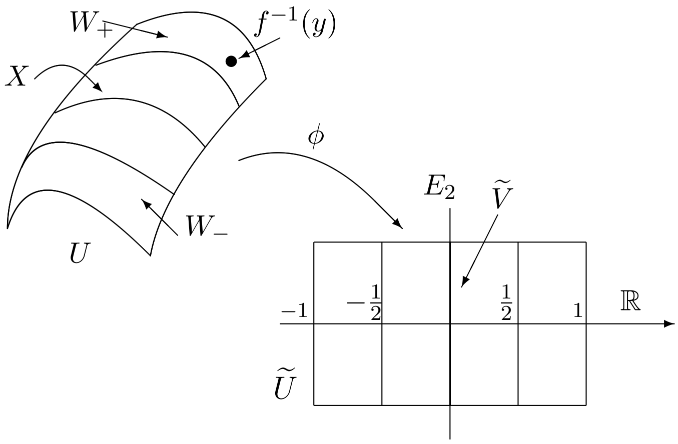

Figure 1. The two diffeomorphic sets U and $\widetilde{U}$.

Call X the submanifold of U given by $X = \phi^{-1}(\{0\} \times \widetilde{V})$. Call U_- and U_+ the open subsets of U given by

$$U_- = \phi^{-1}((-1,0) \times \widetilde{V}) \quad \text{and} \quad U_+ = \phi^{-1}((0,1) \times \widetilde{V}). \tag{5.5}$$

Since Γ is a diffeomorphism on $(-1,0) \times \widetilde{V}$ and on $(0,1) \times \widetilde{V}$, so are the restrictions of g to U_- and U_+.

By Proposition 3.9, g is orientable because its domain is simply connected, being diffeomorphic to $\widetilde{U}$. Since U is connected, again by Proposition 3.9, g admits exactly two orientations, which are uniquely determined by the choice of the orientation of Dg at a chosen point of U.

An useful property concerning the orientations of g is stated in the following lemma.

Lemma 5.5. *Consider any orientation β of g. Then β is the natural orientation of g on U_- if and only if it is the unnatural orientation on U_+.*

Proof. Let us start by introducing the linear operator $A_0 : E \to E$, defined, with respect to the decomposition (5.4), by the matrix

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\widetilde{U}$ is simply connected, then Γ is orientable. Let Γ be oriented in such a way that A_0 is a positive corrector of $D\Gamma(0,0)$ and call α this orientation. Let $\delta > 0$ be such that A_0 is still a positive corrector of $D\Gamma(t,0)$ for all $t \in (-\delta, \delta)$. For any $t \in (-\delta, \delta)$, $t \neq 0$, one has

$$\big(D\Gamma(t,0)\big)^{-1}\big(D\Gamma(t,0) + A_0\big) = \begin{pmatrix} 1 + \frac{1}{\gamma'(t)} & 0 \\ 0 & I_2 \end{pmatrix}.$$

The above composition is a finite dimensional perturbation of the identity of E. Hence, its determinant is well defined (see Section 2) and coincides with $1+\frac{1}{\gamma'(t)}$, which is negative for any negative t sufficiently close to zero, and positive for any positive t. Therefore, recalling the equivalence relation defined in Section 3, the trivial operator is a positive corrector of $D\Gamma(t,0)$ if $t \in (0,\delta)$, whereas it is not a positive corrector if t is negative and close to zero. In other words, $D\Gamma(t,0)$ is naturally oriented if $t \in (0,\delta)$ and unnaturally oriented if t is negative and close to zero. In addition, as the two restrictions of Γ to $(0,1) \times \widetilde{V}$ and $(-1,0) \times \widetilde{V}$ are diffeomorphisms, then $\alpha(t,x)$ is the natural orientation of $D\Gamma(t,x)$ for every $(t,x) \in (0,1) \times \widetilde{V}$, since we have proved that $\alpha(t,0)$ is the natural orientation of $D\Gamma(t,0)$ for some positive t. Analogously, $\alpha(t,x)$ is the unnatural orientation of $D\Gamma(t,x)$ for every $(t,x) \in (-1,0) \times \widetilde{V}$.

Denote $u_0 := \phi^{-1}(0,0)$ and $\psi := \phi f^{-1} : f(U) \to \widetilde{U}$, which is a diffeomorphism. For any $u \in U$, consider the linear operator $B_u : T_uM \to T_{g(u)}N$, defined as

$$B_u = D\psi^{-1}\big(\psi(g(u))\big)\, A_0\, D\phi(u) = D\psi^{-1}\big(\Gamma(\phi(u))\big)\, A_0\, D\phi(u).$$

This operator has clearly finite dimensional image. In addition, a straightforward computation shows that, for any u,

$$D\psi(g(u)) \Big(Dg(u) + B_u\Big)\, D\phi^{-1}(\phi(u)) = D\Gamma(\phi(u)) + A_0.$$

This implies that B_u is a corrector of $Dg(u)$ for u close to u_0.

