GLOBAL CONTINUATION OF FORCED OSCILLATIONS OF RETARDED MOTION EQUATIONS ON MANIFOLDS

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Dedicated to the outstanding mathematician Andrzej Granas whose contribution to the fixed point index theory made the existence of this article possible

Abstract. We investigate T-periodic parametrized retarded functional motion equations on (possibly) noncompact manifolds; that is, constrained second order retarded functional differential equations. For such equations we prove a global continuation result for T-periodic solutions. The approach is topological and is based on the degree theory for tangent vector fields as well as on the fixed point index theory.

Our main theorem is a generalization to the case of retarded equations of an analogous result obtained by the last two authors for second order differential equations on manifolds. As corollaries we derive a Rabinowitz-type global bifurcation result and a Mawhin-type continuation principle. Finally, we deduce the existence of forced oscillations for the retarded spherical pendulum under general assumptions.

1. Introduction

Let $M \subseteq \mathbb{R}^k$ be a (possibly noncompact) boundaryless smooth manifold, and denote by $BU((-\infty,0],M)$ the metric space of bounded and uniformly continuous maps from $(-\infty,0]$ into M with the topology of the uniform convergence. Consider the following parametrized *retarded functional motion equation* on M:

$$x''_\pi(t) = \lambda f(t, x_t) \tag{1.1}$$

where:

- $x''_\pi(t)$ stands for the tangential part of the acceleration $x''(t) \in \mathbb{R}^k$ at the point $x(t) \in M$,
- λ is a nonnegative real parameter,
- $f\colon \mathbb{R} \times BU((-\infty,0],M) \to \mathbb{R}^k$ is such that $f(t,\varphi) \in T_{\varphi(0)}M$ for all (t,φ), where $T_pM \subseteq \mathbb{R}^k$ is the tangent space of M at a point p of M.

We will call *functional field* a continuous map $f\colon \mathbb{R} \times BU((-\infty,0],M) \to \mathbb{R}^k$ verifying the above tangency condition. In addition, we use the standard notation in functional equations: whenever it makes sense, $x_t \in BU((-\infty,0],M)$ denotes

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the function $\theta \mapsto x(t+\theta)$. Roughly speaking, a retarded functional motion equation on M is a motion equation constrained on M in which the active force, λf in this case, takes into account the whole past history of the process.

In this paper we are interested in obtaining *global continuation results* for (1.1). That is, we assume that the functional field f is T-periodic in the first variable and locally Lipschitz in the second one, and we study the topological properties of the set of T-periodic solutions of (1.1). More in detail, we denote by $C^1_T(M)$ the space of the T-periodic C^1 maps $x\colon \mathbb{R} \to M$ with the standard C^1 metric, and we call $(\lambda, x) \in [0, +\infty) \times C^1_T(M)$ a *T-forced pair* of the equation (1.1) if $x\colon \mathbb{R} \to M$ is a T-periodic solution of (1.1) corresponding to λ. Among these pairs we distinguish the *trivial* ones; that is, the elements of the form $(0, p^-)$, where, given $p \in M$, p^- denotes the constant p-valued function defined on $\mathbb{R}$. We call a point $p \in M$ a *bifurcation point* of (1.1) if any neighborhood of $(0, p^-)$ in $[0, +\infty) \times C^1_T(M)$ contains nontrivial T-forced pairs. The main result of this paper, Theorem 4.4 below, is a global continuation result for T-forced pairs of the equation (1.1). That is, given an open subset Ω of $[0, +\infty) \times C^1_T(M)$, Theorem 4.4 provides sufficient conditions for the existence of a *global bifurcating branch in* Ω, meaning a connected subset of Ω of nontrivial T-forced pairs whose closure in Ω is noncompact and intersects the set of trivial T-forced pairs.

The prelude of our approach can be found in some papers of the last two authors (see e.g. [9, 10]). In [10] Furi and Pera proved a global continuation result for the second order parametrized motion equation

$$x''_\pi(t) = \lambda \mathfrak{f}(t, x(t), x'(t)), \quad \lambda \geq 0, \tag{1.2}$$

where $\mathfrak{f}\colon \mathbb{R} \times TM \to \mathbb{R}^k$ is a continuous, T-periodic tangent vector field on M. Such a result is obtained by means of topological methods. In particular, the existence of a global bifurcating branch is given if the degree (in an open subset of M) of the tangent vector field (the average force)

$$\bar{\mathfrak{f}}(p) = \frac{1}{T} \int_0^T \mathfrak{f}(t, p, 0)\, dt, \quad p \in M,$$

is defined and nonzero. Our results improve those of [10] in a natural sense, since differential equations with delay include ODEs as particular cases. On the other hand the extension that we obtain is only partial for the following reasons: first, in (1.2) the active force may depend also on the velocity which is not the case, in the present setting, for (1.1). Second, for technical reasons we are led to assume that the functional field f is locally Lipschitz in the second variable, so we are not able to prove our results with the sole continuity assumption as it was done in [10].

The proof of Theorem 4.4 is obtained proceeding in the spirit of [10]. Such a proof is based on a relation, proved in a previous paper by the authors [4, Lemma 3.8], between the degree (in an open subset of M) of the tangent vector field

$$\bar{f}(p) = \frac{1}{T} \int_0^T f(t, p^-)\, dt, \quad p \in M,$$

and the fixed point index of a sort of Poincaré T-translation operator acting inside the Banach space $C([-T, 0], \mathbb{R}^{2k})$. Notice that (1.1) is equivalent to a first order retarded functional differential equation (RFDE) on the tangent bundle TM. We prove Theorem 4.4 by transforming (1.1) into the first order system

$$\begin{cases} x'(t) = y(t), \\ y'(t) = r(x(t), y(t)) + \lambda f(t, x_t), \quad \lambda \geq 0, \end{cases} \tag{1.3}$$

where $r\colon TM \to \mathbb{R}^k$ is the inertial reaction. On the other hand, observing the above system, one can see that a bifurcation problem associated to equation (1.1) cannot be simply reduced to a RFDE on TM of the form

$$z'(t) = \lambda h(t, z_t)$$

(and system (1.3) is not in this form), since the inertial motion problem does not correspond, in the phase space, to the trivial equation $z'(t) = 0$.

Recently we have devoted some papers to the study of (first- and) second-order RFDEs on differentiable manifolds. The problem that we address in the present paper is characterized by a broad generality and presents some peculiar features which make it very hard and challenging. In a sense, our previous papers turn out to be a preliminary investigation that enable us to better understand how to tackle the problem. In the recent paper [4] we proved a global continuation result for periodic solutions of *first-order* parametrized RFDEs on manifolds. Despite the similarity, as pointed out above, the problem that we address here cannot be handled with the techniques developed in [4] and require a specific study. On the other hand, in [1] we obtained a global continuation result under the assumption that the manifold M is *compact*. Here we present a consequence of our main result (Corollary 4.12) improving the one in [1]. Further, the simultaneous presence of infinite delay and noncompactness of the manifold leads us to work in the space $BU((-\infty, 0], M)$. In the paper [3] we studied general properties of RFDEs with infinite delay on (possibly noncompact) differentiable manifolds since we were not aware of the presence of such results in the literature.

In contrast, the different and related cases of RFDEs with finite delay in Euclidean spaces have been investigated by many authors. For general reference we suggest the monograph by Hale and Verduyn Lunel [17]. We refer also to the works of Gaines and Mawhin [12], Nussbaum [26, 27] and Mallet-Paret, Nussbaum and Paraskevopoulos [21]. For RFDEs with infinite delay in Euclidean spaces we recommend the article of Hale and Kato [16], the book by Hino, Murakami and Naito [18], and the more recent paper of Oliva and Rocha [30]. For RFDEs with finite delay on manifolds we suggest the papers of Oliva [28, 29]. Finally, for RFDEs with infinite delay on manifolds we cite the already mentioned paper [3].

We conclude the paper with some consequences of Theorem 4.4. One is a Rabinowitz-type global bifurcation result (see [31]), obtained by assuming that the degree of the tangent vector field $\bar{f}$ is nonzero on an open subset of M. Another corollary is a Mawhin-type continuation principle [22, 23] which is a partial extension of an analogous result, for a motion differential equation on M, obtained by the last two authors in [10, Corollary 2.1].

A principal motivation in [10] was to investigate the problem about the existence of forced oscillation for constrained systems with compact, topologically nontrivial constraint, such as the spherical pendulum, whose constraint M is S^2. Concerning the *retarded spherical pendulum*, an existence result for forced oscillations has been proved by the authors in [2], assuming the continuity of the functional field f on the space $\mathbb{R} \times C((-\infty, 0], S^2)$, with $C((-\infty, 0], S^2)$ having the too weak topology of the uniform convergence on compact intervals, making the continuity assumption of f

a heavy hypothesis. Our main result allows us to extend this existence theorem to the case in which the functional field is continuous on $\mathbb{R} \times BU((-\infty, 0], S^2)$.

2. Preliminaries

2.1. **Fixed point index.** We recall that a metrizable space $\mathcal{X}$ is an *absolute neighborhood retract* (ANR) if, whenever it is homeomorphically embedded as a closed subset C of a metric space $\mathcal{Y}$, there exist an open neighborhood V of C in $\mathcal{Y}$ and a retraction $r\colon V \to C$ (see e.g. [5, 14]). Polyhedra and differentiable manifolds are examples of ANRs. Let us also recall that a continuous map between topological spaces is called *locally compact* if each point in its domain has a neighborhood whose image is contained in a compact set.

Let $\mathcal{X}$ be a metric ANR and consider a locally compact (continuous) $\mathcal{X}$-valued map k defined on a subset $\mathcal{D}(k)$ of $\mathcal{X}$. Given an open subset U of $\mathcal{X}$ contained in $\mathcal{D}(k)$, if the set of fixed points of k in U is compact, the pair (k, U) is called *admissible*. We point out that such a condition is clearly satisfied if $\overline{U} \subseteq \mathcal{D}(k)$, $\overline{k(U)}$ is compact and $k(p) \neq p$ for all p in the boundary of U. To any admissible pair (k, U) one can associate an integer $\mathrm{ind}_{\mathcal{X}}(k, U)$ –the *fixed point index* of k in U– which satisfies properties analogous to those of the classical Leray–Schauder degree [20]. The reader can see for instance [6, 13, 25, 27] for a comprehensive presentation of the index theory for ANRs. As regards the connection with the homology theory we refer to standard algebraic topology textbooks (e.g. [7, 32]).

We summarize below the main properties of the fixed point index.

