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Periodic Solutions for Nonlinear Systems with p-Laplacian-Like Operators*

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1. INTRODUCTION

Let us consider the so-called one-dimensional p-Laplacian operator $(\phi_p(u'))'$, where $p>1$ and $\phi_p\colon \mathbb{R}\to\mathbb{R}$ is given by $\phi_p(s)=|s|^{p-2}s$ for $s\neq 0$ and $\phi_p(0)=0$. Various separated two-point boundary value problems containing this operator have received a lot of attention with respect to existence and multiplicity of solutions. See, for example, [3, 9, 11, 13, 21, 22, 28, 31, 33], and the references therein. Periodic boundary conditions have been considered in [12, 14]. The case of separated two-point boundary conditions when ϕ_p is replaced by a one-dimensional possibly not homogeneous operator ϕ, has been dealt with in a series of papers, cf. [2, 8, 15, 16, 17, 18, 19].

Our aim in this paper is to study existence of periodic solutions to some *system* cases involving the fairly general vector-valued operator ϕ. Thus we will consider the boundary value problem

$$(\phi(u'))' = f(t,u,u'), \qquad u(0)=u(T), \quad u'(0)=u'(T), \tag{1.1}$$

where the function $\phi\colon \mathbb{R}^N\to\mathbb{R}^N$ satisfies some monotonicity conditions which ensure that ϕ is an homeomorphism onto $\mathbb{R}^N$. As a consequence our results will apply to a large class of nonlinear operators $(\phi(u'))'$, which, for

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example, contain some vector versions of p-Laplacian operators like the case when, for $x=(x_1, ..., x_N) \in \mathbb{R}^N$, $\phi(x)=\psi_p(x) \equiv |x|^{p-2} x$, for $x \neq 0$, $\psi_p(0)=0$, $(p>1)$, and the case when $\phi(x)=(\phi_{p_1}(x_1), ..., \phi_{p_N}(x_N))$, with, for each $i=1, ..., N$, $p_i>1$, and $\phi_{p_i}\colon \mathbb{R} \to \mathbb{R}$ is a one-dimensional p_i-Laplacian.

If $I=[0, T]$, the function $f\colon I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be *Carathéodory*, by this we mean:

- for almost every $t \in I$ the function $f(t, \cdot, \cdot)$ is continuous;
- for each $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ the function $f(\cdot, x, y)$ is measurable on I;
- for each $\rho>0$ there is $\alpha_\rho \in L^1(I, \mathbb{R})$ such that, for almost every $t \in I$ and every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leqslant \rho$, $|x| \leqslant \rho$, one has

$$|f(t, x, y)| \leqslant \alpha_\rho(t).$$

By a *solution* of (1.1) we mean a function $u\colon I \to \mathbb{R}^N$ of class C^1 with $\phi(u')$ absolutely continuous, which satisfies (1.1) a.e. on I.

Throughout the paper $|\cdot|$ will denote absolute value and the Euclidean norm on $\mathbb{R}^N$, while the inner product in $\mathbb{R}^N$ will be denoted by $\langle \cdot, \cdot \rangle$. Also, for $N \geqslant 1$ we will set $C=C(I, \mathbb{R}^N)$, $C^1=C^1(I, \mathbb{R}^N)$, $C_T=\{u \in C \mid u(0)=u(T)\}$, $C_T^1=\{u \in C^1 \mid u(0)=u(T),\ u'(0)=u'(T)\}$, $L^p=L^p(I, \mathbb{R}^N)$, and $W^{1,p}=W^{1,p}(I, \mathbb{R}^N)$, $p \geqslant 1$. The norm in C and C_T will be denoted by $\|\cdot\|_0$, the norm in C^1 and C_T^1 by $\|\cdot\|_1$, and the norm in L^p by $\|\cdot\|_{L^p}$.

This paper is organized as follows. In Section 2 we begin by establishing the monotone-type conditions on the function ϕ, and we will consider and show some important examples of functions ϕ which verify those conditions. We then develop the machinery which will allow reducing (1.1) to a fixed point problem in C_T^1. The corresponding results are of independent interest, and allow us in particular to generalize to the p-Laplacian frame the classical concept of mean value of a periodic function.

In Section 3, combining the Leray–Schauder degree theory with the results of Section 2, we state and prove a first general existence theorem for problem (1.1). This result generalizes to our situation some well-known continuation theorem [23, 24, 25, 29], obtained in the framework of coincidence degree for nonlinear perturbations of linear differential operators with periodic boundary conditions. Indeed, our approach can be viewed as an extension of coincidence degree to some quasi-linear problems. This existence theorem also generalizes a result proved in [20] for nonlinear perturbations of the one-dimensional p-Laplacian and $p \geqslant 2$, to a much wider class, which includes arbitrary homeomorphisms in the scalar case and the operator $(\psi_p(u'))'$, for any $p>1$, in the vector case. Moreover, our generalization is obtained by using classical Leray–Schauder degree theory, instead of the more sophisticated degree theory for mappings of type $(S)_+$ used in [20]. As a consequence of this first continuation theorem, we obtain various existence theorems for quasi-linear systems with a nonlinearity f satisfying some one-sided growth conditions introduced, for semilinear problems, by Ward [32] in the scalar case, and by Cañada–Martinez-Amores [4, 5] and Cañada–Ortega [6] (see also [25], p. 67) in the vector case.

In Section 4, using degree theory for compact vector fields which are invariant under the action of S^1, as developed in [1], we extend to our quasi-linear situation the continuation theorem of [7] and [1], in which a homotopy is made to an arbitrary autonomous system. An application is given to a perturbation of $(\psi_p(u'))'$ by an asymptotically autonomous and $(p-1)$-positive homogeneous system. In all cases, we give explicit examples showing that the assumptions are realistic.

Needless to say, many more applications of those continuation theorems can be done, and some of them will be given in subsequent papers. Furthermore, many of the obtained results have direct generalizations to the study of the T-periodic solutions of systems of *functional-differential equations* of the form

$$(\phi(u'))' = f(t, u_t, (u')_t),$$

when, for $v \in C_T$, v_t denotes, as usual, the function defined on $[-r, 0]$ by $v_t(\theta) = v(t+\theta)$.

2. SOME MONOTONE MAPPINGS AND A FIXED POINT OPERATOR

Let ϕ: $\mathbb{R}^N \to \mathbb{R}^N$ be a continuous function which satisfies the following two conditions:

(H$_1$) For any $x_1, x_2 \in \mathbb{R}^N$, $x_1 \neq x_2$,

$$\langle \phi(x_1) - \phi(x_2), x_1 - x_2 \rangle > 0.$$

(H$_2$) There exists a function α: $[0, +\infty[\to [0, +\infty[$, $\alpha(s) \to +\infty$ as $s \to +\infty$, such that

$$\langle \phi(x), x \rangle \geqslant \alpha(|x|)\,|x|, \qquad \text{for all} \quad x \in \mathbb{R}^N.$$

It is well-known that under these two conditions ϕ is an homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$, satisfies (H$_1$), and that $|\phi^{-1}(y)| \to +\infty$ as $|y| \to +\infty$ (see [10, Ch. 3]).

Let us first give some examples of simple operators ϕ for which conditions (H$_1$) and (H$_2$) are satisfied.

EXAMPLE 2.1. Let ϕ be an homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Then ϕ is either increasing or decreasing. Clearly in the first case ϕ satisfies (H_1) and (H_2) while in the second case $-\phi$ does.