Without loss of generality, we may suppose that B_{u_0} belongs to $\beta(u_0)$. By the property of the orientation recalled in Remark 3.8, one has that $B_u \in \beta(u)$ for any u

in a suitable neighborhood, say O, of u_0 in U. Fix $u \in O$. One has

$$\big(Dg(u)\big)^{-1}\big(Dg(u)+B_u\big) = \big(Dg(u)\big)^{-1}\Big(Dg(u)+B_u\Big)\, D\phi^{-1}(\phi(u))\, D\phi(u) =$$

$$\big(Dg(u)\big)^{-1}\Big[D(f\phi^{-1}\Gamma)(\phi(u)) + \big(D\psi^{-1}(\psi(g(u)))\, A_0\big)\Big]\, D\phi(u) =$$

$$\big(Dg(u)\big)^{-1} D\psi^{-1}(\psi(g(u)))\, D\psi(g(u))$$
$$\Big[D(f\phi^{-1}\Gamma)(\phi(u)) + (D\psi^{-1}(\psi(g(u)))\, A_0)\Big]\, D\phi(u) =$$

$$\big(Dg(u)\big)^{-1} D\psi^{-1}(\psi(g(u)))\, \Big[D\Gamma(\phi(u)) + A_0\Big]\, D\phi(u) =$$

$$\big(D\phi(u)\big)^{-1}\big(D\Gamma(\phi(u))\big)^{-1}\Big[D\Gamma(\phi(u)) + A_0\Big]\, D\phi(u).$$

The equality between the first and the last term of the above formula and Lemma 2.1 say that the determinant of $\big(Dg(u)\big)^{-1}\big(Dg(u)+B_u\big)$ coincides with that of $\big(D\Gamma(\phi(u))\big)^{-1}\big[D\Gamma(\phi(u))+A_0\big]$. Therefore, recalling how Γ is oriented, $\beta(u)$ turns out to be the natural orientation of $Dg(u)$ if $u \in O \cap U_+$, and the unnatural orientation of $Dg(u)$ if $u \in O \cap U_-$. As U_+ and U_- are connected, β is the natural orientation of g on U_+ and the unnatural one on U_-.

Analogously, one can prove that the opposite orientation $\overline{\beta}$ of β is the natural orientation of g on U_- and the unnatural one on U_+. This concludes the proof. □

Consider $W_+ = \phi^{-1}\big((1/2,1)\times\widetilde{V}\big)$ and observe that g coincides with f in W_+, since $\gamma(t) = t$ for every $t \in (1/2,1)$.

Let us choose the orientation of g that coincides in W_+ with the orientation of f. That is, g is unnaturally oriented in W_+. The Localization property implies that $d(g, W_+, y)$ and $d(f, W_+, y)$ coincide.

Let p_+ be the unique element in W_+ such that $g(p_+) = y$. Denote $(t_{p_+}, x_{p_+}) = \phi(p_+)$ and $p_- = \phi^{-1}(-t_{p_+}, x_{p_+})$. By the definition of γ, one immediately has that $g^{-1}(y) = \{p_-, p_+\}$.

Consider $W_- = \phi^{-1}\big((-1,-1/2)\times\widetilde{V}\big)$, which is clearly open in U, disjoint from W_+ and contains p_-. Applying Lemma 5.5, we have that $g|_{W_-}$ is a naturally oriented diffeomorphism. The triple (g, U, y) is admissible and, by the Additivity property,

$$d(g,U,y) = d(g|_{W_+}, W_+, y) + d(g|_{W_-}, W_-, y). \tag{5.6}$$

Let us show that the above value is zero. In fact, consider the homotopy

$$H : U \times [-1,1] \to N, \quad H(u,s) = g(u),$$

and the path

$$\sigma : [-1,1] \to N, \quad \sigma(s) = \big(f\phi^{-1}\big)\,(st_{p_+}, x_{p_+}).$$

The homotopy H is clearly a Fredholm homotopy, which we assume oriented with the orientation induced by that of g.

The set $S = \{(u, s) \in U \times [-1, 1] : H(u, s) = \sigma(s)\}$ is compact, since it coincides with $\phi^{-1}([0, t_{p+}] \times \{x_p\})$.

