Orthogonal geodesic chords on Riemannian manifolds with concave boundary and applications

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On the course

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Method: develop a non smooth Ljusternik–Schnirelmann theory
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- multiple brake orbits for a class of Hamiltonian problems
- multiple homoclinic orbits for a class of Lagrangian systems
Riemannian geometry

$M$ smooth manifold
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$g$ symmetric, positive definite $(2,0)$-tensor on $M$
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$\Sigma \subset M$ hypersurface

$S_n : T_x \Sigma \times T_x \Sigma \to \mathbb{R}$ second fundamental form of $\Sigma$
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$$S_n(v, w) = g(\nabla_v W, n_x)$$ symmetric bilinear form

$W$ extension of $w$, $n_x$ normal vector to $\Sigma$ at $x$. 

School in Nonlinear Analysis and Calculus of Variations – p. 3/6
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Convex and Concave domains

\((M, g)\) Riemannian manifold
\(\Omega \subset M\) open subset, \(\Omega = \Omega \cup \partial \Omega\)
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**Definition.** \(\overline{\Omega}\) is said to be **convex** if for all geodesic \(\gamma : [a, b] \to \overline{\Omega}\) with \(\gamma(a), \gamma(b) \in \Omega\), then \(\gamma([a, b]) \subset \Omega\).

\(\overline{\Omega}\) is **concave** if \(M \setminus \Omega\) is convex.
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$\overline{\Omega}$ is **strongly concave** if $S_n$ is positive definite, where $n$ is inward pointing.
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\(C^2\)-open condition
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**Lemma.** $\overline{\Omega}$ strongly concave $\implies \overline{\Omega}$ concave.
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diagram showing geodesics starting tangentially to \(\partial \Omega\) move inside \(\Omega\)
The boundary of $\Omega$

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4. $|\phi(q)| = \text{dist}(q, \partial \Omega)$ for $q$ near $\partial \Omega$.

Observe: \[\text{Hess}(\phi) = -S_{\nabla \phi} \quad \text{on} \quad T(\partial \Omega).\]
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Observe: $\text{Hess}(\phi) = -S_{\nabla \phi}$ on $T(\partial \Omega)$. 

Back to proof
Def.: An orthogonal geodesic chord (OGC) in $\overline{\Omega}$ is a non constant geodesic $\gamma : [a, b] \rightarrow \overline{\Omega}$ with $\gamma(a), \gamma(b) \in \partial \Omega$ and $\dot{\gamma}(a), \dot{\gamma}(b) \in T(\partial \Omega) \perp$. 

$\Omega$ 

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Def.: An orthogonal geodesic chord (OGC) in $\Omega$ is a non-constant geodesic $\gamma : [a, b] \to \overline{\Omega}$ with $\gamma(a), \gamma(b) \in \partial \Omega$ and $\dot{\gamma}(a), \dot{\gamma}(b) \in T(\partial \Omega)^\perp$.

A weak orthogonal geodesic chord (WOCG). WOGC's do not exist in the convex case.
Some examples – 1

$\Omega \cong$ annulus: $S^{m-1} \times [0, 1]$

An OGC is *crossing* if its endpoints are in distinct connected components of $\partial \Omega$. It is easy to prove the existence of *one* crossing OGC whose length equals the distance between the two connected components of $\partial \Omega$. 
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There may be only one OGC:
If $\overline{\Omega}$ is **convex**, then it is proven the existence of at least **two** crossing OGC’s (Giannoni-Majer, DGA 1997).
If $\overline{\Omega}$ is convex, then it is proven the existence of at least two crossing OGC’s (Giannoni-Majer, DGA 1997).

This is an optimal result (in all dimensions):

$$g(x) = \psi(|x|) \cdot g_0(x), \quad \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$$

convexity of the annulus

$$\frac{1}{2} \psi'(1) + \psi(1) \geq 0, \quad \psi'(2) + \psi(2) \leq 0.$$ 

$$\Omega_\varepsilon = \{ x \in \mathbb{R}^m : 1 < |x|, \ |x - \varepsilon| < 2 \}.$$ 

(Back to the central result)
Getting rid of WOGC’s

Proposition: Assume:

1. \( \partial \Omega \) compact and \( \overline{\Omega} \) strongly concave.
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6. $\overline{\Omega'}$ diffeomorphic to $\overline{\Omega}$, $\partial \Omega'$ smooth;
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- $\overline{\Omega'}$ diffeomorphic to $\overline{\Omega}$, $\partial \Omega'$ smooth;
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- there is no WOGC in $\overline{\Omega}$.
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It suffices to consider the case that there is no WOGC!
Proof

\[ \Omega' = \phi^{-1}(]-\infty, -\delta[), \text{ with } \delta > 0 \text{ small.} \]

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Observe:

- by continuity, \( d\phi \neq 0 \) in \( \phi^{-1}( [\delta, 0] ) \), so \( \partial\Omega' \) is smooth;
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- if \( \delta < \text{foc}(\partial\Omega) \), every OGC in \( \Omega' \) can be extended to an OGC in \( \Omega \).
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Proof

\[ \Omega' = \phi^{-1}(\mathbb{R} - \delta, -\delta], \] with \(\delta > 0\) small. (recall \(\phi\))

Observe:

\(\Omega'\) is strongly concave, by continuity of \(\text{Hess}(\phi)\) and compactness of \(\partial \Omega\);

if \(\delta < \text{foc}(\partial \Omega)\), every OGC in \(\Omega'\) can be extended to an OGC in \(\Omega\);

if by absurd \(\exists \delta_n \to 0\) and a sequence \(\gamma_n\) of WOGC’s in \(\phi^{-1}(\mathbb{R} - \delta, -\delta_n]\), then one would get infinitely many OGC’s in \(\Omega\). QED
The main geometrical result

**Theorem.** (Giambò, Giannoni, Piccione)
Let \((M, g)\) be a Riemannian manifold. Assume:
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Then, there are at least two (geometrically distinct) crossing OCG’s in \(\overline{\Omega}\).
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Obs.: Recall that it suffices to consider the case that there are no WOGC’s.
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**Obs.:** Recall that it suffices to consider the case that there are no WOGC’s.

**Obs.:** Again, the result is optimal. Recall example above (with opposite strict inequalities!).

back to central symmetry
**Central symmetry**

**Def.**: $(M, g)$ Riemannian man., $A \subset M$ is *centrally symmetric around* $x_0 \in M$ if exists an isometry $I : M \to M$, with $I^2 = I$, whose unique fixed pt is $x_0$, and such that $I(A) = A$.

A function $f : M \to \mathbb{R}$ is centrally symmetric if $f \circ I = f$. 
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Theorem. Under the assumptions of the above theorem, if \(\Omega\) is centrally symmetric around some \(x_0\), then there are at least \(m = \dim(M)\) geometrically distinct OGC’s \(\gamma_1, \ldots, \gamma_m\) in \(\Omega\).
A short history of the problem

Two classical results:

- Ljusternik and Schnirelmann, 1932:
  there are at least $n$ principal chords in a compact convex subset of the $n$-dimensional Euclidean space having $C^2$ boundary
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   at least \( n \) OGC’s in convex Riemannian manifolds homeomorphic to an \( n \)-disk.

The topology of the manifold

G & M’s result:

- if the manifold is homeomorphic to an annulus and it is convex, then there are at least two OGC’s;
- if the manifold has compact and convex boundary, and if the $LS$-category of the space of paths with endpoints on the boundary is infinite, then there are infinitely many OGC’s.
G & M’s result:

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Geodesics chords, homoclinics and brake orbits: a short bibliography

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These results will be reviewed later.
More bibliography


More bibliography

More bibliography


More bibliography


... lots more...
R. Giambò, F. Giannoni, P. Piccione,


**Ljusternik–Schnirelman category**

**Def.:** $\mathcal{X}$ top. space, $\mathcal{Y} \subset \mathcal{X}$ is *contractible in* $\mathcal{X}$ if $i : \mathcal{Y} \to \mathcal{X}$ is homotopic to a constant.

**LS-category**

$$\text{cat}_\mathcal{X}(\mathcal{Y}) = \min \{ n : \exists C_1, \ldots, C_n \subset \mathcal{X} \text{ open and contractible,} \quad \mathcal{Y} \subset \bigcup_{k=1}^{n} C_k \in \{0, 1, \ldots, +\infty\} \}. $$
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**Classical result.** If $\mathcal{X}$ is a complete Banach manifold and $f : \mathcal{X} \to \mathbb{R}$ is $C^1$, bounded from below, and satisfies (PS), then $f$ has at least $\text{cat}(\mathcal{X})$ critical points.
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In the case of Riemannian manifolds with *convex boundary*, one can use the *shortening flow* on the space of curves lying *inside* the manifold, and whose endpoints are on the boundary.
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In the case of Riemannian manifolds with convex boundary, one can use the **shortening flow** on the space of curves lying inside the manifold, and whose endpoints are on the boundary.

In the concave case, the shortening flow is not well defined on such space.
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Abstract LS theory

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- a family $\tilde{\mathcal{H}}$ consisting of pairs $(\mathcal{D}, h)$, where $\mathcal{D} \subset \mathcal{C}$ is compact and $h : [0,1] \times \mathcal{D} \to \mathcal{M}$ is a continuous map with $h(0,x) = x$ for all $x$, and satisfying other properties that will be discussed ahead.
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We will reproduce the “ingredients” of the classical LS theory in a *nonsmooth* context:

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We define a suitable notion of **critical pt** for $\mathcal{F}$, in such a way that distinct critical values of $\mathcal{F}$ correspond to **geometrically distinct** OGC’s in $\overline{\Omega}$.

We prove two **deformation lemmas** for the sublevels of $\mathcal{F}$, and we prove a (PS) condition for $\mathcal{F}$, obtaining the existence of $\text{cat}(\mathcal{C}) = \text{cat}(S^{m-1}) = 2$ distinct critical values of $\mathcal{F}$.

For the **symmetric case**, a lower estimate is given by $\text{cat}(\mathbb{R}P^{m-1}) = m$. 

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Deformation Lemmas and critical pts

1DL: noncritical levels of $\mathcal{F}$ can be deformed by homotopies in $\tilde{\mathcal{H}}$ into lower levels;
**Deformation Lemmas and critical pts**

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**2DL:** a similar deformation exists also for critical levels of $\mathcal{F}$, provided that suitable neighborhoods of the critical pts are removed.
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$i = 1, 2$: $\Gamma_i = \{D \in \mathcal{C} : \text{cat}(D) \geq i\}$, $c_i = \inf_{D \in \Gamma_i} \frac{F(D, h)}{(D, h) \in \widetilde{H}}$

One then proves:
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One then proves:

\[ c_i > 0 \text{ and } c_i < +\infty; \]
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$\bullet \quad$ each $c_i$ is a critical value, by 1DL; $c_1 < c_2$ by 2DL.
Basic notations

\[ \overline{\Omega} \cong S^{m-1} \times [0, 1] \text{ strongly concave}, \overline{\Omega} \subset \mathbb{R}^m, D_1, D_2 \cong S^{m-1} \text{ conn. comp. of } \partial \Omega. \]
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$x \in H^1([a, b], \mathbb{R}^m), \|x\|_{a,b} = \left(\frac{1}{2} \left(\|x(a)\| + \int_a^b \|\dot{x}(s)\|^2 \, ds\right)\right)^{\frac{1}{2}}, \|x\|_{L^\infty} \leq \|x\|_{a,b}$. 
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\textbf{Prop.:} If \( \gamma : [a, b] \to \overline{\Omega} \) is a geo with \( \gamma(a), \gamma(b) \in \partial \Omega \), then \( \exists \tilde{s} \in ]a, b[ \) with \( \phi(\gamma(\tilde{s})) < -\delta_0 \).
The path spaces

Define $C_i = \text{connected components of } \phi^{-1}([0, \delta_0]) \text{ containing } D_i, \ i = 1, 2.$

$$\mathcal{M} = \left\{ x \in H^1([0, 1], \mathbb{R}^m) : \phi(x(s)) < \delta_0, \ x(0) \in C_1, \ x(1) \in C_2 \right\}$$
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$x \in \mathcal{M}, \mathcal{J}^0_x = \left\{ [a, b] \subset [0, 1] : x([a, b]) \subset \overline{\Omega}, x(a) \in D_1, x(b) \in D_2 \right\}$

$\mathcal{J}_x = \left\{ [a, b] \in \mathcal{J}^0_x : [a, b] \text{ maximal} \right\}$

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**Def.:** $[a, b] \in \mathcal{J}_x^0$ is an $M_0$-interval if \( \frac{1}{2} \int_a^b g(\dot{x}, \dot{x}) \, ds < M_0 \). 

**Obs.:** $x \in \mathcal{M} \implies \left| \mathcal{J}_x \right| < +\infty$: $[a, b] \in \mathcal{J}_x^0, \ b - a \geq \rho_0^2 \left( \int_a^b g(\dot{x}, \dot{x}) \, ds \right)^{-1}$. 

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Def.: $c \in ]0, M_0[\, N \, is \ a \ geometrically \ critical \ value \ if \ \exists \ a \ crossing \ OGC \ \gamma : [0, 1] \to \Omega \ with \ \frac{1}{2} \int_0^1 g(\dot{\gamma}, \dot{\gamma}) \, dt = c$. A \ geometrically \ regular \ value \ is \ a \ number \ c \ which \ is \ not \ geometrically \ critical.
**Def.:** $c \in \]0, M_0[\text{ is a geometrically critical value if } \exists \text{ a crossing OGC } \gamma : [0, 1] \to \overline{\Omega} \text{ with }$

$$
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A geometrically regular value is a number $c$ which is not geometrically critical.

**Prop.:** If $c_1 \neq c_2$ are GCV’s, then they correspond to geometrically distinct OGC’s.
**Geometrically and variationally critical points**

**Def.:** $c \in ]0, M_0[\text{ is a } \text{geometrically critical value} \text{ if } \exists \text{ a crossing OGC } \gamma : [0, 1] \to \Omega \text{ with }$

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A *geometrically regular value* is a number $c$ which is not geometrically critical.