- (Existence) *If* $\mathrm{ind}_{\mathcal{X}}(k, U) \neq 0$, *then* k *admits at least one fixed point in* U.
- (Normalization) *If* $\mathcal{X}$ *is compact, then* $\mathrm{ind}_{\mathcal{X}}(k, \mathcal{X}) = \Lambda(k)$, *where* $\Lambda(k)$ *denotes the Lefschetz number of* k.
- (Additivity) *Given two disjoint open subsets* U_1, U_2 *of* U, *if any fixed point of* k *in* U *is contained in* $U_1 \cup U_2$, *then* $\mathrm{ind}_{\mathcal{X}}(k, U) = \mathrm{ind}_{\mathcal{X}}(k, U_1) + \mathrm{ind}_{\mathcal{X}}(k, U_2)$.
- (Excision) *Given an open subset* U_1 *of* U, *if* k *has no fixed points in* $U \backslash U_1$, *then* $\mathrm{ind}_{\mathcal{X}}(k, U) = \mathrm{ind}_{\mathcal{X}}(k, U_1)$.
- (Commutativity) *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be metric ANRs. Suppose that* U *and* V *are open subsets of* $\mathcal{X}$ *and* $\mathcal{Y}$ *respectively and that* $k\colon U \to \mathcal{Y}$ *and* $h\colon V \to \mathcal{X}$ *are locally compact maps. Assume that the set of fixed points of either* hk *in* $k^{-1}(V)$ *or* kh *in* $h^{-1}(U)$ *is compact. Then the other set is compact as well and* $\mathrm{ind}_{\mathcal{X}}(hk, k^{-1}(V)) = \mathrm{ind}_{\mathcal{Y}}(kh, h^{-1}(U))$.
- (Generalized homotopy invariance) *Let* I *be a compact real interval and* W *an open subset of* $\mathcal{X} \times I$. *For any* $\lambda \in I$, *denote* $W_\lambda = \{x \in \mathcal{X} : (x, \lambda) \in W\}$. *Let* $H\colon W \to \mathcal{X}$ *be a locally compact map such that the set* $\{(x, \lambda) \in W : H(x, \lambda) = x\}$ *is compact. Then* $\mathrm{ind}_{\mathcal{X}}(H(\cdot, \lambda), W_\lambda)$ *is independent of* λ.

2.2. **Degree of a vector field.** Let us recall some basic notions on degree theory for tangent vector fields on differentiable manifolds. Let $w\colon M \to \mathbb{R}^k$ be a continuous (autonomous) tangent vector field on a smooth manifold M, and let V be an open subset of M. We say that the pair (w, V) is *admissible* (or, equivalently, that w is admissible in V) if $w^{-1}(0) \cap V$ is compact. In this case one can assign to the pair (w, V) an integer, $\deg(w, V)$, called the *degree* (or *Euler characteristic*, or *rotation*) of the tangent vector field w in V which, roughly speaking, counts algebraically the number of zeros of w in V (for general references see e.g. [15, 19, 24, 33]). Notice that the condition for $w^{-1}(0) \cap V$ to be compact is clearly satisfied if V is a relatively compact open subset of M and $w(p) \neq 0$ for all p in the boundary of V.

As a consequence of the Poincaré–Hopf theorem, when M is compact, $\deg(w, M)$ equals $\chi(M)$, the Euler–Poincaré characteristic of M.

In the particular case when V is an open subset of $\mathbb{R}^k$, $\deg(w, V)$ is just the classical Brouwer degree of w in V when the map w is regarded as a vector field; namely, the degree $\deg(w, V, 0)$ of w in V with *target value* $0 \in \mathbb{R}^k$. All the standard properties of the Brouwer degree in the flat case, such as homotopy invariance, excision, additivity, existence, still hold in the more general context of differentiable manifolds. To see this, one can use an equivalent definition of degree of a tangent vector field based on the fixed point index theory as presented in [9] and [10].

Let us stress that, actually, in [9] and [10] the definition of degree of a tangent vector field on M is given in terms of the fixed point index of a Poincaré-type translation operator associated to a suitable ODE on M.

We point out that no orientability of M is required for $\deg(w, V)$ to be defined. This highlights the fact that the extension of the Brouwer degree for tangent vector fields in the non-flat case does not coincide with the one regarding maps between oriented manifolds with a given target value (as illustrated, for example, in [19, 24]). This dichotomy of the notion of degree in the non-flat situation is not evident in $\mathbb{R}^k$: it is masked by the fact that an equation of the type $f(x) = y$ can be written as $f(x) - y = 0$. Anyhow, in the context of RFDEs on manifolds (ODEs included), it is the degree of a vector field that plays a significative role. Moreover, if w has an isolated zero p and V is an isolating (open) neighborhood of p, then $\deg(w, V)$ is called the *index of w at p* or, by abuse of terminology, the *index of p*. The excision property ensures that this is a well-defined integer.

We close this section with two results that will be used in the sequel. Next one is a Wyburn's type topological lemma.

Lemma 2.1 ([10])**.** *Let K be a compact subset of a locally compact metric space $\mathcal{Y}$. Assume that any compact subset of $\mathcal{Y}$ containing K has nonempty boundary. Then $\mathcal{Y} \backslash K$ contains a connected set whose closure is noncompact and intersects K.*

The following lemma, whose elementary proof is given for the sake of completeness, will be used in the proofs of Theorem 4.3 and Lemma 4.7.

Lemma 2.2. *Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map between metric spaces and let $\{\gamma_n\}$ be a sequence of continuous functions from a compact interval $[a, b]$ (or, more generally, from a compact space) into $\mathcal{X}$. If $\{\gamma_n(s)\}$ converges to $\gamma(s)$ uniformly for $s \in [a, b]$, then also $F(\gamma_n(s)) \to F(\gamma(s))$ uniformly for $s \in [a, b]$.*

Proof. Notice that, if C is a compact subset of $\mathcal{X}$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in \mathcal{X}$, $c \in C$, $\mathrm{dist}_{\mathcal{X}}(x, c) < \delta$ imply $\mathrm{dist}_{\mathcal{Y}}(F(x), F(c)) < \varepsilon$. Now, our assertion follows immediately by taking the compact C to be the image of the limit function $\gamma\colon [a, b] \to \mathcal{X}$. □

3. Retarded functional differential equations

Given an arbitrary subset A of $\mathbb{R}^l$, we denote by $BU((-\infty, 0], A)$ the set of bounded and uniformly continuous maps from $(-\infty, 0]$ into A. For brevity, we will use the notation

$$\widetilde{A} := BU((-\infty,0],A).$$

Notice that $\widetilde{\mathbb{R}}^l$ is a Banach space, being closed in the space $BC((-\infty,0],\mathbb{R}^l)$ of the bounded and continuous functions from $(-\infty,0]$ into $\mathbb{R}^l$ (endowed with the standard supremum norm).

Given $T>0$, let

$$\widehat{A} := C([-T,0],A)$$

denote the metric subspace of $C([-T,0],\mathbb{R}^l)$ of the A-valued continuous functions on $[-T,0]$ and set

$$\widehat{A}_* := \big\{\eta \in \widehat{A} : \eta(-T) = \eta(0)\big\}.$$

We observe that, if A is locally compact, then $\widehat{A}$ (but not $\widetilde{A}$) is locally complete. Moreover, $\widehat{A}$ and $\widetilde{A}$ are complete if and only if A is closed.

Throughout the paper, unless otherwise stated, the norm in any Banach space will be denoted by $|\cdot|$ if the dimension of the space is finite and by $\|\cdot\|$ otherwise. Thus, the distance between two elements γ and δ of $\widetilde{A}$ will be denoted by $\|\gamma-\delta\|$, even when $\gamma-\delta$ does not belong to $\widetilde{A}$.

Let N be a boundaryless smooth manifold in $\mathbb{R}^l$ and, given $q\in N$, let $T_qN \subseteq \mathbb{R}^l$ denote the tangent space of N at q. A continuous map

$$g\colon \mathbb{R}\times\widetilde{N} \to \mathbb{R}^l$$

is said to be a *retarded functional tangent vector field over* N if $g(t,\eta)\in T_{\eta(0)}N$ for all $(t,\eta)\in\mathbb{R}\times\widetilde{N}$. In the sequel, any map with this property will be briefly called a *functional field (over N)*.

Let us consider a first order *retarded functional differential equation* (*RFDE*) of the type

$$z'(t) = g(t,z_t), \tag{3.1}$$

where $g\colon\mathbb{R}\times\widetilde{N}\to\mathbb{R}^l$ is a functional field over N. Here, as usual and whenever it makes sense, given $t\in\mathbb{R}$, by $z_t\in\widetilde{N}$ we mean the function $\theta\mapsto z(t+\theta)$.

A *solution* of (3.1) is a function $z\colon J\to N$, defined on an open real interval J with $\inf J=-\infty$, bounded and uniformly continuous on any closed half-line $(-\infty,b]\subset J$, and which verifies eventually the equality $z'(t)=g(t,z_t)$. That is, $z\colon J\to N$ is a solution of (3.1) if $z_t\in\widetilde{N}$ for all $t\in J$ and there exists $\tau\in J$ such that z is C^1 on the interval $(\tau,\sup J)$ and $z'(t)=g(t,z_t)$ for all $t\in(\tau,\sup J)$. Observe that the derivative of a solution z may not exist at $t=\tau$. However, the right derivative $D_+z(\tau)$ of z at τ always exists and is equal to $g(\tau,z_\tau)$. Also, notice that $t\mapsto z_t$ is a continuous curve in $\widetilde{N}$, since z is uniformly continuous on any closed half-line $(-\infty,b]$ of J.

A solution of (3.1) is said to be *maximal* if it is not a proper restriction of another solution. As in the case of ODEs, Zorn's lemma implies that any solution is the restriction of a maximal solution.

Given $\omega\in\widetilde{N}$, let us associate to equation (3.1) the initial value problem

$$\begin{cases} z'(t) = g(t,z_t), \\ z_0 = \omega\,. \end{cases} \tag{3.2}$$

A *solution* of (3.2) is a solution $z\colon J\to N$ of (3.1) such that $\sup J>0$, $z'(t)=g(t,z_t)$ for $t>0$, and $z_0=\omega$.

The continuous dependence of the solutions on initial data is stated in Theorem 3.1 below, which is a particular case of Theorem 4.4 of [3].

Theorem 3.1. *Let N be a boundaryless smooth manifold and $g\colon \mathbb{R}\times\widetilde{N}\to\mathbb{R}^l$ a functional field. Assume, for any $\omega\in\widetilde{N}$, the uniqueness of the maximal solution of problem* (3.2)*. Then, given $T>0$, the set*

$$\mathcal{D}=\big\{\omega\in\widetilde{N}:\ \textit{the maximal solution of } (3.2) \textit{ is defined up to } T\big\}$$

is open and the map $\omega\in\mathcal{D}\mapsto z^\omega_T\in\widetilde{N}$, where $z^\omega(\cdot)$ is the unique maximal solution of problem (3.2)*, is continuous.*

More generally, we will need to consider RFDE depending on parameters of the form

$$z'(t)=h(\alpha,t,z_t), \tag{3.3}$$

where $h\colon\mathbb{R}^s\times\mathbb{R}\times\widetilde{N}\to\mathbb{R}^l$ is a parametrized functional field over N. The continuous dependence on data of the solutions of the initial value problems associated to (3.3) is ensured by the following consequence of Theorem 3.1.