EXAMPLE 2.2. For $p>1$, let $\psi_p: \mathbb{R}^N \to \mathbb{R}^N$ be given by

$$\psi_p(x) = |x|^{p-2}\, x \qquad \text{for} \quad x \neq 0, \quad \psi_p(0) = 0.$$

Then ψ_p is an homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$ with inverse $\psi_{p^*}(x) = |x|^{p^*-2}\, x$, where $p^* = p/(p-1)$. Let now $x, y \in \mathbb{R}^N$, from the inequality

$$\langle \psi_p(x) - \psi_p(y), x - y \rangle \geqslant (|x|^{p-1} - |y|^{p-1})(|x| - |y|) \geqslant 0,$$

it follows immediately that

$$\langle \psi_p(x) - \psi_p(y), x - y \rangle = 0 \qquad \text{implies} \quad x = y,$$

and thus (H_1) holds. Also, from

$$\langle \psi_p(x), x \rangle = |x|^p = |x|^{p-1}\, |x|,$$

(H_2) follows.

EXAMPLE 2.3. More generally, we can consider any $\phi: \mathbb{R}^N \to \mathbb{R}^N$ which is a potential, i.e., $\phi = \nabla \Phi$, $\Phi: \mathbb{R}^N \to \mathbb{R}$ of class C^1 and strictly convex such that ϕ satisfies (H_2). An interesting example of this class is given by

$$\Phi(x) = e^{|x|^2} - |x|^2 - 1.$$

Clearly, $\Phi: \mathbb{R}^N \to \mathbb{R}$ is of class C^1 and strictly convex, and

$$\langle \nabla \Phi(x), x \rangle = 2(e^{|x|^2} - 1)\, |x|^2,$$

and thus (H_2) is satisfied.

EXAMPLE 2.4. Further examples can be obtained from the following proposition.

PROPOSITION 2.1. *For $i = 1, ..., k$ let $N_i \in \mathbb{N}$ and $\psi_i: \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$ be a function which satisfies the following conditions.*

(i) $\langle \psi_i(z) - \psi_i(w), z - y \rangle_i \geqslant 0$, *(with $\langle \cdot, \cdot \rangle_i$ denoting the inner product in $\mathbb{R}^{N_i}$) for any $z, y \in \mathbb{R}^{N_i}$, with equality holding true if and only if $z = y$;*

(ii) *there exists a function* $\alpha_i\colon [0, +\infty) \to [0, +\infty)$, $\alpha_i(s) \to +\infty$ *as* $s \to +\infty$, *such that*

$$\langle \psi_i(z), z \rangle_i \geqslant \alpha_i(|z|)\,|z|, \qquad \textit{for all} \quad z \in \mathbb{R}^{N_i}.$$

Then the function

$$\Psi\colon \prod_{i=1}^{k} \mathbb{R}^{N_i} \to \prod_{i=1}^{k} \mathbb{R}^{N_i},\ x = (x^1, ..., x^k) \mapsto \Psi(x) = (\psi_1(x^1), ..., \psi_k(x^k)),$$

satisfies conditions (H_1) *and* (H_2) *with* $N = \sum_{i=1}^{k} N_i$.

Proof. Let $x = (x^1, x^2, ..., x^k)$, $y = (y^1, y^2, ..., y^k)$ be in $\mathbb{R}^N = \prod_{i=1}^{k} \mathbb{R}^{N_i}$. Then

$$\langle \Psi(x) - \Psi(y), x - y \rangle = \sum_{i=1}^{k} \langle \psi_i(x^i) - \psi_i(y^i), x^i - y^i \rangle_i,$$

and, if $x \neq y$, then $x^i \neq y^i$ for at least one $i \in \{1, ..., k\}$. This implies

$$\langle \Psi(x) - \Psi(y), x - y \rangle \geqslant \langle \psi_i(x^i) - \psi_i(y^i) \rangle_i > 0,$$

and hence (H_1) (with Ψ in the place of ϕ) is satisfied.

Next we will show that

$$\frac{\langle \Psi(x), x \rangle}{|x|} \to \infty \qquad \text{as} \quad |x| \to \infty. \tag{2.1}$$

We argue by contradiction and thus we assume that there exist a positive constant C and a sequence $\{x_n\}$ in $\mathbb{R}^N$ such that $|x_n| \to \infty$, and $(\langle \psi(x_n), x_n \rangle)/(|x_n|) \leqslant C$, for all $n \in \mathbb{N}$. Then for $x_n = (x_n^1, ..., x_n^k)$, we have

$$\alpha_i(|x_n^i|)\, v_n^i \leqslant \frac{\langle \psi_i(x_n^i), x_n^i) \rangle_i}{|x_n|} \leqslant C, \tag{2.2}$$

where $v_n^i = (|x_n^i|)/(|x_n|)$, for all $i = 1, ..., k$, and all $n \in \mathbb{N}$. Clearly $\sum_{i=1}^{k} v_n^i = 1$, for each $n \in \mathbb{N}$. Since the sequences $\{v_n^i\}_n$, $i = 1, ..., k$, are bounded we can assume, passing to subsequences, that $v_n^i \to v^i \geqslant 0$, $i = 1, ..., k$.

Now suppose all the v^i's are not zero, then, from (2.2),

$$\limsup_{n \to \infty} \alpha_i(|x_n^i|) \leqslant \frac{C}{v^i},$$

and the growth conditions on the functions α_i imply that the sequences $\{|x_n^i|\}_n$, $i=1, ..., k$, are bounded. Since this contradicts $|x_n| \to \infty$, we have that in this case (2.1) holds. Next assume that some of the the v^i's are zero. Without loss of generality (modulo a permutation of indices) we can suppose that $v^i=0$, for $i=1, ..., j$, and $v^i>0$, for $i=j+1, ..., k$. Let $\varepsilon>0$ be such that $\varepsilon<(1/j)$, then there is a n_0 such that for all $n>n_0$,

$$\sum_{i=1}^{j} |x_n^i|^2 < j\varepsilon\, |x_n|^2,$$

and thus

$$|x_n|^2 \leqslant \frac{1}{1-j\varepsilon} \sum_{i=j+1}^{k} |x_n^i|^2.$$

Hence there must be $i_0 \in \{j+1, ..., k\}$ such that $|x_n^{i_0}| \to \infty$ as $n \to \infty$. This in turn implies that $\alpha_{i_0}(|x_n^{i_0}|) \to \infty$ as $n \to \infty$, and since $v_n^{i_0} \to v^{i_0} > 0$ as $n \to \infty$, from (2.2), with $i=i_0$, we again obtain a contradiction. Thus (2.1) holds, and it is standard to construct from (2.1) a function α so that (H_2) (with Ψ in the place of ϕ) is verified. ∎

Let us now consider the simple periodic boundary value problem

$$(\phi(u'))' = h(t), \qquad u(0)=u(T), \qquad u'(0)=u'(T), \tag{2.3}$$

where $h \in L^1$ is such that $\int_0^T h(s)\, ds=0$, and let u be a solution to (2.3). By integrating from 0 to $t \in I$, we find that

$$\phi(u'(t)) = a + H(h)(t), \tag{2.4}$$

where

$$H(h)(t) = \int_0^t h(s)\, ds,$$

and $a \in \mathbb{R}^N$ is a constant. The boundary conditions imply that

$$\frac{1}{T} \int_0^T \phi^{-1}(a + H(h)(t))\, dt = 0.$$

For fixed $l \in C$, let us define

$$G_l(a) = \frac{1}{T} \int_0^T \phi^{-1}(a + l(t))\, dt. \tag{2.5}$$

We have

PROPOSITION 2.2. *If ϕ satisfies conditions* (H_1) *and* (H_2), *then the function G_l has the following properties*:

(i) *For any fixed $l \in C$, the equation*

$$G_l(a) = 0, \tag{2.6}$$

has a unique solution $\tilde{a}(l)$.

