

# FORCED OSCILLATIONS FOR DELAY MOTION EQUATIONS ON MANIFOLDS

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ABSTRACT. We prove an existence result for  $T$ -periodic solutions of a  $T$ -periodic second order delay differential equation on a boundaryless compact manifold with nonzero Euler-Poincaré characteristic. The approach is based on an existence result recently obtained by the authors for first order delay differential equations on compact manifolds with boundary.

## 1. INTRODUCTION

Let  $M \subseteq \mathbb{R}^k$  be a smooth boundaryless manifold and let

$$f : \mathbb{R} \times M \times M \rightarrow \mathbb{R}^k$$

be a continuous map which is  $T$ -periodic in the first variable and tangent to  $M$  in the second one; that is,

$$f(t + T, q, \tilde{q}) = f(t, q, \tilde{q}) \in T_q M, \quad \forall (t, q, \tilde{q}) \in \mathbb{R} \times M \times M,$$

where  $T_q M \subseteq \mathbb{R}^k$  denotes the tangent space of  $M$  at  $q$ . Consider the following second order delay differential equation on  $M$ :

$$x''_{\pi}(t) = f(t, x(t), x(t - \tau)) - \varepsilon x'(t), \quad (1.1)$$

where, regarding (1.1) as a motion equation,

- (1)  $x''_{\pi}(t)$  stands for the tangential part of the acceleration  $x''(t) \in \mathbb{R}^k$  at the point  $x(t)$ ;
- (2) the frictional coefficient  $\varepsilon$  is a positive real constant;
- (3)  $\tau > 0$  is the delay.

In this paper we prove that equation (1.1) admits at least one forced oscillation, provided that the constraint  $M$  is compact with nonzero Euler–Poincaré characteristic and that  $T \geq \tau$ . This generalizes a theorem of the last two authors regarding the undelayed case (see [3]). Our result will be deduced from an existence theorem for first order delay equations on compact manifolds with boundary recently obtained by the authors (see [1, Theorem 4.6]). The possibility of reducing (1.1) to the first order equation treated in [1] is due to the fact that any second order differential equation on  $M$  is equivalent to a first order system on the tangent bundle  $TM$  of  $M$ . The difficulty arising from the noncompactness of  $TM$  will be removed by restricting the search for  $T$ -periodic solutions to a convenient compact manifold with boundary contained in  $TM$ . The choice of such a manifold is suggested by a priori estimates on the set of all the possible  $T$ -periodic solutions of equation (1.1).

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These estimates are made possible by the compactness of  $M$  and the presence of the positive frictional coefficient  $\varepsilon$ .

We ask whether or not the existence of forced oscillations holds true even in the frictionless case, provided that the constraint  $M$  is compact with nonzero Euler–Poincaré characteristic. We believe the answer to this question is affirmative; but, as far as we know, this problem is still unsolved even in the undelayed case.

An affirmative answer regarding the special case  $M = S^2$  (the spherical pendulum) can be found in [4] (see also [5] for the extension to the case  $M = S^{2n}$ ).

We point out that the assumption  $T \geq \tau$  is crucial in this paper, since our result is deduced from Theorem 2.1 below, whose proof, given in [1], is based on the fixed point index theory for locally compact maps applied to a Poincaré-type  $T$ -translation operator which is a locally compact map if and only if  $T \geq \tau$ . In a forthcoming paper we will tackle the case  $0 < T < \tau$ , in which this operator is not even locally condensing.