Corollary 3.2 (continuous dependence on data)**.** *Let N be a boundaryless smooth manifold and $h\colon\mathbb{R}^s\times\mathbb{R}\times\widetilde{N}\to\mathbb{R}^l$ a parametrized functional field. For any $\alpha\in\mathbb{R}^s$ and $\omega\in\widetilde{N}$, assume the uniqueness of the maximal solution of the problem*

$$\begin{cases} z'(t)=h(\alpha,t,z_t),\\ z_0=\omega\,. \end{cases} \tag{3.4}$$

Then, given $T>0$, the set

$$\mathcal{D}'=\big\{(\alpha,\omega)\in\mathbb{R}^s\times\widetilde{N}:\ \textit{the maximal solution of } (3.4) \textit{ is defined up to } T\big\}$$

is open and the map $(\alpha,\omega)\in\mathcal{D}'\mapsto z^{(\alpha,\omega)}_T\in\widetilde{N}$, where $z^{(\alpha,\omega)}(\cdot)$ is the unique maximal solution of problem (3.4)*, is continuous.*

Proof. Apply Theorem 3.1 to the problem

$$\begin{cases} \big(\beta'(t),z'(t)\big)=\big(0,h(\beta(t),t,z_t)\big),\\ \big(\beta(0),z_0\big)=\big(\alpha,\omega\big), \end{cases}$$

that can be regarded as an initial value problem of a RFDE on the ambient manifold $\mathbb{R}^s\times N\subseteq\mathbb{R}^{s+l}$. □

In Theorem 3.1 and in Corollary 3.2 above the hypothesis of the uniqueness of the maximal solution of problems (3.2) and (3.4) is essential in order to make their statements meaningful. Sufficient conditions for the uniqueness are presented in Remark 3.3 below.

Remark 3.3. A functional field $g\colon\mathbb{R}\times\widetilde{N}\to\mathbb{R}^l$ is said to be *compactly Lipschitz* (for short, *c-Lipschitz*) if, given any compact subset Q of $\mathbb{R}\times\widetilde{N}$, there exists $L\geq 0$ such that

$$|g(t,\gamma)-g(t,\delta)|\leq L\|\gamma-\delta\|$$

for all $(t,\gamma),(t,\delta)\in Q$. Moreover, we will say that g is *locally c-Lipschitz* if for any $(\tau,\omega)\in\mathbb{R}\times\widetilde{N}$ there exists an open neighborhood of (τ,ω) in which g is c-Lipschitz. In spite of the fact that a locally Lipschitz map is not necessarily (globally) Lipschitz, one could actually show that if g is locally c-Lipschitz, then it is also (globally) c-Lipschitz. As a consequence, if g is locally Lipschitz in the second variable, then it is c-Lipschitz as well. In [3] we proved that, if g is a c-Lipschitz functional field, then problem (3.2) has a unique maximal solution for any $\omega \in \widetilde{N}$. For a characterization of compact subsets of $\widetilde{N}$ see e.g. [8, Part 1, IV.6.5].

In addition to the assumptions of Corollary 3.2, assume that h is T-periodic with respect to t. We will associate to problem (3.4) a parametrized Poincaré-type T-translation operator whose fixed points are the restrictions to the interval $[-T,0]$ of the T-periodic solutions of (3.4). For this purpose, we need to introduce a suitable backward extension of the elements of $\widehat{N} = C([-T,0],N)$. The properties of such an extension are contained in Lemma 3.4 below, obtained in [11]. In what follows, by a T-periodic map on an interval J we mean the restriction to J of a T-periodic map defined on $\mathbb{R}$.

Lemma 3.4. *There exist an open neighborhood U of $\widehat{N}_*$ in $\widehat{N}$ and a continuous map from U to $\widetilde{N}$, $\eta \mapsto \widetilde{\eta}$, with the following properties:*

1) *$\widetilde{\eta}$ is an extension of η;*
2. *$\widetilde{\eta}$ is T-periodic on $(-\infty,-T]$;*
3. *$\widetilde{\eta}$ is T-periodic on $(-\infty,0]$, whenever $\eta \in \widehat{N}_*$.*

Take $U \subseteq \widehat{N}$ as in the previous lemma. Given $\alpha \in \mathbb{R}^s$ and $\eta \in U$, consider the initial value problem

$$\begin{cases} z'(t) = h(\alpha,t,z_t), \\ z_0 = \widetilde{\eta}, \end{cases} \tag{3.5}$$

where $\widetilde{\eta}$ is the extension of η as in Lemma 3.4.

Let

$$\mathcal{E} = \big\{(\alpha,\eta) \in \mathbb{R}^s \times U : \text{the maximal solution of (3.5) is defined up to } T\big\}.$$

By Corollary 3.2 and Lemma 3.4 it follows that $\mathcal{E}$ is open in $\mathbb{R}^s \times \widehat{N}$. Given $(\alpha,\eta) \in \mathcal{E}$, denote by $z^{(\alpha,\widetilde{\eta})}$ the maximal solution of problem (3.5) and define

$$\mathcal{H} \colon \mathcal{E} \to \widehat{N}$$

by

$$\mathcal{H}(\alpha,\eta)(\theta) = z^{(\alpha,\widetilde{\eta})}(\theta + T), \quad \theta \in [-T,0].$$

Observe that $\mathcal{H}(\alpha,\eta)$ is the restriction of $z_T^{(\alpha,\widetilde{\eta})} \in \widetilde{N}$ to the interval $[-T,0]$.

The following theorems regard crucial properties of the operator $\mathcal{H}$. The proof of the first one is standard, while the proof of the second one is similar to that contained in [4, Lemma 3.7] and thus will be omitted.

Theorem 3.5. *The fixed points of $\mathcal{H}(\alpha,\cdot)$ correspond to the T-periodic solutions of the equation* (3.3) *in the following sense: $\eta \in U$ is a fixed point of $\mathcal{H}(\alpha,\cdot)$ if and only if it is the restriction to $[-T,0]$ of a T-periodic solution of* (3.3) *corresponding to α.*

Theorem 3.6. *The operator $\mathcal{H}$ is continuous and locally compact.*

4. Retarded functional motion equations

Let $M \subseteq \mathbb{R}^k$ be a boundaryless smooth m-dimensional manifold and let

$$TM = \left\{(q,v) \in \mathbb{R}^k \times \mathbb{R}^k : \; q \in M, \; v \in T_qM\right\}$$

denote the tangent bundle of M. This is a smooth $2m$-dimensional manifold containing a natural copy of M via the embedding $q \mapsto (q,0)$. The natural projection of TM onto M is just the restriction (to TM as domain and to M as codomain) of the projection of $\mathbb{R}^k \times \mathbb{R}^k$ onto the first factor.

Given $q \in M$, let $(T_qM)^\perp \subseteq \mathbb{R}^k$ denote the normal space of M at q. Since $\mathbb{R}^k = T_qM \oplus (T_qM)^\perp$, any vector $u \in \mathbb{R}^k$ can be uniquely decomposed into the sum of the parallel (or tangential) component $u_\pi \in T_qM$ of u at q and the normal component $u_\nu \in (T_qM)^\perp$ of u at q.

As previously, denote by $\widetilde{M}$ the set $BU((-\infty,0],M)$ and let $f\colon \mathbb{R}\times\widetilde{M} \to \mathbb{R}^k$ be a functional field over M. In this section we will consider the following parametrized *retarded functional motion equation* on the constraint M:

$$x''_\pi(t) = \lambda f(t,x_t), \quad \lambda \geq 0\,, \tag{4.1}$$

where $x''_\pi(t)$ stands for the parallel component of the acceleration $x''(t) \in \mathbb{R}^k$ at the point $x(t)$.

In the case when the active force λf is identically zero, the equation (4.1) reduces to the so-called *inertial equation*

$$x''_\pi(t) = 0,$$

and one obtains the geodesics on M as solutions. Given any $(q,v) \in TM$, there exists one and only one (maximal) geodesic with initial position q and initial velocity $v \in T_qM$. This geodesic can be regarded as a curve in $\mathbb{R}^k$, and if $v \neq 0$ it has a curvature at q that we denote $k(q,v)$. It is convenient to put $k(q,v) = 0$ whenever $v = 0$.

Observe that, given $(q,v) \in TM$ and a real number $s \neq 0$, one has $k(q,v) = k(q,sv)$. This is because the two geodesics with initial position q and initial velocities v and sv have the same image. Obviously, $k(q,v)$ depends continuously on q and $v \in T_qM$ with $|v| = 1$. Therefore, given a compact subset C of M, there exists a constant K such that $k(q,v) \leq K$ for all $(q,v) \in TM$ with $q \in C$.

It is known that the equation (4.1) can be equivalently written as

$$x''(t) = r(x(t),x'(t)) + \lambda f(t,x_t), \quad \lambda \geq 0\,, \tag{4.2}$$

where $r\colon TM \to \mathbb{R}^k$ is a smooth map (the so-called reactive force or inertial reaction) satisfying the following properties:

(a) $r(q,v) \in (T_qM)^\perp$ for any $(q,v) \in TM$;
(b) $r(q,v)$ is quadratic in v and its norm equals $k(q,v)|v|^2$;
(c) given $(q,v) \in TM$, $r(q,v)$ is the unique vector of $\mathbb{R}^k$ such that $(v, r(q,v))$ belongs to $T_{(q,v)}(TM)$;
(d) given any C^2 curve $\gamma\colon (a,b) \to M$, the normal component $\gamma''_\nu(t)$ of $\gamma''(t)$ at $\gamma(t)$ verifies the condition $\gamma''_\nu(t) = r(\gamma(t),\gamma'(t))$, $t \in (a,b)$.

Now, the second order equation (4.2) can be transformed into the first order system

$$\begin{cases} x'(t) = y(t), \\ y'(t) = r(x(t),y(t)) + \lambda f(t,x_t), \quad \lambda \geq 0, \end{cases} \tag{4.3}$$

which is actually a first order parametrized RFDE on the tangent bundle TM, since the map

$$(\lambda, t, (\varphi, \psi)) \in [0, +\infty) \times \mathbb{R} \times \widetilde{TM} \mapsto (\psi(0), r(\varphi(0), \psi(0)) + \lambda f(t, \varphi))$$

is a parametrized functional field over TM. Thus, recalling the notion of solution introduced in Section 3 for first order RFDEs, we get that a C^1 function $x\colon J \to M$, defined on an open real interval J with $\inf J = -\infty$, is a *solution* of (4.1) if the pair $(x, x')\colon J \to TM$ is a solution of (4.3).

From now on, assume in addition that f is T-periodic in the first variable ($T > 0$). Given $\lambda \geq 0$, by a *T-periodic solution*, or *forced oscillation*, of equation (4.1) we mean a solution which is globally defined on $\mathbb{R}$ and is T-periodic. In what follows, it will be useful to consider the forced oscillations of (4.1) as elements of the space $C^1_T(M)$ of the M-valued T-periodic C^1 functions defined on $\mathbb{R}$. This is a metric space, as a subset of the Banach space $C^1_T(\mathbb{R}^k)$ with the standard norm

$$\|x\| = \max_{t\in\mathbb{R}} |x(t)| + \max_{t\in\mathbb{R}} |x'(t)|.$$

The pairs $(\lambda, x) \in [0, +\infty) \times C^1_T(M)$, with $x\colon \mathbb{R} \to M$ a forced oscillation of (4.1) corresponding to λ, will be called *T-forced pairs* (of (4.1)), and we denote by X the set of these pairs.

