Delaunay type hypersurfaces in cohomogeneity one manifolds

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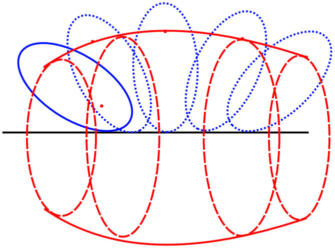
Delaunay surfaces

Delaunay 1841: a rotationally symmetric surface in $\mathbb{R}^3$ has CMC iff its profile curve is a *roulette* of a conic section.

- Delaunay surfaces: spheres, unduloids, nodoids, catenoids and cylinders.
- Similar constructions of rotationally invariant CMC hypersurfaces in $\mathbb{H}^n$, $\mathbb{R}^n$, S^n



Embedded CMC tori in S^3

- CMC Clifford tori in S^3: for each $0 < t < \pi/2$,

$$T^2_t := \Big\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \\ \|x_1\|^2 + \|x_2\|^2 = \cos^2 t, \|x_3\|^2 + \|x_4\|^2 = \sin^2 t\Big\},$$

- T^2_t are orbits of isometric $S^1 \times S^1$-action
- Singular orbits: geodesics S^1 at distance $\pi/2$; limits of T^2_t as $t \to 0$ and $t \to \pi/2$
- Other rotationally symmetric CMC tori: *bifurcating families* of CMC tori of *unduloid type* (classified by Hynd, Park, McCuan 2009 and Perdomo 2010)
- Full classification (announced by Andrews, Li 2012): all embedded CMC tori in S^3 are rotationally symmetric (settles conjecture of Pinkall, Sterling 1989)
- Totally analogous bifurcation theory in higher dimensions: $S^m \times S^k \hookrightarrow S^{m+k+1}$, but classification is wide open

CMC deformation of geodesics tubes

Ye: Pacific J. Math. 1991

Assume that $p \in M$ is a nondegenerate critical point of the scalar curvature on (M, g). Then, a neighborhood of p is foliated by constant mean curvature topological spheres $\Sigma(\rho)$, for $\rho \in \,]0, \rho_0[$.

Mahmoudi, Mazzeo, Pacard: GAFA 2006

For $r > 0$ small, geodesic r-tubes around a nondegenerate minimal submanifold $N^k \subset M^m$ ($k \leq m - 2$) can be deformed to CMC hypersurfaces with $H = \frac{m-1-k}{r(m-1)}$, except for a sequence $r_n \to 0$ of *resonant radii*.

Delaunay-type hypersurfaces:

- bifurcating branches of CMC hypersurfaces issuing from a *natural* 1-parameter family of symmetric CMC embeddings (orbits of isometric actions);
- *partially* preserve the symmetries of the natural branch;
- bifurcating branches *condense* onto a *minimal* submanifold (of higher codimension).

Natural ambient: Manifolds foliated by CMC hypersurfaces, with many symmetries, and condensing on minimal submanifolds.

Cohomogeneity one manifolds

- (M, g) compact Riemannian manifold
- G Lie group acting by isometries on M

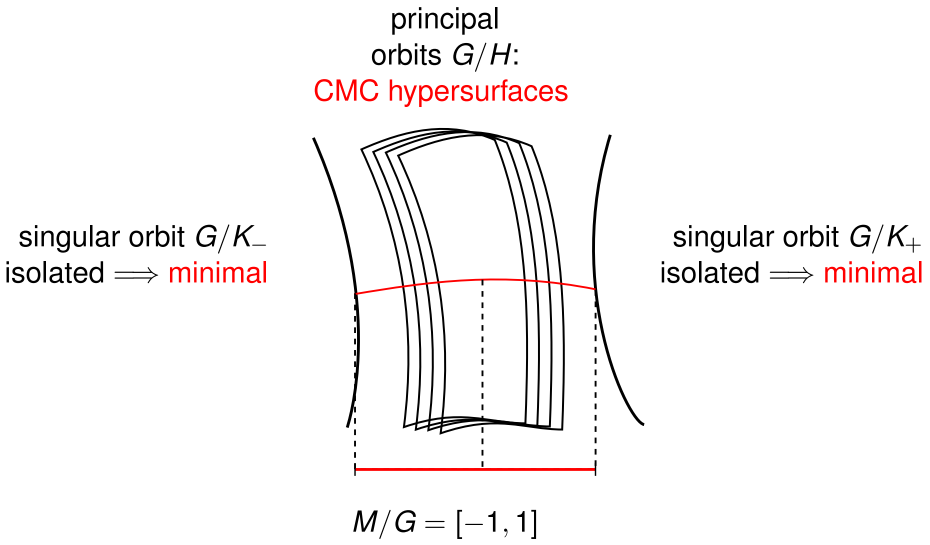
cohomogeneity one: $\dim(M/G) = 1$

$$M/G = \begin{cases} [-1,1] & \iff \quad \text{two non-principal orbits} \\ S^1 & \iff \quad \text{all orbits are principal} \end{cases}$$

