What Is Geometry?

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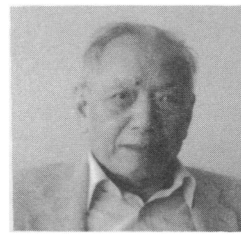
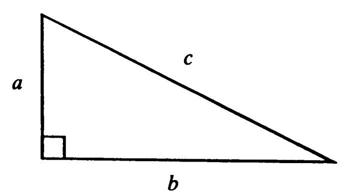


SHIING-SHEN CHERN. Born October 26, 1911 in Kashing, Chekiang Province, China, Dr. Chern was educated at Nankai University, Tsinghua University, and the University of Hamburg, where he received his doctorate in 1936. He has been Professor at UC Berkeley since 1960, Emeritus since 1979, and is now Director Emeritus of MSRI in Berkeley as well as Director of the Nankai Institute of Mathematics in Tianjin, China. He is a member of the National Academy of Sciences and of numerous other Academies, domestic and foreign. He has received the National Medal of Science of the USA, and many other prizes, distinctions and honors.

To avoid misunderstanding I will not give a definition of geometry as in the customary mathematical treatment of a topic. I will only try to discuss its major historical developments.

1. Geometry as a logical system; Euclid. Euclid's "Elements of Geometry" (ca. 300 B.C.) is one of the great achievements of the human mind. It makes geometry into a deductive science and the geometrical phenomena as the logical conclusions of a system of axioms and postulates. The content is not restricted to geometry as we now understand the term. Its main geometrical results are:

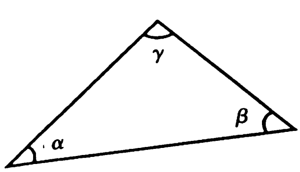
a) Pythagoras' Theorem.



$$c^2 = a^2 + b^2$$

FIG. 1

b) Angle-sum of a triangle.



$$\alpha + \beta + \gamma = 180°$$

FIG. 2

The result b) is derived using the fifth, or the last, postulate, which says: "And that, if a straight line falling on two straight lines make the angles, internal and on

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the same side, less than two right angles, the two straight lines, being produced indefinitely, meet on the side on which are the angles less than two right angles."

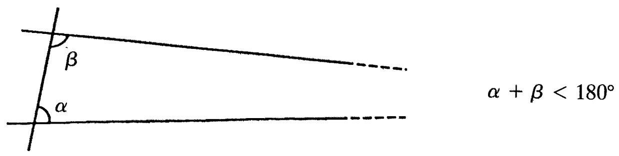

FIG. 3

Euclid realized that the parallel postulate was not as transparent as his other axioms and postulates. Efforts were made to prove it as a consequence. Their failure led to the discovery of non-Euclidean geometry by C. F. Gauss, John Bolyai, and N. I. Lobachevski in the early 19th century.

The "Elements" treated rectilinear figures and the circle. The last three of its thirteen Books were devoted to solid geometry.

2. Coordinatization of space; Descartes. The introduction of coordinates by Descartes (1596–1650) was a revolution in geometry. In the plane it can be described by the following figure:

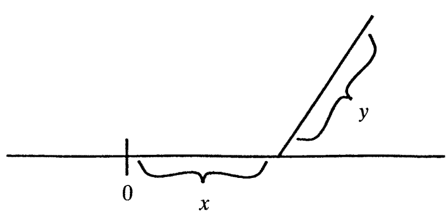

FIG. 4

where the role of the two coordinates x, y is not symmetric. Descartes' work was published in 1637 as an appendix, entitled "La géométrie", to his famous book on philosophy [**6**]. At about the same time Fermat (1601–1665) also found the concept of coordinates and used them to treat successfully geometric problems by algebraic methods. But Fermat's work was published only posthumously [**7**].

One immediate consequence was the study of curves defined by arbitrary equations

$$F(x, y) = 0, \tag{1}$$

thus enlarging the scope of the figures.

Fermat went on to introduce some of the fundamental concepts of the calculus, such as the tangent line and the maxima and minima.

From two dimensions one goes to n dimensions, and to an infinite number of dimensions. In these spaces one studies loci defined by arbitrary systems of equations. Thus a great vista was opened, and geometry and algebra became inseparable.

A mystery is the role of differentiation. The analytic method is most effective when the functions involved are smooth. Hence I wish to quote a philosophical question posed by Clifford Taubes [**15**]: Do humans really take derivatives? Can they tell the difference?