## 2. SECOND ORDER DELAY DIFFERENTIAL EQUATIONS ON MANIFOLDS

Let, as before,  $M$  be a compact smooth boundaryless manifold in  $\mathbb{R}^k$ . Given  $q \in M$ , let  $T_qM$  and  $(T_qM)^\perp$  denote, respectively, the tangent and the normal space of  $M$  at  $q$ . Since  $\mathbb{R}^k = T_qM \oplus (T_qM)^\perp$ , any vector  $u \in \mathbb{R}^k$  can be uniquely decomposed into the sum of the *parallel* (or *tangential*) *component*  $u_\pi \in T_qM$  of  $u$  at  $q$  and the *normal component*  $u_\nu \in (T_qM)^\perp$  of  $u$  at  $q$ . By

$$TM = \{(q, v) \in \mathbb{R}^k \times \mathbb{R}^k : q \in M, v \in T_qM\}$$

we denote the *tangent bundle* of  $M$ , which is a smooth manifold containing a natural copy of  $M$  via the embedding  $q \mapsto (q, 0)$ . The natural projection of  $TM$  onto  $M$  is just the restriction (to  $TM$  as domain and to  $M$  as codomain) of the projection of  $\mathbb{R}^k \times \mathbb{R}^k$  onto the first factor.

Given, as in the Introduction, a continuous map  $f : \mathbb{R} \times M \times M \rightarrow \mathbb{R}^k$  which is  $T$ -periodic in the first variable and tangent to  $M$  in the second one, consider the following delay motion equation on  $M$ :

$$x''_\pi(t) = f(t, x(t), x(t - \tau)) - \varepsilon x'(t), \quad (2.1)$$

where

- i)  $x''_\pi(t)$  stands for the parallel component of the acceleration  $x''(t) \in \mathbb{R}^k$  at the point  $x(t)$ ;
- ii) the frictional coefficient  $\varepsilon$  and the delay  $\tau$  are positive real constants.

By a *solution* of (2.1) we mean a continuous function  $x : J \rightarrow M$ , defined on a (possibly unbounded) real interval, with length greater than  $\tau$ , which is of class  $C^2$  on the subinterval  $(\inf J + \tau, \sup J)$  of  $J$  and verifies

$$x''_\pi(t) = f(t, x(t), x(t - \tau)) - \varepsilon x'(t)$$

for all  $t \in J$  with  $t > \inf J + \tau$ . A *forced oscillation* of (2.1) is a solution which is  $T$ -periodic and globally defined on  $J = \mathbb{R}$ .

It is known that, associated with  $M \subseteq \mathbb{R}^k$ , there exists a unique smooth map  $r : TM \rightarrow \mathbb{R}^k$ , called the *reactive force* (or *inertial reaction*), with the following properties:

- (a)  $r(q, v) \in (T_qM)^\perp$  for any  $(q, v) \in TM$ ;
- (b)  $r$  is quadratic in the second variable;

(c) any  $C^2$  curve  $\gamma : (a, b) \rightarrow M$  verifies the condition

$$\gamma''_\nu(t) = r(\gamma(t), \gamma'(t)), \quad \forall t \in (a, b),$$

i.e., for each  $t \in (a, b)$ , the normal component  $\gamma''_\nu(t)$  of  $\gamma''(t)$  at  $\gamma(t)$  equals  $r(\gamma(t), \gamma'(t))$ .

The map  $r$  is strictly related to the second fundamental form on  $M$  and may be interpreted as the reactive force due to the constraint  $M$ .

By condition (c) above, equation (2.1) can be equivalently written as

$$x''(t) = r(x(t), x'(t)) + f(t, x(t), x(t - \tau)) - \varepsilon x'(t). \quad (2.2)$$

Notice that, if the above equation reduces to the so-called *inertial equation*

$$x''(t) = r(x(t), x'(t)),$$

one obtains the geodesics of  $M$  as solutions.

Equation (2.2) can be written as a first order differential system on  $TM$  as follows:

$$\begin{cases} x'(t) = y(t) \\ y'(t) = r(x(t), y(t)) + f(t, x(t), x(t - \tau)) - \varepsilon y(t). \end{cases}$$

This makes sense since the map

$$g : \mathbb{R} \times TM \times M \rightarrow \mathbb{R}^k \times \mathbb{R}^k, \quad g(t, (q, v), \tilde{q}) = (v, r(q, v) + f(t, q, \tilde{q}) - \varepsilon v) \quad (2.3)$$

verifies the condition  $g(t, (q, v), \tilde{q}) \in T_{(q, v)}TM$  for all  $(t, (q, v), \tilde{q}) \in \mathbb{R} \times TM \times M$  (see, for example, [2] for more details).