Among the T-forced pairs we shall consider as *trivial* those of the type $(0, x)$ with x constant. Henceforth, given any $q \in M$, we will denote by q^- the constant map $t \mapsto q$, $t \in \mathbb{R}$; so that any *trivial T-forced pair* is of the type $(0, q^-)$. In the set X we shall distinguish the subset

$$X^* = \left\{(0, q^-) \in [0, +\infty) \times C^1_T(M) : q \in M\right\} \tag{4.4}$$

consisting of the trivial T-forced pairs. Consequently, its complement in X, $X \backslash X^*$, will be called the set of the *nontrivial T-forced pairs*. Notice that there may exist nontrivial T-forced pairs of the type $(0, x)$, provided that $x\colon \mathbb{R} \to M$ is a non-constant T-periodic geodesic of M.

Once we have a set (usually made up of the solutions of a given equation), a distinguished subset (called the subset of the trivial elements) and a convenient topology on the set, we may define the concept of *bifurcation point*: it is a trivial element (or an alias of it) with the property that any of its neighborhoods contains nontrivial elements.

Thus, in our case, the alias q_* of a trivial T-forced pair $(0, q_*^-)$ is a *bifurcation point* of the equation (4.1) if any neighborhood of $(0, q_*^-)$ in $[0, +\infty) \times C^1_T(M)$ contains nontrivial T-forced pairs.

Let us point out that, if q_* is a bifurcation point, any nontrivial T-forced pair (λ, x) sufficiently close to $(0, q_*^-)$ must have $\lambda > 0$ since, as well known, in a Riemannian manifold there are no non-constant closed geodesics too close to a given point. The same fact could also be deduced from Lemma 4.1 below that will be used in proving our necessary condition for bifurcation.

Lemma 4.1 ([10]). *Given any periodic C^2 curve $x\colon \mathbb{R} \to M$, let*

$$\begin{aligned} d &= \sup\{|x(t_1) - x(t_2)| : t_1, t_2 \in \mathbb{R}\}, \\ u &= \sup\{|x'(t)| : t \in \mathbb{R}\}, \\ F &= \sup\{|x''_\pi(t)| : t \in \mathbb{R}\}, \\ K &= \sup\{k(x(t), x'(t)) : t \in \mathbb{R}\}. \end{aligned}$$

If $Kd < 1$, then

$$u^2 \leq \frac{Fd}{1 - Kd}.$$

Remark 4.2. Lemma 4.1 implies that any non-constant closed geodesic must satisfy $Kd \geq 1$; since otherwise, F being zero, one would get $u = 0$. For example, for a non-constant closed geodesic on a sphere in $\mathbb{R}^3$ one has $Kd = 2$.

In the sequel, we are interested in the existence of branches of nontrivial T-forced pairs that, roughly speaking, emanate from a trivial pair $(0, q_*^-)$, with q_* a bifurcation point of (4.1).

To this end, we introduce the *average force* $\bar{f} \colon M \to \mathbb{R}^k$ given by

$$\bar{f}(q) = \frac{1}{T} \int_0^T f(t, q^-)\, dt,$$

which is clearly a vector field tangent to M. Throughout the paper $\bar{f}$ will play a crucial role in obtaining our continuation results for (4.1).

Observe that, in order to avoid the use of an extra notation, given $q \in M$, we will still denote by q^- the constant q-valued function defined on $\mathbb{R}$ or on any convenient real interval. The actual domain of q^- will be clear from the context (in the above formula, for example, the domain of q^- is $(-\infty, 0]$).

Theorem 4.3 below, provides a necessary condition for $q_* \in M$ to be a bifurcation point. Namely, it asserts that any bifurcation point is contained in the following subset of X^*:

$$X^*_= = \big\{(0, q^-) \in X^* : \bar{f}(q) = 0\big\}. \tag{4.5}$$

In the sequel, to avoid misleading overlaps with the standard notation x_t used in the RFDEs context, we will denote by x^n the n-th term of a sequence of functions in $C^1_T(M)$.

Theorem 4.3. *Let $M \subseteq \mathbb{R}^k$ be a boundaryless smooth manifold and $f \colon \mathbb{R} \times \widetilde{M} \to \mathbb{R}^k$ a functional field on M, T-periodic in the first variable. If $q_* \in M$ is a bifurcation point of* (4.1)*, then $\bar{f}(q_*) = 0$. Consequently, the set*

$$X^*_{\neq} = \{(0, q^-) \in X^* : \bar{f}(q) \neq 0\} \tag{4.6}$$

is a relatively open subset of the set X of the T-forced pairs.

Proof. Let $q_* \in M$ be a bifurcation point of (4.1). Thus, there exists a sequence $\big\{(\lambda_n, x^n)\big\}$ of nontrivial T-forced pairs of (4.1) converging to $(0, q_*^-)$ in $[0, +\infty) \times C^1_T(M)$. Since any (λ_n, x^n) is a nontrivial pair, as previously observed we have $\lambda_n > 0$ for $n \in \mathbb{N}$ sufficiently large. We also have $x^n(t) \to q_*$ uniformly on $\mathbb{R}$, so that there exists a compact subset of M containing $x^n(t)$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence, there exists a positive constant K such that

$$|r(x^n(t), (x^n)'(t))| = k(x^n(t), (x^n)'(t))\, |(x^n)'(t)|^2 \leq K |(x^n)'(t)|^2.$$

By integrating on $[0, T]$ the equality

$$(x^n)''(t) = r(x^n(t), (x^n)'(t)) + \lambda_n f(t, x^n_t),$$

we get

$$0 = (x^n)'(T) - (x^n)'(0) = \int_0^T r(x^n(t), (x^n)'(t))\, dt + \lambda_n \int_0^T f(t, x^n_t)\, dt$$

which implies

$$\lambda_n \left| \int_0^T f(t, x_t^n)\, dt \right| \leq \int_0^T K|(x^n)'(t)|^2\, dt. \tag{4.7}$$

Now, observe that the sequence of curves $t \mapsto (t, x_t^n) \in \mathbb{R} \times \widetilde{M}$ converges uniformly to $t \mapsto (t, q_*^-)$ for $t \in [0,T]$. Hence, because of Lemma 2.2, $f(t, x_t^n) \to f(t, q_*^-)$ uniformly for $t \in [0,T]$ and, thus, there exists $F > 0$ such that $|f(t, x_t^n)| \leq F$. Let $d_n = \max\{|x^n(t_1) - x^n(t_2)| : t_1, t_2 \in [0,T]\}$. Since $d_n \to 0$ as $n \to +\infty$, one has $Kd_n < 1$ for n sufficiently large. Therefore, by applying Lemma 4.1, we obtain

$$|(x^n)'(t)|^2 \leq \lambda_n \frac{Fd_n}{1 - Kd_n},$$

for n sufficiently large and all $t \in [0,T]$. Consequently, by (4.7),

$$\left| \int_0^T f(t, x_t^n)\, dt \right| \leq KT \frac{Fd_n}{1 - Kd_n}.$$

Thus, passing to the limit, we get the necessary condition for q_* to be a bifurcation point: $\bar{f}(q_*) = 0$.

Finally, the last assertion follows easily from the necessary condition, taking into account that the tangent vector field $\bar{f} \colon M \to \mathbb{R}^k$ is continuous. □

Let now Ω be an open subset of $[0, +\infty) \times C_T^1(M)$. Our main result (Theorem 4.4 below) provides a sufficient condition for the existence of a bifurcation point q_* in M with $(0, q_*^-) \in \Omega$. More precisely, we give conditions which ensure the existence in Ω of a connected subset of nontrivial T-forced pairs of equation (4.1) (an Ω-*global bifurcating branch* for short), whose closure in Ω is noncompact and intersects the set of trivial T-forced pairs contained in Ω.

Theorem 4.4. *Let $M \subseteq \mathbb{R}^k$ be a boundaryless smooth manifold, $f \colon \mathbb{R} \times \widetilde{M} \to \mathbb{R}^k$ a functional field on M, T-periodic in the first variable and locally Lipschitz in the second one, and $\bar{f} \colon M \to \mathbb{R}^k$ the autonomous tangent vector field*

$$\bar{f}(q) = \frac{1}{T} \int_0^T f(t, q^-)\, dt\,.$$

Let Ω be an open subset of $[0, +\infty) \times C_T^1(M)$ and let $j \colon M \to [0, +\infty) \times C_T^1(M)$ be the embedding $q \mapsto (0, q^-)$. Assume that $\deg(\bar{f}, j^{-1}(\Omega))$ is defined and nonzero. Then, Ω contains a connected subset of nontrivial T-forced pairs of the equation (4.1) whose closure in Ω is noncompact and meets the set $\big\{(0, q^-) \in \Omega : \bar{f}(q) = 0\big\}$.

The proof of Theorem 4.4 requires some preliminaries and will start on page 16.

As in Section 3, denote by $\widehat{TM}$ the set $C([-T,0], TM)$ and by $\widehat{TM}_*$ the subset of $\widehat{TM}$ of those elements (φ, ψ) such that $(\varphi(-T), \psi(-T)) = (\varphi(0), \psi(0))$.

Let U be an open neighborhood of $\widehat{TM}_*$ in $\widehat{TM}$ as in Lemma 3.4. Given $\lambda \geq 0$ and $(\varphi, \psi) \in U$, consider the initial value problem

$$\begin{cases} x'(t) = y(t), \\ y'(t) = r(x(t), y(t)) + \lambda f(t, x_t), \\ x_0 = \widetilde{\varphi}, \\ y_0 = \widetilde{\psi}, \end{cases} \tag{4.8}$$

where $(\widetilde{\varphi}, \widetilde{\psi}) \in \widetilde{TM}$ is the extension of (φ, ψ) as in Lemma 3.4.

Since f is locally Lipschitz in the second variable, for any $\lambda \in [0, +\infty)$ the uniqueness of the maximal solution of problem (4.8) is ensured (recall Remark 3.3) and, because of Corollary 3.2 and Lemma 3.4, the set

$$D = \big\{(\lambda, (\varphi, \psi)) \in [0, +\infty) \times U : \text{the maximal solution of (4.8)} \\ \text{is defined up to } T\big\} \tag{4.9}$$

is an open subset of $[0, +\infty) \times \widehat{TM}$. Also observe that D_0, the slice of D at $\lambda = 0$, contains the set $\big\{(q^-, 0) \in \widehat{TM} : q \in M\big\}$, which is a natural copy of M in $\widehat{TM}$.

Define $P\colon D \to \widehat{TM}$ by

$$P(\lambda, (\varphi, \psi)(\theta) = (x^{(\lambda,(\widetilde{\varphi},\widetilde{\psi}))}(\theta + T), y^{(\lambda,(\widetilde{\varphi},\widetilde{\psi}))}(\theta + T)), \quad \theta \in [-T, 0],$$

where $(x^{(\lambda,(\widetilde{\varphi},\widetilde{\psi}))}, y^{(\lambda,(\widetilde{\varphi},\widetilde{\psi}))})$ denotes the maximal solution of problem (4.8). We observe that, given $\lambda \geq 0$, the partial map $P(\lambda, \cdot)$ is, in some sense, a Poincaré-type T-translation operator of system (4.3) corresponding to the frozen parameter λ. In fact, if the solution of problem (4.8) is defined up to T, $P(\lambda, (\varphi, \psi))$ is the back-shift to the interval $[-T, 0]$ of the restriction to $[0, T]$ of this solution.

