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A brief introduction to topological degree theory

1 Introduction

In these few pages we present an introduction to the concept of topological degree from an analytic viewpoint and, in particular, we summarize two of the most relevant constructions of the degree in literature: the Brouwer degree for continuous maps between Euclidean spaces of finite dimension and the Leray–Schauder degree for compact perturbations of the identity in real Banach spaces.

2 A first approach

The reader who meets this topic for the first time could perhaps start by asking the following question: *what is the topological degree*? As a rough answer, the degree is a tool, precisely a number, which gives information about the solutions of particular equations. In a great generality, consider the equation

$$f(x) = y, \quad x \in U, \tag{2.1}$$

where

i) $f : X \to Y$ is a given function, supposed at least continuous,

ii a) X and Y are Euclidean spaces or real, finite dimensional, differentiable manifolds or

ii b) Banach spaces or manifolds of possibly infinite dimension,

iii) y is a fixed element of Y,

iv) U is an open subset of X.

In the cases when a direct computation does not solve an equation as the (2.1) above, neither give suitable approximations of the solutions, we can looking for other methods to get information about the set of solutions. For example we can ask if the set of solutions is not empty, finite or infinite, where the solutions, or some of them, are localized in U, if they are stable with respect to perturbations of f or y, and even other more complicated issues.

Consider for a moment the finite dimensional setting of Euclidean spaces. A family of problems as (2.1) could be seen as a set $\mathcal{T}$ of *admissible triples*,

$$\mathcal{T} = \{(f, U, y)\},$$

where, given an open subset Ω of $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ is a continuous function, U is an open subset of Ω and $f^{-1}(y) \cap U$ is compact. A *topological degree*, simply a *degree*, is a map

$$\deg : \mathcal{T} \to \mathbb{Z}$$

such that some particular properties are verified. Among them let us mention here the following two:

1. (*Existence*) given an admissible triple (f, U, y), if

$$\deg(f, U, y) \neq 0,$$

then the equation $f(x) = y$ has at least one solution in U.

2. (*Homotopy invariance*) given two admissible triples (f, U, y) and (g, U, y), and a continuous map $H : U \times [0, 1] \to \mathbb{R}^n$ such that $H^{-1}(0)$ is compact and

$$H(x, 0) = f(x), \quad H(x, 1) = g(x), \quad \forall x \in U,$$

then

$$\deg(f, U, y) = \deg(g, U, y).$$

Let us briefly explain the importance of these two properties. If we are able to compute the degree of a triple (f, U, y) and this is different from zero, the existence property tells us that the equation $f(x) = y$ has, at least, one solution in U. In other words, the computation of the degree allows us to answer the question of the existence of solutions[1]. If $\deg(f, U, y)$ is difficult to be calculated, we can use the homotopy invariance property and look for another triple (g, U, y) where g is a 'continuous deformation' of f (the deformation is the homotopy H). If $\deg(g, U, y)$ is known, we obtain also the value of $\deg(f, U, y)$.

3 The finite dimensional case: the Brouwer degree

3.1 The Brouwer degree

After a pioneering work of Kronecker [5] of 1869, the first definition of degree for maps between Euclidean spaces is due to Brouwer [2] in 1912. In 1951 Nagumo [11] redefines the concept, today commonly known as Brouwer degree, by an analytical approach, which is different from the original Brouwer construction and uses Sard's Theorem[2]. We give in this section a short summary, in the analytical and modern version, of the Brouwer degree with its most important properties.

Definition 3.1. Let Ω be an open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ a continuous map. Given an open subset U of Ω and an element $y \in \mathbb{R}^n$, the triple (f, U, y) is said to be *admissible for the Brouwer degree* if $f^{-1}(y) \cap U$ is compact.