(ii) *The function $\tilde{a}: C \to \mathbb{R}^N$, defined in* (i), *is continuous and sends bounded sets into bounded sets.*

Proof. (i) By (H_1), it is immediate that

$$\langle G_l(a_1) - G_l(a_2), a_1 - a_2 \rangle > 0, \qquad \text{for} \quad a_1 \neq a_2,$$

and hence if (2.6) has a solution then it is unique. To prove its existence we will show that $\langle G_l(a), a \rangle > 0$ for $|a|$ sufficiently large. Indeed we have

$$\begin{aligned} \langle G_l(a), a \rangle &= \frac{1}{T} \int_0^T \langle \phi^{-1}(a + l(t)), a \rangle \, dt \\ &= \frac{1}{T} \int_0^T \langle \phi^{-1}(a + l(t)), a + l(t) \rangle \, dt \\ &\quad - \frac{1}{T} \int_0^T \langle \phi^{-1}(a + l(t)), l(t) \rangle \, dt, \end{aligned}$$

and thus

$$\begin{aligned} \langle G_l(a), a \rangle \geqslant & \frac{1}{T} \int_0^T \langle \phi^{-1}(a + l(t)), a + l(t) \rangle \, dt \\ & - \frac{\|l\|_0}{T} \int_0^T |\phi^{-1}(a + l(t))| \, dt. \end{aligned} \tag{2.7}$$

Now from (H_2), for any $y \in \mathbb{R}^N$, we have that

$$\langle \phi^{-1}(y), y \rangle \geqslant \alpha(|\phi^{-1}(y)|) \, |\phi^{-1}(y)|. \tag{2.8}$$

Thus from (2.7) and (2.8),

$$\langle G_l(a), a \rangle \geqslant \frac{1}{T} \int_0^T (\alpha(|\phi^{-1}(a + l(t))|) - \|l\|_0) \, |\phi^{-1}(a + l(t))| \, dt. \tag{2.9}$$

Since $|a| \to \infty$ implies that $|\phi^{-1}(a + l(t))| \to \infty$, uniformly for $t \in I$, we find from (2.9) that there exists an $r > 0$ such that

$$\langle G_l(a), a \rangle > 0 \qquad \text{for all} \quad a \in \mathbb{R}^N \qquad \text{with} \qquad |a| = r.$$

It follows by an elementary topological degree argument that the equation $G_l(a) = 0$ has a solution for each $l \in C$, which by our previous argument is unique. In this way we define a function $\tilde{a}$: $C \to \mathbb{R}^N$ which satisfies

$$\int_0^T \phi^{-1}(\tilde{a}(l) + l(t))\, dt = 0, \qquad \text{for any} \quad l \in C. \tag{2.10}$$

To prove (ii) let Λ be a bounded subset of C and let $l \in \Lambda$. Then, from (2.10)

$$\int_0^T \langle \phi^{-1}(\tilde{a}(l) + l(t)), \tilde{a}(l) \rangle\, dt = 0,$$

and hence

$$\int_0^T \langle \phi^{-1}(\tilde{a}(l) + l(t)), \tilde{a}(l) + l(t) \rangle\, dt = \int_0^T \langle \phi^{-1}(\tilde{a}(l) + l(t)), l(t) \rangle\, dt. \tag{2.11}$$

Assume next that $\{\tilde{a}(l), l \in \Lambda\}$ is not bounded. Then for an arbitrary $A > 0$ there is $l \in \Lambda$ with $\|l\|_0$ sufficiently large so that

$$A \leqslant \alpha(|\phi^{-1}(\tilde{a}(l) + l(t))|),$$

uniformly in $t \in I$. Hence by using (2.8) and (2.11), we find that

$$A \int_0^T |\phi^{-1}(\tilde{a}(l) + l(t))|\, dt \leqslant \int_0^T \alpha(|\phi^{-1}(\tilde{a}(l) + l(t))|)\, |\phi^{-1}(\tilde{a}(l) + l(t))|\, dt$$

$$\leqslant \|l\|_0 \int_0^T |\phi^{-1}(\tilde{a}(l) + l(t))|\, dt.$$

Thus $A \leqslant \|l\|_0$, which is a contradiction. Therefore $\tilde{a}$ sends bounded sets in C into bounded sets in $\mathbb{R}^N$.

Finally to show the continuity of $\tilde{a}$, let $\{l_n\}$ be a convergent sequence in C, say $l_n \to l$, as $n \to \infty$. Since $\{a(l_n)\}$ is a bounded sequence, any subsequence of it contains a convergent subsequence denoted by $\{a(l_{n_j})\}$. Let $a(l_{n_j}) \to \hat{a}$, as $j \to \infty$. By letting $j \to \infty$ in

$$\int_0^T \phi^{-1}(\tilde{a}(l_{n_j}) + l_{n_j}(t))\, dt = 0,$$

we find that

$$\int_0^T \phi^{-1}(\hat{a} + l(t))\, dt = 0,$$

and hence $\tilde{a}(l) = \hat{a}$, which shows the continuity of $\tilde{a}$. ∎

Let now $a: L^1 \to \mathbb{R}^N$ be defined by

$$a(h) = \tilde{a}(H(h)). \tag{2.12}$$

Then it is clear that a is a continuous function which sends bounded sets of L^1 into bounded sets of $\mathbb{R}^N$, and hence it is a completely continuous mapping.

We continue now with our argument previous to Proposition 2.2. By solving for u' in (2.4) and integrating we find

$$u(t) = u(0) + H\{\phi^{-1}[a(h) + H(h)]\}(t). \tag{2.13}$$

Here ϕ^{-1} is understood as the the operator $\phi^{-1}: C \to C$ defined by $\phi^{-1}(v)(t) = \phi^{-1}(v(t))$. It is clear that ϕ^{-1} is continuous and sends bounded sets into bounded sets.

Let us define

$$P: C_T^1 \to C_T^1, \qquad u \mapsto u(0), \qquad Q: L^1 \to L^1, \qquad h \mapsto \frac{1}{T} \int_0^T h(s)\, ds.$$

Then it is clear that if $u \in C_T^1$ solves (2.3), then u satisfies the abstract equation

$$u = Pu + Qh + \mathscr{K}(h), \tag{2.14}$$

where the (in general nonlinear) operator $\mathscr{K}: L^1 \to C_T^1$ is given by

$$\mathscr{K}(h)(t) = H\{\phi^{-1}[a((I-Q)\,h) + H((I-Q)\,h)]\}(t), \qquad \text{for all } t \in I. \tag{2.15}$$

Conversely, since, by definition of the mapping a,

$$H\{\phi^{-1}[a((I-Q)\,h) + H((I-Q)\,h)]\}(T) = 0,$$

it is a simple matter to see that if u satisfies (2.14) then u is a solution to (2.3).

Note that since $a(0) = \tilde{a}(0)$, we have, by (2.15) and (2.10), that

$$\mathscr{K}(0) = 0.$$

Lemma 2.1. *The operator $\mathscr{K}$ is continuous and sends equi-integrable sets in L^1 into relatively compact sets in C_T^1.*

Proof. The continuity of $\mathscr{K}$ in C follows immediately by observing that this operator is a composition of continuous operators. Also, we have that

$$\mathscr{K}(h)'(t) = \phi^{-1}[a((I-Q)h) + H((I-Q)h)](t),$$

which is also a composition of continuous operators and hence continuous.