$\gamma\colon [-1,1] \to M$ horizontal geodesic, section $\Longrightarrow$ polar action

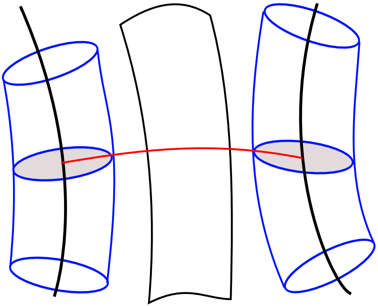
- $H := G_{\gamma(t)}$ principal isotropy, $t \in \,]-1,1[$
- $K_\pm := G_{\gamma(\pm 1)}$ singular isotropies
- $H \subset \{K_-, K_+\} \subset G$

Note: M simply connected $\Longrightarrow$ non-principal orbits are *singular*.



principal
orbits G/H:
CMC hypersurfaces
singular orbit G/K_-
isolated $\Longrightarrow$ minimal
singular orbit G/K_+
isolated $\Longrightarrow$ minimal
$M/G = [-1, 1]$

Geometry of cohomogeneity one manifolds – 2



Tubular neighborhood of singular orbit

- $K_\pm \circlearrowright D_\pm$ slice representation
- $D(G/K_\pm) := G \times_{K_\pm} D_\pm$
  Fiber bundle with fiber $D_\pm$ associated to $K_\pm$-principal bundle $K_\pm \to G \to G/K_\pm$
- $M \cong D(G/K_-) \bigcup_{G/H} D(G/K_+)$ is obtained by gluing the two tubular neighborhoods along a principal orbit G/H.

Geometry of cohomogeneity one manifolds – 3

$$
\begin{tikzcd}
 & G & \\
K_- \arrow[ur] & & K_+ \arrow[ul] \\
 & H \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Group diagram:

- $S_\pm^\perp = \partial D_\pm$ normal sphere to $G/K_\pm$
- $S_\pm^\perp = K_\pm/H$
- $K_\pm \circlearrowright S_\pm^\perp$ transitive action

M is determined by data

$$\boxed{H \subset \{K_-, K_+\} \subset G}$$

with $K_\pm/H$ diffeomorphic to spheres.

Collapse of singular orbits

- $x_t\colon G/H \hookrightarrow M$ family of principal orbits, $t \in \left]-1, 1\right[$
- $S \cong G/K_+$ singular orbit at $t = +1$
- $x_t(G/H)$ is a geodesic tube around S
- $x_t(G/H)$ is the total space of a homogeneous fibration:

$$K/H \longrightarrow x_t(G/H) \longrightarrow S \cong G/K$$

- As $t \to 1$, $x_t(G/H)$ converges to S in the Hausdorff metric, i.e., the fibers (normal spheres) collapse to a point:

$$\boxed{x_t(G/H)\ \textit{condeses}\ \text{on}\ S\ \text{as}\ t \to 1}$$

- $\lim\limits_{t\to 1} H_t = +\infty$, however, S is *minimal*!

Discuss minimality of limit submanifold.

Adapted metric

G-invariant metric on a G-manifold of cohomogeneity one:

$$g = g_t + \mathrm{d}t^2, \quad t \in \left]-1, 1\right[$$

g_t is a G-invariant metric on $x_t(G/H)$, with some conditions as $t \to \pm 1$. (Back-Hsiang 1987, ..., Mendes 2012 for polar actions)

Definition

g is *adapted* near $S_\pm$ if the projection $(G/H, g_t) \xrightarrow{\pi} (G/K_\pm, \check{g}_{\pm 1})$ is a Riemannian submersion for t near ± 1 (up to a factor $\alpha(t) \to 1$ as $t \to \pm 1$), i.e.:

$$\pi^*\left(\check{g}_{\pm 1}\right) = \alpha(t)\, g_t$$

Existence of adapted metrics

Lie algebras: $\boxed{\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}}$ Choose complements