Coordinate geometry paved the way to applications to physics. An example was Newton's derivation of the Kepler laws from his law of gravitation. Kepler's first law says that the planetary orbits are ellipses with the sun as their common focus. The proof was possible only after an analytic theory of conics had been established.

3. Space based on the group concept; Klein's Erlanger Programm. Works on geometry led to the development of projective geometry, among whose founders were: J. V. Poncelet (1788–1867), A. F. Möbius (1790–1868), M. Chasles (1793–1880), and J. Steiner (1796–1863). Projective geometry studies the geometrical properties arising from the linear subspaces of a space and the transformations generated by projections and sections. Other geometries resulted, the most notable ones being affine geometry and conformal geometry.

In 1872 Felix Klein formulated his Erlanger Programm [**1**], [**11**], which defines geometry as the study of the properties of a space that are invariant under a group of transformations. Thus there is a geometry corresponding to every group of transformations acting on a space. The basic notion is "group" and the notion of a space is now greatly expanded. In a certain sense the group of projective collineations is the most encompassing group and projective geometry occupies a dominant position.

The most important application of the Erlanger Programm was the treatment of non-Euclidean geometry by the so-called Cayley-Klein projective metric [**12**]. The hyperbolic space can be identified with the interior of a hypersphere and the non-euclidean motions with the group of projective collineations leaving invariant the hypersphere. The same group may appear as a group of transformations in different spaces. As a result the same algebraic argument could give entirely different geometric theorems. For example, everybody knows that the three medians of a triangle meet in a point. By using Study's dual numbers this translates into the following theorem of J. Petersen and F. Morley: Let *ABCDEF* be a skew hexagon such that consecutive sides are perpendicular. The three common perpendiculars of the pairs of opposite sides AB, DE; BC, EF; CD, FA have a common perpendicular. See [**13**].

Sophus Lie founded a theory of general transformation groups, which became a fundamental tool of all geometry.

4. Localization of geometry; Gauss and Riemann. In his monograph on surface theory published in 1827 [**8**], Gauss (1777–1855) developed the geometry on a surface based on its fundamental form. This was generalized by B. Riemann (1826–1866) to n dimensions in his Habilitationschrift in 1854 [**14**]. Riemannian geometry is the geometry based on the quadratic differential form

$$ds^2 = \sum g_{ik}(u)\, du^i\, du^k, \qquad g_{ik} = g_{ki}, \qquad 1 \le i, k \le n \tag{2}$$

in the space of the coordinates $u^1, \ldots, u^n$, where the form is positive definite, or at least non-degenerate. Given ds^2, one can define the arc length of a curve, the angle between two intersecting curves, the volume of a domain, and other geometrical concepts.

The main characteristic of this geometry is that it is local: it is valid in a neighborhood of the u-space. Because of this feature it fits well with field theory in physics. Einstein's general theory of relativity interprets the physical universe as a four-dimensional Lorentzian space (with a ds^2 of signature $+++-$) satisfying the field equations

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi\kappa T_{ik}, \tag{3}$$

where R_{ik} is the Ricci curvature tensor, R is the scalar curvature, κ is a constant, and T_{ik} is the energy-stress tensor.

It is soon observed that most properties of Riemannian geometry derive from its Levi-Civita parallelism, an infinitesimal transport of the tangent spaces. In other words, Riemannian geometry studies the tangent bundle of a Riemannian space with the Levi-Civita connection.

5. Globalization; topology. Riemannian geometry and its generalizations in differential geometry are local in character. It seems a mystery to me that we do need a whole space to piece the neighborhoods together. This is achieved by topology. The notion of a differentiable manifold is one of the most sophisticated concepts in mathematics. The idea was clear to Riemann. The first mathematical formulation of a topological manifold was made by D. Hilbert in 1902 **[10]**, **[17]**. Hermann Weyl identified the Riemann surfaces with one-dimensional complex manifolds and used it as the central theme of his epoch-making book *Die Idee der Riemannschen Fläche* **[16]**. On the topological side "neighborhood" became the basic concept in Hausdorff's topology **[9]**.

Hassler Whitney saw the merit of establishing an imbedding theorem on differentiable manifolds (1936), thus beginning the serious study of differential topology. That derivatives play a role in topology came as a shock when J. Milnor discovered the exotic differentiable structures on the seven-dimensional sphere (1956). By studying the Yang-Mills equations on a four-dimensional manifold, S. Donaldson found in 1983 a remarkable theorem on the intersection-form, which led to the existence of an infinite number of differentiable structures on R^4.