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\[ \mathcal{V}^+(x) = \left\{ V \text{ vector field along } x : g(V(s), \nabla \phi(x(s))) \geq 0 \text{ when } x(s) \in \phi^{-1}([0, \delta_0/2]) \right\} \]
**Def.:** \( c \in ]0, M_0[ \) is a **geometrically critical value** if \( \exists \) a crossing OGC \( \gamma : [0, 1] \to \overline{\Omega} \) with \( \frac{1}{2} \int_0^1 g(\dot{\gamma}, \dot{\gamma}) \, dt = c \). A **geometrically regular value** is a number \( c \) which is not geometrically critical.

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**Prop.:** If $c_1 \neq c_2$ are **GCV's**, then they correspond to geometrically distinct **OGC**'s.

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**Def.:** $x \in \mathcal{M}$, $[a, b] \subset [0, 1]$; then $x|_{[a, b]}$ is a **variationally critical portion** of $x$ if $x|_{[a, b]}$ is not constant and if $\int_a^b g(\dot{x}, \frac{D}{dt}V) \, dt \geq 0$ for all $V \in \mathcal{V}^+(x)$. 
Geometrically and variationally critical points

**Def.** \( c \in ]0, M_0[ \) is a geometrically critical value if \( \exists \) a crossing OGC \( \gamma : [0, 1] \rightarrow \overline{\Omega} \) with \( \frac{1}{2} \int_0^1 g(\dot{\gamma}, \dot{\gamma}) \, dt = c \). A geometrically regular value is a number \( c \) which is not geometrically critical.

**Prop.** If \( c_1 \neq c_2 \) are GCV’s, then they correspond to geometrically distinct OGC’s.

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**Def.** \( x \in \mathcal{M}, [a, b] \subset [0, 1] \); then \( x|_{[a, b]} \) is a variationally critical portion of \( x \) if \( x|_{[a, b]} \) is not constant and if \( \int_a^b g(\dot{x}, \frac{D}{dt} V) \, dt \geq 0 \) for all \( V \in \mathcal{V}^+(x) \).

Variationally critical portions of \( x \) are curves whose geodesic energy is not decreased by “infinitesimal variations” with curves stretching outwards from \( \overline{\Omega} \).

first variation of the geodesic action functional
Lem.: \( x \in \mathcal{M}, \, [\alpha, \beta] \subset 0, 1 \) and \( t \in ]\alpha, \beta[ \) such that \( x(\alpha), x(\beta) \in \partial \Omega, \, \phi(x(t)) \leq -\delta_0 \).

Then \( \beta - \alpha \geq \frac{\delta_0^2}{K_0^2} \left( \int_{\alpha}^{\beta} g(\dot{x}, \ddot{x}) \, dt \right)^{-1} \).
Classification of variationally critical portions

Lem.: \( x \in \mathcal{M}, [\alpha, \beta] \subset 0, 1 \) and \( \bar{t} \in ]\alpha, \beta[ \) such that \( x(\alpha), x(\beta) \in \partial \Omega, \phi(x(\bar{t})) \leq -\delta_0 \).
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Prop.: \( x \in \mathcal{M}, x|_{[a,b]} \) var. critical portion of \( x \) with \( x(a), x(b) \in \partial \Omega, x([a,b]) \subset \overline{\Omega} \). Then:
Classification of variationally critical portions

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- \( x \) is constant on each connected component of \( x^{-1}(\partial \Omega) \);
Lem.: $x \in \mathcal{M}$, $[\alpha, \beta] \subset [0, 1]$ and $\bar{t} \in ]\alpha, \beta[$ such that $x(\alpha), x(\beta) \in \partial \Omega$, $\phi(x(\bar{t})) \leq -\delta_0$. Then $\beta - \alpha \geq \frac{\delta_0^2}{K_0} \left( \int_{\alpha}^{\beta} g(\dot{x}, \dot{x}) \, dt \right)^{-1}$.

Prop.: $x \in \mathcal{M}$, $x|_{[a, b]}$ var. critical portion of $x$ with $x(a), x(b) \in \partial \Omega$, $x([a, b]) \subset \overline{\Omega}$. Then:

- $x^{-1}(\partial \Omega)$ consists of a finite number of closed intervals and isolated pts;
- $x$ is constant on each connected component of $x^{-1}(\partial \Omega)$;
- $x|_{[a, b]}$ is piecewise $C^2$; the discontinuities of $\dot{x}$ may occur on $\partial \Omega$;
Classification of variationally critical portions

Lem.: \( x \in \mathcal{M}, [\alpha, \beta] \subset 0, 1 \) and \( \bar{t} \in ]\alpha, \beta[ \) such that \( x(\alpha), x(\beta) \in \partial \Omega, \phi(x(\bar{t})) \leq -\delta_0. \)
Then \( \beta - \alpha \geq \frac{\delta_0^2}{K_0} \left( \int_{\alpha}^{\beta} g(\dot{x}, \dot{x}) \, dt \right)^{-1}. \)

Prop.: \( x \in \mathcal{M}, \ x|_{[a,b]} \) var. critical portion of \( x \) with \( x(a), x(b) \in \partial \Omega, x([a, b]) \subset \overline{\Omega}. \) Then:

1. \( x^{-1}(\partial \Omega) \) consists of a finite number of closed intervals and isolated pts;
2. \( x \) is constant on each connected component of \( x^{-1}(\partial \Omega); \)
3. \( x|_{[a,b]} \) is piecewise \( C^2; \) the discontinuities of \( \dot{x} \) may occur on \( \partial \Omega; \)
4. each \( C^2 \) portion of \( x \) is a geodesic in \( \overline{\Omega}; \)
Lem.: $x \in \mathcal{M}$, $[\alpha, \beta] \subset 0, 1$ and $\bar{t} \in ]\alpha, \beta[$ such that $x(\alpha), x(\beta) \in \partial \Omega$, $\phi(x(\bar{t})) \leq -\delta_0$. Then $\beta - \alpha \geq \frac{\delta_0^2}{K_0^2} \left( \int_{\alpha}^{\beta} g(\dot{x}, \dot{x}) \, dt \right)^{-1}$.

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4. each $C^2$ portion of $x$ is a geodesic in $\overline{\Omega}$;
5. $\min \left\{ \phi(x(s)) : s \in [a,b] \right\} < -\delta_0$. 

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**Lem.** \( x \in \mathcal{M}, [\alpha, \beta] \subset (0, 1] \) and \( \bar{t} \in ]\alpha, \beta[ \) such that \( x(\alpha), x(\beta) \in \partial \Omega, \phi(x(\bar{t})) \leq -\delta_0 \). Then \( \beta - \alpha \geq \frac{\delta_0^2}{K_0^2} \left( \int_\alpha^\beta g(\dot{x}, \dot{x}) \, dt \right)^{-1} \).

**Prop.** \( x \in \mathcal{M}, x|_{[a,b]} \) var. critical portion of \( x \) with \( x(a), x(b) \in \partial \Omega, x([a,b]) \subset \overline{\Omega} \). Then:

1. \( x^{-1}(\partial \Omega) \) consists of a finite number of closed intervals and isolated pts;
2. \( x \) is constant on each connected component of \( x^{-1}(\partial \Omega) \);
3. \( x|_{[a,b]} \) is piecewise \( C^2 \); the discontinuities of \( \dot{x} \) may occur on \( \partial \Omega_{[a]} \);
4. each \( C^2 \) portion of \( x \) is a geodesic in \( \overline{\Omega} \);
5. \( \min \{ \phi(x(s)) : s \in [a,b] \} < -\delta_0 \).
Def.: A VCP of $x \in \mathcal{M}$ is *regular* if it is $C^1$, *irregular* otherwise.
Def.: A VCP of $x \in \mathcal{M}$ is regular if it is $C^1$, irregular otherwise.

Prop.: $x \in \mathcal{M}$, $[a, b] \in J_x^0$ such that $x|_{[a, b]}$ is an irregular VCP.
Regular and irregular variationally critical portions

**Def.:** A VCP of \( x \in \mathcal{M} \) is *regular* if it is \( C^1 \), *irregular* otherwise.

**Prop.:** \( x \in \mathcal{M}, [a, b] \in \mathcal{J} \) such that \( x|_{[a,b]} \) is an irregular VCP. Then, \( \exists [\alpha, \beta] \subset [a, b] \) s.t. \( x|_{[\alpha,a]} \) and \( x|_{[b,\beta]} \) are constant in \( \partial \Omega \), \( \dot{x}(\alpha^+), \dot{x}(\beta^-) \in T(\partial \Omega)^\perp \), and one of the two occurs:
**Def.:** A VCP of $x \in \mathcal{M}$ is *regular* if it is $C^1$, *irregular* otherwise.

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- $\exists$ a finite number of intervals $[t_1, t_2] \subset [\alpha, \beta]$ s.t. $x([t_1, t_2]) \subset \partial \Omega$ that are maximal w.r. to this property; moreover, $x|_{[t_1,t_2]}$ is constant, and $\dot{x}(t^-_1) \neq \dot{x}(t^+_2)$. 

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**Regular and irregular variationally critical portions**

**Def.:** A VCP of \( x \in \mathcal{M} \) is *regular* if it is \( C^1 \), *irregular* otherwise.

**Prop.:** \( x \in \mathcal{M}, [a, b] \in \mathcal{J}^0_x \) such that \( x|_{[a,b]} \) is an irregular VCP. Then, \( \exists [\alpha, \beta] \subset [a, b] \) s.t. \( x|_{[\alpha,a]} \) and \( x|_{[b,\beta]} \) are constant in \( \partial \Omega \), \( \dot{x}(\alpha^+), \dot{x}(\beta^-) \in T(\partial \Omega)^\perp \), and one of the two occurs:

1. \( \exists \) a finite number of intervals \( [t_1, t_2] \subset [\alpha, \beta] \) s.t. \( x([t_1, t_2]) \subset \partial \Omega \) that are maximal w.r. to this property; moreover, \( x|_{[t_1,t_2]} \) is constant, and \( \dot{x}(t^-_1) \neq \dot{x}(t^+_2) \).

2. \( x|_{[\alpha,\beta]} \) is a crossing OGC in \( \Omega \).  

*second type*  
*first type*
**Def.**: A VCP of \( x \in \mathbb{M} \) is *regular* if it is \( C^1 \), *irregular* otherwise.

**Prop.**: \( x \in \mathbb{M} \), \([a, b] \in J^0_x\) such that \( x|_{[a, b]} \) is an irregular VCP. Then, \( \exists [\alpha, \beta] \subset [a, b] \) s.t. \( x|_{[\alpha, a]} \) and \( x|_{[b, \beta]} \) are constant in \( \partial \Omega \), \( \dot{x}(\alpha^+), \dot{x}(\beta^-) \in T(\partial \Omega)^\perp \), and one of the two occurs:

1. \( \exists \) a finite number of intervals \([t_1, t_2] \subset [\alpha, \beta] \) s.t. \( x([t_1, t_2]) \subset \partial \Omega \) that are maximal w.r. to this property; moreover, \( x|_{[t_1, t_2]} \) is constant, and \( \dot{x}(t_1^-) \neq \dot{x}(t_2^+) \).
2. \( x|_{[\alpha, \beta]} \) is a crossing OGC in \( \bar{\Omega} \). **second type**

**Note**: if \( x|_{[a, b]} \) is a regular VCP, with \([a, b] \in J^0_x\), then \( x|_{[a, b]} \) is a crossing OGC.
**Regular and irregular variationally critical portions**

**Def.:** A VCP of \( x \in \mathcal{M} \) is *regular* if it is \( C^1 \), *irregular* otherwise.

**Prop.:** \( x \in \mathcal{M}, [a, b] \in \mathcal{J}_x^0 \) such that \( x|_{[a,b]} \) is an irregular VCP. Then, \( \exists [\alpha, \beta] \subset [a, b] \) s.t. \( x|_{[\alpha,a]} \) and \( x|_{[b,\beta]} \) are constant in \( \partial \Omega \), \( \dot{x}(\alpha^+), \dot{x}(\beta^-) \in T(\partial \Omega)^\perp \), and one of the two occurs:

1. \( \exists \) a finite number of intervals \([t_1, t_2] \subset [\alpha, \beta]\) s.t. \( x([t_1, t_2]) \subset \partial \Omega \) that are maximal w.r. to this property; moreover, \( x|_{[t_1, t_2]} \) is constant, and \( \dot{x}(t^-_1) \neq \dot{x}(t^+_{2}) \).

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\([t_1, t_2]\) cusp interval of the irregular variationally critical portion of \( x \)

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Obs.: strong concavity $\implies$ number of cusp intervals on an $M_0$-int. is *unif. bounded.*
More on irregular VCP’s

**Obs.**: strong concavity $\implies$ number of cusp intervals on an $M_0$-int. is *unif. bounded.*

$[t_1, t_2]$ cusp interval of $x|_{[a,b]}$, $\Theta_x(t_1, t_2) =$ angle between $\dot{x}(t_1^-)$ and $\dot{x}(t_2^+)$.  

**Obs.**: the tangential components along $\partial \Omega$ of $\dot{x}(t_1^-)$ and $\dot{x}(t_2^+)$ are equal, hence, if $\Theta_x(t_1, t_2) > 0$, $x$ is not tangent to $\partial \Omega$ at $t_1$.  

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More on irregular VCP's

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**Prop.:** If $(x_n) \subset \mathcal{M}$, $[a_n, b_n] \in \mathcal{J}_{x_n}^0$ are $M_0$-intervals s.t. $x_n|_{[a_n, b_n]}$ is a VCP of $x_n$, then (up to subsequences) $a_n \to a$, $b_n \to b$, $x_n|_{[a_n, b_n]} \to x|_{[a, b]}$, where $x|_{[a, b]}$ is a VCP of $x$. 