$\boxed{\mathfrak{g} = \mathfrak{k} + \mathfrak{m},\ [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}}$ $\boxed{\mathfrak{k} = \mathfrak{h} + \mathfrak{p},\ [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}}$ $\boxed{\mathfrak{n} := \mathfrak{m} + \mathfrak{p}}$

Then, $g = \mathrm{d}t^2 + g_t$ is adapted iff g_t is of the form on $\mathfrak{n}$:

$$g_t(\cdot,\cdot) = \alpha(t)\, A(\cdot,\cdot) + B_t(\cdot,\cdot),$$

- A: K-invariant inn. prod. on $\mathfrak{m}$ coming from $S_\pm = G/K_\pm$
- B_t: any H-invariant inn. prod. on $\mathfrak{p}$.

Using a bi-invariant metric on G one proves easily:

Proposition

Every cohomogeneity one G-manifold M with $M/G = [-1, 1]$ admits a metric that is adapted near both of its singular orbits.

A criterion in non-negative curvature

Criterion

Let M be a cohomogeneity one manifold with an invariant metric g of nonnegative sectional curvature. If (M, g) has a totally geodesic principal orbit N, then the metric g is adapted near both singular orbits (with $\alpha_\pm \equiv 1$).

Proof.

Assume N disconnects M (general case follows).

- $N \subset M$ totally geodesic & $\sec \geq 0 \Longrightarrow \mathrm{dist}(\cdot, N)$ concave.
- Each component $C_\pm$ of $M \setminus N$ is a loc. convex subset of M.
- $S_\pm = \{\text{points at maximal distance from } N\}$ soul of $C_\pm$
- By Perelman, the Sharafutdinov retraction onto the soul (projection from each principal orbit G/H onto $S_\pm$) is a Riemannian submersion. $\square$

Main result

Theorem

M cohomogeneity one G-manifold, H principal isotropy, singular orbit $S = G/K$. Assume:

- *S is not a fixed point*
- *metric adapted near S*
- *either of the two normality assumptions (N1) or (N2) below.*

Then, there are infinitely many bifurcating branches of CMC embeddings of G/H in M issuing from principal orbits arbitrarily close to S. Such embeddings are K-invariant, but not G-invariant.

(N1) K normal in G

(N2) H normal in K, and K-invariant metric g_t on G/H w.r.t. to a modified action.

On the normality assumptions

(N1) implies:

(P) K-orbits (inside principal orbits) coincide with the fibers $(gK)H$ of homogeneous fibration:

$$K/H \longrightarrow G/H \longrightarrow G/K.$$

Under (N2), consider a different action:

$$K \times G/H \ni (k, gH) \longmapsto gk^{-1}H \in G/H.$$

Extends to a smooth isometric action of K on regular part $M_0 = M \setminus \{S_\pm\}$ and (P) holds

(P) yields:

- Eigenvalues of the Jacobi operator for the K-symmetric CMC variational problem come from *basic* eigenvalues of the total space of the fibration $G/H \longrightarrow G/K$.

(Besson, Bordoni, 1991)

Some consequences of the normality assumptions

- (N1) or (N2) implies S totally geodesic (fixed point set of K)
- (N1) implies that K-action is *fixed-point homogeneous*
- (N2) implies $\mathrm{codim}(S) = 2, 4$

On the normality assumption (N2)

H normal in K, K/H = sphere $\Longrightarrow K/H \cong S^1$ or $K/H \cong S^3$.
Conversely:

Proposition

Let K be a connected group and $H \subset K$ be a compact subgroup such that $K/H \cong S^1$. Then, H is normal in K.

Proof.

- H compact $\Longrightarrow \exists$ K-invariant metric on $K/H \cong S^1$
- all Riemannian metrics on S^1 are round $\Longrightarrow$ K-action given by a homomorphism $\varphi\colon K \to \mathrm{O}(2)$
- K connected $\Longrightarrow \varphi(K) \subset \mathrm{SO}(2)$
- $\mathrm{SO}(2)$ acts freely on $S^1 \Longrightarrow H$ = stabilizer = $\mathrm{Ker}(\varphi)$.