The Brouwer degree is a map from the set of admissible triples into $\mathbb{Z}$. To define it, consider first the particular triples (f, U, y) such that $f|_U$ is C^∞ and y is a regular value for f in U (see Definition 3.2 below). In this case $f^{-1}(y) \cap U$ is a finite set, being compact, and the *Brouwer degree* of (f, U, y), in symbols $\deg_B(f, U, y)$, is given by

$$\deg_B(f, U, y) = \sum_{x \in f^{-1}(y) \cap U} \operatorname{sign} f'(x), \tag{3.1}$$

where $\operatorname{sign} f'(x)$ is the sign of the determinant of the Jacobian matrix $f'(x)$.

Exercise 1. Prove that, if y is a regular value for f in U and $f^{-1}(y) \cap U$ is compact, then it is a finite set.

Then, as a second step in the construction, we remove the assumption that y is regular, f still being C^∞. This step in the construction will be based on the Sard Theorem.

[1] The converse is, generally, not true: if the degree is zero, we do not know if the equation $f(x) = y$ has a solution in U.
[2] The original idea of Brouwer uses a simplicial geometric approach. On the other hand, Sard's Theorem appears in 1942.

Definition 3.2. Recall that, given a C^1 map $g : U \subseteq \mathbb{R}^k \to \mathbb{R}^s$, $x \in U$ is called a *regular point* for g if the derivative $g'(x) : \mathbb{R}^k \to \mathbb{R}^s$ is onto. Otherwise, x is called a *critical point.* An element $y \in \mathbb{R}^s$ is called a *critical value* for g in U if $g^{-1}(y) \cap U$ contains at least a critical point. Otherwise, it is called a *regular value.*

Theorem 3.3 (Sard, 1942)**.** *Let $f : U \to \mathbb{R}^s$ be a C^n map defined on an open subset of $\mathbb{R}^k$. If $n > \max\{0, k-s\}$, then the set of critical values of f has (s-dimensional) Lebesgue measure zero.*

Consider an open bounded neighborhood D of $f^{-1}(y) \cap U$ such that its closure (in $\mathbb{R}^n$) is contained in U. Let $d = \operatorname{dist}(y, f(\partial D))$ be the distance between y and $f(\partial D)$. As ∂D is bounded and closed, $f(\partial D)$ is bounded and closed as well, so d is positive. Sard's Theorem ensures that any neighborhood of y contains regular values for $f|_U$; thus define

$$\deg_B(f, U, y) = \deg_B(f, D, y'), \tag{3.2}$$

where y' is any regular value of $f|_D$ such that $\|y - y'\| < d$. The definition is well posed because it is possible to prove that the right hand side of the above formula is independent of f, y' and D.

Remark 3.4. The independence of y' in the above equality is not easy to be proven. This fact uses Sard's Theorem and actually could not be proven if f were only C^1. In other words: the reader can observe that the right hand side of formula (3.1) makes sense also in the weaker assumption that f is C^1 (and, of course, y is regular for $f|_U$). Actually, equality (3.2) is proven using Sard's Theorem, which requires, in this particular case, that f is C^2, not only C^1 (we put C^∞ for a sake of simplicity). Consequently, the degree for a triple (f, U, y), where f is C^1 and y is regular is defined by formula (3.3) below. On the other hand, one can prove that equality (3.1) remains valid in the C^1 case.

Exercise 2. Show one of the above cited facts: as ∂D is bounded and closed, $f(\partial D)$ is bounded and closed as well.

Finally, if (f, U, y) is a general admissible triple, let D and d be as above. By classical approximation results, there exists a C^∞ map, $g : \overline{D} \to \mathbb{R}^n$, closer than d to f on $\overline{D}$ in the sup-norm. Then, (g, D, y) is admissible and define

$$\deg_B(f, U, y) = \deg_B(g, D, y). \tag{3.3}$$

This definition is well posed since the right hand side of the above equality does not depend on the choice of D and g.

The theorem below lists the most important properties of the Brower degree.