Let now $\mathscr{E}$ be an equi-integrable set in L^1. Then if $h \in \mathscr{E}$, there is $\eta \in L^1$ such that

$$|h(t)| \leqslant \eta(t) \qquad \text{a.e. in} \quad I.$$

We want to show that $\overline{\mathscr{K}(\mathscr{E})} \subset C^1_T$ is a compact set. For this it suffices to prove that if $\{v_n\}$ is a subsequence in $\mathscr{K}(\mathscr{E})$, then it contains a convergent subsequence in C^1_T. Let $\{h_n\}$ be a sequence in L^1 such that $v_n = \mathscr{K}(h_n)$. For $t, t' \in I$, we have that

$$\begin{aligned} |H(I-Q)(h_n)(t) - H(I-Q)(h_n)(t')| &\leqslant \left| \int_{t'}^{t} h(s)\, ds \right| + |Q(h)|\, |t-t'| \\ &\leqslant \left| \int_{t'}^{t} \eta(s)\, ds \right| + |t-t'| \int_0^T \eta(s)\, ds. \end{aligned}$$

Hence the sequence $\{H(I-Q)(h_n)\}$ is uniformly bounded and equicontinuous. By the Ascoli–Arzela theorem there is a subsequence of $\{H(I-Q)(h_n)\}$, which we rename the same, which is convergent in C. Then, passing to a subsequence if necessary, we obtain that the sequence $\{a((I-Q)(h_n)) + H(I-Q)(h_n)\}$ is convergent in C. Using that $\phi^{-1}: C \to C$ is continuous it follows from

$$(\mathscr{K}(h_n))'(t) = \phi^{-1}[a((I-Q)(h_n)) + H(I-Q)(h_n)]$$

that the sequence $\{(\mathscr{K}(h_n))'\}$ is convergent in C and hence so does the sequence $\{(\mathscr{K}(h_n))\}$. ∎

Another consequence of Proposition 2.2 is of independent interest.

PROPOSITION 2.3. *For each $u \in C$, there exists a unique $\overline{u_p} \in \mathbb{R}^N$ such that the function $\widetilde{u_p} := u - \overline{u_p}$ satisfies the relation*

$$\int_0^T \psi_p(\widetilde{u_p}(t))\, dt = 0.$$

Furthermore, the mapping $u \mapsto \overline{u_p}$ is continuous and takes bounded sets of C into bounded sets of $\mathbb{R}^N$.

Proof. The result consists in finding some $\overline{u_p} \in \mathbb{R}^N$ such that

$$\int_0^T \psi_p(u(t) - \overline{u_p})\, dt = 0.$$

It is a consequence of Proposition 2.2 with $\phi(v) = |v|^{p^*-2}\, v$, $p^* = p/(p-1)$, $l = u$, $a = -\overline{u_p}$. ∎

Remark 2.1. For $p = 2$, $\overline{u_p}$ reduces to the usual mean value $\bar{u} = Qu = (1/T) \int_0^T u(t)\, dt$ of u. Therefore, we can refer to $\overline{u_p}$ as the *p-mean value of u*.

The following properties of the *p-mean value of a scalar function u* extend the standard ones for $p = 2$.

PROPOSITION 2.4. *If $u \in C(I, \mathbb{R})$, there exists $\overline{t_p} \in I$ such that*

$$\overline{u_p} = u(\overline{t_p}). \tag{2.16}$$

Proof. As $\overline{u_p} = u(t) - \widetilde{u_p}(t)$ and

$$\int_0^T \psi_p(\widetilde{u_p}(t))\, dt = 0,$$

there exists $\overline{t_p}$ such that

$$\psi_p(\widetilde{u_p}(\overline{t_p})) = 0,$$

i.e., $\widetilde{u_p}(\overline{t_p}) = 0$, so that

$$\overline{u_p} = u(\overline{t_p}). \quad ∎$$

PROPOSITION 2.5. *If $u \in W^{1,p}(I, \mathbb{R})$, then one has the inequality*

$$\|\widetilde{u_p}\|_{L^\infty} \leqslant T^{1/p^*} \|u'\|_{L^p}, \tag{2.17}$$

where p^* is conjugate to p.

Proof. If $\bar{t}$ is given by Proposition 2.4, we have, for all $t \in I$, using Hölder's inequality

$$|\tilde{u}(t)| = |u(t) - u(\bar{t})| = \left| \int_{\bar{t}}^t u'(s)\, ds \right| \leqslant T^{1/p^*} \|u'\|_{L^p}. \quad ∎$$

Let us consider finally the abstract functional differential periodic problem

$$(\phi(u'))' = N(u, \lambda), \qquad u(0) = u(T), \qquad u'(0) = u'(T), \tag{2.18}$$

where $\lambda \in [0, 1]$, and $N: C^1_T \times [0, 1] \to L^1$ is continuous, and send bounded sets into equi-integrable sets. Thus, defining $\mathscr{G}: C^1_T \times [0, 1] \to C^1_T$ by

$$\mathscr{G}(u, \lambda) := Pu + QN(u, \lambda) + (\mathscr{K} \circ N)(u, \lambda), \tag{2.19}$$

we obtain that $\mathscr{G}$ is a completely continuous operator. Furthermore, problem (2.18) is equivalent to the problem

$$u = \mathscr{G}(u, \lambda). \tag{2.20}$$

In particular, if $g: I \times \mathbb{R}^N \times \mathbb{R}^N \times [0, 1]$ is Carathéodory and if we denote by $N_g: C^1_T \times [0, 1] \to L^1$ the *Nemistki operator associated to* g defined by

$$N_g(u, \lambda)(t) = g(t, u(t), u'(t), \lambda), \qquad \text{a.e. on} \quad I,$$

it is known that N_g is continuous and sends bounded sets into equi-integrable sets.

We will apply Leray–Schauder's degree to (2.20) by choosing N in such a way that the Leray–Schauder degree of $I - \mathscr{G}(\cdot, 0)$ with respect to a suitable open bounded set of C^1_T exists and is easy to compute. Such situations will be considered in the following sections.

3. HOMOTOPY TO THE AVERAGED NONLINEARITY

We will suppose in this section that $\phi: \mathbb{R}^N \to \mathbb{R}^N$ is continuous and satisfies the conditions (H_1)–(H_2) of Section 2. Our first aim in this section is to extend a continuation theorem proved in [23] for semilinear equations (see also [24, 26, 27]) to the quasilinear problem (1.1), which we repeat here for convenience of the reader,

$$(\phi(u'))' = f(t, u, u'), \qquad u(0) = u(T), \qquad u'(0) = u'(T), \tag{3.1}$$

where $f: I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory.

THEOREM 3.1. *Assume that* Ω *is an open bounded set in* C^1_T *such that the following conditions hold.*

(1) *For each* $\lambda \in]0, 1[$ *the problem*

$$(\phi(u'))' = \lambda f(t, u, u'), \qquad u(0) = u(T), \qquad u'(0) = u'(T), \tag{3.2}$$

has no solution on $\partial\Omega$.

(2) *The equation*

$$F(a) := \frac{1}{T} \int_0^T f(t, a, 0) = 0, \tag{3.3}$$

has no solution on $\partial\Omega \cap \mathbb{R}^N$.

(3) *The Brouwer degree*

$$d_B[F, \Omega \cap \mathbb{R}^N, 0] \neq 0. \tag{3.4}$$

Then problem (3.1) *has a solution in* $\bar{\Omega}$.

Proof. Let us embed problem (3.1) into the one parameter family of problems

$$(\phi(u'))' = \lambda N_f(u) + (1 - \lambda)\, QN_f(u), \qquad u(0) = u(T), \qquad u'(0) = u'(T), \tag{3.5}$$

where $N_f \colon C_T^1 \to \mathbb{R}^N$ is the Nemytski operator associated to f. Explicitly,

$$(\phi(u'))' = \lambda f(t, u, u') + (1 - \lambda) \left[\frac{1}{T} \int_0^T f(s, u(s), u'(s))\, ds \right],$$

$$u(0) = u(T), \qquad u'(0) = u'(T).$$

For $\lambda \in]0, 1]$, observe that, in both cases, u is a solution to problem (3.2) or u is a solution to problem (3.5). we have necessarily

$$\frac{1}{T} \int_0^T f(s, u(s), u'(s))\, ds = 0.$$

It follows that, for $\lambda \in]0, 1]$, problems (3.2) and (3.5) have the same solutions. Furthermore it is easy to see that f Carathéodory implies that $N \colon C_T^1 \times [0, 1] \to L^1$ defined by

$$N(u, \lambda) = \lambda N_f(u) + (1 - \lambda)\, QN_f(u)$$

is continuous and takes bounded sets into equi-integrable sets. Also problem (3.5) can be written in the equivalent form

$$u = \mathcal{G}_f(u, \lambda) \tag{3.6}$$

with

$$\mathscr{G}_f(u, \lambda) = Pu + QN_f(u) + (\mathscr{K} \circ [\lambda N_f + (1-\lambda)\, QN_f])(u)$$
$$= Pu + QN_f(u) + (\mathscr{K} \circ [\lambda(I-Q)\, N_f])(u).$$

