Teichmüller theory, collapse of flat manifolds and applications to the Yamabe problem

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Outiline of this talk

- **The Yamabe problem**
  - Compact manifolds. Aubin inequality.
  - Noncompact manifolds. Example: $\mathbb{S}^n \setminus \mathbb{S}^k$.
  - Two arguments for the existence of multiple solutions.
- **Compact flat manifolds**
  - Bieberbach theorems.
  - Teichmüller space of flat metrics.
  - Algebraic description.
  - Existence of flat deformations.
- **Boundary of the flat Teichmüller space**
  - Collapse of flat manifolds;
  - Flat orbifolds.
  - Examples of 3-D collapse.

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FIELDS

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FAPESP

Finding the *best* metric on a manifold

- M a compact manifold;
- a metric g on M is a smooth choice of measuring length and angles of tangent vectors to M: $g_p\colon T_pM \times T_pM \to \mathbb{R}$;
- metrics give a way of computing length of curves and distances;
- metrics determine curvature in M. Best metrics are those that have the simplest curvature formulas. **Constant.**
- When $n = \dim(M) > 2$, there are several notions of curvature:

sectional curvature	**Ricci** curvature	**scalar** curvature
constant: only if $M = \mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n$	constant: Einstein field equations in vacuum	constant: Yamabe problem

The Yamabe problem

- Two metrics g_1 and g_2 are conformal if they give the same *angles* between vectors.
- Conformal class of g: $[g] := \{\text{metrics conformal to } g\}$

Yamabe problem

Given a compact manifold M^n ($n \geq 3$) and a metric g on M, does there exist $h \in [g]$ with scal_h constant?

Solutions h to the Yamabe problem are critical points of the Hilbert–Einstein functional $\mathcal{A}\colon [g] \to \mathbb{R}$:

$$\mathcal{A}(h) = \mathrm{vol}(h)^{\frac{2-n}{n}} \int_M \mathrm{scal}_h \, \mathrm{d}M_h$$

Solution (Yamabe, Trudinger, Aubin, Schoen)

$\mathcal{A}$ always attains its *minimum* in $[g]$: $Y(M,g)$ (Yamabe constant)

Yamabe problem on round spheres

- Consider the round sphere $(\mathbb{S}^n, g_{\mathrm{round}})$

Two special properties

- $[g_{\mathrm{round}}]$ contains infinitely many constant scalar curvature metrics (infact, a *noncompact* set!)
- (Aubin inequality) for *any* compact manifold M^n and *any* metric g on M:

$$\boxed{Y(M, g) \le Y(\mathbb{S}^n, g_{\mathrm{round}})}$$

The Yamabe problem on noncompact manifolds

- *Asymptotic condition:* completeness.

Yamabe problem on noncompact manifolds

Given a noncompact (M^n, g), $n \geq 3$, does there exist a *complete* metric $h \in [g]$ with scal_h constant?

Counterexample (Jin Zhiren, 1980)

$\widetilde{M}$ compact, $\quad M = \widetilde{M} \setminus \{p_1, \ldots, p_k\}$.
Choose g on $\widetilde{M}$ with $\mathrm{scal}_g < 0$ (Aubin). Then:

- there is no $h \in [g]$ with $\mathrm{scal}_h \geq 0$;
- If $h \in [g]$ and $\mathrm{scal}_h < 0$, then h is noncomplete (a priori estimates on an elliptic PDE).

A nice example: $\mathbb{S}^n \setminus \mathbb{S}^k$

- Consider $M = \mathbb{S}^n \setminus \mathbb{S}^k \qquad (0 \geq k < n)$
- metric: g_{round}

$$\mathbb{S}^n \setminus \mathbb{S}^k \cong \mathbb{R}^n \setminus \mathbb{R}^k \text{ (stereographic projection)}$$
$$g_{\text{round}} \cong g_{\text{flat}}$$

$$\begin{aligned} \mathbb{R}^n \setminus \mathbb{R}^k &= \left(\mathbb{R}^{n-k} \setminus \{0\}\right) \times \mathbb{R}^k \\ g_{\text{flat}} &= r^2 \mathrm{d}\theta^2 + \mathrm{d}r^2 + \mathrm{d}y^2 \\ &\cong \mathrm{d}\theta^2 + \frac{1}{r^2}\left(\mathrm{d}r^2 + \mathrm{d}y^2\right) \\ &= \mathbb{S}^{n-k-1} \times \mathbb{H}^{k+1} \Longleftarrow \end{aligned}$$

complete and constant scalar curvature

Infinitely many solutions!