Theorem 3.5. *The Brouwer degree verifies the following properties.*

1. (Normalization) *Let I denote the identity of $\mathbb{R}^n$. If U is open in $\mathbb{R}^n$ and $y \in U$, then*

$$\deg_B(I, U, y) = 1.$$

2. (Additivity) *If (f, U, y) is an admissible triple and U_1 and U_2 are open in U, disjoint and such that $f(x) \neq y$ if $x \in U \backslash (U_1 \cup U_2)$, then*

$$\deg_B(f, U, y) = \deg_B(f, U_1, y) + \deg_B(f, U_2, y).$$

3. (Homotopy invariance) *Given $H : U \times [0,1] \to \mathbb{R}^n$ continuous, if $H^{-1}(y)$ is compact, then $\deg_B(H(\cdot, t), U, y)$ is independent of $t \in [0,1]$.*

4. (Existence) *If $\deg_B(f, U, y) \neq 0$ then $f^{-1}(y) \cap U \neq \emptyset$.*

5. (Excision) *Given* (f, U, y) *admissible, if* D *is open in* U *and* $f^{-1}(y) \cap U \subseteq D$*, then*

$$\deg_B(f, U, y) = \deg_B(f, D, y).$$

6. (Local constance with respect to the map) *Let* (f, U, y) *be an admissible triple such that* U *is bounded,* f *is defined on* $\overline{U}$ *and* $f^{-1}(y) \cap \partial U = \emptyset$*. If* $g : \overline{U} \to \mathbb{R}^n$ *is continuous and* $\sup_{x \in \overline{U}} \|f(x) - g(x)\| < dist(y, f(\partial U))$*, then*

$$\deg_B(f, U, y) = \deg_B(g, U, y).$$

7. (Local constance with respect to the target point) *Let* (f, U, y) *be an admissible triple where* $\overline{U}$ *is bounded and contained in the domain of* f*, and* $f^{-1}(y) \cap \partial U = \emptyset$*. If* z *belongs to the same component of* y *in* $\mathbb{R}^n \backslash f(\partial U)$*, then*

$$\deg_B(f, U, y) = \deg_B(f, U, z).$$

8. (Dependence on the boundary) *Let* (f, U, y) *and* (g, U, y) *be two admissible triples, with* U *bounded,* f *and* g *defined on* $\overline{U}$*,* $f = g$ *on* ∂U*, and* $f(x) = g(x) \neq y$ *for all* $x \in \partial U$*. Then*

$$\deg_B(f, U, y) = \deg_B(g, U, y).$$

4 Some topological applications

Proposition 4.1. *Let* $f_n : \mathbb{C} \to \mathbb{C}$ *be a polynomial of algebraic degree* $n > 0$*. Consider, with an abuse of notation,* f_n *as a map from* $\mathbb{R}^2$ *into itself. Then,* $\deg_B(f_n, \mathbb{R}^2, 0)$ *is well defined and takes value* n*.*

The idea of the proof is the following. First, observe that $\deg_B(f_n, \mathbb{R}^2, y)$ is well defined for any $y \in \mathbb{R}^2$ since $f^{-1}(y)$ is finite. Then, write $f_n(z) = az^n + q(z)$, with $a \neq 0$ and $q(z)$ a polynomial of degree less then n. Consider the homotopy

$$H(z, \lambda) = az^n + \lambda q(z)$$

and observe that

$$\lim_{|z| \to +\infty} |H(z, \lambda)| = +\infty,$$

uniformly with respect to $\lambda \in [0, 1]$. Thus H is a proper map (see Exercise 4 below) and, consequently, given $g_n : z \mapsto az^n$, we have

$$\deg_B(f_n, \mathbb{R}^2, 0) = \deg_B(g_n, \mathbb{R}^2, 0).$$

To conclude that the Brouwer degree of f_n is n, observe that the equation $az^n = a$ has exactly n solutions $z_1, ..., z_n$ each of them verifying $\operatorname{sign} g_n'(z_i) = 1$ (as the sign of the determinant in $\mathbb{R}^2$).