□

Ex. 1: Delaunay-type spheres S^{2n+1} in $\mathbb{C}P^{n+1}$

- $(M, g) = (\mathbb{C}P^{n+1}, g_{FS})$, g_{FS} Fubini-Study metric

$$\begin{tikzcd}
& \mathrm{U}(n+1) & \\
\mathrm{U}(n+1) \arrow[ur] & & \mathrm{U}(n)\mathrm{U}(1) \arrow[ul] \\
& \mathrm{U}(n) \arrow[ul] \arrow[ur, "\triangle"'] &
\end{tikzcd}$$

- Singular orbits: $S_- = \{p\}$, $S_+ = \mathrm{Cut}(p) \cong \mathbb{C}P^n$
- Principal orbits: $S_t^{2n+1} = (\mathrm{U}(n+1)/\mathrm{U}(n), g_t)$, $t \in]0, \pi/2[$, geodesic spheres of radius t centered at p, metrically Berger spheres
- $K/H \to G/H \to G/K$ is Hopf fibration $tS^1 \to S_t^{2n+1} \to \mathbb{C}P^n$
- g_{FS} is adapted near S_+, $\alpha(t) = \sin^2 t$
- (N2) is satisfied: $\mathrm{U}(n) \triangleleft \mathrm{U}(n)\mathrm{U}(1)$, $\mathrm{U}(n)\mathrm{U}(1)/\mathrm{U}(n) \cong S^1$

Example 2: Delaunay-type spheres S^{4n+3} in $\mathbb{H}P^{n+1}$

- $(M,g) = (\mathbb{H}P^{n+1}, g_{FS})$, g_{FS} Fubini-Study metric

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{Sp}(n+1) & \\
\mathrm{Sp}(n+1) \arrow[ur] & & \mathrm{Sp}(n)\mathrm{Sp}(1) \arrow[ul] \\
 & \mathrm{Sp}(n) \arrow[ul] \arrow[ur, "\triangle"'] &
\end{tikzcd}
$$

- Singular orbits: $S_- = \{p\}$, $S_+ = \mathrm{Cut}(p) \cong \mathbb{H}P^n$
- Principal orbits: $S_t^{4n+3} = (\mathrm{Sp}(n+1)/\mathrm{Sp}(n), g_t)$, $t \in]0, \pi/2[$, geodesic spheres of radius t centered at p, metrically Berger spheres
- $K/H \to G/H \to G/K$ is Hopf fibration $tS^3 \to S_t^{4n+3} \to \mathbb{H}P^n$
- g_{FS} is adapted near S_+, $\alpha(t) = \sin^2 t$
- (N2) is satisfied: $\mathrm{Sp}(n) \triangleleft \mathrm{Sp}(n)\mathrm{Sp}(1)$, $\mathrm{Sp}(n)\mathrm{Sp}(1)/\mathrm{Sp}(n) \cong S^3$

Ex. 3: Other Delaunay-type hypersurfaces in CROSS

Grove, Wilking, Ziller JDG 2008: full description of cohom 1 actions on CROSS

Essential cohom 1 actions on CROSS with (N2) with $H \triangleleft K_-$

M	G	K_-	K_+	H
S^{2k+3}	$\mathrm{SO}(2)\mathrm{SO}(k+2)$	$\Delta\mathrm{SO}(2)\mathrm{SO}(k)$	$\mathbb{Z}_2\cdot\mathrm{SO}(k+1)$	$\mathbb{Z}_2\cdot\mathrm{SO}(k)$
S^{15}	$\mathrm{SO}(2)\mathrm{Spin}(7)$	$\Delta\mathrm{SO}(2)\mathrm{SU}(3)$	$\mathbb{Z}_2\cdot\mathrm{Spin}(6)$	$\mathbb{Z}_2\cdot\mathrm{SU}(3)$
S^{13}	$\mathrm{SO}(2)\cdot\mathrm{G}_2$	$\Delta\mathrm{SO}(2)\mathrm{SU}(2)$	$\mathbb{Z}_2\cdot\mathrm{SU}(3)$	$\mathbb{Z}_2\cdot\mathrm{SU}(2)$
S^7	$\mathrm{SO}(4)$	$\mathrm{S}(\mathrm{O}(2)\mathrm{O}(1))$	$\mathrm{S}(\mathrm{O}(1)\mathrm{O}(2))$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$
S^4	$\mathrm{SO}(3)$	$\mathrm{S}(\mathrm{O}(2)\mathrm{O}(1))$	$\mathrm{S}(\mathrm{O}(1)\mathrm{O}(2))$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$
$\mathbb{C}P^{k+1}$	$\mathrm{SO}(k+2)$	$\mathrm{SO}(2)\mathrm{SO}(k)$	$\mathrm{O}(k+1)$	$\mathbb{Z}_2\cdot\mathrm{SO}(k)$
$\mathbb{C}P^6$	G_2	$\mathrm{U}(2)$	$\mathbb{Z}_2\cdot\mathrm{SU}(3)$	$\mathbb{Z}_2\cdot\mathrm{SU}(2)$
$\mathbb{C}P^7$	$\mathrm{Spin}(7)$	$S^1\cdot\mathrm{SU}(3)$	$\mathbb{Z}_2\cdot\mathrm{Spin}(6)$	$\mathbb{Z}_2\cdot\mathrm{SU}(3)$