Theorem

When $2k < n - 2$, there are infinitely many solutions of the Yamabe problem in $\mathbb{S}^n \setminus \mathbb{S}^k$.

Proof.

Two different arguments:

(A) Topology (covering spaces)

(B) Bifurcation theory (when $k = 0, 1$). □

This theorem produces *periodic solutions*, i.e., coming from compact quotients.

The topological argument

1. $\mathbb{H}^{k+1}$ has compact quotients that give an infinite tower of finite-sheeted Riemannian coverings:

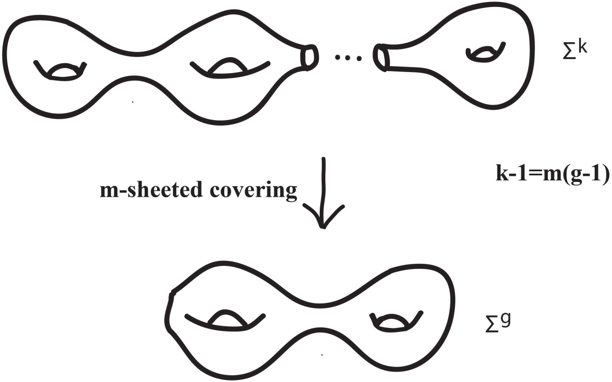
$$(\mathbb{H}^{k+1}, g_{\mathrm{hyp}}) \to \ldots \to (\Sigma_2, g_2) \to (\Sigma_1, g_1) \to (\Sigma_0, g_0)$$

2. Multiply by $(S^{n-k-1}, g_{\mathrm{round}})$, product metrics:

$$\ldots \to (S^{n-k-1} \times \Sigma_1, g_{\mathrm{round}} \oplus g_1) \to (S^{n-k-1} \times \Sigma_0, g_{\mathrm{round}} \oplus g_0)$$

3. pull-back Yamabe metric in $[g_{\mathrm{round}} \oplus g_0]$: Hilbert–Einstein energy $\mathcal{A}$ diverges! (uses assumption $2k < n - 2$)
4. By Aubin inequality, minimum of $\mathcal{A}$ must be attained at some other metric in the conformal class of the product.
5. Iterate.

Coverings of hyperbolic surfaces



Profinite completion and residually finite groups

Infinite tower of finite-sheeted coverings:

$$\ldots \to M_k \to M_{k-1} \to \ldots \to M_1 \to M_0$$

iff $G = \pi_1(M_0)$ has infinite *profinite completion* $\widehat{G}$.

Def. $\widehat{G} = \varprojlim G/\Gamma, \qquad \Gamma \trianglelefteq G, \quad \left[G : \Gamma\right] < +\infty.$

Canonical homomorphism $\iota : G \to \widehat{G}$

$$\mathrm{Ker}(i) = \bigcap_{\substack{\Gamma \trianglelefteq G \\ [G:\Gamma] < +\infty}} \Gamma$$

Def. G is *residually finite* if: $\displaystyle\bigcap_{\substack{\Gamma \trianglelefteq G \\ [G:\Gamma] < +\infty}} \Gamma = \{1\}$

The bifurcation argument ($k = 1$)

1. Choose a compact quotient of $\mathbb{H}^2$: a compact surface $\Sigma^{\mathbf{g}}$ of genus $\mathbf{g} \geq 2$.
2. $\Sigma^{\mathbf{g}}$ has many nonisometric hyperbolic metrics. Deformations: Teichmüller space $\mathcal{T}(\Sigma^{\mathbf{g}}) \cong \mathbb{R}^{6\mathbf{g}-6}$.
3. For $h \in \mathcal{T}(\Sigma^{\mathbf{g}})$, $g_{\mathrm{round}} \oplus h$ is a solution of the Yamabe problem in $\mathbb{S}^{n-2} \times \Sigma^{\mathbf{g}}$
4. Each of these solutions has a Morse index, computed in terms of the spectrum of Δ_h.
5. Jump of Morse index when Δ_h has many *small* eigenvalues $\Longrightarrow$ bifurcation must occur along paths in $\mathcal{T}(\Sigma^{\mathbf{g}})$.