Exercise 3. The reader can verify the above final property concerning the sign of the derivative.

Exercise 4. Let $X \subseteq \mathbb{R}^k$ be closed and unbounded. Prove that a map $f : X \to \mathbb{R}^s$ is proper if and only if

$$\lim_{x \in X,\ |x| \to +\infty} |f(x)| = +\infty.$$

Exercise 5. Prove that, if $f : \mathbb{R} \to \mathbb{R}$ is proper, then $\deg(f, \mathbb{R}, y)$ may assume only three values: $-1, 0, 1$.

An easy consequence of Proposition 4.1 is the Fundamental Theorem of Algebra.

Fundamental Theorem of Algebra. *Any nonconstant polynomial with complex coefficients admits at least one root.*

The following is another famous topological result that can be deduced from degree theory (even if it can be proven without the use of the degree).

Brouwer Fixed Point Theorem. *Let U be the open unit ball in $\mathbb{R}^k$ and let $f : \overline{U} \to \mathbb{R}^k$ be continuous and such that $f(\overline{U}) \subseteq \overline{U}$ (or, more generally, $f(\partial U) \subseteq \overline{U}$). Then f has a fixed point in $\overline{U}$.*

Hairy Ball Theorem. *Let S^{2n} be the unit sphere in $\mathbb{R}^{2n+1}$ and let $v : S^{2n} \to \mathbb{R}^{2n+1}$ be a continuous tangent vector field, that is, such that $v(x) \cdot x = 0$ for any x (the above is the scalar product). Then, v vanishes in at least one point.*

5 An application

In the following example the degree can be applied to prove the existence of solutions of a nonlinear equation in $\mathbb{R}^2$. We omit the details giving the main ideas.

Example 5.1. Consider the following nonlinear system:

$$\begin{cases} x^2 - 2y^2 - \sin(xy) & = & 0 \\ xy + 2\cos x + \frac{1}{1+y^2} & = & 0 \end{cases} \tag{5.1}$$

We claim that this system has at least one solution. The reader can prove as an exercise that the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x^2 - 2y^2, xy)$ is proper. Indeed, let c be a positive constant and consider the inequalities

$$|x^2 - 2y^2| \leq c \quad \text{and} \quad |xy| \leq c.$$

The first one implies that when $|x|$ is large, so is $|y|$; but this is in contrast with the second one. Thus, if $f(x, y)$ belongs to a compact set, also (x, y) must stay in a compact set. Hence, $\deg_B(f, \mathbb{R}^2, p)$ is well defined and independent of p. To compute it, observe that the system

$$\begin{cases} x^2 - 2y^2 & = & 1 \\ xy & = & 0 \end{cases}$$

has the following two solutions: $(1, 0)$ and $(-1, 0)$. One can check that these solutions are both regular and 1 is the sign of the derivatives. Thus, $\deg_B(f, \mathbb{R}^2, (1, 0)) = 2$, and this implies the existence of at least one solution of system (5.1), as claimed, because we could use the homotopy invariance property to "join" f with the map $(x, y) \mapsto (x^2 - 2y^2 - \sin(xy), xy + 2\cos x + \frac{1}{1+y^2})$.

Actually, since $\deg_B(f, \mathbb{R}^2, p) = 2$, for any p, applying Sard's Lemma (and the definition of degree for a regular triple) one can say more: for almost all $(a, b) \in \mathbb{R}^2$ the system

$$\begin{cases} x^2 - 2y^2 - \sin(xy) & = & a \\ xy + 2\cos x + \frac{1}{1+y^2} & = & b \end{cases}$$

has at least two solutions (and, of course, at least one for all $(a, b) \in \mathbb{R}^2$).