We assume that for $\lambda = 1$, (3.6) does not have a solution on $\partial\Omega$ since otherwise we are done with the proof. Now by hypothesis (1) it follows that (3.6) has no solutions for $(u, \lambda) \in \partial\Omega \times]0, 1]$. For $\lambda = 0$, (3.5) is equivalent to the problem

$$(\phi(u'))' = \frac{1}{T} \int_0^T f(s, u(s), u'(s))\, ds, \qquad u(0) = u(T), \qquad u'(0) = u'(T), \quad (3.7)$$

and thus if u is a solution to this problem, we must have

$$\int_0^T f(s, u(s), u'(s))\, ds = 0. \tag{3.8}$$

Hence

$$u'(t) = \phi^{-1}(c),$$

where $c \in \mathbb{R}^N$ is a constant. Integrating this last equation on I we obtain that $\phi^{-1}(c) = 0$, and thus $u(t) = d$, a constant. Thus, by (3.8)

$$\int_0^T f(s, d, 0)\, ds = 0,$$

which, together with hypothesis 2, implies that $u = d \notin \partial\Omega$. Thus we have proved that (3.6) has no solution $(u, \lambda) \in \partial\Omega \times [0, 1]$. Then we have that for each $\lambda \in [0, 1]$, the Leray–Schauder degree $d_{LS}[I - \mathscr{G}_f(\cdot, \lambda), \Omega, 0]$ is well-defined and, by the properties of that degree, that

$$d_{LS}[I - \mathscr{G}_f(\cdot, 1), \Omega, 0] = d_{LS}[I - \mathscr{G}_f(\cdot, 0), \Omega, 0]. \tag{3.9}$$

Now it is clear that problem

$$u = \mathscr{G}_f(u, 1) \tag{3.10}$$

is equivalent to problem (3.1), and (3.9) tells us that problem (3.10) will have a solution if we can show that $d_{LS}[I - \mathscr{G}_f(\cdot, 0), \Omega, 0] \neq 0$. This we do next.

We have that

$$\mathscr{G}_f(u, 0) = Pu + QN_f(u) + \mathscr{K}(0) = Pu + QN_f(u).$$

Thus we obtain

$$u - \mathcal{G}_f(u, 0) = u - Pu - \frac{1}{T} \int_0^T f(s, u(s), u'(s))\, ds.$$

Hence by the properties of the Leray–Schauder degree we have that

$$d_{LS}[I - \mathcal{G}_f(\cdot, 0), \Omega, 0] = (-1)^N d_B[F, \Omega \cap \mathbb{R}^N, 0],$$

where the function F is defined in (3.3) and d_B denotes the Brouwer degree. Since by hypothesis (3) this last degree is different from zero, the theorem is proved. ∎

Our next theorem is a consequence of Theorem 3.1. We need first the following definition.

Let $f = (f_1, ..., f_N): I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function. We will say that f satisfies a *generalized Villari condition* if there is an $\rho_0 > 0$ such that for all $u \in C_T^1$, $u = (u_1, ..., u_N)$, with $\min_{t \in I} |u_j(t)| > \rho_0$, for some $j \in \{1, ..., N\}$ it holds that

$$\int_0^T f_i(t, u(t), u'(t))\, dt \neq 0, \tag{3.11}$$

for some $i \in \{1, ..., N\}$.

Remark 3.1. A condition of this type was introduced for the scalar case by Villari in [30].

Let $b(R)$ denotes the open ball in $\mathbb{R}^N$ with center zero and radius R. The following theorem extends a result of Cañada and Ortega [6] to our class of quasi-linear equations

THEOREM 3.2. *Assume that the following conditions hold.*

(1) *There exist* $n \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ *and* $h \in L^1(I, \mathbb{R}_+)$ *such that*

$$\langle \phi(y), n'(x)\, y \rangle \geqslant 0, \tag{3.12}$$

and

$$|f(t, x, y)| \leqslant \langle f(t, x, y), n(x) \rangle + h(t), \tag{3.13}$$

for all $x, y \in \mathbb{R}^N$, *and a.e.* $t \in I$.

(2) *f satisfies a generalized Villari condition.*

(3) *There is $R_0 > 0$, such that all the possible solutions to the equation*

$$F(a) := \frac{1}{T} \int_0^T f(t, a, 0) = 0, \tag{3.14}$$

belong to $b(R_0)$.

(4) *The Brouwer degree*

$$d_B[F, b(R_0), 0] \neq 0. \tag{3.15}$$

Then problem (3.1) *has at least one solution.*

Proof. Let (u, λ), $u \in C_T^1$, $\lambda \in]0, 1[$, be a solution to (3.2), then necessarily, using (3.12), we have

$$\begin{aligned} 0 &\geqslant -\int_0^T \langle \phi'(u'(t)), n'(u(t))\, u'(t) \rangle \, dt = \int_0^T \langle (\phi'(u'(t)))', n(u(t)) \rangle \, dt \\ &= \lambda \int_0^T \langle f(t, u(t), u'(t)), n'(ut)) \rangle \, dt. \end{aligned} \tag{3.16}$$

For simplicity of the exposition let us set $\phi(u'(t)) = \tilde{b}(t) + \bar{b}$, with $\int_0^T \tilde{b}(t)\, dt = 0$, and $\bar{b} = (1/T) \int_0^T \phi(u'(t))\, dt$. Then from (3.16) and (3.13), we get

$$\begin{aligned} \|\tilde{b}'\|_{L^1} &\leqslant \int_0^T |f(t, u(t), u'(t))| \, dt \\ &\leqslant \int_0^T \langle f(t, u(t), u'(t)), n(u) \rangle \, dt + \|h\|_{L^1} \leqslant \|h\|_{L^1}, \end{aligned} \tag{3.17}$$

which yields

$$\|\tilde{b}\|_0 \leqslant NT \ \|h\|_{L^1}.$$

We next find an a priori bound for $\bar{b}$. We have that

$$u'(t) = \phi^{-1}(\tilde{b}(t) + \bar{b}). \tag{3.18}$$

Hence, by integrating on I and using the boundary conditions, we obtain

$$\int_0^T \phi^{-1}(\tilde{b}(t) + \bar{b})\, dt = 0. \tag{3.19}$$

By Proposition 2.2 it follows that $\bar{b} = \tilde{a}(\tilde{b})$, where the function $\tilde{a}$ is defined in that proposition. Recalling that $\tilde{a}$ sends bounded sets into bounded sets, we have that there is a positive constant C_1 such that $|\bar{b}| \leqslant C_1$. Hence, from (3.18) and the fact that ϕ^{-1} seen as an operator from C into C sends bounded sets into bounded sets, we obtain a positive constant C_2 such that

$$\|u'\|_0 \leqslant C_2, \tag{3.20}$$

which in turn implies that

$$\left|\int_0^t u'(s)\, ds\right| \leqslant \int_0^T |u'(s)|\, ds \leqslant C_2 T. \tag{3.21}$$

Next, the solution u satisfies

$$0 = \frac{1}{T} \int_0^T f(t, u(t), u'(t))\, dt = \frac{1}{T} \int_0^T f\left(t, u(0) + \int_0^t u'(s)\, ds, u'(t)\right) dt. \tag{3.22}$$