With the foundation of differentiable manifolds laid, geometrical structures can now be defined on them, such as the Riemannian structure, the complex structure, the conformal structure, the projective structure based on a system of paths, etc. Tools are developed for their treatment, of which the most important are the exterior differential calculus and the tensor analysis.

A fundamental notion is "curvature," in its different forms. Its simplest manifestation is the circle in plane Euclidean geometry. It could also be the force of a physical system or the strength of a gravitational or electro-magnetic field. In mathematical terms it measures the non-commutativity of covariant differentiation.

It is remarkable that suitable algebraic combinations of curvature give topological invariants. To illustrate this we wish to state the Gauss-Bonnet theorem. Let D be a domain with a sectionally smooth boundary on a two-dimensional Riemannian manifold. Then the Gauss-Bonnet theorem is the formula

$$\sum(\pi-\alpha)+\int_{\partial D}k_g\,ds+\int_D\!\!\int K\,dA=2\pi\chi(D), \tag{4}$$

where the first term is the sum of the exterior angles at the corners, the second term is the integral of the geodesic curvature along the sides, the third term is the integral of the Gaussian curvature over D, and $\chi(D)$ is the Euler characteristic of D. For a rectilinear triangle in the Euclidean plane this is the theorem on the angle-sum stated in §1. For higher dimensions we will only give, for the sake of simplicity, the theorem for a compact oriented Riemannian manifold M of dimension $2n$ without boundary. Let R_{ijkl} be the Riemann-Christoffel tensor and let

$$\Omega_{ij}=\sum R_{ijkl}\,du^k\wedge du^l \tag{5}$$

be the "curvature form". Let

$$Pf=\sum\epsilon_{i_1\cdots i_{2n}}\Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{2n-1}i_{2n}} \tag{6}$$

be the pfaffian, where $\epsilon_{i_1\cdots i_{2n}}$ is $+1$ or -1 according as its indices form an even or odd permutation of $1,\dots,2n$, and is otherwise zero, and the sum is extended over all indices from 1 to $2n$. Then the Gauss-Bonnet theorem says

$$(-1)^n\frac{1}{2^{2n}\pi^n n!}\int_M Pf=\chi(M), \tag{7}$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M.

6. Connections in a fiber bundle; Elie Cartan. A notion which includes both Klein's homogeneous spaces and Riemann's local geometry is Cartan's generalized spaces (espaces généralisés). In modern terms it is called "a connection in a fiber bundle." It is a straightforward generalization of the Levi-Civita parallelism, which is a connection in the tangent bundle of a Riemannian manifold. In general, we have a fiber bundle $\pi: E\to M$, whose fibers $\pi^{-1}(x)$, $x\in M$, are homogeneous spaces acted on by a Lie group G. A connection is an infinitesimal transport of the fibers compatible with the group action by G.

I wish to illustrate this more precisely in the case of a complex vector bundle, where the fibers are complex vector spaces C_q of dimension q and $G=GL(q;\mathbb{C})$ [**4**]. The importance of complex numbers in geometry is a mystery to me. It is well organized and complete. One manifestation is the simple behaviour of the group $GL(q;\mathbb{C})$: its maximal compact subgroup $U(q)$ has no torsion and has as Weyl group the group of all permutations on q letters.

We shall call a frame an ordered set of linearly independent vectors $e_1,\dots,e_q\in\pi^{-1}(x)$, $x\in M$. In a neighborhood U where a frame field $e_1(x),\dots,e_q(x)$, $x\in U$, is defined, a connection is given by the infinitesimal displacement

$$De_\alpha=\sum\omega_\alpha^\beta e_\beta,\qquad 1\le\alpha,\beta\le q, \tag{8}$$

where ω_α^β are linear differential forms in U. We call ω_α^β the connection forms and the matrix

$$\omega = (\omega_\alpha^\beta) \tag{9}$$

the connection matrix. Under a change of the frame field

$$e'_\alpha = \sum a_\alpha^\beta e_\beta, \qquad A = (a_\alpha^\beta), \tag{10}$$

the connection matrix is changed as follows:

$$\omega' A = dA + A\omega. \tag{11}$$

We introduce the curvature matrix

$$\Omega = d\omega - \omega \wedge \omega, \tag{12}$$

which is a matrix of exterior two-forms. By exterior differentiation of (11) we get

$$\Omega' = A\Omega A^{-1}. \tag{13}$$

It follows that the exterior polynomial

$$\det\left(I + \frac{i}{2\pi}\Omega\right) = 1 + c_1(\Omega) + \cdots + c_q(\Omega), \tag{14}$$