6 The uniqueness of the Brouwer degree

As we said above, the topological degree in Euclidean spaces is a map which to any admissible triple (f, U, y) assigns an integer, $\deg(f, U, y)$, satisfying some important properties. The first three stated in Theorem 3.5, *Normalization*, *Additivity* and *Homotopy Invariance*, are usually called *fundamental properties* for the following reason. A famous result of Amann and Weiss in 1973, see [1], establishes *the uniqueness of the topological degree* proving that there exists at most one real-valued map, defined on the class of admissible triples, which verifies the three fundamental properties. Moreover, this map, assuming its existence, must be integer valued.

There are several methods for the construction of the degree:

1. by simplicial geometry (the orignal work of Brouwer [2]),

2. by algebraic geometry (see [6]),

3. by an analytical approach with the use of Sard's Theorem (see [7, 9]),

4. by an analytical approach without the use of Sard's Theorem (see [12]).

In any case, because of the Amann–Weiss result, given any method, what is important is proving the three fundamental properties, that one may regard as *axioms of the degree*: all the other classical properties are a consequence of these axioms.

7 The infinite dimensional degree by Jean Leray and Juliusz Schauder

In the attempt to construct a degree theory for functions acting between infinite dimensional spaces, a possible generalization of the Brouwer degree could involve triples (f, U, y) where $f : \Omega \to F$ is a continuous map defined on an open subset of a real Banach space E into a real Banach space F, U is an open subset of Ω, and $f^{-1}(y) \cap U$ is compact. Actually, a strategy following, in an infinite dimensional context, an analogous outline of the Brouwer degree meets crucial obstructions. Let us mention three:

– if f is C^1 and x is a regular point, it is not clear how we can define a sign for the Fréchet derivative $f'(x)$ and thus it is not clear how to generalize formula (3.1);

– a general result ensuring the approximation of continuous maps by smooth maps does not exist;

– if U is bounded, f is continuous on $\overline{U}$ and y is a given element in F, then $f^{-1}(y) \cap \overline{U}$ is not necessarily compact.

One of course could have the following objection: can the above obstacles be overcamed simply by a different approach in the construction? In principle the answer may be affirmative, but we have the following serious problem, independent of any method

– In the textbook [8] we can find an example wich says the following: given a Banach space E, an open subset U containing the origin, the identity map $I : U \to E$ and a continuous map $f : U \to E$ such that $f(x) = 0$ has no solution in U. In addition, and it is crucial, we can define a $H : \overline{U} \times [0, 1] \to E$ between I and g, such that $H(x, \lambda) \neq 0$ for any $x \in \partial U$ and any $\lambda \in [0, 1]$. This homotopy says what follows:

  (a) the (possible) degrees of $(f, U, 0)$ and $(I, U, 0)$ should coincide by a sort of hopotopy invariance

  (a) degrees of $(f, U, 0)$ should be zero because $f(x) = 0$ has no solution in U and the degrees of $(I, U, 0)$ should be 1.

As a consequence of the above last fact, no degree theory in infinite dimension can be defined (and will never be defined!) for the whole family of continuous maps, but for special classes of them.

The first construction of a degree theory in infinite dimension is due to Leray and Schauder [6] in 1934. We summarize here the construction. The degree introduced by Leray and Schauder is defined for the maps of the form
$$f : \Omega \to E, \qquad f(x) = x - k(x),$$
where E is a real Banach space, Ω is an open subset of E and k is *completely continuous*, that is, it sends bounded sets into sets with compact closure (called *relatively compact*). The *admissible triples* are those (f, U, y) such that f is as above, y belongs to E and U is an open subset of Ω with $f^{-1}(y) \cap U$ compact.

The following properties of completely continuous maps play a crucial role in the construction of the degree, justifying the choice of this class of maps.

(P1) Given a completely continuous map $k : E \to E$ and a closed bounded subset B of E, for any $\varepsilon > 0$ there exists a continuous map $k_1 : B \to E$ such that

1. $k_1(B)$ is contained in a finite dimensional subspace of E,
2. $\sup_{x\in B} \|k(x) - k_1(x)\| < \varepsilon$.