By the generalized Villari condition we find that for each $j \in \{1, ..., N\}$ there exists a $t_j \in I$ such that $|u_j(t_j)| \leqslant \rho_0$. Since

$$u_j(t) = u_j(t_j) + \int_{t_j}^t u_j'(s)\, ds,$$

we get, by (3.20),

$$|u_j(t)| \leqslant |u_j(t_j)| + C_2 T \leqslant \rho_0 + C_2 T.$$

Thus there is a constant C_3 such that $\|u\|_0 \leqslant C_3$. It follows that we can find $R > 0$ such that if (u, λ) is a solution to (P_λ) then $\|u\|_1 \leqslant R$. We define next the set $\Omega \subset C_T^1$ that appears in Theorem 3.1 as $\Omega = B(R)$, where $B(R)$ is the open ball in C_T^1 center 0 and radius R. Thus condition (1) of Theorem 3.1 is satisfied with $\Omega = B(R)$, and since the rest of the conditions of Theorem 3.1 are also satisfied, the proof is complete. ▮

Remark 3.2. In the case where $\phi(x) = \psi_p(x)$, $(p > 1)$, condition (3.12) reduces to the assumption that $n'(x) \in \mathscr{L}(\mathbb{R}^N, \mathbb{R}^N)$ *is positive semidefinite for each* $x \in \mathbb{R}^N$.

Remark 3.3. Condition (3.12) is trivially satisfied for a *constant* mapping n, in which case one recovers assumptions introduced in [32] for $N = 1$ and in [4, 5] for $N > 1$. In particular, let $f(t, x, y) = (f_1(t, x, y), ..., f_N(t, x, y))$. Following [4, 5], we have that if there is a vector $n = (n_1, ..., n_N)$, with $n_i \geqslant 0$, for all $i = 1, ..., N$, and $h_i \in L^1(I, \mathbb{R}_+)$ such that

$$|f_i(t, x, y)| \leqslant n_i f_i(t, x, y) + h_i(t), \qquad \text{for each} \quad i = 1, ..., N, \tag{3.23}$$

then condition (3.13) is satisfied.

COROLLARY 3.1. *Suppose that the following conditions are satisfied.*

(1) *There is a mapping* $n \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ *such that conditions* (3.12) *and* (3.13) *hold.*

(2) *There exists a function* $\tilde{h} \in L^1(I, \mathbb{R}_+)$ *and a continuous function* $\eta\colon [0, +\infty) \to [0, +\infty)$, *such that*

$$\eta(s) \to +\infty \qquad \text{as} \quad s \to +\infty,$$

and

$$\eta(|x|) - \tilde{h}(t) \leqslant |f(t, x, y)|, \tag{3.24}$$

for almost all $t \in I$, *and all* $x, y \in \mathbb{R}^N$.

(3) *Condition* (3.15) *holds.*

Then problem (3.1) *has at least one solution.*

Proof. Let (u, λ), $\lambda \in]0, 1[$ be a solution to (3.2). As in the proof of Theorem 3.2, it follows from conditions (3.12) and (3.13), that there is a positive constant C_2 such that $\|u'\|_0 \leqslant C_2$. We claim that conditions (3.13) and (3.24) imply that there is a constant C_3 such that $\|u\|_0 \leqslant C_3$. Indeed, from (3.17), we have that

$$\int_0^T |f(t, u(t), u'(t))| \, dt \leqslant \|h\|_{L^1}.$$

Then by (3.24)

$$\int_0^T \eta(|u(t)|) \, dt \leqslant \|h\|_{L^1} + \|\tilde{h}\|_{L^1}. \tag{3.25}$$

Since (3.20) holds by the reasoning of the previous theorem and $\eta(s) \to \infty$ as $s \to \infty$, from (3.25) we find the required bound for $\|u\|_0$.

Now let $a \in \mathbb{R}^N$ be such that $\int_0^T f(t, a, 0) \, dt = 0$. Then (3.24) implies that $\eta(|a|) \leqslant C_2$, and hence $|a| \leqslant C_3$, for some positive constants C_2 and C_3. Thus there is $R_0 > 0$ such that all solutions to (3.3) belongs to $b(R_0)$. The rest of the proof follows from the last theorem. ∎

As a simple example for Corollary 3.1, we consider the system of two equations

$$\begin{gathered}
(|x'|^{p-2} x_1')' + x_2(1 + x_2^2) + \sin x_1 = e_1(t), \\
(|x'|^{p-2} x_2')' - x_1(1 + x_1^2) + \cos x_2 = e_2(t), \\
x_1(0) = x_1(T), \quad x_1'(0) = x_1'(T), \quad x_2(0) = x_2(T), \quad x_2'(0) = x_2'(T),
\end{gathered} \tag{3.26}$$

where $e = (e_1, e_2) \in L^1(I, \mathbb{R}^2)$. Letting

$$f(t, x) = (-x_2(1 + x_2^2) - \sin x_1 + e_1(t), x_1(1 + x_1^2) - \cos x_2 + e_2(t)),$$
$$n(x) = (-2x_2, 2x_1),$$
$$\eta(|x|) = |x|,$$

we have, for a.e. $t \in I$ and all $x \in \mathbb{R}^2$,

$$\begin{aligned} -|e(t)| - 1 + \eta(|x|) &\leqslant |f(t, x)| \\ &\leqslant [x_1^2(1 + x_1^2)^2 + x_2^2(1 + x_2^2)^2]^{1/2} + |e(t)| + 1 \\ &\leqslant |x|^3 + |e(t)| + 1, \end{aligned}$$

whenever $|x_1| \geqslant 1$ and $|x_2| \geqslant 1$, and hence, for some $h_1 \in L^1(I, \mathbb{R}_+)$, we have

$$-|e(t)| - 1 + \eta(|x|) \leqslant |f(t, x)| \leqslant |x|^3 + h_1(t), \tag{3.27}$$

for a.e. $t \in I$ and all $x \in \mathbb{R}^2$. On the other hand,

$$\langle f(t, x), n(x) \rangle \geqslant |x|^4 - 2\,|x|[\,|e(t)| + 1], \tag{3.28}$$

for a.e. $t \in I$ and all $x \in \mathbb{R}^2$. Consequently, (3.27) and (3.28) imply that, for a.e. $t \in I$ and all $x \in \mathbb{R}^2$ we have

$$|f(t, x)| \leqslant \langle f(t, x), n(x) \rangle + h(t),$$

if we choose for h the positive part of the L^1-function defined for a.e. $t \in I$ by

$$\max_{x \in \mathbb{R}_+} [-x^4 + x^3 + 2[\,|e(t)| + 1]\,x + h_1(t)].$$

Now,

$$F(a_1, a_2) = (-a_2(1 + a_2^2) - \sin a_1 + \overline{e_1}, a_1(1 + a_1^2) \cos a_2 + \overline{e_2}),$$

where

$$\overline{e_j} = \frac{1}{T} \int_0^T e_j(t)\, dt, \qquad (j = 1, 2).$$

By an easy homotopy and the product formula for degree, we have, for all sufficiently large $R > 0$,

$$d_B[F, b(R), 0] = 1,$$

and the problem (3.26) has at least one solution.

We now apply Theorem 3.2 to the problem

$$(\phi(u'))' = g(u) - e(t), \qquad u(0) = u(T), \qquad u'(0) = u'(T), \tag{3.29}$$

where $g\colon \mathbb{R}^N \to \mathbb{R}^N$ is continuous and $e \in L^1$, and extends in several ways a result of [6]. We write again

$$\bar{e} = \frac{1}{T} \int_0^T e(t)\, dt.$$

COROLLARY 3.2. *Assume that the following conditions hold.*

(1) *There is a mapping* $n \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ *and* $h \in L^1(I, \mathbb{R}_+)$ *such that* (3.12) *holds and*

$$|g(x) - e(t))| \leqslant \langle g(x) - e(t), n(x) \rangle + h(t), \tag{3.30}$$

for all $x \in \mathbb{R}^N$, *and a.e.* $t \in I$.