Natural candidates to replace $\mathbb{H}^{k+1}$

Simply-connected space (X, g) such that:

(a) scal_g constant;

(b) (X, g) admits an infinite tower of finite sheeted compact Riemannian coverings X/Γ (Γ has infinite profinite completion)

(c) A rich space of metrics in X/Γ (Teichmüller space) that are locally isometric to g, with small Laplacian eigenvalues.

Theorem (Borel)

Symmetric spaces of noncompact type X admit irreducible compact quotients X/Γ.

X/Γ loc. symmetric $\quad\Longrightarrow\quad$ constant scalar curvature

Selberg–Malcev lemma

Finitely generated linear groups are residually finite.

Corollary. $\Gamma = \pi_1(X/\Gamma)$ has infinite profinite completion.

Simplest example: $(X, g) = (\mathbb{R}^n, g_{\mathrm{flat}})$

- $\Gamma \subset \mathrm{Iso}(\mathbb{R}^n) \cong \mathbb{R}^n \ltimes \mathrm{O}(n)$ is a Bieberbach group.
- $\mathbb{R}^n/\Gamma$ is a compact flat manifold/orbifold.

Compact flat manifolds and orbifolds

- $\Gamma \subset \mathrm{Iso}(\mathbb{R}^n)$ is a *Bieberbach* group:
  (a) discrete;
  (b) co-compact;
  (c) torsion-free.

Theorem

(M, g) compact flat manifold $\Longleftrightarrow$ $\boxed{M = \mathbb{R}^n/\Gamma}$ Γ Bieberbach.

Algebraic structure:

$$0 \longrightarrow L \longrightarrow \Gamma \longrightarrow H \longrightarrow 1$$

- $L \subset \mathbb{R}^n$ is a co-compact lattice
- $H \subset \mathrm{O}(n)$ is a finite group (holonomy)

Orbifolds: compact flat orbifolds have a similar structure:
$\Gamma \subset \mathrm{Iso}(\mathbb{R}^n)$ is a *crystallographic*, possibly with torsion.

Small eigenvalues

Theorem (Cheng)

(F^d, g_F) closed manifold with nonnegative Ricci curvature and *unit volume*. Then:

$$\lambda_j(F, g_F) \leq 2\, j^2 \frac{d(d+4)}{\mathrm{diam}(F, g_F)^2}.$$

For bifurcation purposes, need flat metrics with volume 1 and arbitrarily large diameter. Equivalently, fixed diameter and arbitrarily small volume. **Collapse flat metrics!**

Gromov–Hausdorff convergence and collapse

Hausdorff distance: $X, Y \subset Z$:

$$d_{\mathrm{H}}^{Z}(X,Y) = \inf\left\{\varepsilon : X \subset B(Y,\varepsilon) \text{ and } Y \subset B(X,\varepsilon)\right\}$$

Gromov–Hausdorff distance: $\boxed{d_{\mathrm{GH}}(X,Y) = \inf_{X,Y \hookrightarrow Z} d_{\mathrm{H}}^{Z}(X,Y)}$

Gromov

$M = \{\text{compact metric spaces}\}/\text{isometries}.$

(M, d_{GH}) is a complete metric space.

- GH-limits can change topology, dimension...
- Diameter is a continuous function in (M, d_{GH}).
- Collapse of compact Riemannian manifolds:
  $\mathrm{GH} - \lim(M_i, g_i) = (X, d)$, with $\lim \mathrm{vol}(M_i, g_i) = 0$.

Bieberbach's theorems (geometric version)

Theorem

(I) *A compact flat n-orbifold* $(\mathcal{O}, g)$ *is isometrically covered by a flat n-torus.*

(II) *Compact flat orbifolds of the same dimension and with isomorphic fundamental groups are affinely diffeomorphic.*

(III) *For all n, there is only a finite number of affine equivalence classe of compact flat n-orbifolds.*

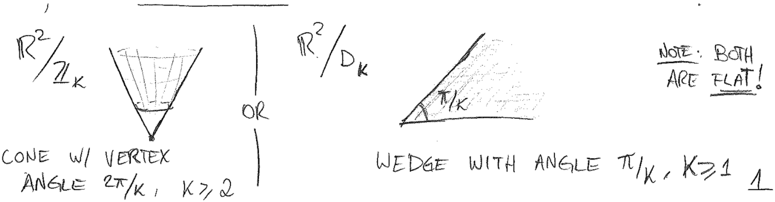
- $n = 2$:
  - **manifolds**: torus $\mathbb{T}^2$ and Klein bottle $\mathbb{K}^2$
  - **orbifolds**: 17 affine classes (wallpaper groups).
- $n = 3$: $\#\text{mnfld} = 10$, $\#\text{orbfld} = 219$
- $n = 4$: $\#\text{mnfld} = 74$, $\#\text{orbfld} = 4783$