(P2) $I - k$ is proper on closed bounded subsets of E, that is, for any closed (in E) and bounded subset B of Ω and any compact subset C of E, then $(I - k)^{-1}(C) \cap B$ is compact. As a byproduct, $I - k$ sends closed bounded sets into closed sets.

Exercise 6. Prove the (P1) as seen in the lesson of october 17, and prove the (P2).

Given an admissible triple (f, U, y), let D be an open bounded subset of U containing $f^{-1}(y) \cap U$, such that $\overline{D} \subseteq U$ and $f(x) \neq y$ for any $y \in \partial D$. Let d be the distance between y and $f(\partial D)$, which is positive since $f(\partial D)$ is closed and does not contain y. Then, let $k_1 : \overline{D} \to E$ be continuous and such that

1. $k_1(\overline{D}) \subseteq E_1$, where E_1 is a finite dimensional subspace of E,

2. $\sup_{x\in\overline{D}} \|k(x) - k_1(x)\| < d$.

Assume, without loss of generality, that E_1 contains y. Denote $D_1 = D \cap E_1$ and consider

$$f_1 : \overline{D_1} \to E_1, \qquad f_1(x) = x - k_1(x).$$

We have

$$f_1^{-1}(y) \cap \overline{D_1} = f_1^{-1}(y) \cap D_1 = f^{-1}(y) \cap U.$$

Moreover, consider E_1 oriented with the same orientation as source and target space. Then, the Brouwer degree of the triple (f_1, D_1, y) is well defined and independent of the orientation of E_1. The *Leray-Schauder degree* of (f, U, y), $\deg_{LS}(f, U, y)$ in symbols, is defined as

$$\deg_{LS}(f, U, y) = \deg_B(f_1, D_1, y). \tag{7.1}$$

The Leray-Schauder degree verifies the same properties as those shown in Theorem 3.5 for the Brower degree, with the necessary changes in the statement: $\mathbb{R}^n$ is replaced by a Banach space E, and, in the homotopy invariance property, $H(x, t)$ is of the form $x - k(x, t)$, with k completely continuous.

8 An application of the Leray-Schauder degree.

Since the publication of the paper of 1934, the Leray-Schauder degree has been recognized as a very important tool for the study of many problems in ordinary and partial differential equations. An interesting and accurated description of important applications of the Leray-Schauder degree is due to Mawhin in [9]. In this section we summarize a particular application of the Leray-Schauder degree to an existence problem for periodic solutions of a differential equation. We give a simplified and more understandble version of the results given in [8, 10]). Consider the system

$$u'' = f(t, u, u'), \quad u(0) = u(T), \;\; u'(0) = u'(T), \tag{8.1}$$

where $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. By a solution of (8.1) we mean a C^2 function $u : [0, T] \to \mathbb{R}^n$, satisfying the boundary conditions, such that $u''(t) = f(t, u(t), u'(t))$ for every $t \in [0, T]$.

In order to study problem (8.1), we can start by considering the following simplified equation

$$u'' = h, \quad u(0) = u(T), \;\; u'(0) = u'(T), \tag{8.2}$$

where h belongs to the closed subspace $\mathcal{C}^0_m$ of $C^0([0,T],\mathbb{R}^n)$ of the maps k such that $\int_0^T k(t)dt = 0$. If a C^2 function $u : [0,T] \to \mathbb{R}^n$ solves the equation $u''(t) = h(t)$, there exists $a \in \mathbb{R}^n$ such that