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**Cor.:** $\exists d_0 > 0$ such that $\max \Theta_x(t_1, t_2) \geq d_0$, the max being taken over all $x \in \mathcal{M}$, all $M_0$-intervals $[a, b] \in \mathcal{J}_{x}^0$ s.t. $x|_{[a, b]}$ is an irregular VCP of $x$, and all $[t_1, t_2] \subset [a, b]$ cusp interval.
More on irregular VCP’s

**Obs.:** strong concavity $\implies$ number of cusp intervals on an $M_0$-int. is *unif. bounded.*

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**Proof.** Uses in a crucial way the fact that there is no WOGC.
More on irregular VCP's

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**Cor.**: $\exists d_0 > 0$ such that $\max \Theta_x(t_1, t_2) \geq d_0$, the max being taken over all $x \in \mathcal{M}$, all $M_0$-intervals $[a, b] \in \mathcal{J}^0_x$ s.t. $x|_{[a, b]}$ is an irregular VCP of $x$, and all $[t_1, t_2] \subset [a, b]$ cusp interval.

**Proof.** Uses in a crucial way the fact that there is no WOGC.

The corollary tells us, in particular, that the (VCP)'s of first and of second type are *far from each other*. 

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The classical Palais–Smale condition

Let $X$ be a smooth Banach manifold, and let $f : X \rightarrow \mathbb{R}$ be a $C^1$-map.

$f$ satisfies the (classical) Palais–Smale condition if every sequence $(x_n) \subset X$ such that:

1. $f(x_n)$ is bounded;
2. $df(x_n) \rightarrow 0$ as $n \rightarrow \infty$,

admits a converging subsequence in $X$. 
The Palais–Smale condition

For \([a, b] \subset [0, 1]\), consider the set \(Z_{a,b}\) of curves in \(\mathcal{M}\) s.t. \(x|_{[a,b]}\) is a VCP, not necessarily contained in \(\overline{\Omega}\):

\[
Z_{a,b} = \left\{ y : [a, b] \rightarrow \phi^{-1}(-\infty, \delta_0[) : \int_a^b g(y, \frac{D}{dt} V) \, dt \geq 0 \ \forall V \in \mathcal{V}^+(y) \right\}
\]
The Palais–Smale condition

For \([a, b] \subset [0, 1]\), consider the set \(\mathcal{Z}_{a,b}\) of curves in \(\mathcal{M}\) s.t. \(x|_{[a,b]}\) is a VCP, not necessarily contained in \(\overline{\Omega}\):

\[
\mathcal{Z}_{a,b} = \left\{ y : [a, b] \to \phi^{-1}(]-\infty, \delta_0[) : \int_a^b g(y, \frac{D_t V}{A_1} V) \, dt \geq 0 \; \forall V \in V^+(y) \right\}
\]

The following result plays the role of the classical Palais–Smale condition in our context:

**Proposition (PS):** For all \(r > 0\), \(\exists \theta(r), \mu(r) > 0\) with the following properties: for all \(x \in \mathcal{M}\) and all \([a, b] \in \mathcal{J}_x^0\) s.t.

(a) \(\frac{1}{2} \int_a^b g(\dot{x}, \dot{x}) \, dt \leq M_0\),

(b) \(\|x|_{[a,b]} - y\|_{a,b} \geq r\) for all \(y \in \mathcal{Z}_{a,b}\),

there exists a vector field \(V_x : [a, b] \to \mathbb{R}^m\) such that:
The Palais–Smale condition

For \([a, b] \subset [0, 1]\), consider the set \(Z_{a,b}\) of curves in \(\mathcal{M}\) s.t. \(x|_{[a,b]}\) is a VCP, not necessarily contained in \(\overline{\Omega}\):

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there exists a vector field \(V_x : [a, b] \to \mathbb{R}^m\) such that:

\[
g(\nabla \phi(x(s)), V_x(s)) \geq \theta(r)\|V_x\|_{a,b} \text{ for all } s \in [a, b] \text{ with } \phi(x(s)) = 0;
\]
The Palais–Smale condition

For \([a, b] \subset [0, 1]\), consider the set \(Z_{a,b}\) of curves in \(M\) s.t. \(x|_{[a,b]}\) is a VCP, not necessarily contained in \(\overline{\Omega}\):

\[
Z_{a,b} = \left\{ y : [a, b] \rightarrow \phi^{-1}([-\infty, \delta_0]) : \int_a^b g(\dot{y}, \frac{D}{dt}V) \, dt \geq 0 \, \forall V \in V^+(y) \right\}
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The following result plays the role of the classical Palais–Smale condition in our context:

**Proposition (PS):** For all \(r > 0\), \(\exists \theta(r), \mu(r) > 0\) with the following properties: for all \(x \in M\) and all \([a, b] \in J_x^0\) s.t.

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there exists a vector field \(V_x : [a, b] \rightarrow \mathbb{R}^m\) such that:

1. \(g(\nabla \phi(x(s)), V_x(s)) \geq \theta(r) \|V_x\|_{a,b}\) for all \(s \in [a, b]\) with \(\phi(x(s)) = 0\);

2. \(\int_a^b g(\dot{x}, \frac{D}{dt}V_x) \, dt \leq -\mu(r) \|V_x\|_{a,b}\).
By the compactness of $\phi^{-1}([-\infty, \delta_0])$, $\exists \ell_0, L_0 > 0$ s.t., denoting by $\| \cdot \|_E$ the Euclidean norm and by $\| \cdot \|$ the $g$-norm,

$$\ell_0 \| v \|_E^2 \leq \| v \|^2 \leq L_0 \| v \|_E^2, \quad \forall x \in \phi^{-1}([-\infty, \delta_0]), \forall v \in \mathbb{R}^m.$$
By the compactness of \( \phi^{-1}([-\infty, \delta_0]) \), \( \exists \ell_0, L_0 > 0 \) s.t., denoting by \( \| \cdot \|_E \) the Euclidean norm and by \( \| \cdot \| \) the \( g \)-norm,

\[
\ell_0 \| v \|_E^2 \leq \| v \| \leq L_0 \| v \|_E^2, \quad \forall x \in \phi^{-1}([-\infty, \delta_0]), \forall v \in \mathbb{R}^m.
\]

Moreover, \( \exists G_0, L_1 = L_1(M_0) > 0 \) s.t.

\[
|g_x(v_1, v) - g_z(v_2, v)| \leq G_0 \left( \| v_1 - v_2 \|_E \| v \|_E + \| x - z \|_E \| v_1 \|_E \| v \|_E \right),
\]

for all \( x, z \in \phi^{-1}([-\infty, \delta_0]) \) and for any \( v_1, v_2, v \in \mathbb{R}^m \), and

\[
\left( \int_a^b \| \frac{D}{ds} V \|_E^2 \, ds \right)^{1/2} \leq L_1 \| V \|_{a,b}
\]

for all \( x \in \mathcal{M} \) s.t. \( \frac{1}{2} \int_a^b g(\dot{x}, \ddot{x}) \, ds \leq M_0 \), for all \( V \in H^1([a, b], \mathbb{R}^N) \) along \( x \), and for any \( [a, b] \subset [0, 1] \).
For $a, b \in [0, 1]$, denote by $I_{a,b}$ the interval $[a, b]$ if $b \geq a$ and the interval $[b, a]$ if $b < a$; set:

$$D(x, \alpha, \beta, a, b) = \frac{1}{2} \int_{I_{a,\alpha} \cup I_{b,\beta}} g(\dot{x}, \dot{x}) \, dt.$$
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**Lem.:** Fix $K > 0$. For any $x, z \in \mathcal{M}$, $[a, b] \subset [0, 1]$, $[a_z, b_z] \subset [0, 1]$, and $V \in H^1([0, 1], \mathbb{R}^N)$, then if $\frac{1}{2} \int_a^b g(\dot{x}, \dot{x}) \, dt \leq M_0$ and $D(x, a_z, b_z, a, b) \leq K$, it is

$$\left| \int_a^b g_x(\dot{x}, \frac{D}{dt} V) \, dt - \int_{a_z}^{b_z} g_z(\dot{z}, \frac{D}{dt} V) \, dt \right| \leq \sqrt{2} \left( \sqrt{L_0 K} + G_0 \|x - z\|_{a_z, b_z} \left( 1 + \sqrt{\frac{M_0 + K}{\ell_0}} \right) \right) L_1 \|V\|_{0,1},$$
For $a, b \in [0, 1]$, denote by $I_{a,b}$ the interval $[a, b]$ if $b \geq a$ and the interval $[b, a]$ if $b < a$; set:

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$$\left| \int_a^b g_x(\dot{x}, \frac{D}{dt} V) \, dt - \int_{a_z}^{b_z} g_z(\dot{z}, \frac{D}{dt} V) \, dt \right| \leq$$

$$\sqrt{2} \left( \sqrt{L_0 K + G_0} \|x - z\|_{a_z, b_z} \left( 1 + \sqrt{\frac{M_0 + K}{\ell_0}} \right) \right) L_1 \|V\|_{0,1},$$

Define:

$$E(r) = \frac{\mu(r)^2}{32L_1^2 L_0}. $$
Prop.: For $r > 0$, let $\theta(r), \mu(r) > 0$ be as in PS. For all $x \in \mathcal{M}$ and for all $[a, b] \in \mathcal{J}_x$ for which (a) and (b) of PS hold, let $V_x$ be the vector field in PS. Extend $V_x$ to $[0, 1]$ making it constant outside $[a, b]$. 
Construction of local vector fields

**Prop.:** For \( r > 0 \), let \( \theta(r), \mu(r) > 0 \) be as in PS. For all \( x \in \mathcal{M} \) and for all \([a, b] \in \mathcal{J}_x^0\) for which (a) and (b) of PS hold, let \( V_x \) be the vector field in PS. Extend \( V_x \) to \([0, 1]\) making it constant outside \([a, b]\). Then, \( \exists [\alpha_x, \beta_x] \supset [a, b] \) and \( \rho(x) > 0 \) such that:
Construction of local vector fields

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Then, $\exists [\alpha_x, \beta_x] \supset [a, b]$ and $\rho(x) > 0$ such that:

$\alpha_x < a$ if $a > 0$ and $\beta_x > b$ if $b < 1$, $\frac{1}{2} \int_{\alpha_x}^{\beta_x} g(\dot{x}, \ddot{x}) \, dt \leq M_0 + 1$;
Construction of local vector fields

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Then, \( \exists [\alpha_x, \beta_x] \supset [a, b] \) and \( \rho(x) > 0 \) such that:

1. \( \alpha_x < a \) if \( a > 0 \) and \( \beta_x > b \) if \( b < 1 \), \( \frac{1}{2} \int_{\alpha_x}^{\beta_x} g(\dot{x}, \ddot{x}) \, dt \leq M_0 + 1; \)

2. \( \sup_{s \in [\alpha_x, \beta_x]} \phi(x(s)) \leq \frac{1}{4} \delta_0; \)
**Construction of local vector fields**

**Prop.:** For \( r > 0 \), let \( \theta(r), \mu(r) > 0 \) be as in PS. For all \( x \in \mathcal{M} \) and for all \([a, b] \in J_x^0\) for which (a) and (b) of PS hold, let \( V_x \) be the vector field in PS.

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1. \( \alpha_x < a \) if \( a > 0 \) and \( \beta_x > b \) if \( b < 1 \), \( \frac{1}{2} \int_{\alpha_x}^{\beta_x} g(\dot{x}, \dot{x}) \, dt \leq M_0 + 1; \)
2. \( \sup_{s \in [\alpha_x, \beta_x]} \phi(x(s)) \leq \frac{1}{4} \delta_0; \)
3. \( z \in \mathcal{M} \) and \( \|x - z\|_{L^\infty} < \rho(x) \) imply the following:
   (i) \( g(\nabla \phi(z(s)), V_x(s)) \geq \frac{1}{2} \theta(r) \|V_x\|_{\alpha_x, \beta_x} \) for all \( s \in [\alpha_x, \beta_x] \) with \( \|V_x\|_{\alpha_x, \beta_x} \); 
   (ii) \( \sup_{s \in [\alpha_x, \beta_x]} \phi(z(s)) \leq \frac{1}{2} \delta_0; \)
Construction of local vector fields

Prop.: For $r > 0$, let $\theta(r), \mu(r) > 0$ be as in PS. For all $x \in \mathcal{M}$ and for all $[a, b] \in J^0_x$ for which (a) and (b) of PS hold, let $V_x$ be the vector field in PS. Extend $V_x$ to $[0, 1]$ making it constant outside $[a, b]$. Then, $\exists [\alpha_x, \beta_x] \supset [a, b]$ and $\rho(x) > 0$ such that:

1. $\alpha_x < a$ if $a > 0$ and $\beta_x > b$ if $b < 1$, $\frac{1}{2} \int_{\alpha_x}^{\beta_x} g(\dot{x}, \dot{x}) \, dt \leq M_0 + 1$;
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3. $z \in \mathcal{M}$ and $\|x - z\|_{L^\infty} < \rho(x)$ imply the following:
   (i) $g(\nabla \phi(z(s)), V_x(s)) \geq \frac{1}{2} \theta(r) \|V_x\|_{\alpha_x, \beta_x}$ for all $s \in [\alpha_x, \beta_x]$ with $0 \leq \phi(z(s)) \leq \frac{1}{2} \delta_0$;
   (ii) $\sup_{s \in [\alpha_x, \beta_x]} \phi(z(s)) \leq \frac{1}{2} \delta_0$;
4. for all $z \in \mathcal{M}$, for all $[a_z, b_z] \in J_z$ with $[a_z, b_z] \subset [\alpha_x, \beta_x]$, with $\|x - z\|_{a_z, b_z} < \rho(x)$ and with $D(x, a_z, b_z, a, b) < E(r)$, then:
   $$\int_{a_z}^{b_z} g(\dot{z}, \frac{d}{dt} V_x) \, dt \leq -\frac{1}{2} \mu(r) \|V_x\|_{\alpha_x, \beta_x}.$$
Interpretation of the constant $E(r)$

By the definition of $D(x, a_z, b_z, a, b)$, the number $E(r)$ gives a bound on the admissible difference between the energy of $x|_{[a,b]}$ and $x|_{[a_z,b_z]}$, to obtain a rate of decrease $\mu(r)/2$ for the quantity $\frac{1}{2} \int_{a_z}^{b_z} g(\dot{z}, \ddot{z}) \, ds$, when $\|x - z\|_{a_z,b_z} < \rho(x)$. 
"Genuine" crossing intervals

**Def.:** $\mathcal{D} \subset \mathcal{C}$, $h : [0, 1] \times \mathcal{D} \xrightarrow{C^0} \mathcal{M}$, $\gamma \in \mathcal{D}$, $\tau \in [0, 1]$. An interval $[a_\tau, b_\tau] \in \mathcal{J}_{h(\tau, \gamma)}$ is $h$-genuine if for all $\tau' \in [0, \tau]$ there exists $[a_{\tau'}, b_{\tau'}] \in \mathcal{J}_{h(\tau', \gamma)}$ such that $[a_\tau, b_\tau] \subset [a_{\tau'}, b_{\tau'}]$. 