Ex. 4: Delaunay hypersurfaces in Kervaire spheres

- $M_d^{2n-1} \subset \mathbb{C}^{n+1}$ defined by:

$$\begin{cases} z_0^d + z_1^2 + \cdots + z_n^2 = 0, \\ \|z_0\|^2 + \|z_1\|^2 + \cdots + \|z_n\|^2 = 1 \end{cases}$$

n odd, d odd $\Rightarrow M_d^{2n-1}$ homeom. to S^{2n-1};
$2n - 1 \equiv 1 \mod 8 \Rightarrow M_d^{2n-1} = \Sigma^{2n-1}$ exotic (Kervaire) spheres
Cohom 1 action ($n = 3$: Calabi, $n \geq 3$: Hsiang-Hsiang, 1967):

$$\begin{tikzcd}
& \mathrm{SO}(2)\mathrm{SO}(n) & \\
\mathrm{SO}(2)\mathrm{SO}(n-2) \arrow[ur] & & \mathrm{O}(n-1) \arrow[ul] \\
& \mathbb{Z}_2 \times \mathrm{SO}(n-2) \arrow[ul, "\triangle"] \arrow[ur] &
\end{tikzcd}$$

- Singular orbit: $S_- = \mathrm{SO}(n)/\mathrm{SO}(n-2)$
- Principal orbits: $S^1 \times \mathrm{SO}(n)/\mathrm{SO}(n-2)$
- (N2) is satisfied: $\mathbb{Z}_2 \times \mathrm{SO}(n-2) \triangleleft \mathrm{SO}(2)\mathrm{SO}(n-2)$

Constructions

Extensions:

- M cohom 1 mfld, diagram $H \subset \{K_-, K_+\} \subset G$
- $G \hookrightarrow \widetilde{G}$ extension of G
- Get cohom 1 bundle $\widetilde{M}$ with $\widetilde{G}$-action,
  $M \to \widetilde{M} \to \widetilde{G}/G$
- M has (N2) $\Rightarrow$ $\widetilde{M}$ has (N2)

Products:

- (H, K_+) pair of Lie groups with $K_+/H = S^n$
- $K_- := H \times S^1$ (or $K_- := H \times S^3$)
- G any Lie group containing $K_\pm$
- E.g., $G = K_+ \times S^1$ (or $K_+ \times S^3$), $M = S^{n+2}$ sphere, principal orbits are $G/H = S^n \times S^1$ (or $S^n \times S^3$), singular orbits are $S_- = S^n$ and $S_+ = S^1$ (or S^3)
- (N2) is trivially satisfied

Ingredients of Proof

- Variational bifurcation theory: t-spectral flow of Jacobi operators

$$J_t(\psi) = \Delta_{g_t}\psi - \big(\mathrm{Ric}(\vec{n}) + \|\mathcal{S}_t\|^2\big)\psi, \quad \psi\colon G/H \to \mathbb{R}$$

- Space of (unparameterized) K-invariant embeddings $x\colon G/H \to M$
- Area functional with volume constraint & Palais' symmetric criticality principle
- Eigenvalues of the Jacobi operators related to eigenvalues of Laplacian of a collapsing homogeneous fibration

Delaunay and Yamabe

Delaunay CMC problem	$\Longleftrightarrow$	Yamabe problem in homogeneous fibration

Orbits of isometric actions are:

- CMC embeddings
- solutions of the Yamabe problem (constant scalar curvature)

Fact. Jacobi operators of the area functional and of the Yamabe functional are both Schrödinger operators with potential given by curvatures.

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