$$u'(t) = a + H(h)(t), \tag{8.3}$$

where H is the integral operator

$$H(h)(t) = \int_0^t h(s)ds.$$

Thus any C^2 solution u can be written as

$$u(t) = u(0) + \int_0^t a + H(h)(s)ds.$$

The boundary condition $u(0) = u(T)$ implies that

$$\int_0^T a + H(h)(t)dt = 0. \tag{8.4}$$

It is possible to prove that, for any $h \in \mathcal{C}^0_m$, the element $a \in \mathbb{R}^n$ such that (8.4) holds is unique and then it is well defined the map $a(h)$ which is, in addition, continuous and bounded. Therefore, problem (8.2) has infinite solutions which differ by a constant and can be written as

$$u(t) = u(0) + H\left(a(h) + H(h)\right)(t).$$

Denote $\mathcal{C}^0 = C^0([0,T],\mathbb{R}^n)$, $\mathcal{C}^1 = C^1([0,T],\mathbb{R}^n)$ and $\mathcal{C}^1_T = \{u \in \mathcal{C}^1 : u(0) = u(T),\ u'(0) = u'(T)\}$. Define

$$P : \mathcal{C}^1_T \to \mathcal{C}^1_T, \quad Pu = u(0), \quad Q : \mathcal{C}^0 \to \mathcal{C}^0, \quad Qh = \frac{1}{T}\int_0^T h(t)dt.$$

Then, consider the nonlinear operator $K : \mathcal{C}^0 \to \mathcal{C}^1_T$, defined as

$$K(\widehat{h})(t) = H\left(a((I-Q)\widehat{h}) + H((I-Q)\widehat{h})\right)(t).$$

If a C^1 function u is a solution of (8.2), for a given $h \in \mathcal{C}^0_m$, of course u solves the equation

$$u = Pu + Qh + K(h). \tag{8.5}$$

Conversely, if $u \in \mathcal{C}^1_T$ is a solution of (8.5), for a given $h \in \mathcal{C}^0$, it is possible to prove that h actually belongs to $\mathcal{C}^0_m$ and u solves (8.2).

The idea of studying equation (8.5), in order to find a solution of (8.2), is particularly important if we consider the periodic problem (8.1). In fact, consider the operator $N : \mathcal{C}^1 \to \mathcal{C}^0$ (usually called *Nemitski operator*), associated to problem (8.1)

$$N(u)(t) = f(t, u(t), u'(t)).$$

if we define $\mathcal{G} : \mathcal{C}^1_T \to \mathcal{C}^1_T$ by

$$\mathcal{G}(u) = Pu + QN(u) + K(N(u)),$$

we observe that problem (8.1) is equivalent to the fixed point problem

$$u = \mathcal{G}(u).$$

The crucial point is that $\mathcal{G}$ is completely continuous, as it can be proved by the properties of K, applying Ascoli-Arzelà Theorem. Following this idea, one can apply the Leray-Schauder degree to prove the following existence theorem for (8.1).

Theorem 8.1. *Let Ω be a bounded open subset of $\mathcal{C}^1_T$ such that the following conditions hold:*

1. *for each $\lambda \in (0,1)$ the problem*
$$u'' = \lambda f(t,u,u'), \quad u(0) = u(T), \;\; u'(0) = u'(T) \tag{8.6}$$
has no solution on $\partial\Omega$;

2. *the equation*
$$F(a) := \int_0^T f(t,a,0)dt = 0 \tag{8.7}$$
has no solution on $\partial\Omega_2$, where $\Omega_2 := \Omega \cap E_2$ and E_2 is the subspace of $\mathcal{C}^1_T$ of constant maps;

3. *the Brouwer degree*
$$\deg_B(F, \Omega_2, 0)$$
is well defined and nonzero.