By Theorem 3.5 with $\widehat{N} = \widehat{TM}$, $\mathcal{E} = D$, $\mathcal{H} = P$, $\eta = (\varphi, \psi)$, $\alpha = \lambda$, $z = (x, y)$, we get that the fixed points of $P(\lambda, \cdot)$ correspond to the T-periodic solutions of system (4.3). More precisely, the coincidence set

$$S = \big\{(\lambda, (\varphi, \psi)) \in D : P(\lambda, (\varphi, \psi)) = (\varphi, \psi)\big\}$$

is the image of the set X of the T-forced pairs under the injective linear operator

$$\rho\colon \mathbb{R} \times C^1_T(\mathbb{R}^k) \to \mathbb{R} \times C([-T, 0], \mathbb{R}^k \times \mathbb{R}^k)$$

given by $(\lambda, x) \mapsto (\lambda, (\varphi, \psi))$, where φ and ψ are the restrictions of x and x' to the interval $[-T, 0]$ respectively.

As for the set X, also for S we may introduce the subsets labeled with the symbols $*$, $=$, $\neq$ (see formulas (4.4), (4.5), (4.6)). Each one of these sets is the image under ρ of the analogous subset of X with the same label. Actually, as we shall see, if two subsets, one of X and one of S, have the same label, they correspond homeomorphically under ρ.

Remark 4.5. The $(\lambda = 0)$-slice S_0 of S contains the following copy of M:

$$\big\{(q^-, 0) \in \widehat{TM} : q \in M\big\},$$

made up of the *trivial fixed points* of $P(0, \cdot)$. Obviously S_0 contains as well the set $\mathcal{G}_T$ of the nontrivial fixed points of $P(0, \cdot)$; that is, the elements $(\varphi, \psi) \in \widehat{TM}$ such that φ and ψ are, respectively, the restrictions to the interval $[-T, 0]$ of a non-constant T-periodic geodesic $x\colon \mathbb{R} \to M \subseteq \mathbb{R}^k$ and of its derivative $x'\colon \mathbb{R} \to \mathbb{R}^k$.

The following crucial lemma shows that the nontrivial fixed points of $P(0, \cdot)$ do not contribute in the computation of the fixed point index of the operator $P(\lambda, \cdot)$, provided that $\lambda > 0$ is sufficiently small.

Lemma 4.6. *Let V be an open subset of $\widehat{TM}$. Assume that for some $\delta > 0$ the set $[0, \delta] \times \overline{V}$ is contained in the domain D of the Poincaré operator P and its image under P is relatively compact in $\widehat{TM}$. If $P(0, \cdot)$ has no fixed points along ∂V and no trivial fixed points in V, then for $\lambda \geq 0$ sufficiently small one has $\mathrm{ind}_{\widehat{TM}}(P(\lambda, \cdot), V) = 0$.*

Proof. Observe first that, because of the compactness of the restriction of P to the set $[0,\delta]\times\overline{V}$ and by virtue of the assumption that $P(0,\cdot)$ is fixed point free along ∂V, $P(\lambda,\cdot)$ remains fixed point free along ∂V also for $\lambda>0$ sufficiently small. Taking δ smaller, if necessary, we may assume that this holds for $0\le\lambda\le\delta$. Thus $\mathrm{ind}_{\widehat{TM}}(P(\lambda,\cdot),V)$ is defined for any $\lambda\in[0,\delta]$ and, as a consequence of the homotopy invariance, this index does not depend on λ. Therefore, it is sufficient to prove that $\mathrm{ind}_{\widehat{TM}}(P(0,\cdot),V)=0$. To this purpose consider the equation

$$x''_\pi(t)=-\varepsilon x'(t), \tag{4.10}$$

obtained by adding to the inertial motion equation on M a frictional force with coefficient $\varepsilon\ge 0$.

As was done for the definition of P, associated to the equation (4.10) one may consider a Poincaré-type translation operator $G\colon[0,+\infty)\times\overline{V}\to\widehat{TM}$. Precisely, given $(\varepsilon,(\varphi,\psi))\in[0,+\infty)\times\overline{V}$, $G(\varepsilon,(\varphi,\psi))$ is the back-shift to the interval $[-T,0]$ of the restriction to the interval $[0,T]$ of the solution of the initial value problem

$$\begin{cases} x'(t)=y(t),\\ y'(t)=r(x(t),y(t))-\varepsilon y(t),\\ x_0=\widetilde{\varphi},\\ y_0=\widetilde{\psi}, \end{cases}$$

provided that this solution is defined up to T. Clearly, (4.10) being an ordinary differential equation, the solution of the above problem depends only on the initial point $(\varphi(0),\psi(0))\in TM$.

We claim that the operator G is defined on the whole set $[0,+\infty)\times\overline{V}$. To see this, observe first that $G(0,\cdot)=P(0,\cdot)$. Consequently, $G(0,\cdot)$ is defined on $\overline{V}$, since so is $P(0,\cdot)$. Now, the bigger is ε, the slower is the motion on M. Thus, when $\varepsilon=0$, if a solution $x(t)$ of (4.10) is defined up to T, the same property is *a fortiori* verified, when $\varepsilon>0$, by the solution $u(t)$ with the same initial condition $(x(0),x'(0))$ as $x(t)$. Actually, as one can check, for $t\ge 0$ one has $u(t)=x(\alpha(t))$, where $\alpha\colon\mathbb{R}\to\mathbb{R}$ is the unique function such that $\alpha''(t)=-\varepsilon\alpha'(t)$, $\alpha(0)=0$, $\alpha'(0)=1$. This proves our claim since $\alpha(t)=\frac{1}{\varepsilon}(1-e^{-\varepsilon t})<t$ for $t>0$.

If $\varepsilon>0$, the only T-periodic solutions of (4.10) are the trivial ones; that is, the constant functions $q^-\in C^1_T(M)$ with $q\in M$. This can be easily deduced, for example, from the fact that if $u(t)$ is a non-stationary solution of (4.10), then its kinetic energy $\frac{1}{2}|u'(t)|^2$, when $\varepsilon>0$, is strictly decreasing. Consequently, when $\varepsilon>0$, the fixed points of the operator $G(\varepsilon,\cdot)$ are all trivial, i.e. elements of $\widehat{TM}$ of the type $(q^-,0)$, with $q\in M$. Since, by assumption, the set $\overline{V}$ does not contain any one of these points, then $\mathrm{ind}_{\widehat{TM}}(G(\varepsilon,\cdot),V)=0$ for all $\varepsilon>0$. Hence, recalling that $G(0,\cdot)=P(0,\cdot)$ and that $P(0,\cdot)$ is fixed point free along ∂V, the homotopy invariance property of the fixed point index implies

$$0=\mathrm{ind}_{\widehat{TM}}(G(0,\cdot),V)=\mathrm{ind}_{\widehat{TM}}(P(0,\cdot),V),$$

and the assertion follows applying the homotopy invariance to the family of operators $P(\lambda,\cdot)$, $\lambda\in[0,\delta]$. □

The following lemma states another crucial property of the set of the T-forced pairs.

Lemma 4.7. *The set X of the T-forced pairs of* (4.1) *is closed in* $[0,+\infty)\times C^1_T(M)$.

Proof. Let $\{(\lambda_n, x^n)\}$ be a sequence in X that converges to an element $(\bar\lambda, \bar x)$ in $[0,+\infty) \times C^1_T(M)$. Observe that the sequence of functions $t \mapsto (t, x^n_t) \in \mathbb{R} \times \widetilde{M}$ converges uniformly to $t \mapsto (t, \bar x_t)$ for $t \in \mathbb{R}$. Hence, because of Lemma 2.2 and the T-periodicity of the above functions, $\lambda_n f(t, x^n_t) \to \bar\lambda f(t, \bar x_t)$ uniformly for $t \in \mathbb{R}$. Analogously, $r(x^n(t), (x^n)'(t))$ converges uniformly to $r(\bar x(t), \bar x'(t))$. Thus $(x^n)''(t)$ converges uniformly in $\mathbb{R}$ to $\alpha(t) := r(\bar x(t), \bar x'(t)) + \bar\lambda f(t, \bar x_t)$. Therefore, taking into account that $(x^n)'(t)$ converges to $\bar x'(t)$ for $t \in \mathbb{R}$, we get that $\bar x'(t)$ is differentiable and its derivative, $\bar x''(t)$, coincides with $\alpha(t)$. □

We turn now our attention to the following two-parameter family of retarded functional motion equations on M:

$$x''_\pi(t) = \lambda\left((1-\mu)f(t,x_t) + \mu\bar f(x(t))\right), \quad \lambda \geq 0,\ \mu \in [0,1]. \tag{4.11}$$

Notice that for $\mu = 0$ this equation coincides with (4.1), while for $\mu = 1$ one gets a second order autonomous ordinary differential equation on M.

The triples $(\lambda, x, \mu) \in [0,+\infty) \times C^1_T(M) \times [0,1]$, with $x\colon \mathbb{R} \to M$ a forced oscillation of (4.11) corresponding to λ and μ, will be called *T-forced triples* of (4.11). Denote by Ξ the set of these triples and by

$$\Xi^* = \left\{(0, q^-, \mu) : q \in M,\ \mu \in [0,1]\right\}$$

the distinguished subset of Ξ of the *trivial T-forced triples.* These sets are metric spaces, as subsets of the Banach space $\mathbb{R} \times C^1_T(\mathbb{R}^k) \times \mathbb{R}$ with norm

$$\|(\lambda, x, \mu)\| = |\lambda| + \max_{t\in\mathbb{R}} |x(t)| + \max_{t\in\mathbb{R}} |x'(t)| + |\mu|. \tag{4.12}$$

Notice that the slices at $\mu = 0$ of Ξ and Ξ^* are, respectively, the set X of the T-forced pairs of the equation (4.1) and its subset X^* of the trivial ones.

The two-parameter equation (4.11) can be transformed into the following family of first order systems on TM:

$$\begin{cases} x'(t) = y(t), \\ y'(t) = r(x(t), y(t)) + \lambda\left((1-\mu)f(t,x_t) + \mu\bar f(x(t))\right), \ \lambda \geq 0,\ \mu \in [0,1]. \end{cases} \tag{4.13}$$

Given $\lambda \geq 0$, $\mu \in [0,1]$ and $(\varphi, \psi) \in U$, with U as above, consider the initial value problem

$$\begin{cases} x'(t) = y(t), \\ y'(t) = r(x(t), y(t)) + \lambda\left((1-\mu)f(t,x_t) + \mu\bar f(x(t))\right), \\ x_0 = \widetilde\varphi, \\ y_0 = \widetilde\psi, \end{cases} \tag{4.14}$$

where $(\widetilde\varphi, \widetilde\psi) \in \widetilde{TM}$ is the extension of (φ, ψ) as in Lemma 3.4.