Theorem 2.1 below, which is a straightforward consequence of Theorem 4.6 in [1], will play a crucial role in the proof of our result (Theorem 2.2). Its statement needs some preliminary definitions.

Let  $X \subseteq \mathbb{R}^s$  be a smooth manifold with (possibly empty) boundary  $\partial X$ . Following [1], we say that a continuous map  $F : \mathbb{R} \times X \times X \rightarrow \mathbb{R}^s$  is *tangent to  $X$  in the second variable* or, for short, that  $F$  is a *vector field (on  $X$ )* if  $F(t, p, \tilde{p}) \in T_p X$  for all  $(t, p, \tilde{p}) \in \mathbb{R} \times X \times X$ . A vector field  $F$  will be said *inward* (to  $X$ ) if for any  $(t, p, \tilde{p}) \in \mathbb{R} \times \partial X \times X$  the vector  $F(t, p, \tilde{p})$  points inward at  $p$ . Recall that, given  $p \in \partial X$ , the set of the tangent vectors to  $X$  pointing inward at  $p$  is a closed half-subspace of  $T_p X$ , called *inward half-subspace* of  $T_p X$  (see e.g. [6]) and here denoted  $T_p^- X$ .

**Theorem 2.1.** *Let  $X \subseteq \mathbb{R}^s$  be a compact manifold with (possibly empty) boundary, whose Euler–Poincaré characteristic  $\chi(X)$  is different from zero. Let  $\tau > 0$  and let  $F : \mathbb{R} \times X \times X \rightarrow \mathbb{R}^s$  be an inward vector field on  $X$  which is  $T$ -periodic in the first variable, with  $T \geq \tau$ . Then, the delay differential equation*

$$x'(t) = F(t, x(t), x(t - \tau)) \quad (2.4)$$

*has a  $T$ -periodic solution.*

The result of this paper is the following.

**Theorem 2.2.** *Assume that the period  $T$  of  $f$  is not less than the delay  $\tau$  and that the Euler–Poincaré characteristic of  $M$  is different from zero. Then, the equation (2.1) has a forced oscillation.*

*Proof.* As we already pointed out, the equation (2.1) is equivalent to the following first order system on  $TM$ :

$$\begin{cases} x'(t) = y(t) \\ y'(t) = r(x(t), y(t)) + f(t, x(t), x(t - \tau)) - \varepsilon y(t). \end{cases} \quad (2.5)$$

Define  $F : \mathbb{R} \times TM \times TM \rightarrow \mathbb{R}^k \times \mathbb{R}^k$  by

$$F(t, (q, v), (\tilde{q}, \tilde{v})) = (v, r(q, v) + f(t, q, \tilde{q}) - \varepsilon v).$$

Notice that the map  $F$  is a vector field on  $TM$  which is  $T$ -periodic in the first variable.

Given  $c > 0$ , set

$$X_c = (TM)_c = \{(q, v) \in M \times \mathbb{R}^k : v \in T_q M, \|v\| \leq c\}.$$

It is not difficult to show that  $X_c$  is a compact manifold in  $\mathbb{R}^k \times \mathbb{R}^k$  with boundary

$$\partial X_c = \{(q, v) \in M \times \mathbb{R}^k : v \in T_q M, \|v\| = c\}.$$

Observe that

$$T_{(q,v)}(X_c) = T_{(q,v)}(TM)$$

for all  $(q, v) \in X_c$ . Moreover,  $\chi(X_c) = \chi(M)$  since  $X_c$  and  $M$  are homotopically equivalent ( $M$  being a deformation retract of  $TM$ ).