(2) g *is one-to-one and* $g(\mathbb{R}^N)$ *is convex.*

(3) *For each* $y \in g(\mathbb{R}^N)$ *there exists* $r > 0$ *such that for each* $x \in C^1_T$ *with* $\min_{t \in I} |x_j(t)| \geqslant r$ *for some* $j \in \{1, ..., N\}$, *one has*

$$\frac{1}{T} \int_0^T g(x(t))\, dt \neq y.$$

Then problem (3.29) *has a solution if and only if*

$$\bar{e} \in g(\mathbb{R}^N).$$

Proof. If (3.29) has a solution u, then

$$\bar{e} = \frac{1}{T} \int_0^T g(u(t))\, dt \in \overline{co}(g(u(I)) = co(g(u(I)) \subset co(g(\mathbb{R}^N)) = g(\mathbb{R}^N),$$

where co denotes the convex hull.

Conversely, to apply Theorem 3.2, it remains to show that, for $\bar{e} \in g(\mathbb{R}^N)$, all the possible solutions of the equation in $\mathbb{R}^N$,

$$F(a) := g(a) - \bar{e} = 0,$$

belong to $b(R)$ for sufficiently large R and that $d_B[F, b(R), 0] \neq 0$. The first part follows from Villari's condition 3 with $y = \bar{e}$, and the second one from the fact that g is one-to-one and $\bar{e} \in g(\mathbb{R}^N)$. ∎

For example, taking $n(u) = (-1, 1)$, it is easy to see that Corollary 3.2 implies that the problem

$$\pm(|u'|^{p-2}u_1')' = -\exp u_2 - e_1(t), \qquad \pm(|u'|^{p-2}u_2')' = \exp u_1 - e_2(t),$$
$$u_1(0) = u_1(T),\ u_2(0) = u_2(T), \qquad u_1'(0) = u_1'(T),\ u_2'(0) = u_2'(T),$$

has a solution if and only if $\overline{e_1} < 0$ and $\overline{e_2} > 0$.

Better results can of course be obtained for (3.29) in the scalar case $N = 1$ with ϕ an arbitrary homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

COROLLARY 3.3. *Let the continuous function $g: \mathbb{R} \to \mathbb{R}$ satisfy the following conditions.*

(1) *There exists $h \in L^1(I, \mathbb{R}_+)$ such that*

$$|g(u)| \leqslant g(u) + h(t)$$

for a.e. $t \in I$ and all $u \in \mathbb{R}$.

(2) *There exist $A < B$ and $r > 0$ such that either*

$$g(u) \leqslant A \quad \text{for} \quad u \leqslant -r \quad \text{and} \quad g(u) \geqslant B \quad \text{for} \quad u \geqslant r$$

or

$$g(u) \leqslant A \quad \text{for} \quad u \geqslant r \quad \text{and} \quad g(u) \geqslant B \quad \text{for} \quad u \leqslant -r.$$

Then (3.29) *has as least one solution for each $e \in L^1$ such that*

$$\bar{e} \in\,]A, B[\,.$$

Proof. It suffices to apply Theorem 3.2 with $n(u) = 1$. Villari's and degree's conditions follow easily from the assumptions. ∎

For example, the problem

$$\pm((\exp|u'| - 1)\operatorname{sgn} u')' + \exp u = e(t), \qquad u(0) = u(T), \qquad u'(0) = u'(T),$$

has at least one solution if (and only if) $\bar{e} > 0$.

4. HOMOTOPY TO AN AUTONOMOUS SYSTEM

Let us consider now the problem

$$(\phi(u'))' = g(t, u, u', \lambda), \qquad u(0) = u(T), \qquad u'(0) = u'(T), \tag{4.1}$$

where $\lambda \in [0, 1]$, and $g: I \times \mathbb{R}^N \times \mathbb{R}^N \times [0, 1] \to \mathbb{R}^N$ is Carathéodory. The following result extends to our quasi-linear situation a continuation theorem first proved in [7] for periodic solutions of semilinear systems. We follow here the simpler approach of [1] and [26]. In this continuation theorem, the homotopy is made to an autonomous system and one takes advantage of the S^1-invariance of the corresponding periodic problem to compute the associated Leray–Schauder degree.

THEOREM 4.1. *Assume that*

$$g(t, u, v, 0) = g_0(u, v) \tag{4.2}$$

is independent of t, and that Ω is an open bounded set in C^1_T such that the following conditions hold.

(1) *For each $\lambda \in [0, 1[$ the problem* (4.1) *has no solution on $\partial\Omega$.*

(2) *The Brouwer degree*

$$d_B[g_0(\cdot, 0), \Omega \cap \mathbb{R}^N, 0] \neq 0. \tag{4.3}$$

Then problem (4.1) *with $\lambda = 1$ has at least one solution in $\bar{\Omega}$.*

Proof. Problem (4.1) can be written in the equivalent form (2.20), i.e.,

$$u = \mathscr{G}_g(u, \lambda), \tag{4.4}$$

where,

$$\mathscr{G}_g(u, \lambda) = Pu + QN_g(u, \lambda) + (\mathscr{K} \circ N_g)(u, \lambda),$$

and $N_g: C^1_T \times [0, 1] \to L^1$ is the Nemytski operator associated to g. We assume that for $\lambda = 1$, (4.4) does not have a solution on $\partial\Omega$ since otherwise we are done with the proof. Now by hypothesis (1) it follows that (4.4) has no solutions for $(u, \lambda) \in \partial\Omega \times [0, 1]$. Then we have that for each $\lambda \in [0, 1]$, the Leray–Schauder degree $d_{LS}[I - \mathscr{G}_g(\cdot, \lambda), \Omega, 0]$ is well-defined and by the properties of that degree that

$$d_{LS}[I - \mathscr{G}_g(\cdot, 1), \Omega, 0] = d_{LS}[I - \mathscr{G}_g(\cdot, 0), \Omega, 0]. \tag{4.5}$$

Now it is clear that problem

$$u = \mathscr{G}_g(u, 1) \tag{4.6}$$

is equivalent to problem (4.1) with $\lambda=1$, and (4.5) tells us that this problem will have a solution if we can show that $d_{LS}[I-\mathscr{G}_g(\cdot,0),\Omega,0]\neq 0$. This we do next. We have that

$$\mathscr{G}_g(u,0)=Pu+QN_g(u,0)=(\mathscr{K}\circ N_g)(u,0),$$

where

$$N_g(u,0)(t)=g(t,u(t),u'(t),0)=g_0(u(t),u'(t)).$$

Hence, because g_0 is independent of t, $\mathscr{G}_g(\cdot,0)$ is invariant under the action of the group S^1 acting on C^1_T through the linear isometry $T_\tau u=u(\cdot+\tau)$. We can then use Theorem 2 of [1] to compute $d_{LS}[I-\mathscr{G}_g(\cdot,0),\Omega,0]$ when Ω is invariant under the action of S^1. If Ω is not invariant we replace it by $\tilde{\Omega}=\{u\in\Omega:\operatorname{dist}(u,\Omega\cap\operatorname{Fix}\mathscr{G}_g(\cdot,0))<\varepsilon\}$ with $0<\varepsilon<\operatorname{dist}(\Omega\cap\operatorname{Fix}\mathscr{G}_g(\cdot,0),\partial\Omega)$. Here $\operatorname{Fix}\mathscr{G}_g(\cdot,0)$ denotes the set of fixed points of $\mathscr{G}_g(\cdot,0)$. $\tilde{\Omega}$ is invariant since S^1 is path-connected, hence $\partial\Omega\cap\operatorname{Fix}\mathscr{G}_g(\cdot,0)=\varnothing$ implies that $\Omega\cap\operatorname{Fix}\mathscr{G}_g(\cdot,0)$ is invariant. We also have used here that S^1 acts through isometries. Since $\operatorname{Fix}\mathscr{G}_g(\cdot,0)\cap\Omega\subset\tilde{\Omega}\subset\Omega$, the excision property of the Leray–Schauder degree yields

$$d_{LS}[I-\mathscr{G}_g(\cdot,0),\Omega,0]=d_{LS}[I-\mathscr{G}_g(\cdot,0),\tilde{\Omega},0].$$