This argument allows us to apply the Homotopy invariance property. Thus, $d(g, U, \sigma(s))$ is well defined and independent of s. Since $g^{-1}(\sigma(-1))$ is empty and $\sigma(1) = y$, by the Existence property one has

$$d(g, U, \sigma(-1)) = 0.$$

Hence, $d(g, U, y) = 0$ and then, recalling formula (5.6),

$$d(g|_{W_+}, W_+, y) = -d(g|_{W_-}, W_-, y).$$

By the Normalization property we have $\deg(g|_{W_-}, W_-, y) = 1$. In addition, since the restrictions of f and g to W_+ coincide, then $d(f|_{W_+}, W_+, y) = d(g|_{W_+}, W_+, y)$. Hence

$$d(f|_{W_+}, W_+, y) = -1.$$

Finally, by Localization and Excision, we obtain $d(f, M, y) = -1$.

Step 3. We are now in the position to prove formula (5.2). Let (f, U, y) be a regular triple. We know that $f^{-1}(y) \cap U$ is a finite set, say $\{x_1, ..., x_n\}$. Since $Df(x_i)$ is an isomorphism for any $i = 1, ..., n$, we can apply the Inverse Function Theorem, obtaining that there exist n pairwise disjoint neighborhoods $U_1, ..., U_n$ of $x_1, ..., x_n$, respectively, such that each restriction $f|_{U_i}$ is a diffeomorphism onto an open subset of N. By the Additivity property we have

$$d(f, U, y) = \sum_{i=1}^{n} d(f|_{U_i}, U_i, y).$$

On the other hand, by the above step 2 and the Normalization property, it follows

$$d(f|_{U_i}, U_i, y) = \operatorname{sign} Df(x_i)$$

and this proves formula (5.2).

Step 4. In this final step we conclude the proof of the uniqueness of d. As a consequence we obtain that $d = \deg$ on the whole family $\mathcal{T}$.

Let (f, U, y) be an admissible triple. Since f is locally proper, we can consider an open subset W of U, containing $f^{-1}(y) \cap U$, such that $\overline{W} \subseteq U$ and f is proper on $\overline{W}$. By the Excision property (Proposition 5.2) we have

$$d(f, U, y) = d(f, W, y).$$

Being f a closed map on $\overline{W}$, we can take a regular value z of $f|_W$ and a continuous path $\sigma : [0, 1] \to N$ with $\sigma(0) = y$, $\sigma(1) = z$ and such that the set

$\{(u,t) \in W \times [0,1] : f(u) = y(t)\}$ is compact. By the Homotopy invariance property and formula (5.2) one has

$$d(f, W, y) = d(f, W, z) = \sum_{x \in f^{-1}(y) \cap W} \operatorname{sign} Df(x),$$

and this shows the uniqueness of d.

At the end of this procedure we have obtained that any map defined on the class $\mathbb{T}$ of all admissible triples and verifying the three fundamental properties is actually the degree.

REFERENCES

[1] H. Amann and S.A. Weiss, *On the uniqueness of the topological degree,* Math. Z., **130** (1973), 39–54.

[2] P. Benevieri and M. Furi, *A simple notion of orientability for Fredholm maps of index zero between Banach manifolds and degree theory,* Ann. Sci. Math. Québec, **22** (1998), 131–148.

[3] P. Benevieri and M. Furi, *On the concept of orientability for Fredholm maps between real Banach manifolds,* Topol. Methods Nonlinear Anal. **16** (2000), 279–306.

[4] P. Benevieri, M. Furi, M.P. Pera, M. Spadini, *About the sign of oriented Fredholm operators between Banach spaces* Z. Anal. Anwendungen **22**, n. 3 (2003), 619–645.

[5] A. Dold, *Lectures on algebraic topology,* Springer- Verlag, Berlin, 1972.

[6] K.D. Elworthy and A.J. Tromba, *Differential structures and Fredholm maps on Banach manifolds,* in “Global Analysis”, S. S. Chern and S. Smale Eds., Proc. Symp. Pure Math., Vol. 15, 1970, 45–94.

[7] K.D. Elworthy and A.J. Tromba, *Degree theory on Banach manifolds,* in “Nonlinear Functional Analysis”, F. E. Browder Ed., Proc. Symp. Pure Math., Vol. 18 (Part 1), 1970, 86–94.

[8] M.P. Fitzpatrick, J. Pejsachowicz and P.J. Rabier, *Orientability of Fredholm Families and Topological Degree for Orientable Nonlinear Fredholm Mappings*, Journal of Functional Analysis **124** (1994), 1–39.

[9] L. Führer, *Theorie des Abbildungsgrades in endlichdimensionalen Räumen*, Dissertation, Freie Univ. Berlin, Berlin, 1972.

[10] L. Führer, *Ein elementarer analytischer Beweis zur Eindeutigkeit des Abbildungsgrades im* R^n, Math. Nachr. **54** (1972), 259–267.

[11] T. Kato, *Perturbation Theory for Linear Operators*, Grundlehren der Mathematischen Wissenschaften, 132, Springer-Verlag, Berlin, 1980.

[12] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math., **87** (1965), 861–866.