Flat 2-orbifolds

Underlying top. space: $D^2, \mathbb{S}^2, \mathbb{R}P^2, \mathbb{M}^2, \mathbb{K}^2, \mathbb{S}^1 \times [0,1]$

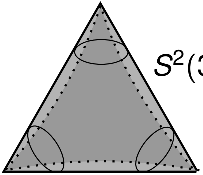
Cyclic or dihedral local groups (Leonardo da Vinci)

Cone points and corner reflections.



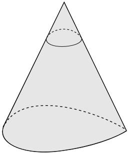
Notation: $S(n_1, \ldots, n_k; m_1, \ldots, m_\ell)$

A few pictures



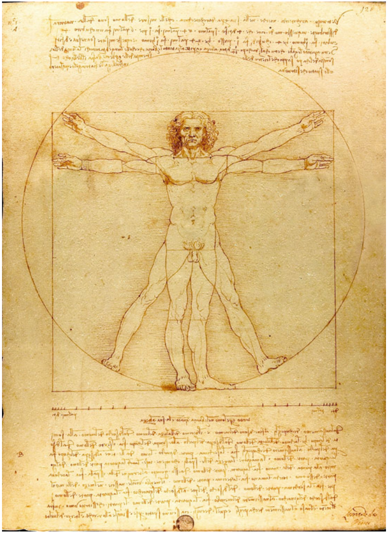
$S^2(3, 3, 3;)$

$D^2(; 4, 4, 4, 4)$



$D^2(4; 2)$

Symmetries of the Vitruvian Man



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Teichmüller space of flat metrics

- M compact manifold (orbifold) admitting a flat metric;
- $\mathrm{Flat}(M)$ the set of all flat metrics on M;
- $\mathfrak{M}_{\mathrm{flat}} = \mathrm{Flat}(M)/\mathrm{Diff}(M)$ moduli space of flat metrics on M;
- $\mathcal{T}_{\mathrm{flat}}(M) = \mathrm{Flat}(M)/\mathrm{Diff}_0(M)$ space of *deformations* of flat metrics (Teichmüller space).

Theorem

- *$\mathcal{T}_{flat}(M)$ is a real-analytic manifold (homogeneous space) diffeomorphic to some $\mathbb{R}^d$.*
- *$\mathfrak{M}_{flat}(M) = \mathcal{T}_{flat}(M)/\mathrm{MCG}(M)$, where the mapping class group $\mathrm{MCG}(M)$ is countable and discrete.*

Algebraic description

- $H \subset \mathrm{O}(n)$ holonomy representation
- $\mathbb{R}^n = \bigoplus_{i=1}^{\ell} W_i$, where W_i isotypical component
- W_i direct sum of m_i copies of an irreducible
- $\mathbb{K}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}$ type of W_i

Theorem

$\mathcal{T}_{flat}(M) = \prod_{i=1}^{\ell} \frac{\mathrm{GL}(m_i, \mathbb{K}_i)}{\mathrm{O}(m_i, \mathbb{K}_i)}$

$$\frac{\mathrm{GL}(m_i, \mathbb{K}_i)}{\mathrm{O}(m_i, \mathbb{K}_i)} \cong \mathbb{R}^{d_i}, \qquad d_i = \begin{cases} \frac{1}{2} m_i(m_i + 1), & \text{if } \mathbb{K}_i = \mathbb{R}, \\ m_i^2, & \text{if } \mathbb{K}_i = \mathbb{C}, \\ m_i(2m_i - 1), & \text{if } \mathbb{K}_i = \mathbb{H}. \end{cases}$$

The above builds on previous work by Wolf, Thurston, Baues...

Existence of nontrivial flat deformations

Theorem (Hiss–Szczepański)

Holonomy repn of a compact flat manifold is never irreducible.

Corollary

Compact flat manifolds admit nonhomothetic flat deformations.