For $(\mathcal{D}, h) \in \tilde{\mathcal{H}}$ and $z \in h(1, \mathcal{D})$, set:

$$\mathcal{J}_z^h = \{ [a, b] \in \mathcal{J}_z : [a, b] \text{ is } h\text{-genuine} \}$$
Def.: \( \mathcal{D} \subset \mathcal{C} \), \( h : [0, 1] \times \mathcal{D} \xrightarrow{C^0} \mathcal{M} \), \( \gamma \in \mathcal{D} \), \( \tau \in [0, 1] \). An interval \([a_\tau, b_\tau] \in J_{h(\tau, \gamma)}\) is \( h \)-genuine if for all \( \tau' \in [0, \tau] \) there exists \([a_{\tau'}, b_{\tau'}] \in J_{h(\tau', \gamma)}\) such that \([a_\tau, b_\tau] \subset [a_{\tau'}, b_{\tau'}] \).

For \((\mathcal{D}, h) \in \tilde{\mathcal{H}}\) and \( z \in h(1, \mathcal{D}) \), set:

\[
J^h_z = \{ [a, b] \in J_z : [a, b] \text{ is } h \text{-genuine} \}
\]

\[
\widehat{J}^h_z(\mathcal{D}) = \left\{ [a, b] \subset [0, 1] : \forall s \in [a, b] \exists [\alpha, \beta] \subset [a, b] \text{ such that } s \in [\alpha, \beta] \text{ and there exists } z_n \subset h(1, \mathcal{D}) \text{ and } [\alpha_n, \beta_n] \in J^h_{z_n} \text{ such that } \right\}
\]

\[z_n |_{[\alpha_n, \beta_n]} \rightarrow z |_{[\alpha, \beta]}, \text{ and } [a, b] \text{ is maximal w.r. to such property} \}
Def.: \( D \subset \mathcal{C}, \ h : [0, 1] \times D \stackrel{C^0}{\rightarrow} \mathcal{M}, \ \gamma \in D, \ \tau \in [0, 1]. \) An interval \([a_\tau, b_\tau] \in J_{h(\tau, \gamma)}\) is \( h \)-genuine if for all \( \tau' \in [0, \tau] \) there exists \([a_{\tau'}, b_{\tau'}] \in J_{h(\tau', \gamma)}\) such that \([a_\tau, b_\tau] \subset [a_{\tau'}, b_{\tau'}]\).

For \((D, h) \in \tilde{\mathcal{H}}\) and \(z \in h(1, D)\), set:

\[
J^h_z = \{ [a, b] \in J_z : [a, b] \text{ is } h \text{-genuine} \}
\]

\[
\hat{J}^h_z(D) = \{ [a, b] \subset [0, 1] : \forall s \in [a, b] \exists [\alpha, \beta] \subset [a, b] \text{ such that } s \in [\alpha, \beta] \\
\text{and there exists } (z_n) \subset h(1, D) \text{ and } [\alpha_n, \beta_n] \in J^h_{z_n} \text{ such that } \\
z_n|_{[\alpha_n, \beta_n]} \to z|_{[\alpha, \beta]}, \text{ and } [a, b] \text{ is maximal w.r. to such property} \}
\]

Obs.: \( \hat{J}^h_z(D) \) is always non empty. If \( z \in h(1, D) \) and \([a, b] \in J^h_z\), then \([a, b] \in \hat{J}^h_z(D)\).
Admissible homotopies

Def.: A set of *admissible homotopies* $\mathcal{H}$ of our variational problem (that will be used in a crucial preparatory deformation result) consists of all continuous maps $h : [0, 1] \times \mathcal{D} \to \mathcal{M}$, with $\mathcal{D}$ closed subset of $\mathcal{C}$, such that:
**Def.** A set of *admissible homotopies* \( \mathcal{H} \) of our variational problem (that will be used in a crucial preparatory deformation result) consists of all continuous maps \( h : [0, 1] \times \mathcal{D} \rightarrow \mathcal{M} \), with \( \mathcal{D} \) closed subset of \( \mathcal{C} \), such that:

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1. $h(0, \cdot)$ is the inclusion of $\mathcal{D}$ into $\mathcal{M}$;

2. if $h(\tau_0, \gamma)(s) \notin \overline{\Omega}$, then $h(\tau, \gamma)(s) \notin \overline{\Omega}$ for all $\tau \geq \tau_0$;
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3. for all $x \in h(1, \mathcal{D})$, every $[a, b] \in \overline{\mathcal{J}}_z^h(\mathcal{D})$ is an $M_0$-interval, i.e., $\frac{1}{2} \int_a^b g(\dot{x}, \dot{x}) \, dt < M_0$. 
**Admissible homotopies**

**Def.:** A set of *admissible homotopies* $\mathcal{H}$ of our variational problem (that will be used in a crucial preparatory deformation result) consists of all continuous maps $h : [0, 1] \times D \to \mathcal{M}$, with $D$ closed subset of $\mathcal{C}$, such that:

1. $h(0, \cdot)$ is the inclusion of $D$ into $\mathcal{M}$;
2. if $h(\tau_0, \gamma)(s) \not\in \Omega$, then $h(\tau, \gamma)(s) \not\in \Omega$ for all $\tau \geq \tau_0$;
3. for all $x \in h(1, D)$, every $[a, b] \in \mathcal{J}^h_a(D)$ is an $M_0$-interval, i.e.,
   \[ \frac{1}{2} \int_a^b g(\dot{x}, \dot{x}) \, dt < M_0. \]

$$\mathcal{H} = \left\{ (D, h) : D \text{ is a closed subset of } \mathcal{C}, \text{ and } h : [0, 1] \times D \to \mathcal{M} \right\}$$

is a continuous homotopy satisfying (1), (2), (3) above.
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3. for all $x \in h(1, \mathcal{D})$, every $[a, b] \in \tilde{\mathcal{J}}_{\mathcal{D}}(\mathcal{D})$ is an $M_0$-interval, i.e., $\frac{1}{2} \int_a^b g(\dot{x}, \dot{x}) \, dt < M_0$.

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**Obs. 1:** Defining the *constant homotopy*: $h_0(\tau, x) \equiv x$ for all $x \in \mathcal{C}$, $\mathcal{H}$ contains $(\mathcal{C}, h_0)$.
Admissible homotopies

**Def.**: A set of *admissible homotopies* \( \mathcal{H} \) of our variational problem (that will be used in a crucial preparatory deformation result) consists of all continuous maps \( h : [0, 1] \times D \to M \), with \( D \) closed subset of \( \mathcal{C} \), such that:

1. \( h(0, \cdot) \) is the inclusion of \( D \) into \( M \);
2. if \( h(\tau_0, \gamma)(s) \not\in \overline{\Omega} \), then \( h(\tau, \gamma)(s) \not\in \overline{\Omega} \) for all \( \tau \geq \tau_0 \);
3. for all \( x \in h(1, D) \), every \( [a, b] \in \mathcal{J}_z^h(D) \) is an \( M_0 \)-interval, i.e.,
\[
\frac{1}{2} \int_a^b g(\dot{x}, \dot{x}) \, dt < M_0.
\]

\[ \mathcal{H} = \left\{ (D, h) : D \text{ is a closed subset of } \mathcal{C}, \text{ and } h : [0, 1] \times D \to M \right\} \]

is a continuous homotopy satisfying (1), (2), (3) above.

**Obs. 1**: Defining the *constant homotopy*: \( h_0(\tau, x) \equiv x \) for all \( x \in \mathcal{C} \), \( \mathcal{H} \) contains \((\mathcal{C}, h_0)\).

**Obs. 2**: There exists \( N > 0 \) (independent of \( x, D \) and \( h \)) such that \( |\mathcal{J}_z^h(D)| \leq N \).
Concatenation of homotopies

\[ F_1, F_2 \subset \mathcal{M} \] closed sets

\[ h_i : [0, 1] \times F_i \xrightarrow{C^0} \mathcal{M}, \ i = 1, 2 \]

If \( h_1(1, F_1) \subset F_2 \), then one defines the concatenation:

\[ h_1 \star h_2 : [0, 1] \times F_1 \longrightarrow \mathcal{M} \]

\[
h_1 \star h_2(t, x) = \begin{cases} 
  h_1(2t, x), & \text{if } t \in [0, \frac{1}{2}]; \\
  h_2(2t - 1, h_1(1, x)), & \text{if } t \in \left] \frac{1}{2}, 1 \right]. 
\end{cases}
\]
Consider the following functional $F : \mathcal{H} \to \mathbb{R}^+$:

$$F(\mathcal{D}, h) = \sup \left\{ \frac{b-a}{2} \int_a^b g(\dot{x}, \ddot{x}) \, dt : x \in h(1, \mathcal{D}), \ [a, b] \in \mathcal{J}_x^h(\mathcal{D}) \right\}$$
The functional $F$

Consider the following functional $F : \mathcal{H} \to \mathbb{R}^+$:

$$F(D, h) = \sup \left\{ \frac{b-a}{2} \int_a^b g(\dot{x}, \dot{x}) \, dt : x \in h(1, D), \ [a, b] \in \mathcal{J}_x^h(D) \right\}$$

**Obs. 1:** $\frac{b-a}{2} \int_a^b g(\dot{y}, \dot{y}) \, dt = \frac{1}{2} \int_0^1 g(y_{a,b}, \dot{y}_{a,b}) \, ds$, where $y_{a,b}$ is the affine reparameterization of $y|_{[a,b]}$ on the interval $[0, 1]$. 
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**Obs. 1:** $\frac{b-a}{2} \int_a^b g(\dot{y}, \dot{y}) \, dt = \frac{1}{2} \int_0^1 g(y_{a,b}, y_{a,b}) \, ds$, where $y_{a,b}$ is the affine reparameterization of $y|_{[a,b]}$ on the interval $[0, 1]$.

**Obs. 2:** for all $(D, h) \in \mathcal{H}$, $\frac{1}{2} \rho_0^2 \leq F(D, h) \leq \frac{1}{2} M_0$. 
An “outward pushing” deformation

Lemma

\[ Z_{a,b}^1 = \{ y \in H^1([a,b], \phi^{-1}([-\infty, \delta_0[) ) : y|_{[a,b]} \text{ is an OGC,} \]

or \[ y|_{[a,b]} \text{ is an irregular variational portion of first type} \] \}
Prop.: Let \( r > 0 \) and \( 0 < c_1 < c < c_2 \) be fixed. Then there exists \( \varepsilon_0 = \varepsilon_0(r, c) > 0 \) such that, for all \((D, h) \in \mathcal{H}\) satisfying:
Prop.: Let $r > 0$ and $0 < c_1 < c < c_2$ be fixed. Then there exists $\varepsilon_0 = \varepsilon_0(r, c) > 0$ such that, for all $(D, h) \in \mathcal{H}$ satisfying:

\[ F(D, h) \leq c_2; \]
An “outward pushing” deformation

Lemma

**Prop.:** Let \( r > 0 \) and \( 0 < c_1 < c < c_2 \) be fixed. Then there exists \( \varepsilon_0 = \varepsilon_0(r, c) > 0 \) such that, for all \((D, h) \in \mathcal{H}\) satisfying:

1. \( \mathcal{F}(D, h) \leq c_2 \);
2. \( \inf \{ \| x_{[a,b]} - y \|_{a,b} \} \geq r \), \( x = h(1, \gamma), \gamma \in D, \frac{b-a}{2} \int_a^b g(\dot{x}, \dot{x}) \, dt \in [c_1, c_2] \), \( [a, b] \in \mathcal{J}^h_x(D), y \in Z^1_{a,b} \).
An “outward pushing” deformation

**Lemma**

**Prop.:** Let $r > 0$ and $0 < c_1 < c < c_2$ be fixed. Then there exists $\varepsilon_0 = \varepsilon_0(r, c) > 0$ such that, for all $(D, h) \in \mathcal{H}$ satisfying:

1. $\mathcal{F}(D, h) \leq c_2$;
2. $\inf \left\{ \| x_{[a,b]} - y \|_{a,b} \right\} \geq r$,

then there exists a continuous map $H_{\varepsilon} : [0, 1] \times h(1, D) \rightarrow \mathcal{M}$ with the following properties:
Prop.: Let $r > 0$ and $0 < c_1 < c < c_2$ be fixed. Then there exists $\epsilon_0 = \epsilon_0(r, c) > 0$ such that, for all $(\mathcal{D}, h) \in \mathcal{H}$ satisfying:

1. $\mathcal{F}(\mathcal{D}, h) \leq c_2$;
2. $\inf \left\{ \| x|_{[a,b]} - y \|_{a,b} \right\} \geq r, \quad x = h(1, \gamma), \gamma \in \mathcal{D}, \frac{b-a}{2} \int_a^b g(\dot{x}, \dot{x}) \, dt \in [c_1, c_2], [a,b] \in \widehat{\mathcal{J}}_x^h(\mathcal{D}), y \in \mathcal{Z}^1_{a,b}$

and for all $\epsilon \in ]0, \epsilon_0[$ there exists a continuous map $H_\epsilon : [0, 1] \times h(1, \mathcal{D}) \to \mathcal{M}$ with the following properties:

1. $(\mathcal{D}, H_\epsilon \star h) \in \mathcal{H}$;
**Prop.:** Let $r > 0$ and $0 < c_1 < c < c_2$ be fixed. Then there exists $\varepsilon_0 = \varepsilon_0(r, c) > 0$ such that, for all $(D, h) \in \mathcal{H}$ satisfying:

1. $\mathcal{F}(D, h) \leq c_2$;
2. $\inf \left\{ \|x|_{[a, b]} - y\|_{a, b} \right\} \geq r,$

and for all $\varepsilon \in ]0, \varepsilon_0[$ there exists a continuous map $H_\varepsilon : [0, 1] \times h(1, D) \to \mathcal{M}$ with the following properties:

1. $(D, H_\varepsilon \star h) \in \mathcal{H};$
2. if $c \leq \mathcal{F}(D, h) \leq c_2$ then $\mathcal{F}(D, H_\varepsilon \star h) \leq \mathcal{F}(D, h) - \varepsilon;$
Prop.: Let $r > 0$ and $0 < c_1 < c < c_2$ be fixed. Then there exists $\varepsilon_0 = \varepsilon_0(r, c) > 0$ such that, for all $(D, h) \in \mathcal{H}$ satisfying:

1. $\mathcal{F}(D, h) \leq c_2$;
2. $\inf \left\{ \|x|_{[a,b]} - y\|_{a,b} \right\} \geq r, \quad x = h(1, \gamma), \gamma \in D, \frac{b-a}{2} \int_a^b g(\dot{x}, \dot{x}) \, dt \in [c_1, c_2], \quad [a, b] \in \tilde{J}_x^h(D), \ y \in Z^1_{a,b}$

and for all $\varepsilon \in ]0, \varepsilon_0[$ there exists a continuous map $H_\varepsilon : [0, 1] \times h(1, D) \to \mathcal{M}$ with the following properties:

1. $(D, H_\varepsilon \ast h) \in \mathcal{H}$;
2. if $c \leq \mathcal{F}(D, h) \leq c_2$ then $\mathcal{F}(D, H_\varepsilon \ast h) \leq \mathcal{F}(D, h) - \varepsilon$;
3. there exists $T_\varepsilon > 0$, with $T_\varepsilon \to 0$ as $\varepsilon \to 0$, such that for all $z \in h(1, D)$, $\|H_\varepsilon(\tau, z) - z\|_{a,b} \leq \tau T_\varepsilon$ for all $\tau \in [0, 1]$, for all $[a, b] \in \tilde{J}_z^h(D)$.
An “outward pushing” deformation

Lemma

Prop.: Let \( r > 0 \) and \( 0 < c_1 < c < c_2 \) be fixed. Then there exists \( \varepsilon_0 = \varepsilon_0(r, c) > 0 \) such that, for all \((D, h) \in \mathcal{H}\) satisfying:

1. \( F(D, h) \leq c_2 \);
2. \( \inf \{ \|x|_{[a,b]} - y\|_{a,b} \} \geq r, \quad x = h(1, \gamma), \gamma \in D, \frac{b-a}{2} \int_a^b g(\dot{x}, \dot{x}) \, dt \in [c_1, c_2] \),

and for all \( \varepsilon \in [0, \varepsilon_0[ \) there exists a continuous map \( H_{\varepsilon} : [0, 1] \times h(1, D) \to \mathcal{M} \) with the following properties:

1. \((D, H_{\varepsilon} \star h) \in \mathcal{H}\);
2. if \( c \leq F(D, h) \leq c_2 \) then \( F(D, H_{\varepsilon} \star h) \leq F(D, h) - \varepsilon \);
3. there exists \( T_{\varepsilon} > 0 \), with \( T_{\varepsilon} \to 0 \) as \( \varepsilon \to 0 \), such that for all \( z \in h(1, D) \),

\[
\|H_{\varepsilon}(\tau, z) - z\|_{a,b} \leq \tau T_{\varepsilon} \quad \text{for all } \tau \in [0, 1], \text{ for all } [a, b] \in \mathcal{J}^h_x(D).
\]

Interpretation: far from crossing OGC’s and irregular VCP, the functional \( F \) decreases along homotopies of \( \mathcal{H} \).
On the proof of the outward pushing deformation Lemma

in a small neighborhood of portions of curves that are far from VCP, one uses integral curves of vector fields in $V^+$, discussed above;
On the proof of the outward pushing deformation Lemma

- in a small neighborhood of portions of curves that are far from VCP, one uses integral curves of vector fields in $\mathcal{V}^+$, discussed above;
- in a small neighborhood of irregular VCP’s of second type, one uses suitable reparameterization flows;
On the proof of the outward pushing deformation Lemma

- in a small neighborhood of portions of curves that are far from VCP, one uses integral curves of vector fields in $\mathcal{V}^+$, discussed above;

- in a small neighborhood of irregular VCP’s of second type, one uses suitable reparameterization flows;

- one uses the methods of Degiovanni–Marzocchi (AMPA 1994) to build a global flow using local flows.
Flows far from VCP of first type

In order to obtain existence and multiplicity results for crossing OGC’s in the strictly concave case, we must construct nonincreasing flows that *fasten* from the irregular VCP of first type.
In order to obtain existence and multiplicity results for crossing OGC’s in the strictly concave case, we must construct nonincreasing flows that *fasten* from the irregular VCP of first type.

This can be done thanks to the following crucial regularity result, due to Marino and Scolozzi (Boll. UMI 1982):
Flows far from VCP of first type

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This can be done thanks to the following crucial regularity result, due to Marino and Scolozzi (Boll. UMI 1982):

**THM.** Let $y \in H^1([a, b], \Omega)$ be such that

$$\int_a^b g(\dot{y}, \frac{D}{dt} V) \, dt \geq 0, \quad \forall V \in \mathcal{V}^-(y) \text{ with } V(a) = V(b) = 0.$$  

Then $y \in H^{2, \infty}([a, b], \Omega)$, and in particular $y$ is of class $C^1$. 
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Irregular VCP’s of first type are not $C^1$, thus if a portion of curve is *close* to one of them it is *far* to VCP w.r. to $\mathcal{V}^\rightarrow$.

$\tilde{\mathcal{H}}$ consists of pairs $(\mathcal{D}, h)$, where $\mathcal{D} \subset \mathcal{C}$ is closed, and $h : \mathcal{D} \times [0, 1] \to \mathcal{C}$ is such that portions of curves near *cusps* of amplitude $\Theta \geq d_0$ are deformed into curves that remains *inside* $\Omega$. 
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Such homotopies $h$ are constructed using vector fields in $\mathcal{V}^-$: they deform into curves far from irregular VCP’s of first type, and the functional is not increasing by concatenation.
Moving away from irregular VCP’s of first type.

**Prop.** There exist $\bar{T}$ and $\bar{r} > 0$ with the following property:
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Prop.: There exist $\bar{T}$ and $\bar{r} > 0$ with the following property: for all $(D, h) \in \tilde{H}$ there exists a continuous homotopy $H_0 : [0, 1] \times h(1, D) \to M$ such that:

1. $(D, H_0 \ast h) \in \tilde{H}$;

2. $\mathcal{F}(D, H_0 \ast h) \leq \mathcal{F}(D, h)$;

3. $\|H_0(\tau, x) - x\|_{0,1} \leq \tau \bar{T}$, for all $x \in h(1, D)$;

4. for every $x \in h(1, D)$, and for every $[a, b] \in \hat{\mathcal{J}}_x^h$, it is $\|H_0(1, x)\|_{[a,b]} - y\|_{[a,b]} \geq \bar{r}$ for any $y \in M$ such that $y\|_{[a,b]}$ is an irregular VCP of first type.
Combining the previous deformation Lemmas, one obtains:

**1st Deformation Lemma:** Let \( c \) be geometrically regular value. There exists \( \varepsilon = \varepsilon(c) > 0 \) such that, for all \( (D, h) \in \tilde{H} \) with \( \mathcal{F}(D, h) \leq c + \varepsilon \), there exists a continuous map \( \eta : [0, 1] \times h(1, D) \to \mathcal{M} \) such that \( (D, \eta \star h) \in \tilde{H} \) and \( \mathcal{F}(D, \eta \star h) \leq c - \varepsilon \).
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\Gamma_i = \left\{ \mathcal{D} \subset \mathcal{C} \text{ closed} : \text{cat}_\mathcal{C}(\mathcal{D}) \geq i \right\} \neq \emptyset \quad i = 1, 2
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c_i = \inf_{\substack{D \in \Gamma_i \\ (D,h)\in\tilde{\mathcal{H}}}} \mathcal{F}(D,h)
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Corollary: Each \( c_i \) is a geometrically critical value.
Let $r_\star > 0$ be fixed and $(\mathcal{D}, h) \in \tilde{\mathcal{H}}$; consider the set:

$$
\mathcal{W} = \mathcal{W}(\mathcal{D}, h, r_\star) = \left\{ x \in \mathcal{M} : \exists [a, b] \in \mathcal{J}_x^h(\mathcal{D}) \text{ and a crossing OGC } \gamma : [a, b] \to \Omega \right\}
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s.t. $\max_{s \in [a, b]} \text{dist}(x(s), \gamma([a, b])) \leq r_\star$
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\( \mathcal{W} \) is closed in \( \mathcal{M} \).
Preparation for the 2nd Def. Lemma

Let \( r_* > 0 \) be fixed and \((D, h) \in \tilde{\mathcal{H}}\); consider the set:

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1. for all $x \in \mathcal{W}$, for all $[a, b] \in \mathcal{J}_x^0$ there exists at most one OGC $\gamma$ satisfying

2. the set $\left\{ A \in D_1 : \|A - \gamma(0)\| < 2r_* \text{ for some OGC } \gamma \text{ from } D_1 \text{ to } D_2 \right\}$ is contractible in $D_1$. 

(back to 2DL)
The 2nd Deformation Lemma

**Prop. 1:** Let \( c \) be a geometrically critical value. Then, there exists \( \varepsilon_* = \varepsilon_* (c) > 0 \) such that, for all \( (\mathcal{D}, h) \in \tilde{\mathcal{H}} \) with \( \mathcal{F}(\mathcal{D}, h) \leq c + \varepsilon_* \), there exists a continuous map \( \eta : [0, 1] \times h(1, \mathcal{D}) \to \mathcal{M} \) such that \( (\mathcal{D}, \eta \ast h) \in \tilde{\mathcal{H}} \) and

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The 2nd Deformation Lemma

Prop. 1: Let $c$ be a geometrically critical value. Then, there exists $\varepsilon_* = \varepsilon_*(c) > 0$ such that, for all $(\mathcal{D}, h) \in \tilde{\mathcal{H}}$ with $\mathcal{F}(\mathcal{D}, h) \leq c + \varepsilon_*$, there exists a continuous map $\eta : [0, 1] \times h(1, \mathcal{D}) \rightarrow \mathcal{M}$ such that $(\mathcal{D}, \eta \ast h) \in \tilde{\mathcal{H}}$ and

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Using the transversality of the OGC’s, and the fact that $\overline{\Omega}$ can be retracted onto one of the connected components of its boundary, one proves the following:
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Prop. 1: Let $c$ be a geometrically critical value. Then, there exists $\varepsilon_* = \varepsilon_*(c) > 0$ such that, for all $(D, h) \in \mathcal{H}$ with $F(D, h) \leq c + \varepsilon_*$, there exists a continuous map $\eta : [0, 1] \times h(1, D) \rightarrow \mathcal{M}$ such that $(D, \eta \star h) \in \mathcal{H}$ and

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Using the transversality of the OGC's, and the fact that $\overline{\Omega}$ can be retracted onto one of the connected components of its boundary, one proves the following:

Prop. 2: Assume that there are only a finite number of crossing OGC's from $D_1$ to $D_2$, and assume that $r_* > 0$ is so small so that properties (1) and (2) in the page above are satisfied. Then, for all $(D, h) \in \mathcal{H}$ there exists an open set $\mathcal{A}$ of $\mathcal{C}$, with $h(1, \cdot)^{-1}(\mathcal{W}) \subset \mathcal{A}$, that is contractible in $D_1$. 

School in Nonlinear Analysis and Calculus of Variations – p. 44/68
**Prop. 1:** Let $c$ be a geometrically critical value. Then, there exists $\varepsilon^* = \varepsilon^*(c) > 0$ such that, for all $(D, h) \in \tilde{\mathcal{H}}$ with $\mathcal{F}(D, h) \leq c + \varepsilon^*$, there exists a continuous map 

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**Corollary:** Assume that there is only a finite number of crossing OGC’s from $D_1$ to $D_2$. Then $c_1 < c_2$. 