Since f is locally Lipschitz in the second variable, then it is easy to see that $\bar f$ is locally Lipschitz as well. Hence, for any $\lambda \in [0,+\infty)$ and $\mu \in [0,1]$, the uniqueness of the maximal solution of problem (4.14) is ensured (recall Remark 3.3) and, because of Corollary 3.2 and Lemma 3.4, the set

$$E = \big\{(\lambda, (\varphi,\psi), \mu) \in [0,+\infty) \times U \times [0,1] : \text{the maximal solution of (4.14)} \\ \text{is defined up to } T\big\}$$

is an open subset of $[0,+\infty) \times \widehat{TM} \times [0,1]$. Also observe that E_0, the slice of E at $\lambda = 0$, contains the set

$$\left\{((q^-, 0), \mu) \in \widehat{TM} \times [0,1] : q \in M,\ \mu \in [0,1]\right\}.$$

Define $H\colon E \to \widehat{TM}$ by

$$H(\lambda, (\varphi, \psi), \mu)(\theta) = (x^{(\lambda,(\widetilde{\varphi},\widetilde{\psi}),\mu)}(\theta + T), y^{(\lambda,(\widetilde{\varphi},\widetilde{\psi}),\mu)}(\theta + T)), \quad \theta \in [-T, 0],$$

where $(x^{(\lambda,(\widetilde{\varphi},\widetilde{\psi}),\mu)}, y^{(\lambda,(\widetilde{\varphi},\widetilde{\psi}),\mu)})$ denotes the maximal solution of problem (4.14). Notice that, given $\lambda \geq 0$ and $\mu \in [0, 1]$, the map $H(\lambda, \cdot, \mu)$ is a Poincaré-type T-translation operator of system (4.13) corresponding to the frozen parameters λ and μ. In fact, if the solution of problem (4.14) is defined up to T, $H(\lambda, (\varphi, \psi), \mu)$ is the back-shift to the interval $[-T, 0]$ of the restriction to $[0, T]$ of this solution.

By Theorem 3.5, with $\widehat{N} = \widehat{TM}$, $\mathcal{E} = E$, $\mathcal{H} = H$, $\eta = (\varphi, \psi)$, $\alpha = (\lambda, \mu)$, $z = (x, y)$, we get that the fixed points of $H(\lambda, \cdot, \mu)$ correspond to the T-periodic solutions of system (4.13). More precisely, the coincidence set

$$\Sigma = \big\{(\lambda, (\varphi, \psi), \mu) \in E : H(\lambda, (\varphi, \psi), \mu) = (\varphi, \psi)\big\}$$

is the image of the set Ξ of the T-forced triples under the injective linear operator

$$\sigma\colon \mathbb{R} \times C^1_T(\mathbb{R}^k) \times \mathbb{R} \to \mathbb{R} \times C([-T, 0], \mathbb{R}^k \times \mathbb{R}^k) \times \mathbb{R}$$

given by $(\lambda, x, \mu) \mapsto (\lambda, (\varphi, \psi), \mu)$, where φ and ψ are the restrictions of x and x' to the interval $[-T, 0]$ respectively.

We have already introduced a norm in the domain of σ (see formula (4.12)). As regards the codomain $\mathbb{R} \times C([-T, 0], \mathbb{R}^k \times \mathbb{R}^k) \times \mathbb{R}$, defining

$$\|(\lambda, (\varphi, \psi), \mu)\| = |\lambda| + \max_{t\in[-T,0]} |\varphi(t)| + \max_{t\in[-T,0]} |\psi(t)| + |\mu|,$$

the map σ establishes an isometry between the two metric spaces Ξ and Σ. Anyhow, what is important in the sequel is that σ maps Ξ onto Σ, homeomorphically.

Observe that the $(\mu = 0)$-slices of Ξ and Σ are the sets X and S, that have been already introduced and denoted, respectively, with the Latin letters corresponding to Ξ and Σ. Obviously X and S correspond, homeomorphically, under the partial map $\rho = \sigma(\cdot, \cdot, 0)$ of σ obtained by freezing $\mu = 0$. Roughly speaking, the elements of S are the initial conditions of all the T-forced pairs of (4.1).

Finally, Theorem 3.6 implies that the operator H is locally compact. Therefore, taking into account that the projection $(\lambda, (\varphi, \psi), \mu) \mapsto (\varphi, \psi)$ is a locally proper map, the coincidence set Σ is locally compact. Consequently, so is the set Ξ of the T-forced triples, being homeomorphic to Σ. Hence the set X of the T-forced pairs is locally compact, as a slice of a locally compact space. In fact, it is homeomorphic to the closed subset $X \times \{0\}$ of Ξ.

Proof of Theorem 4.4. Let, as before, X denote the set of all the T-forced pairs of the equation (4.1). Recall that this set is closed in $[0, +\infty) \times C^1_T(M)$, as stated in Lemma 4.7, and locally compact, as observed above. As in (4.4), denote by X^* the distinguished subset of X of the trivial T-forced pairs, and consider the subset $X^*_=$, defined in (4.5), of the elements of X^* that satisfy the necessary condition for bifurcation. According to this notation, to prove Theorem 4.4 we need to show that there exists a connected subset of $(X\backslash X^*) \cap \Omega$ whose closure in Ω is noncompact and intersects $X^*_=$.

Now, define

$$\check{X} = (X\backslash X^*) \cup X^*_=$$

and observe that this set is closed in $[0, +\infty) \times C^1_T(M)$. In fact, $\check{X}$ coincides with $X\backslash X^*_{\neq}$, where $X^*_{\neq}$, defined in (4.6), is a relatively open subset of X because of

Theorem 4.3. Thus, the closure in $\check{X} \cap \Omega$ of a subset of $\check{X} \cap \Omega$ is the same as in Ω. Consequently, to prove the assertion of Theorem 4.4 it is enough to show that there exists a connected subset of $(X \backslash X^*) \cap \Omega$ whose closure in $\check{X} \cap \Omega$ is noncompact and intersects $X^*_=$. Now, since X is locally compact, the closed subset $\check{X}$ of X is locally compact as well. Hence, so is the open subset $\check{X} \cap \Omega$ of $\check{X}$.

Put $Y = \check{X} \cap \Omega$ and observe that the subset $K = X^*_= \cap \Omega$ of Y is compact because of the assumption that $\deg(\bar{f}, j^{-1}(\Omega))$ is well defined. Therefore, according to Lemma 2.1, to prove Theorem 4.4 it is enough to show that any compact subset of Y containing K has nonempty boundary in Y. By contradiction, suppose that there exists a compact subset C of Y containing K with empty boundary in Y. This implies that C is relatively open in Y and, therefore, in $\check{X}$ as well. Consequently, the image C' of C under the isometry ρ is a compact, relatively open subset of

$$\check{S} = (S \backslash S^*) \cup S^*_= = \rho(\check{X}).$$

Thus, there exists an open subset W of $[0, +\infty) \times \widehat{TM}$ such that $C' = \check{S} \cap W$ and $\check{S} \cap \partial W = \emptyset$.

Notice that the coincidence set S is contained in the open set D defined in (4.9). Further, observe that

$$D = \big\{(\lambda, (\varphi, \psi)) \in [0, +\infty) \times U : (\lambda, (\varphi, \psi), 0) \in E\big\}$$

and $P(\lambda, (\varphi, \psi)) = H(\lambda, (\varphi, \psi), 0)$. Therefore, we may assume that the closure $\overline{W}$ of W is contained in D, so that the Poincaré-type T-translation operator $P \colon D \to \widehat{TM}$ as in (4.9) is well defined on $\overline{W}$. We recall that, due to Theorem 3.5, any fixed point of $P(\lambda, \cdot)$ is a "starting element" of a T-periodic solution of (4.1). Observe also that the set of fixed points of $P(\lambda, \cdot)$ in the slice W_λ of W is the slice C'_λ of C'.

Since C' is compact, for any $\lambda > 0$ the fixed point index $\operatorname{ind}_{\widehat{TM}}(P(\lambda, \cdot), W_\lambda)$ is defined and independent of λ because of the generalized homotopy invariance. Now, C'_λ being empty for some $\lambda > 0$, we get

$$\operatorname{ind}_{\widehat{TM}}(P(\lambda, \cdot), W_\lambda) = 0 \tag{4.15}$$

for all $\lambda > 0$. Therefore, the assertion will follow, by contradiction, if we show that

$$\operatorname{ind}_{\widehat{TM}}(P(\lambda, \cdot), W_\lambda) \neq 0$$

for some $\lambda > 0$. We will prove that this holds for $\lambda > 0$ sufficiently small. To this purpose, it is convenient to assume that $\overline{W}$ is mapped by P into a relatively compact subset of $\widehat{TM}$. This is possible because of the local compactness of P and the compactness of C'.

Notice that $\operatorname{ind}_{\widehat{TM}}(P(0, \cdot), W_0)$ could be undefined. In fact, the set of fixed points of $P(0, \cdot)$ coincides with $W_0 \cap S_0$ and, as already pointed out (see Remark 4.5) the ($\lambda = 0$)-slice S_0 of S contains a copy of the possibly noncompact manifold M, made up of the trivial fixed points of $P(0, \cdot)$. Recall that S_0 contains as well the set $\mathcal{G}_T$ of the nontrivial fixed points of $P(0, \cdot)$.

Let $V \subseteq \widehat{TM}$ be an open neighborhood of the ($\lambda = 0$)-slice C'_0 of C' with the property that $[0, \lambda_+] \times \overline{V}$ is contained in W for some $\lambda_+ > 0$. The existence of such an open set is ensured by the compactness of C'_0.

Since the closed subset $\overline{W}_0 \backslash V$ of $\widehat{TM}$ does not contain elements of C'_0, the compactness of C' implies that there are no fixed points of $P(\lambda, \cdot)$ in $\overline{W}_\lambda \backslash V$ for $\lambda > 0$ sufficiently small (observe that, for $\lambda > 0$, the set of fixed points in $\overline{W}_\lambda$ of

$P(\lambda,\cdot)$ coincides with C'_λ, even if it is not so for $\lambda = 0$). Therefore, for these values of λ, the excision property of the fixed point index implies

$$\mathrm{ind}_{\widehat{TM}}(P(\lambda,\cdot), W_\lambda) = \mathrm{ind}_{\widehat{TM}}(P(\lambda,\cdot), V). \tag{4.16}$$

Recalling that non-constant geodesics cannot accumulate to a given point, we may assume that V is the disjoint union of two open sets. One of them containing the set $\mathcal{G}_T \cap V$ of all the nontrivial fixed points of $P(0,\cdot)$ in V, and the other one, say V_-, containing all the remaining elements of C'_0 (that is, among the trivial fixed points $(q^-, 0) \in V$, only those satisfying the necessary condition for bifurcation $\bar{f}(q) = 0$).