Then problem (8.1) *has a solution in $\overline{\Omega}$.*

We give here an idea of the proof. Let N denote (as previously) the Nemitski operator associated to f, that is,
$$N : \mathcal{C}^1_T \to \mathcal{C}^0, \quad N(u)(t) = f(t,u(t),u'(t)).$$
Consider the problem
$$\begin{cases} u'' = \lambda N(u) + (1-\lambda)QN(u) \\ u(0) = u(T) \\ u'(0) = u'(T). \end{cases} \tag{8.8}$$
For $\lambda \in (0,1]$, if u is a solution of (8.6), then, as seen before, condition $u'(0) = u'(T)$ implies $QN(u) = 0$ and hence u solves problem (8.8) as well. Conversely, if u is a solution of problem (8.8), then $QN(u) = 0$ since it is easy to see that
$$Q[\lambda N(u) + (1-\lambda)QN(u)] = QN(u),$$
and thus u solves problem (8.6) (λ still belongs to $(0,1]$). Let us now consider problem (8.8). It can be written in the equivalent form
$$u = \mathcal{K}(u,\lambda), \tag{8.9}$$
where
$$\begin{aligned} \mathcal{K}(u,\lambda) &= Pu + QN(u) + (K \circ [\lambda N + (1-\lambda)QN])(u) \\ &= Pu + QN(u) + (K \circ [\lambda(I-Q)N])(u) \end{aligned}$$
is well defined in $\overline{\Omega} \times [0,1]$. Suppose that (8.8) has no solution on $\partial\Omega$ for $\lambda = 1$, since, otherwise, the theorem is proved. Take $\lambda = 0$. Problem (8.8) becomes
$$\begin{cases} u'' = \frac{1}{T} \int_0^T f(t,u(t),u'(t))dt \\ u(0) = u(T) \\ u'(0) = u'(T). \end{cases} \tag{8.10}$$
It follows that $\int_0^T f(t,u(t),u'(t))dt = 0$ and this implies that u is a constant function, say $u(t) = c$. Therefore, we have
$$\int_0^T f(t,c,0)dt = 0.$$
By assumption (2), $c \notin \partial\Omega_2$. Therefore we obtain that the equation
$$u - \mathcal{K}(u,\lambda) = 0$$

has no solution on $\partial\Omega \times [0,1]$. In addition, the nonlinear map $\mathcal{N} : \mathcal{C}^1_T \times [0,1] \to \mathcal{C}^0$, defined by

$$\mathcal{N}(u,\lambda) = \lambda N(u) + (1-\lambda)QN(u), \tag{8.11}$$

is completely continuous. We can apply the homotopy invariance property of the Leray-Schauder degree to the map $(u,\lambda) \mapsto u - \mathcal{K}(u,\lambda)$, obtaining

$$\deg_{LS}(I - \mathcal{K}(\cdot,0),\Omega,0) = \deg_{LS}(I - \mathcal{K}(\cdot,1),\Omega,0). \tag{8.12}$$

We can now say that problem (8.1) has a solution in $\overline{\Omega}$ if we prove that $\deg_{LS}(I - \mathcal{K}(\cdot,0),\Omega,0) \neq 0$. To see this observe first that $K(0) = 0$, then

$$\mathcal{K}(u,0) = Pu + QN(u).$$

To compare the Leray-Schauder degree of the triple $(I-\mathcal{K}(\cdot,0),\Omega,0)$ with the Brouwer degree of $(F,\Omega_2,0)$, consider the splitting

$$\mathcal{C}^1_T = E_1 \oplus E_2, \tag{8.13}$$

where E_1 contains the maps $\widetilde{u}$ such that $\widetilde{u}(0) = 0$ and E_2 is the N-dimensional subspace of constant maps. The operator $I - \mathcal{K}(\cdot,0)$ can be represented in block-matrix form as

$$I - \mathcal{K}(\cdot,0) = \begin{pmatrix} I_{E_1} & -\mathcal{K}_{12} \\ 0 & -F \end{pmatrix}.$$

By the properties of the Leray-Schauder degree we have a sort o connection (finite-dimensional reduction property of the degree) we the Brouwer degree. One has that

$$\deg_{LS}(I - \mathcal{K}(\cdot,0),\Omega,0) = (-1)^N \deg_B(F,\Omega_2,0)$$

and this completes the proof.

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