School in Nonlinear Analysis and Calculus of Variations – p. 44/68
Some old and new results

We will now review some old and new results on periodic solutions of conservative dynamical systems.
Euler, Maupertuis, Jacobi, XVIII century:
consider the \textbf{conservative system}:

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\frac{D}{dt} \dot{x} = \nabla V(x)
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If \( x \) is a solution, then \( E = \frac{1}{2} g(\dot{x}, \dot{x}) + V(x) \) is constant: energy of the solution
Principle of least action

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Variational principle: Orbits of the conservative system having energy \( E \) are \( g_E \)-geodesics in \( \Omega_E \) (up to reparameterization).
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**Obs.:** \( g_E \) degenerate on \( \partial \Omega_E = V^{-1}(E) \).
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**Periodic solutions** \(\iff\) \[
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The existence of closed geodesics is clear on an **intuitive ground:** rest position of an elastic string whose initial position is a non null-homotopic closed curve.
**Curve shortening method**

**Birkhoff (1917):** formalization of the method of curve shortening on a Riemannian manifold:
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\[ c \text{ closed curve} \quad \mapsto \quad D(c) = \text{inscribed geodesic polygon}; \]
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- $c$ closed curve $\mapsto D(c)=$ inscribed geodesic polygon;
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If $c = c_0$ is a non null-homotopic curve, then the iterates $c_{n+1} = D(c_n)$ must have a subsequence converging to $c_\infty.$ By continuity:

$D(c_\infty) = D(\lim c_n) = \lim D(c_n) = \lim c_{n+1} = c_\infty,$ hence $c_\infty$ is a closed geodesic.
**Curve shortening method**

**Birkhoff (1917):** formalization of the method of curve shortening on a Riemannian manifold:

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If $c = c_0$ is a non null-homotopic curve, then the iterates $c_{n+1} = D(c_n)$ must have a subsequence converging to $c_∞$. By continuity:

$D(c_∞) = D(\lim c_n) = \lim D(c_n) = \lim c_{n+1} = c_∞$, hence $c_∞$ is a closed geodesic.

**Minimax method:** existence of a *closed geodesic on a sphere* (with arbitrary metric)

- apply the shortening method to a family of closed curves that cover simply a sphere;
- consider the **longest** curve of the family after each shortening process;
- a subsequence to this must converge to a closed geodesic, which is *not trivial*, because the sphere is not contractible.
Fet, Ljusternik (1957): observe that the minimax method can be used to prove the existence of a closed geodesic on any closed (i.e., compact with no boundary) manifold $M$. 
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Let $k > 0$ be the first integer such that $\pi_k(M) \neq 0$ (this exists by Hurewicz’s theorem, $k \leq \dim(M)$;)

\textbf{Topological methods}
Fet, Ljusternik (1957): observe that the minimax method can be used to prove the existence of a closed geodesic on any closed (i.e., compact with no boundary) manifold $M$.

1. Let $k > 0$ be the first integer such that $\pi_k(M) \neq 0$ (this exists by Hurewicz’s theorem, $k \leq \dim(M)$);

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2. take an essential map $f : S^k \rightarrow M$ and transfer to $M$ a family of closed curve covering $S^k$;
3. apply the curve shortening method to this family, and obtain a closed geodesic in $M$ which is not trivial, due to the assumption that $f$ represents a non zero element in $\pi_k(M)$. 
Classical Hamiltonian Systems

\[ H : \mathbb{R}^{2n} \to \mathbb{R}, \quad H(q, p) = \frac{1}{2} \sum_{i, j=1}^{n} g^{ij} p_i p_j + V(q), \quad V : \mathbb{R}^n \to \mathbb{R}, \ g^{ij} \text{ positive definite.} \]
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Classical Hamiltonian Systems

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**Obs.:** By the conservation of energy, \( p(0) = p(T) = 0 \). Since \( H \) is even in \( p \), the solution can be continued to a \( 2T \)-periodic solution according to the formulas: \( q(-t) = q(t) \), \( q(T + t) = q(T - t) \), \( p(-t) = -p(t) \), \( p(T - t) = -P(T - t) \) brake orbit.

Its image in the configuration space oscillates back and forth along a curve in \( D \) with endpoints in \( \partial D \).
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**Proof:** apply the shortening method to a family of diameters of \( D \). The main difficulty here is the fact that \( g_E \) vanishes on \( \partial D \), and a limit procedure is employed to control the behaviour of geodesics near \( \partial D \).
Hamiltonians of classical type

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**Case $n = 2$:** the result follows from another famous result by Seifert:

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\[ P : \Sigma \xrightarrow{\cong} S^3 \text{ radial projection (picture)}, \]

\[ \vec{H} = \sum \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right), \]

\[ dP(\vec{H}) \] is nowhere orthogonal to the Hopf vector field $\sum \left( p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i} \right)$
Theorem 1 $\implies$ Theorem 2

First prove that the solutions of the Hamilton equations *only depend on $\Sigma$* (not on $H$):
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**“Doubling trick”:** periodic solutions $(x, y)$ of $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ with period $2T$ correspond to pairs $(\alpha, \beta)$ and $(\xi, \eta)$ of solutions resp. of:

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\Omega_{ij} = \begin{cases} 
1 & \text{if } j = i + n; \\
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0 & \text{otherwise}
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$$

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**Proof of THM 1**: curve shortening method in Finsler geometry.
The results of Gluck and Ziller (1983)

Observe that the relative Hurewicz’s theorem can be used to generalize the arguments of Seifert and Weinstein.
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**Proof.** Curve shortening method in Finsler geometry with free boundary.
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**Proof.** Curve shortening method in Finsler geometry with free boundary. Need a convex boundary: enlarge $M$ to a larger manifold $\tilde{M}$ constructed by adding a convex collar.
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**THM 2:** If $H : T^* M \to \mathbb{R}$ is a Hamiltonian of classical type, and if $E$ is a regular value of $H$ such that $H^{-1}$ is non empty and compact, then there is a periodic solution of the Hamiltonian equation having energy $E$.

**Proof.** Curve shortening method in Finsler geometry with free boundary. Need a convex boundary: enlarge $M$ to a larger manifold $\tilde{M}$ constructed by adding a convex collar. Then, take limit as the size of the collar goes to 0.
The results of Gluck and Ziller (1983)

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They also obtain a multiplicity result in the case that the $E$-sublevel of the potential is homeomorphic to a disk, under a certain nonresonance assumption: the maximum diameter of the disk should have $g_E$-length smaller than twice the length of the shortest $g_E$-geodesic chord.
**Natural Hamiltonian:** \( H \in C^2(\mathbb R^{2m}, \mathbb R),\):

\[
H(p, q) = \frac{1}{2} \sum_{i,j=1}^{m} a^{ij}(q) p_i p_j + V(q)
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\( V \in C^2(\mathbb R^m, \mathbb R), \)

\( A(q) = (a^{ij}(q)) \) positive definite quadratic form on \( \mathbb R^m \):

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\sum_{i,j=1}^{m} a^{ij}(q) p_i p_j \geq \nu(q) |q|^2, \quad \nu : \mathbb R^m \to \mathbb R^+ \text{ continuous.}
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The corresponding Hamiltonian system (HS) is:

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\dot{p} &= -\frac{\partial H}{\partial q} \\
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- $p = \mathcal{L}(q)q$

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Def.: A brake orbit is a non constant periodic sol. of (HS)

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Choose $E > \inf V$ regular value of $V$; set:

$$\Omega_E = V^{-1}(\left]-\infty, E\right[) = \{ x \in \mathbb{R}^m : V(x) < E \}$$ open
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*Jacobi metric* in $\Omega_E$:

$$g_E(x) = (E - V(x)) \cdot \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(x) \, dx^i \, dx^j, \quad (a_{ij}) = (a^{ij})^{-1}. $$
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$q(t) \in \overline{\Omega}_E$ for all $t$, $q(0), q(T) \in \partial \Omega_E$. 

School in Nonlinear Analysis and Calculus of Variations – p. 56/68
Maupertuis integral $f_{a,b} : H^1(\Omega_E, \mathbb{R}) \to \mathbb{R}$:

$$f_{a,b}(x) = \frac{1}{2} \int_a^b (E - V(x)) g(\dot{x}, \ddot{x}) \, dt.$$. 
Maupertuis–Jacobi principle

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Euler–Lagrange equations:

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(E - V(x)) \frac{d}{dt} \dot{x} - g(\nabla V, \dot{x}) \dot{x} + \frac{1}{2} g(\dot{x}, \ddot{x}) \nabla V = 0.
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Maupertuis–Jacobi principle:

Critical points of $f_{a,b} \iff$ solutions of (HS)

We want to extend the MJ-principle to brake orbits.
Maupertuis–Jacobi principle for brake orbits

Thm.: $E$ regular value of $V$, $x : ]a, b[ \rightarrow \Omega_E$ s.t.: $C^0 \cap H^1_{loc}$

\[
\int_a^b \left[ (E - V)g(\dot{x}, \frac{D}{dt} W) - \frac{1}{2} g(\dot{x}, \dot{x}) g(\nabla V, W) \right] dt = 0, \quad \forall W \in C^\infty_0,
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$$

and $V(x(a)), V(x(b)) = E$. Then $\exists c_x, T \in \mathbb{R}^+$ and a diffeo $\sigma : [0, T] \rightarrow [a, b]$ with:
Maupertuis–Jacobi principle for brake orbits

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6. \( x(a) \neq x(b) \) and \( (E - V(x)) g(\dot{x}, \dot{x}) \equiv c_x \);
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1. $x(a) \neq x(b)$ and $(E - V(x))g(\dot{x}, \dot{x}) \equiv c_x;$
2. $(p, q) : [0, T] \rightarrow \mathbb{R}^m$ solution of $(HS)$, $q = x \circ \sigma$, $p = \mathcal{L}(q) \dot{q};$
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- $(p, q)$ can be extended to a $2T$-periodic brake orbit of energy $E$. 

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The Lagrangian problem

Let \((M, g)\) be a Riemannian manifold \(V : M \rightarrow \mathbb{R}\) a \(C^2\)-map (potential).
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The **Lagrangian problem** \((LP)\) is the 2nd order equation:

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\frac{D}{dt} \dot{q} + \nabla V(q) = 0 \quad q : \mathbb{R} \rightarrow M.
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\((HS) \iff (LP)\) (Legendre transform)
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same energy
Consider the Lagrangian problem. Let $x_0$ be a **critical point** of $V$: $\nabla V(x_0) = 0$. 

*Homoclinic horbits*
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2. $\lim_{t \to -\infty} \dot{q}(t) = \lim_{t \to +\infty} \dot{q}(t) = 0$.

Observe: $V(x_0) = \lim_{t \to \infty} \left[ \frac{1}{2} g(\dot{q}, \dot{q}) + V(q) \right] = E$; moreover, $x_0$ must be a critical point of $V$. 
Consider the Lagrangian problem.  
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We need a Maupertuis–Jacobi principle for homoclinics.
**M–J principle for homoclinics**

**Thm.:** $(M, g)$ Riemannian manifold, $V \in C^2(M, \mathbb{R})$, $x_0 \in M$ a nondegenerate max of $V$, $E = V(x_0)$. 
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**M–J principle for homoclinics**

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then \(\exists\) a diffeo \(\sigma : [0, +\infty[ \rightarrow [a, b[,\ \text{s.t. } q = x \circ \sigma\) is a solution of \((LP)\) with:

3. \(q(0) = x(a)\)
4. \(\lim_{t \to +\infty} q(t) = x_0, \ \lim_{t \to +\infty} \dot{q}(t) = 0.\)
Jacobi distance from $\partial \Omega_E$

If $E$ reg. value of $V$, $\overline{\Omega_E}$ compact, set $d_E : \Omega \to [0, +\infty[$:

$$d_E(Q) = \inf \left\{ \int_0^1 (E - V)g(\dot{x}, \ddot{x}) \frac{1}{2} \, dt : x \in H^1([0, 1], \overline{\Omega_E}), \ x(0) = Q, \ x(1) \in \partial \Omega \right\}.$$
Jacobi distance from $\partial \Omega_E$

If $E$ reg. value of $V$, $\Omega_E$ compact, set $d_E : \Omega \to [0, +\infty[$:

$$d_E(Q) = \inf \left\{ \int_0^1 (E - V)g(\dot{x}, \dot{x}) \frac{1}{2} dt : x \in H^1([0, 1], \Omega_E), x(0) = Q, x(1) \in \partial \Omega \right\}.$$

**Lem 1:** $d_E(Q)$ attained on some $\gamma_Q \in H^1([0, 1], \Omega_E) \cap C^2([0, 1]).$ Such curve satisfies

$$\int_0^1 (E - V)g(\dot{\gamma}_Q, \frac{D}{dt} W) - \frac{1}{2} g(\dot{\gamma}_Q, \dot{\gamma}_Q)g(\nabla V, W) dt = 0, \forall W \in C_0^\infty.$$
Jacobi distance from $\partial \Omega_E$

If $E$ reg. value of $V$, $\Omega_E$ compact, set $d_E : \Omega \to [0, +\infty[$:

$$d_E(Q) = \inf \{ \int_0^1 (E - V)g(\dot{x}, \dot{x}) \frac{1}{2} \, dt : x \in H^1([0, 1], \Omega_E), x(0) = Q, \, x(1) \in \partial \Omega \}.$$ 

**Lem 1:** $d_E(Q)$ attained on some $\gamma_Q \in H^1([0, 1], \Omega_E) \cap C^2([0, 1])$. Such curve satisfies

$$\int_0^1 (E - V)g(\dot{\gamma}_Q, \frac{D}{Dt} W) - \frac{1}{2} g(\dot{\gamma}_Q, \dot{\gamma}_Q) g(\nabla V, W) \, dt = 0, \quad \forall W \in C_0^\infty.$$ 

**Lem 2:** The map $d_E : \Omega_E \to [0, +\infty[$ is continuous, and it admits a continuous extension to $\Omega_E$ by setting $d_E = 0$ on $\partial \Omega_E$. 

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Jacobi distance from $\partial \Omega_E$

If $E$ reg. value of $V$, $\overline{\Omega_E}$ compact, set $d_E: \Omega \to [0, +\infty[$:

$$d_E(Q) = \inf \{ \int_0^1 (E - V) g(\dot{x}, \dot{x}) \frac{1}{2} \, dt : x \in H^1([0, 1], \overline{\Omega_E}, x(0) = Q, x(1) \in \partial \Omega \}.$$  

**Lem 1:** $d_E(Q)$ attained on some $\gamma_Q \in H^1([0, 1], \overline{\Omega_E}) \cap C^2([0, 1])$. Such curve satisfies

$$\int_0^1 (E - V) g(\dot{\gamma}_Q, \frac{D}{dt} W) - \frac{1}{2} g(\dot{\gamma}_Q, \dot{\gamma}_Q) g(\nabla V, W) \, dt = 0, \forall W \in C_0^\infty.$$  

**Lem 2:** The map $d_E: \Omega_E \to [0, +\infty[ $ is continuous, and it admits a continuous extension to $\overline{\Omega_E}$ by setting $d_E = 0$ on $\partial \Omega_E$.  