Using the additivity property of the fixed point index and applying Lemma 4.6, for $\lambda > 0$ small we get

$$\mathrm{ind}_{\widehat{TM}}(P(\lambda,\cdot), V) = \mathrm{ind}_{\widehat{TM}}(P(\lambda,\cdot), V_-),$$

and from (4.16) one has

$$\mathrm{ind}_{\widehat{TM}}(P(\lambda,\cdot), W_\lambda) = \mathrm{ind}_{\widehat{TM}}(P(\lambda,\cdot), V_-), \tag{4.17}$$

for the above values of λ.

We claim that, taking both V_- and λ_+ smaller if necessary, we have

$$\mathrm{ind}_{\widehat{TM}}(P(\lambda,\cdot), V_-) = \mathrm{ind}_{\widehat{TM}}(H_\lambda(\cdot,0), V_-) = \mathrm{ind}_{\widehat{TM}}(H_\lambda(\cdot,1), V_-). \tag{4.18}$$

for all $\lambda \in (0, \lambda_+]$. In fact, consider the set

$$\mathcal{O} = \big\{(\lambda,(\varphi,\psi)) \in [0,+\infty) \times U : (\lambda,(\varphi,\psi),\mu) \in E,\ \forall\, \mu \in [0,1]\big\},$$

which is clearly open, due to the compactness of $[0,1]$. Moreover, it contains the compact set

$$K' = \{0\} \times (C'_0 \cap V_-) = \{(0,(q^-,0)) \in W : \bar{f}(q) = 0\} = \rho(K).$$

Thus, if necessary taking both V_- and λ_+ smaller, we may assume that $[0,\lambda_+] \times \overline{V}_-$ is contained in $\mathcal{O}$, so that the homotopy

$$H_\lambda \colon \overline{V} \times [0,1] \to \widehat{TM},$$

given by $((\varphi,\psi),\mu) \mapsto H(\lambda,(\varphi,\psi),\mu)$, is well defined for any $\lambda \in [0,\lambda_+]$.

Since, as already pointed out, the map $H\colon E \to \widehat{TM}$ is locally compact, without loss of generality we may suppose that the image under H of the box

$$\Delta = [0,\lambda_+] \times \overline{V}_- \times [0,1]$$

is relatively compact in $\widehat{TM}$. This is possible due to the compactness of K' and $[0,1]$, again taking V_- and λ_+ smaller, if necessary. Of course, after having chosen V_-, λ_+ can be taken so small that (4.17) holds for all $\lambda \in (0,\lambda_+]$.

Since, in $\widehat{TM}$, $H(\Delta)$ is relatively compact and Δ is closed, the coincidence set

$$\Sigma \cap \Delta = \big\{(\lambda,(\varphi,\psi),\mu) \in \Delta : H(\lambda,(\varphi,\psi),\mu) = (\varphi,\psi)\big\}$$

is compact. Recalling that $\Sigma^*_{\neq}$ is an open subset of Σ and that $\check{\Sigma} = \Sigma \backslash \Sigma^*_{\neq}$, we get that $\check{\Sigma} \cap \Delta$ is closed in $\Sigma \cap \Delta$ and consequently compact. Therefore, so is its image $\check{F}$ under the projection of Δ onto $[0,\lambda_+] \times \overline{V}_-$.

Notice that, given $\lambda \in (0,\lambda_+]$, for the slice $\check{F}_\lambda = \{(\varphi,\psi) \in \overline{V}_- : (\lambda,(\varphi,\psi)) \in \check{F}\}$ of $\check{F}$ one has

$$\check{F}_\lambda = \{(\varphi,\psi) \in \overline{V}_- : (\varphi,\psi) \text{ is a fixed point of } H_\lambda(\cdot,\mu) \text{ for some } \mu \in [0,1]\}. \tag{4.19}$$

We point out that this is not so for the $(\lambda = 0)$-slice of $\check{F}$ since, because of the choice of V_-, $\check{F}_0$ coincides only with the set of trivial fixed points $(q^-, 0)$ of $P(0, \cdot)$ in V_- satisfying the necessary condition for bifurcation $\bar{f}(q) = 0$. Since $\check{F}_0$ does not intersect the closed set ∂V_-, the compactness of $\check{F}$ implies that the same holds for any slice $\check{F}_\lambda$ with $\lambda > 0$ sufficiently small. Taking λ_+ smaller, if necessary, we may assume that this happens for all $\lambda \in (0, \lambda_+]$.

Now, fix any $\lambda \in (0, \lambda_+]$ and observe that the homotopy invariance property of the fixed point index applies to the compact homotopy $H_\lambda \colon \overline{V}_- \times [0,1] \to \widehat{TM}$ since, because of (4.19), the condition $\check{F}_\lambda \cap \partial V_- = \emptyset$ means that $H_\lambda((\varphi, \psi), \mu) \neq (\varphi, \psi)$ for all $((\varphi, \psi), \mu) \in \partial V_- \times [0, 1]$. Thus,

$$\mathrm{ind}_{\widehat{TM}}(H_\lambda(\cdot, 0), V_-) = \mathrm{ind}_{\widehat{TM}}(H_\lambda(\cdot, 1), V_-)$$

and, recalling that $H_\lambda(\cdot, 0)$ coincides with the Poincaré operator $P(\lambda, \cdot)$, we get that the equality (4.18) holds, as claimed.

Consequently, taking into account (4.17), we obtain

$$\mathrm{ind}_{\widehat{TM}}(P(\lambda, \cdot), W_\lambda) = \mathrm{ind}_{\widehat{TM}}(H_\lambda(\cdot, 1), V_-),$$

for all $\lambda \in (0, \lambda_+]$. Now, as we already pointed out, the assertion of the theorem follows, by contradiction, if we show that $\mathrm{ind}_{\widehat{TM}}(P(\lambda, \cdot), W_\lambda) \neq 0$ for some $\lambda > 0$. To this purpose, we shall compute $\mathrm{ind}_{\widehat{TM}}(H_\lambda(\cdot, 1), V_-)$ for $\lambda > 0$ small.

Let

$$V' = \{(q, v) \in TM : (q, v)^- \in V_-\}$$

and let

$$Q \colon [0, +\infty) \times V' \to TM$$

be the classical (finite dimensional) Poincaré T-translation operator

$$Q(\lambda, (q, v)) = (x^{(\lambda,(q,v))}(T), y^{(\lambda,(q,v))}(T)),$$

where $(x^{(\lambda,(q,v))}, y^{(\lambda,(q,v))})$ denotes the maximal solution of the undelayed system

$$\begin{cases} x'(t) = y(t), \\ y'(t) = r(x(t), y(t)) + \lambda \bar{f}(x(t)), \quad \lambda \geq 0\,, \\ x(0) = q, \\ y(0) = v. \end{cases}$$

An argument (based on the commutativity property of the fixed point index) analogous to that used in the proof of Lemma 3.8 (Step 5) of [4] shows that, for small values of λ, $\mathrm{ind}_{TM}(Q(\lambda, \cdot), V')$ is defined and

$$\mathrm{ind}_{\widehat{TM}}(H_\lambda(\cdot, 1), V_-) = \mathrm{ind}_{TM}(Q(\lambda, \cdot), V')\,.$$

Now, as shown in [10, Lemma 2.2], for $\lambda > 0$ sufficiently small we get

$$\mathrm{ind}_{TM}(Q(\lambda, \cdot), V') = (-1)^m \deg(\bar{f}, i^{-1}(V'))$$

where $m = \dim M$ and $i \colon M \to TM$ is the embedding $q \mapsto (q, 0)$.

Finally, by the excision property of the degree, we obtain

$$\deg(\bar{f}, i^{-1}(V')) = \deg(\bar{f}, j^{-1}(\Omega))$$

that, by assumption, is nonzero. Consequently, by taking into account (4.15) and the assumption $\deg(\bar{f}, j^{-1}(\Omega)) \neq 0$, we get a contradiction. This completes the proof. □

Remark 4.8 (On the meaning of global bifurcating branch)**.** In addition to the hypotheses of Theorem 4.4, assume that f sends bounded subsets of $\mathbb{R} \times \widetilde{M}$ into bounded subsets of $\mathbb{R}^k$, and that M is closed in $\mathbb{R}^k$ (or, more generally, that the closure $\overline{\Omega}$ of Ω in $[0, +\infty) \times C^1_T(M)$ is complete).

Then a connected subset Γ of Ω as in Theorem 4.4 is either unbounded or, if bounded, its closure $\overline{\Gamma}$ in $\overline{\Omega}$ reaches the boundary $\partial\Omega$ of Ω.

To see this, assume that $\overline{\Gamma}$ is bounded. Then, being $f(\overline{\Gamma})$ bounded, because of Ascoli's theorem, Γ is actually totally bounded. Thus, $\overline{\Gamma}$ is compact, being totally bounded and, additionally, complete since $\overline{\Gamma}$ is contained in $\overline{\Omega}$. On the other hand, according to Theorem 4.4, the closure $\overline{\Gamma} \cap \Omega$ of Γ in Ω is noncompact. Consequently, the subset $\overline{\Gamma} \setminus \Omega$ of $\overline{\Omega} \setminus \Omega = \partial\Omega$ is nonempty.

We give now some consequences of Theorem 4.4. The first one is in the spirit of a celebrated result due to P. H. Rabinowitz [31].

Corollary 4.9 (Rabinowitz-type global bifurcation result)**.** *Let M, f and $\bar{f}$ be as in Theorem 4.4. Assume that M is closed in $\mathbb{R}^k$ and that f sends bounded subsets of $\mathbb{R} \times \widetilde{M}$ into bounded subsets of $\mathbb{R}^k$. Let V be an open subset of M such that $\deg(\bar{f}, V) \neq 0$. Then, equation (4.1) has a connected subset of nontrivial T-forced pairs whose closure contains some $(0, q_0^-)$, with $q_0 \in V$, and is either unbounded or goes back to some $(0, q^-)$, where $q \notin V$.*

Proof. Let Ω be the open set obtained by removing from $[0, +\infty) \times C^1_T(M)$ the closed set $\{(0, q^-) : q \notin V\}$. In other words,

$$\Omega = \big([0, +\infty) \times C^1_T(M)\big) \setminus \big(\{0\} \times (M \setminus V)^-\big).$$

Observe that $\overline{\Omega}$ is complete, because of the closedness of M. Consider, by Theorem 4.4, a connected set $\Gamma \subseteq \Omega$ of nontrivial T-periodic pairs with noncompact closure (in Ω) and intersecting $\{0\} \times C^1_T(M)$ in a subset of $\{(0, q^-) \in \Omega : \bar{f}(q) = 0\}$. Suppose that Γ is bounded. From Remark 4.8 it follows that $\overline{\Gamma} \setminus \overline{\Gamma}_\Omega$, where $\overline{\Gamma}_\Omega$ denotes the closure of Γ in Ω, is nonempty and hence contains a point $(0, q^-)$ which does not belong to Ω, that is, such that $q \notin V$. □

Remark 4.10. The assumption of Corollary 4.9 above on the existence of an open subset V of M such that $\deg(\bar{f}, V) \neq 0$ is clearly satisfied in the case when $\bar{f}$ has an isolated zero with nonzero index. For example, if $\bar{f}(q) = 0$ and $\bar{f}$ is C^1 with injective derivative $\bar{f}'(q) \colon T_qM \to \mathbb{R}^k$, then q is an isolated zero of $\bar{f}$ and its index is either 1 or -1. In fact, in this case, $\bar{f}'(q)$ sends T_qM into itself and, consequently, its determinant is well defined and nonzero. The index of q is just the sign of this determinant (see e.g. [24]).