We claim that, if  $c > 0$  is large enough, then  $F$  is an inward vector field on  $X_c$ . To see this, let  $(q, v) \in \partial X_c$  be fixed, and observe that the inward half-subspace of  $T_{(q,v)}(X_c)$  is

$$T_{(q,v)}^-(X_c) = \{(\dot{q}, \dot{v}) \in T_{(q,v)}(TM) : \langle v, \dot{v} \rangle \leq 0\},$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{R}^k$ . We have to show that if  $c$  is large enough then  $F(t, (q, v), (\tilde{q}, \tilde{v}))$  belongs to  $T_{(q,v)}^-(X_c)$  for any  $t \in \mathbb{R}$  and  $(\tilde{q}, \tilde{v}) \in TM$ ; that is,

$$\langle v, r(q, v) + f(t, q, \tilde{q}) - \varepsilon v \rangle = \langle v, r(q, v) \rangle + \langle v, f(t, q, \tilde{q}) \rangle - \varepsilon \langle v, v \rangle \leq 0$$

for any  $t \in \mathbb{R}$  and  $(\tilde{q}, \tilde{v}) \in TM$ . Now,  $\langle v, r(q, v) \rangle = 0$  since  $r(q, v)$  belongs to  $(T_q M)^\perp$ . Moreover,  $\langle v, v \rangle = c^2$  since  $(q, v) \in \partial X_c$ , and

$$\langle v, f(t, q, \tilde{q}) \rangle \leq \|v\| \|f(t, q, \tilde{q})\| \leq K \|v\|,$$

where

$$K = \max \{ \|f(t, q, \tilde{q})\| : (t, q, \tilde{q}) \in \mathbb{R} \times M \times M \}.$$

Thus,

$$\langle v, r(q, v) + f(t, q, \tilde{q}) - \varepsilon v \rangle \leq Kc - \varepsilon c^2.$$

This shows that, if we choose  $c > K/\varepsilon$ , then  $F$  is an inward vector field on  $X_c$ , as claimed. Therefore, given  $c > K/\varepsilon$ , Theorem 2.1 implies that system (2.5) admits a  $T$ -periodic solution in  $X_c$ , and this completes the proof.  $\square$

It is evident from this proof that the result holds true even if we replace

$$f(t, q, \tilde{q}) - \varepsilon v$$

by a  $T$ -periodic force  $g(t, (q, v), (\tilde{q}, \tilde{v})) \in T_q M$  satisfying the following assumption:

*there exists  $c > 0$  such that  $\langle g(t, (q, v), (\tilde{q}, \tilde{v})), v \rangle \leq 0$  for any*

$$(t, (q, v), (\tilde{q}, \tilde{v})) \in \mathbb{R} \times TM \times TM$$

*such that  $\|v\| = c$ .*

We point out that, in the above theorem, the condition  $\chi(M) \neq 0$  cannot be dropped. Consider for example the equation

$$\theta''(t) = a - \varepsilon\theta'(t), \quad t \in \mathbb{R}, \quad (2.6)$$

where  $a$  is a nonzero constant and  $\varepsilon > 0$ . Equation (2.6) can be regarded as a second order ordinary differential equation on the unit circle  $S^1 \subseteq \mathbb{C}$ , where  $\theta$  represents an angular coordinate. In this case, a solution  $\theta(\cdot)$  of (2.6) is periodic of period  $T > 0$  if and only if for some  $k \in \mathbb{Z}$  it satisfies the boundary conditions

$$\begin{cases} \theta(T) - \theta(0) = 2k\pi, \\ \theta'(T) - \theta'(0) = 0. \end{cases}$$

Notice that the applied force  $a$  represents a nonvanishing autonomous vector field on  $S^1$ . Thus, it is periodic of arbitrary period. However, simple calculations show that there are no  $T$ -periodic solutions of (2.6) if  $T \neq 2\pi\varepsilon/a$ .

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