**Lem 3:** For $Q$ sufficiently near $\partial \Omega_E$, the minimizer $\gamma_Q$ is unique.
Jacobi distance from $\partial \Omega_E$

If $E$ reg. value of $V$, $\overline{\Omega_E}$ compact, set $d_E : \Omega \to [0, +\infty[$:

$$d_E(Q) = \inf \{ \int_0^1 (E - V)g(\dot{x}, \dot{x})^{\frac{1}{2}} \, dt : x \in H^1([0, 1], \overline{\Omega_E}, x(0) = Q, x(1) \in \partial \Omega \}.$$  

**Lem 1:** $d_E(Q)$ **attained** on some $\gamma_Q \in H^1([0, 1], \overline{\Omega_E}) \cap C^2([0, 1])$. Such curve satisfies

$$\int_0^1 (E - V)g(\dot{\gamma}_Q, \frac{D}{dt}W) - \frac{1}{2} g(\dot{\gamma}_Q, \dot{\gamma}_Q)g(\nabla V, W) \, dt = 0, \forall W \in C_0^\infty.$$  

**Lem 2:** The map $d_E : \Omega_E \to [0, +\infty[$ is continuous, and it admits a continuous extension to $\overline{\Omega_E}$ by setting $d_E = 0$ on $\partial \Omega_E$.

**Lem 3:** For $Q$ sufficiently near $\partial \Omega_E$, the minimizer $\gamma_Q$ is **unique**.

**Lem 4:** Set $\psi = \frac{1}{2} d_E^2 : \Omega_E \to \mathbb{R}^+; $ for $y$ near $\partial \Omega_E$:

$$\text{Hess}(\psi)_y[v, v] > 0, \quad \text{for } v \neq 0 \text{ with } d\psi_y[v] = 0.$$
THM: $E$ reg. value of $V$, $\Omega_E$ compact. Then, exists $\delta_* > 0$ s.t., setting $\Omega_* = \{x \in \Omega_E : d_E(x) > \delta_*\}$ the following hold:
**THM:** $E$ reg. value of $V$, $\Omega_E$ compact. Then, exists $\delta_* > 0$ s.t., setting $\Omega_* = \{ x \in \Omega_E : d_E(x) > \delta_* \}$ the following hold:

- $\partial \Omega_*$ of class $C^2$, 
**THM:** $E$ reg. value of $V$, $\Omega_E$ compact. Then, exists $\delta_* > 0$ s.t., setting $\Omega_* = \{ x \in \Omega_E : d_E(x) > \delta_* \}$ the following hold:

- $\partial \Omega_*$ of class $C^2$, $\overline{\Omega_*}$ diffeomorphic to $\overline{\Omega_E}$;
**THM:** $E$ reg. value of $V$, $\Omega_E$ compact. Then, exists $\delta_* > 0$ s.t., setting $\Omega_* = \{x \in \Omega_E : d_E(x) > \delta_*\}$ the following hold:

1. $\partial \Omega_*$ of class $C^2$, $\Omega_*$ diffeomorphic to $\overline{\Omega}_E$;
2. $\overline{\Omega}_*$ is *strongly concave* w.r. to $g_E$;
**THM:** $E$ reg. value of $V$, $\Omega_E$ compact. Then, exists $\delta_* > 0$ s.t., setting $\Omega_* = \{ x \in \Omega_E : d_E(x) > \delta_* \}$ the following hold:

1. $\partial \Omega_*$ of class $C^2$, $\overline{\Omega_*}$ diffeomorphic to $\overline{\Omega}_E$;
2. $\overline{\Omega_*}$ is *strongly concave* w.r. to $g_E$;
3. If $x : [0, 1] \rightarrow \overline{\Omega_*}$ is a $g_E$-OGC, then $\exists [\alpha, \beta] \supset [0, 1]$ and a unique extension $\hat{x} : [\alpha, \beta] \rightarrow \overline{\Omega}$ of $x$ such that:
OGC’s and the Maupertuis Integral

**THM:** $E$ reg. value of $V$, $\Omega_E$ compact. Then, exists $\delta_0 > 0$ s.t., setting $\Omega_* = \{ x \in \Omega_E : d_E(x) > \delta_* \}$ the following hold:

- $\partial \Omega_*$ of class $C^2$, $\overline{\Omega_*}$ diffeomorphic to $\overline{\Omega_E}$;
- $\overline{\Omega_*}$ is *strongly concave* w.r. to $g_E$;
- if $x : [0, 1] \rightarrow \overline{\Omega_*}$ is a $g_E$-OGC, then $\exists [\alpha, \beta] \supset [0, 1]$ and a unique extension $\hat{x} : [\alpha, \beta] \rightarrow \overline{\Omega}$ of $x$ such that:
  \[
  \frac{1}{2} \int_0^1 (E - V) g(\dot{x}', \frac{d}{dt} W) - \frac{1}{2} g(\dot{x}', \dot{x}') g(\nabla V, W) \, dt = 0, \forall W \in C_0^\infty;
  \]
**THM:** $E$ reg. value of $V$, $\Omega_E$ compact. Then, exists $\delta_* > 0$ s.t., setting $\Omega_* = \{x \in \Omega_E : d_E(x) > \delta_*\}$ the following hold:

1. $\partial \Omega_*$ of class $C^2$, $\overline{\Omega_*}$ diffeomorphic to $\overline{\Omega_E}$;
2. $\overline{\Omega_*}$ is **strongly concave** w.r. to $g_E$;
3. if $x : [0, 1] \rightarrow \overline{\Omega_*}$ is a $g_E$-OGC, then $\exists [\alpha, \beta] \supset [0, 1]$ and a unique extension $\hat{x} : [\alpha, \beta] \rightarrow \overline{\Omega}$ of $x$ such that:

   \[ \frac{1}{0} \int (E - V) g(\hat{x}', \frac{d}{dt} W) - \frac{1}{2} g(\hat{x}', \hat{x}') g(\nabla V, W) \, dt = 0, \forall W \in C^\infty_0; \]

   \[ \hat{x}(s) \in d_{E}^{-1}(\cdot - \delta_*, 0[\cdot] \text{ for } s \in ]\alpha, 0[ \cup ]1, \beta[; \]
OGC’s and the Maupertuis Integral

**THM:** \( E \) reg. value of \( V \), \( \Omega_E \) compact. Then, exists \( \delta_* > 0 \) s.t., setting \( \Omega_* = \{ x \in \Omega_E : d_E(x) > \delta_* \} \) the following hold:

- \( \partial \Omega_* \) of class \( C^2 \), \( \Omega_* \) diffeomorphic to \( \Omega_E \);
- \( \Omega_* \) is *strongly concave* w.r. to \( g_E \);
- if \( x : [0, 1] \to \Omega_* \) is a \( g_E \)-OGC, then \( \exists [\alpha, \beta] \supset [0, 1] \) and a unique extension \( \hat{x} : [\alpha, \beta] \to \Omega \) of \( x \) such that:
  - \( \int_0^1 (E - V)g(\hat{x}', \frac{d}{dt}W) - \frac{1}{2}g(\hat{x}', \hat{x}')g(\nabla V, W)\, dt = 0, \forall W \in C_0^\infty \);
  - \( \hat{x}(s) \in d_{E}^{-1}(]-\delta_*, 0[) \) for \( s \in ]\alpha, 0[ \cup ]1, \beta[ \);
  - \( V(\hat{x}(\alpha)) = V(\hat{x}(\beta)) = 0. \)
**THM:** $E$ reg. value of $V$, $\Omega_E$ compact. Then, exists $\delta_*>0$ s.t., setting $\Omega_* = \{ x \in \Omega_E : d_E(x) > \delta_* \}$ the following hold:

- $\partial \Omega_*$ of class $C^2$, $\overline{\Omega_*}$ diffeomorphic to $\overline{\Omega}_E$;
- $\overline{\Omega_*}$ is *strongly concave* w.r. to $g_E$;
- if $x : [0, 1] \to \overline{\Omega_*}$ is a $g_E$-OGC, then $\exists [\alpha, \beta] \supset [0, 1]$ and a unique extension $\hat{x} : [\alpha, \beta] \to \overline{\Omega}$ of $x$ such that:
  - $\frac{1}{0} \int (E - V)g(\hat{x}', \frac{d}{dt}W) - \frac{1}{2}g(\hat{x}', \hat{x}')g(\nabla V, W) dt = 0, \forall W \in C_0^\infty$;
  - $\hat{x}(s) \in d_{E}^{-1}(\cdot] - \delta_*, 0[)$ for $s \in ]\alpha, 0[ \cup ]1, \beta[$;
  - $V(\hat{x}(\alpha)) = V(\hat{x}(\beta)) = 0$.
- if $\overline{\Omega}$ is *centrally symmetric*, also $\overline{\Omega_*}$ is cent. symmetric.
$x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ reg. value of $V$, $V^{-1}(]-\infty, E])$ compact.
$x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ reg. value of $V$, $V^{-1}([-\infty, E])$ compact. Choose $\delta > 0$ small so that $\Omega_\delta = V^{-1}(E - \delta, +\infty]$ has two connected components.
Jacobi distance from a nondegenerate max

$x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ reg. value of $V$, $V^{-1}([-\infty, E])$ compact.

Choose $\delta > 0$ small so that $\Omega_\delta = V^{-1}(E - \delta, +\infty]$ has two connected components.

$$\lambda_E(Q) = \inf \left\{ \left[ \int_0^1 (E - V)g(\dot{x}, \dot{x})\,dt \right]^{\frac{1}{2}} : x \in C^0 \cap H^1_{\text{loc}}([0, 1], \overline{\Omega_\delta}), x(0) = Q, x(1) = x_0 \right\}$$
Jacobi distance from a nondegenerate max

$x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ reg. value of $V$, $V^{-1}([-\infty, E])$ compact.

Choose $\delta > 0$ small so that $\Omega_\delta = V^{-1}([E-\delta, +\infty[)$ has two connected components.

$$\lambda_E(Q) = \inf \left\{ \left[ \int_0^1 (E - V) g(\dot{x}, \dot{x}) \, dt \right]^{\frac{1}{2}} : x \in C^0 \cap H^1_{loc}([0, 1], \Omega_\delta), x(0) = Q, x(1) = x_0 \right\}$$

**Lem 1:** $\lambda_E(Q)$ is attained on some $\gamma_Q, g_E(\dot{\gamma}_Q, \dot{\gamma}_Q)$ constant, $\gamma_Q([0, 1[) \subset \overline{\Omega_Q \setminus \{x_0\}}$. 
Jacobi distance from a nondegenerate max

\( x_0 \) nondegenerate max of \( V \), \( V(x_0) = E \), \( E \) reg. value of \( V \), \( V^{-1}([-\infty, E]) \) compact. Choose \( \delta > 0 \) small so that \( \Omega_\delta = V^{-1}(E - \delta, +\infty] \) has two connected components.

\[
\lambda_E(Q) = \inf \left\{ \left[ \int_0^1 (E - V)g(\dot{x}, \dot{x}) \, dt \right]^{\frac{1}{2}} : x \in C^0 \cap H^1_{\text{loc}}([0, 1], \overline{\Omega_\delta}), x(0) = Q, x(1) = x_0 \right\}
\]

**Lem 1:** \( \lambda_E(Q) \) is attained on some \( \gamma_Q, g_E(\dot{\gamma}, \dot{\gamma}) \) constant, \( \gamma_Q([0, 1]) \subset \overline{\Omega_Q} \setminus \{x_0\} \).

\[
\lim_{Q \to x_0} \lambda_E(Q) = 0, \quad \lim_{Q \to x_0} \left[ \sup_{s \in [0, 1]} \text{dist}(\gamma_Q(s), x_0) \right] = 0;
\]
Jacobi distance from a nondegenerate max

$x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ reg. value of $V$, $V^{-1}([-\infty, E])$ compact.

Choose $\delta > 0$ small so that $\Omega_{\delta} = V^{-1}(]E - \delta, +\infty[)$ has two connected components.

$$\lambda_E(Q) = \inf \left\{ \left[ \int_0^1 (E - V)g(\dot{x}, \dot{x})dt \right]^{\frac{1}{2}} : x \in C^0 \cap H^1_{loc}([0, 1], \overline{\Omega_{\delta}}), x(0) = Q, x(1) = x_0 \right\}$$

Lem 1: $\lambda_E(Q)$ is attained on some $\gamma_Q$, $g_E(\dot{\gamma_Q}, \dot{\gamma_Q})$ constant, $\gamma_Q([0, 1]) \subset \overline{\Omega_Q} \setminus \{x_0\}$.

$$\lim_{Q \to x_0} \lambda_E(Q) = 0, \lim_{Q \to x_0} \left[ \sup_{s \in [0,1]} \text{dist} (\gamma_Q(s), x_0) \right] = 0;$$

For $Q$ near $x_0$, $\gamma_Q([0, 1]) \subset \Omega_{\delta}$, $\gamma_Q$ is of class $C^2$ and it satisfies:

$$\int_0^1 (E - V)g(\dot{\gamma_Q}, \frac{D}{dt}W) - \frac{1}{2} g(\dot{\gamma_Q}, \dot{\gamma_Q})g(\nabla V, W)dt = 0, \forall W \in C_0^\infty$$
nondegenerate max of $V$, $V(x_0) = E$, $E$ reg. value of $V$, $V^{-1}(]-\infty, E])$ compact. Choose $\delta > 0$ small so that $\Omega_\delta = V^{-1}(]E - \delta, +\infty[)$ has two connected components.

\[ \lambda_E(Q) = \inf \left\{ \left[ \int_0^1 (E - V)g(\dot{x}, \dot{x})d\tau \right]^{\frac{1}{2}} : x \in C^0 \cap H^1_{\text{loc}}([0, 1], \overline{\Omega_\delta}), x(0) = Q, x(1) = x_0 \right\} \]

Lem 1: $\lambda_E(Q)$ is attained on some $\gamma_Q$, $g_E(\dot{\gamma}_Q, \dot{\gamma}_Q)$ constant, $\gamma_Q([0, 1]) \subset \overline{\Omega_Q} \setminus \{x_0\}$.

\[ \lim_{Q \to x_0} \lambda_E(Q) = 0, \lim_{Q \to x_0} \sup_{s \in [0, 1]} \text{dist}(\gamma_Q(s), x_0) = 0; \]

For $Q$ near $x_0$, $\gamma_Q([0, 1]) \subset \Omega_\delta$, $\gamma_Q$ is of class $C^2$ and it satisfies:

\[ \int_0^1 (E - V)g(\dot{\gamma}_Q, \frac{D}{dt}W) - \frac{1}{2}g(\dot{\gamma}_Q, \dot{\gamma}_Q)g(\nabla V, W)d\tau = 0, \forall W \in C^\infty_0 \]

Lem 2: $\lambda_E : \Omega_E \to [0, +\infty[$ is continuous.
Jacobi distance from a nondegenerate max

$x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ reg. value of $V$, $V^{-1}(]-\infty, E])$ compact.