Corollary 4.11 below is a kind of continuation principle in the spirit of a well known result due to Jean Mawhin for ODEs in $\mathbb{R}^k$ [22, 23], and extends, partially, an analogous one obtained by the last two authors in [10] for ODEs on differentiable manifolds. In what follows, by a *T-periodic orbit* of $x''_\pi(t) = \lambda f(t, x_t)$ we mean the image of a T-periodic solution of this equation.

Corollary 4.11 (Mawhin-type continuation principle)**.** *Let M, f and $\bar{f}$ be as in Theorem 4.4. Assume that f sends bounded subsets of $\mathbb{R} \times \widetilde{M}$ into bounded subsets of $\mathbb{R}^k$. Let V be a relatively compact open subset of M and $R > 0$ such that*

(1) $\bar{f}(q) \neq 0$ *along the boundary ∂V of V;*

(2) $\deg(\bar{f}, V) \neq 0$;

(3) *for any* $\lambda \in [0,1]$*, the non-constant* T*-periodic orbits of* $x''_\pi(t) = \lambda f(t, x_t)$ *lying in* $\overline{V}$ *do not meet* ∂V *and satisfy* $|x'(t)| \leq R$*, for all* $t \in \mathbb{R}$.

Then, the equation

$$x''_\pi(t) = f(t, x_t),$$

has a T*-periodic orbit in* V.

Proof. Define $\Omega = [0, +\infty) \times C^1_T(V)$ and observe that the closure $\overline{\Omega}$ of Ω in the metric space $[0, +\infty) \times C^1_T(M)$ coincides with $[0, +\infty) \times C^1_T(\overline{V})$, which is clearly complete, since so is the closure $\overline{V}$ of V in M.

Because of conditions (1) and (2), Theorem 4.4 ensures the existence of a connected set $\Gamma \subseteq \Omega$ of nontrivial T-forced pairs whose closure in Ω is noncompact and meets the set $\{(0, q^-) \in \Omega : \bar{f}(q) = 0\}$.

It is enough to show that Γ contains a pair of the form $(1, x)$. Suppose not. Thus, due to the connectedness of Γ, any $(\lambda, x) \in \Gamma$ is such that $\lambda < 1$. In addition to this, the boundedness of V and the a priori estimate on the derivative of the forced oscillations ensured by condition (3) imply that Γ is bounded. Consequently, according to Remark 4.8, there exists at least one element $(\lambda_*, x_*) \in \partial\Omega \cap \overline{\Gamma}$. This implies $\lambda_* \in [0,1]$ and $x_*(t) \in \partial V$ for some $t \in \mathbb{R}$.

Since, as stated in Lemma 4.7, the set X of the T-forced pairs is closed in $[0, +\infty) \times C^1_T(M)$, we get that x_* is a forced oscillation of the equation $x''_\pi(t) = \lambda_* f(t, x_t)$. Thus, due to condition (3), x_* must be a constant function of the type q_*^- with $q_* \in \partial V$. Regarding λ_* there are two possibilities: $\lambda_* = 0$ and $\lambda_* \in (0,1]$. The first case can be excluded, since otherwise q_* would be a bifurcation point and, because of Theorem 4.3, we would get $\bar{f}(q_*) = 0$, which is excluded by condition (1). The second case is also impossible. In fact, if this were the case, the function $t \mapsto f(t, q_*^-)$ would be identically zero and, consequently, $\bar{f}$ would vanish at q_*, again contradicting condition (1).

In conclusion, the assumption that Γ does not contain a pair of the type $(1, x)$ turns out to be false, and the assertion follows. □

The following continuation result improves an analogous one obtained in [1], in which the map f is continuous on $\mathbb{R} \times C((-\infty, 0], M)$, with the compact-open topology in $C((-\infty, 0], M)$. In fact, such a coarse topology makes the assumption of the continuity of f a more restrictive condition than the one we require here.

Corollary 4.12. *Let* M *and* f *be as in Theorem 4.4. Assume that* M *is compact with nonzero Euler–Poincaré characteristic* $\chi(M)$ *and that* f *sends bounded subsets of* $\mathbb{R} \times \widetilde{M}$ *into bounded subsets of* $\mathbb{R}^k$*. Then, the equation* (4.1) *admits an unbounded connected set of nontrivial* T*-forced pairs whose closure meets the set*

$$\{(0, q^-) \in [0, +\infty) \times C^1_T(M) : q \in M\}$$

of the trivial T*-forced pairs.*

Proof. Apply Corollary 4.9 with $V = M$. By the Poincaré–Hopf theorem we have

$$\deg(\bar{f}, M) = \chi(M) \neq 0.$$

Thus, equation (4.1) has a connected subset of nontrivial T-forced pairs that, according to Corollary 4.9, turns out to be necessarily unbounded. □

We conclude with a result about the existence of forced oscillations for the retarded spherical pendulum. The proof is omitted since it is based on the existence of a global branch as in Corollary 4.12, as well as on the argument, that can be found in [2], about the forced motion of the spherical pendulum.

Theorem 4.13. *Let $f\colon \mathbb{R}\times BU((-\infty,0],S^2)\to\mathbb{R}^3$ be a T-periodic functional field on the unit sphere of $\mathbb{R}^3$. Suppose that f is bounded and locally Lipschitz in the second variable. Then, the motion equation*

$$x''_\pi(t) = f(t,x_t)$$

admits a forced oscillation.

We do not know whether or not the above result can be extended just replacing S^2 with any compact smooth manifold $M\subset\mathbb{R}^k$ with nonzero Euler–Poincaré characteristic.

References

[1] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *A continuation result for forced oscillations of constrained motion problems with infinite delay*, Adv. Nonlinear Stud. **13** (2013), 263–278.

[2] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *On the Existence of Forced Oscillations for the Spherical Pendulum Acted on by a Retarded Periodic Force*, J. Dyn. Diff. Equat. **23** (2011), 541–549, WOS:000295175300009, ISSN: 1040-7294 DOI: 10.1007/s10884-010-9201-2

[3] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *On General Properties of Retarded Functional Differential Equations on Manifolds*, Discrete Contin. Dynam. Syst. **33** (2013), no. 1, 27–46 (doi:10.3934/dcds.2013.33.27).

[4] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *Global continuation of periodic solutions for retarded functional differential equations on manifolds*, Special Issue on "Jean Mawhin's Achievements in Nonlinear Analysis", Boundary Value Problems 2013, **2013:21** (2013), pp. 19, doi:10.1186/1687-2770-2013-21.

[5] K. Borsuk, *Theory of retracts*, Polish Sci. Publ., Warsaw, 1967.

[6] R.F. Brown, *The Lefschetz fixed point theorem*, Scott, Foresman and Co., Glenview, Ill.-London, 1971.

[7] A. Dold, *Lectures on algebraic topology*, Springer-Verlag, Berlin, 1972.

[8] N. Dunford and J.T. Schwartz, *Linear Operators*, Wiley & Sons, Inc., New York, 1957.

[9] M. Furi and M.P. Pera, *A continuation principle for forced oscillations on differentiable manifolds*, Pacific J. Math. **121** (1986), 321–338.

[10] M. Furi and M.P. Pera, *A continuation principle for periodic solutions of forced motion equations on manifolds and applications to bifurcation theory*, Pacific J. Math. **160** (1993), 219–244.

[11] M. Furi, M.P. Pera and M. Spadini, *Periodic Solutions of Functional Differential Perturbations of Autonomous Differential Equations*, Commun. Appl. Anal. **15** (2011), 381–394.

[12] R. Gaines and J. Mawhin, *Coincidence degree and nonlinear differential equations*, Lecture Notes in Math., **568**, Springer Verlag, Berlin, 1977.

[13] A. Granas, *The Leray-Schauder index and the fixed point theory for arbitrary ANR's*, Bull. Soc. Math. France **100** (1972), 209–228.

[14] A. Granas and J. Dugundji, *Fixed point theory*, Springer-Verlag, New York, 2003.

[15] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974.

[16] J.K. Hale and J. Kato, *Phase Space for Retarded Equations with Infinite Delay*, Funkc. Ekvac. **21** (1978), 11–41.

[17] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer Verlag, New York, 1993.

[18] Y. Hino, S. Murakami and T. Naito, *Functional-differential equations with infinite delay*, Lecture Notes in Math., **1473**, Springer Verlag, Berlin, 1991.

[19] M.W. Hirsch, *Differential Topology*, Graduate Texts in Math., Vol. 33, Springer Verlag, Berlin, 1976.

[20] J. Leray and J. Schauder, *Topologie et équations fonctionnelles*, Ann. Sci. École Norm. Sup. **51** (1934), 45–78.
[21] J. Mallet-Paret, R.D. Nussbaum and P. Paraskevopoulos, *Periodic solutions for functional-differential equations with multiple state-dependent time lags*, Topol. Methods Nonlinear Anal. **3** (1994), 101–162.
[22] J. Mawhin, *Équations intégrales et solutions périodiques des systèmes différentiels non linéaires,* Acad. Roy. Belg. Bull. Cl. Sci. (5) **55** (1969), 934–947.
[23] J. Mawhin, *Periodic solutions of nonlinear functional differential equations* J. Differential Equations, **10** (1971), 240–261.
[24] J.M. Milnor, *Topology from the differentiable viewpoint*, Univ. Press of Virginia, Charlottesville, 1965.
[25] R.D. Nussbaum, *The fixed point index for local condensing maps*, Ann. Mat. Pura Appl. **89** (1971), 217–258.
[26] R.D. Nussbaum, *Periodic solutions of some nonlinear autonomous functional differential equations*, Ann. Mat. Pura Appl. **101** (1974), 263–306.
[27] R.D. Nussbaum, *The fixed point index and fixed point theorems*, Topological methods for ordinary differential equations (Montecatini Terme, 1991), 143–205, Lecture Notes in Math., **1537**, Springer, Berlin, 1993.
[28] W.M. Oliva, *Functional differential equations on compact manifolds and an approximation theorem*, J. Differential Equations **5** (1969), 483–496.
[29] W.M. Oliva, *Functional differential equations–generic theory*. Dynamical systems (Proc. Internat. Sympos., Brown Univ., Providence, R.I., 1974), Vol. I, pp. 195–209. Academic Press, New York, 1976.
[30] W.M. Oliva and C. Rocha, *Reducible Volterra and Levin–Nohel Retarded Equations with Infinite Delay*, J. Dyn. Diff. Equat., **22** (2010), 509–532.
[31] P.H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7** (1971), 487–513.
[32] E. Spanier, *Algebraic Topology*, Mc Graw-Hill Series in High Math., New York, 1966.
[33] A.J. Tromba, *The Euler characteristic of vector fields on Banach manifolds and a globalization of Leray-Schauder degree*, Advances in Math., **28** (1978), 148–173.

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