in which $c_\alpha(\Omega)$ is a 2α-form, is independent of the choice of the frame field and is hence globally defined in M. Moreover, each c_α is closed, i.e.,

$$dc_\alpha = 0. \tag{15}$$

The form $c_\alpha(\Omega)$ has been called the αth Chern form of the connection and its cohomology class $\{c_\alpha(\Omega)\}$ in the sense of de Rham cohomology is an element of the cohomology group $H^{2\alpha}(M; \mathbb{Z})$ and is called the αth Chern class of the bundle E. These characteristic classes are the simplest and most fundamental global invariants of a complex vector bundle. They have the advantage of possessing a local representation, by curvature.

As in the Gauss-Bonnet formula such a representation is of great importance, because the forms $c_\alpha(\Omega)$ themselves have a geometrical significance. Moreover, let π': $P \to M$ be the bundle of frames of the complex vector bundle. Then the pull-back $\pi'^* c_\alpha$ becomes a derived form, i.e.,

$$\pi'^* c_\alpha = dTc_\alpha, \tag{16}$$

where Tc_α, a form of degree $2\alpha - 1$ in P, is uniquely determined by certain properties. This operation is called transgression and Tc_α have been called the Chern-Simons forms [**5**]. These forms have played a role in three-dimensional topology and in recent works of E. Witten on quantum field theory [**20**].

This theory can be developed for any fiber bundle; see [**3**]. The above provides the geometrical basis of gauge field theory in physics. Here M is a four-dimensional Lorentzian manifold, so that the Hodge $*$-operator is defined, and we define the codifferential

$$\delta = * d * \tag{17}$$

There is a discrepancy of terminology and notation, as given by the following table:

mathematics	physics
connection ω	gauge potential A
curvature Ω	strength F

Maxwell's theory is based on a $U(1)$-bundle over M, and his field equations can be written

$$dA = F, \quad \delta F = J, \tag{18}$$

where J is the current vector. Actually, Maxwell wrote the first equation as

$$dF = 0, \tag{19}$$

which is a consequence. For most applications (19) is sufficient. But a critical study of an experiment proposed by Boehm and Aharanov and performed by Chambers shows that (18) are the correct equations [**21**]. A generalization of (18) to an $SU(2)$ bundle over M gives the Yang-Mills equations

$$DA = F, \quad \delta F = J. \tag{20}$$

It is indeed remarkable that developments in geometry have been consistently parallel to those in physics.

7. An application to biology. So far the most far-reaching applications of geometry are to physics, from which it is indeed inseparable. I wish to mention an application to biology, namely, to the structures of DNA molecules. This is known to be a "double helix", which geometrically means a pair of closed curves. Their geometrical invariants will clearly be of significance in biology. The following three are most important: 1) The linking number introduced by Gauss; 2) the total twist, which is essentially the integral of the torsion; 3) the writhing number.

James White proved that between these invariants there is the relation [**18**]

$$Lk = Tw + Wr. \tag{21}$$

This formula is of fundamental importance in molecular biology.

8. Conclusion. Contemporary geometry is thus a far cry from Euclid. To summarize, I would like to consider the following as the major developments in the history of geometry:

1) Axioms (Euclid);
2) Coordinates (Descartes, Fermat);
3) Calculus (Newton, Leibniz);
4) Groups (Klein, Lie);
5) Manifolds (Riemann);
6) Fiber bundles (Elie Cartan, Whitney).

A property is geometric, if it does not deal directly with numbers or if it happens on a manifold, where the coordinates themselves have no meaning. Going to several variables, algebra and analysis have a tendency to be involved with geometry.

This story is clearly one-sided and incomplete, representing only my personal viewpoint, and my limitations. It is clear that the story will not end here. Recent developments in theoretical physics, such as geometric quantum field theory, string theory, etc, are pushing for a much more general definition of geometry [**19**].

It is satisfying to note that so far almost all the sophisticated notions introduced in geometry have been found useful.

Finally, I wish to call attention to an early paper of mine [**2**], which could be read as a companion to this one.

詩一首

陳省身

一九八〇年九月訪問科學院理論物理研究所，歸前賦此：

物理幾何是一家　共同携手到天涯
黑洞單極窮奧秘　纖維連絡織錦霞
進化方程孤立異　對偶曲率瞬息空
疇算竟有天人用　拈花一笑欲無言

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