Choose $\delta > 0$ small so that $\Omega_\delta = V^{-1}(]E - \delta, +\infty[)$ has two connected components.

$$\lambda_E(Q) = \inf \left\{ \left[ \int_0^1 (E - V) g(\dot{x}, \dot{x}) \, dt \right]^{\frac{1}{2}} : x \in C^0 \cap H^1_{\text{loc}}([0, 1], \overline{\Omega_\delta}), x(0) = Q, x(1) = x_0 \right\}$$

**Lem 1:** $\lambda_E(Q)$ is attained on some $\gamma_Q$, $g_E(\dot{\gamma}_Q, \dot{\gamma}_Q)$ constant, $\gamma_Q([0, 1[ \subset \overline{\Omega_Q} \setminus \{x_0\}$.

$$\lim_{Q \to x_0} \lambda_E(Q) = 0, \quad \lim_{Q \to x_0} \left[ \sup_{s \in [0, 1]} \text{dist}(\gamma_Q(s), x_0) \right] = 0;$$

For $Q$ near $x_0$, $\gamma_Q([0, 1[ \subset \Omega_\delta$, $\gamma_Q$ is of class $C^2$ and it satisfies:

$$\int_0^1 (E - V) g(\dot{\gamma}_Q, \frac{D}{dt} W) - \frac{1}{2} g(\dot{\gamma}_Q, \dot{\gamma}_Q) g(\nabla V, W) \, dt = 0, \quad \forall W \in C^\infty_0$$

**Lem 2:** $\lambda_E : \Omega_E \to [0, +\infty[ \text{ is continuous}.$

**Lem 3:** $\exists \hat{\rho} > 0$ s.t., setting $\psi(y) = \frac{1}{2} \lambda_Q(y)^2$, for $\text{dist}(y, x_0) \leq \hat{\rho}$:

$$\text{Hess}(\psi)_y[v, v] > 0, \quad \text{for } v \neq 0 \text{ with } d\psi_y[v] = 0.$$
**M–J principle for homoclinics**

**THM:** $x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ regular value of $V$, $V^{-1}(-\infty, E] \cup \{x_0\}$ homeomorphic to an open ball in $\mathbb{R}^m$. 
**M–J principle for homoclinics**

**THM:** $x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ regular value of $V$, $V^{-1}([-\infty, E[) \cup \{x_0\}$ homeomorphic to an open ball in $\mathbb{R}^m$.

\[ \exists \delta_* > 0 \text{ s.t., setting: } \Omega_* = \{ x \in \mathbb{R}^m : \text{dist}_E(x, V^{-1}(E)) > \delta_* \} \]

denoting by $D_0$ the connected component of $\partial \Omega_*$ near $x_0$, by $D_1$ the connected component of $\partial \Omega_*$ near $V^{-1}(E) \setminus \{0\}$, the following hold:
**M–J principle for homoclinics**

**THM:** $x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ regular value of $V$, $V^{-1}(-\infty, E[\{x_0\}]$ homeomorphic to an open ball in $\mathbb{R}^m$.

$\exists \delta_* > 0$ s.t.

setting: $\Omega_* = \{ x \in \mathbb{R}^m : \text{dist}_E(x, V^{-1}(E)) > \delta_* \}$

denoting by $D_0$ the connected component of $\partial \Omega_*$ near $x_0$,

by $D_1$ the connected component of $\partial \Omega_*$ near $V^{-1}(E) \setminus \{0\}$,

the following hold:

$\exists$ \quad $\partial \Omega_*$ is of class $C^2$, $\overline{\Omega}_*$ is homeomorphic to an annulus;
**M–J principle for homoclinics**

**THM:** \( x_0 \) nondegenerate max of \( V \), \( V(x_0) = E \), \( E \) regular value of \( V \), \( V^{-1}(\mathbb{R}^{-\infty}) \bigcup \{x_0\} \) homeomorphic to an open ball in \( \mathbb{R}^m \).

\( \exists \delta_* > 0 \) s.t., setting: \( \Omega_* = \{ x \in \mathbb{R}^m : \text{dist}_E(x, V^{-1}(E)) > \delta_* \} \)

denoting by \( D_0 \) the connected component of \( \partial \Omega_* \) near \( x_0 \),

by \( D_1 \) the connected component of \( \partial \Omega_* \) near \( V^{-1}(E) \setminus \{0\} \),

the following hold:

1. \( \partial \Omega_* \) is of class \( C^2 \), \( \overline{\Omega_*} \) is homeomorphic to an annulus;
2. \( \overline{\Omega_*} \) is \( g_E \)-strongly concave;
**M–J principle for homoclinics**

**THM:** $x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ regular value of $V$, $V^{-1}(]-\infty, E[) \cup \{x_0\}$ homeomorphic to an open ball in $\mathbb{R}^m$. 

$\exists \delta_* > 0$ s.t., setting: $\Omega_* = \{x \in \mathbb{R}^m : \text{dist}_E(x, V^{-1}(E)) > \delta_*\}$

denoting by $D_0$ the connected component of $\partial \Omega_*$ near $x_0$,

by $D_1$ the connected component of $\partial \Omega_*$ near $V^{-1}(E) \setminus \{0\}$,

the following hold:

1. $\partial \Omega_*$ is of class $C^2$, $\overline{\Omega_*}$ is homeomorphic to an annulus;
2. $\overline{\Omega_*}$ is $g_E$-strongly concave;
3. if $x : [0, 1] \rightarrow \overline{\Omega_*}$ is an OGC with $x(0) \in D_0$, $x(1) \in D_1$, then there exists $\alpha, \beta \supset [0, 1]$ and a unique extension $\hat{x} : [\alpha, \beta] \rightarrow \overline{\Omega_E}$, $x \in C^0 \cap H^1_{\text{loc}}$, satisfying:
**M–J principle for homoclinics**

**THM:** $x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ regular value of $V$, $V^{-1}(-\infty, E[\cdot] \cup \{x_0\}$ homeomorphic to an open ball in $\mathbb{R}^m$.

$\exists \delta_* > 0$ s.t., setting: $\Omega_* = \{x \in \mathbb{R}^m : \text{dist}_E(x, V^{-1}(E)) > \delta_*\}$
denoting by $D_0$ the connected component of $\partial \Omega_*$ near $x_0$, by $D_1$ the connected component of $\partial \Omega_*$ near $V^{-1}(E) \setminus \{0\}$, the following hold:

1. $\partial \Omega_*$ is of class $C^2$, $\overline{\Omega}_*$ is homeomorphic to an annulus;
2. $\overline{\Omega}_*$ is $g_E$-strongly concave;
3. if $x : [0, 1] \to \overline{\Omega}_*$ is an OGC with $x(0) \in D_0$, $x(1) \in D_1$, then there exists $]\alpha, \beta] \supset [0, 1]$ and a unique extension $\tilde{x} : [\alpha, \beta] \to \overline{\Omega}_E$, $x \in C^0 \cap H^1_{\text{loc}}$, satisfying:
   - $\tilde{x}$ is a $g_E$-geodesic;
**THM:** \(x_0\) nondegenerate max of \(V\), \(V(x_0) = E\), \(E\) regular value of \(V\), \(V^{-1}(-\infty, E[\cup\{x_0\})\) homeomorphic to an open ball in \(\mathbb{R}^m\).

\(\exists \delta_* > 0\) s.t., setting: \(\Omega_* = \{x \in \mathbb{R}^m : \text{dist}_E (x, V^{-1}(E)) > \delta_*\}\)

denoting by \(D_0\) the connected component of \(\partial \Omega_*\) near \(x_0\),
by \(D_1\) the connected component of \(\partial \Omega_*\) near \(V^{-1}(E) \setminus \{0\}\),
the following hold:

1. \(\partial \Omega_*\) is of class \(C^2\), \(\overline{\Omega_*}\) is homeomorphic to an annulus;
2. \(\overline{\Omega_*}\) is \(g_E\)-strongly concave;
3. if \(x : [0, 1] \rightarrow \overline{\Omega_*}\) is an OGC with \(x(0) \in D_0\), \(x(1) \in D_1\), then there exists \([\alpha, \beta] \supset [0, 1]\) and a unique extension \(\hat{x} : [\alpha, \beta] \rightarrow \overline{\Omega_E}, x \in C^0 \cap H^1_{\text{loc}}\), satisfying:
   - \(\hat{x}\) is a \(g_E\)-geodesic;
   - \(\text{dist}(\hat{x}(s), V^{-1}(E)) \in ]-\delta_*, 0[\) for \(s \in ]\alpha, 0[\cup 1, \beta[\).
**M–J principle for homoclinics**

**THM:** $x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ regular value of $V$, $V^{-1}(-\infty, E[) \cup \{x_0\}$ homeomorphic to an open ball in $\mathbb{R}^m$.

$\exists \delta_* > 0$ s.t., setting: $\Omega_* = \{x \in \mathbb{R}^m : \text{dist}_E(x, V^{-1}(E)) > \delta_* \}$

denoting by $D_0$ the connected component of $\partial \Omega_*$ near $x_0$,

by $D_1$ the connected component of $\partial \Omega_*$ near $V^{-1}(E) \setminus \{0\}$,

the following hold:

1. $\partial \Omega_*$ is of class $C^2$, $\overline{\Omega}_*$ is homeomorphic to an annulus;
2. $\overline{\Omega}_*$ is $g_E$-strongly concave;
3. if $x : [0, 1] \to \overline{\Omega}_*$ is an OGC with $x(0) \in D_0$, $x(1) \in D_1$, then there exists $]\alpha, \beta] \supset [0, 1]$ and a unique extension $\hat{x} : [\alpha, \beta] \to \overline{\Omega}_E$, $x \in C^0 \cap H^1_{\text{loc}}$, satisfying:
   - $\hat{x}$ is a $g_E$-geodesic;
   - $\text{dist}(\hat{x}(s), V^{-1}(E)) \in ]-\delta_*, 0[ \text{ for } s \in ]\alpha, 0[ \cup ]1, \beta[;$
   - $\hat{x}(\alpha) = x_0$, $\hat{x}(\beta) \in V^{-1}(E) \setminus \{x_0\};$
**THM:** $x_0$ nondegenerate max of $V$, $V(x_0) = E$, $E$ regular value of $V$, 
$V^{-1}(]-\infty, E[) \cup \{x_0\}$ homeomorphic to an open ball in $\mathbb{R}^m$.

$\exists \delta_* > 0$ s.t., setting: 

$$\Omega_* = \{x \in \mathbb{R}^m : \text{dist}_E(x, V^{-1}(E)) > \delta_*\}$$

denoting by $D_0$ the connected component of $\partial \Omega_*$ near $x_0$, 
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the following hold:

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   - $\hat{x}(\alpha) = x_0$, $\hat{x}(\beta) \in V^{-1}(E) \setminus \{x_0\}$;
4. if $V^{-1}(]-\infty, E[) \cup \{x_0\}$ and $V$ are centrally symmetric around $x_0$, then so is $\overline{\Omega}_*$. 
Theorem 1: Let $H \in C^2(\mathbb{R}^{2m}, \mathbb{R})$ be a natural Hamiltonian.
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Theorem 1: Let $H \in C^2(\mathbb{R}^{2m}, \mathbb{R})$ be a natural Hamiltonian. Let $E$ be a regular value of the potential $V$, and assume

$$\Omega_E = V^{-1}(-\infty, E]$$

is homeomorphic to an $m$-dimensional annulus. Then, the Hamiltonian system (HS) has at least two geometrically distinct brake orbits of energy $E$, whose endpoints are in different connected components of $V^{-1}(E)$. 
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**Theorem 2:** Under the assumptions of THM 1, if the functions $a_{ij}$ and $V$ are *centrally symmetric* w. resp. to some $y_0 \not\in V^{-1}(-\infty, E]$), then there are at least $m$ geometrically distinct brake orbits for (HS) with energy $E$. 
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Theorem 3: \((M, g)\) Riemannian manifold, \(V : M \overset{C^2}{\to} \mathbb{R}\), \(x_0 \in M\) a nondegenerate maximum of \(V\). Assume:

- \(V^{-1}(-\infty, E[\cdot]) \cup \{x_0\}\) is homeomorphic to an open ball of \(\mathbb{R}^m\);
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Then, there are at least two geometrically distinct homoclinic orbits for the Lagrangian problem (LP) emanating from $x_0$.

Theorem 4: Under the assumptions of THM 3, if $(M, g)$ and $V$ are centrally symmetric around $x_0$, then there are at least $m$ geometrically distinct homoclinics of (LP) emanating from $x_0$. 
Gluing a convex collar to a manifold with boundary