Now the fixed point set $(C^1_T)^{S^1}=\{u\in C^1_T:u(\cdot+\tau)=u(\cdot)\text{ for all }\tau\in I\}$ is the set of constant u in C^1_T and, for such a constant c, $Pc=c$, $Q\mathscr{N}_g(c,0)=g_0(c,0)$, and

$$(\mathscr{K}\circ N_g)(c,0)(t)=H\{\phi^{-1}[\tilde{a}(0)]\}(t)=t\phi^{-1}[\tilde{a}(0)].$$

But, by definition of $\tilde{a}$, we have

$$0=G_0(\tilde{a}(0))=\frac{1}{T}\int_0^T\phi^{-1}[\tilde{a}(0)]\,dt=\phi^{-1}[\tilde{a}(0)],$$

so that $(\mathscr{K}\circ N_g)(c,0)=0$. Thus

$$c-\mathscr{G}_g(c,0)=-g_0(c,0).$$

Consequently, as the Leray–Schauder degree in finite dimensional spaces reduces to the Brouwer degree, we get, using Theorem 2 of [1] and excision,

$$\begin{aligned}d_{LS}[I-\mathscr{G}_g(\cdot,0),\tilde{\Omega},0]&=d_{LS}[(I-\mathscr{G}_g(\cdot,0))|_{(C^1_T)^{S^1}},\tilde{\Omega}\cap(C^1_T)^{S^1},0]\\&=d_B[-g_0(\cdot,0)|_{\mathbb{R}^N},\tilde{\Omega}\cap\mathbb{R}^N,0]\\&=(-1)^N\,d_B[g_0(\cdot,0)|_{\mathbb{R}^N},\tilde{\Omega}\cap\mathbb{R}^N,0]\\&=(-1)^N\,d_B[g_0|_{\mathbb{R}^N},\Omega\cap\mathbb{R}^N,0]\end{aligned}$$

as $\tilde{\Omega} \cap \mathbb{R}^N$ contains all constant T-periodic solutions of (4.8) with $\lambda = 0$ contained in Ω, i.e., all zeros of $g_0(\cdot, 0)|_{\mathbb{R}^N}$. By assumption (2), this last degree is different from zero, and the proof is complete. ∎

As an application of Theorem 4.1, let us consider the problem

$$(\psi_p(u'))' = h(u, u') + e(t, u, u'), \qquad u(0) = u(T), \qquad u'(0) = u'(T), \tag{4.7}$$

where $h: \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and $e: I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory. The following result extends Corollary 9 of [7] to the p-Laplacian case.

THEOREM 4.2. *Assume that the following conditions hold.*

(1) $h(ku, kv) = k^{p-1} h(u, v)$ *for all* $k > 0$ *and all* $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$.

(2) $\lim_{|u| + |v| \to \infty} (e(t, u, v))/((|u| + |v|)^{p-1}) = 0$, *uniformly a.e. in* $t \in I$.

(3) *The problem*

$$(\psi_p(y'))' = h(y, y'), \qquad y(0) = y(T), \;\; y'(0), \;\; y'(T),$$

has only the trivial solution $y = 0$.

(4) $d_B[h(\cdot, 0), b(R_0), 0] \neq 0$ *for some* $R_0 > 0$.

Then problem (4.7) *has at least one solution.*

Proof. We apply Theorem 4.1 to the homotopy

$$\begin{gathered} (\psi_p(u'))' = h(u, u') + \lambda e(t, u, u'), \\ u(0) = u(T), \qquad u'(0) = u'(T), \qquad \lambda \in [0, 1], \end{gathered} \tag{4.8}$$

and show that there exists some $R > 0$ such that, for each $\lambda \in [0, 1]$ and each possible solution u of (4.8), one has $\|y\|_1 < R$, with $\|y\|_1 = \|y\|_0 = \|y'\|_0$. The result will then follow from Theorem 4.1 by taking $\Omega = B(R)$. If it is not the case, one can find a sequence $\{\lambda_n\}$ in $[0, 1]$ and a sequence $\{u_n\}$ of solutions of (4.8) with $\lambda = \lambda_n$ such that $\|u_n\|_1 \to \infty$ when $n \to \infty$. If we set

$$y_n = \frac{u_n}{\|u_n\|_1}, \qquad n = 1, 2, ...,$$

it follows from assumption (1) that

$$\begin{gathered} (\psi_p(y_n'))' = h(y_n, y_n') + \lambda_n \frac{e(t, \|u_n\|_1 \, y_n, \|u_n\|_1 \, y_n')}{\|u_n\|_1^{p-1}}, \\ y_n(0) = y_n(T), \qquad y_n'(0) = y_n'(T), \quad n = 1, 2, \ldots . \end{gathered} \tag{4.9}$$

As $\|y_n\|_1 = 1$ for all n, we can assume, going if necessary to a subsequence, that $y_n \to y$ uniformly in I, for some $y \in C_T$. Letting $z_n = \psi_p(y'_n)$, it is clear that $\{z_n\}$ is bounded in C_T and it follows from Eq. (4.9) and from assumption (2) that $\{z'_n\}$ is bounded in C_T as well. Thus, up to a further subsequence, we can assume that $\{z_n\}$ converges uniformly on I to some $z \in C_T$. Notice, then, that $\{y'_n\}$ converges uniformly on I to $\psi_{p^*}(z)$, and that

$$\|y\|_0 + \|\psi_{p^*}(z)\|_0 = 1, \tag{4.10}$$

where p^* is conjugate to p. Now, problem (4.9) is equivalent to

$$\begin{aligned} &y'_n = \psi_{p^*}(z_n), \\ &z'_n = h(y_n, \psi_{p^*}(z_n)) + \lambda_n \frac{e(t, \|u_n\|_1\, y_n, \|u_n\|_1\, \psi_{p^*}(z_n))}{\|u_n\|_1^{p-1}}, \\ &\qquad y_n(0) = y_n(T), \quad z_n(0) = z_n(T), \quad n = 1, 2, \ldots . \end{aligned} \tag{4.11}$$

Using the above convergence results and an integrated form of (4.11), it is easy to see that (y, z) will be a solution of the problem

$$\begin{aligned} y' &= \psi_{p^*}(z), & z' &= h(y, \psi_{p^*}(z)) \\ y(0) &= y(T), & z(0) &= z(T), \end{aligned}$$

and hence y will be a solution of the problem

$$(\psi_p(y'))' = h(y, y'), \qquad y(0) = y(T), \qquad y'(0) = y'(T).$$

But then using assumption (3), it follows that $y = 0$, and hence $\psi_{p^*}(z) = 0$, a contradiction to (4.10). ∎

As a special case of Theorem 4.2, we get of course the following classical result. Recall that $\mu \in \mathbb{R}$ is an eigenvalue of the p-Laplacian ψ_p with T-periodic boundary conditions if the problem

$$(\psi_p(u'))' + \mu\psi_p(u) = 0, \qquad u(0) = u(T), \qquad u'(0) = u'(T),$$

has a nontrivial solution.

COROLLARY 4.1. *If μ is not an eigenvalue of the p-Laplacian ψ_p with T-periodic boundary conditions, then, for each $e \in L^1$, the problem*

$$(\psi_p(u'))' + \mu\psi_p(u) = e(t), \qquad u(0) = u(T), \qquad u'(0) = u'(T),$$

has at least one solution.

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