Proof. $\dim\big(\mathcal{T}_{\mathrm{flat}}(M)\big) \geq 2$. □

Obs. Flat orbifolds can be *rigid*!! ($\Longleftrightarrow$ irreducible holonomy)

Theorem

- (M_0, g_0) *closed Riemannian manifold with* $\mathrm{scal}_{g_0} > 0$
- $\Gamma \subset \mathrm{Iso}(\mathbb{R}^d)$ *Bieberbach,* $d \geq 2$.

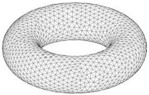
Then there exist infinitely many branches of Γ-periodic solutions to the Yamabe problem on $\big(M_0 \times \mathbb{R}^d, g_0 \oplus g_{\mathrm{flat}}\big)$.

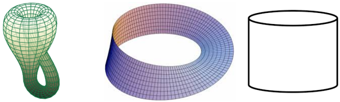
Examples of Teichmüller space

n-torus

$\mathcal{T}_{\text{flat}}(T^n) \cong \mathrm{GL}(n)/\mathrm{O}(n) \cong \mathbb{R}^{\frac{1}{2}n(n+1)}$

$\mathrm{MCG}(T^n) = \mathrm{GL}(n, \mathbb{Z})$.





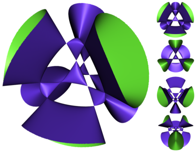
Klein bottle, Möbius band, cylinder

$H = \mathbb{Z}_2$ (reflection). $\mathcal{T}_{\text{flat}}(\mathcal{O}) \cong \mathbb{R}^2$

Kummer surface

$\mathcal{O} = T^4/\mathbb{Z}_2$ (antipodal map on each coordinate).
16 conical singularities
Holonomy rep: 4 copies of nontrivial $\mathbb{Z}_2$-rep

$\mathcal{T}_{\text{flat}}(\mathcal{O}) \cong \mathrm{GL}(4, \mathbb{R})/\mathrm{O}(4) \cong \mathbb{R}^{10}$



Joyce orbifolds

6-dim flat orbifolds (desingularized to Calabi–Yau mnflds)

- $\mathcal{O}_1 = T^6/\mathbb{Z}_4$, holonomy generated by $\mathrm{diag}(-1, i, i)$ of $\mathbb{C}^3 \cong \mathbb{R}^6$.
  $\mathcal{T}_{\text{flat}}(\mathcal{O}_1) \cong \mathrm{GL}(2, \mathbb{R})/\mathrm{O}(2) \times \mathrm{GL}(2, \mathbb{C})/\mathrm{U}(2) \cong \mathbb{R}^7$
- $\mathcal{O}_2 = T^6/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Holonomy generated by $\mathrm{diag}(1, -1, -1)$ and $\mathrm{diag}(-1, 1, -1)$ of $\mathbb{C}^3 \cong \mathbb{R}^6$
  $\mathcal{T}_{\text{flat}}(\mathcal{O}_2) \cong \mathrm{GL}(2, \mathbb{R})/\mathrm{O}(2) \times \mathrm{GL}(2, \mathbb{R})/\mathrm{O}(2) \times \mathrm{GL}(2, \mathbb{R})/\mathrm{O}(2) \cong \mathbb{R}^9$.

Boundary of the flat Teichmüller space – 1

Theorem

The Gromov–Hausdorff limit of a sequence (M^n, g_i) of compact flat manifolds is a compact flat orbifold.

Proof.

- Result true for flat tori (Mahler's compactness thm)
- can assume all holonomy groups equal: $H_i = H$
- (M, g_i) is the quotient of a flat torus $(\mathbb{T}^n, \widetilde{g}_i)$ by an isometric free action of H.
- By Fukaya–Yamaguchi: $\lim(M, g_i) = \lim(\mathbb{T}^n/H, g_i) = \lim(\mathbb{T}^n, g_i)/H$. $\square$

Boundary of the Teichmüller space – 2

Flat orbifolds admit flat desingularization
(through higher dimensional manifolds):

Theorem

Every compact flat orbifold is the limit of a sequence of compact flat manifolds.

Proof.

- $\mathcal{O}^n = \mathbb{R}^n/\Gamma$, $\quad \Gamma \subset \mathrm{Iso}(\mathbb{R}^n)$ crystallographic
- $H \subset \mathrm{O}(n)$ holonomy, $\mathcal{O} = \mathbb{T}^n/H$ (possibly nonfree action).
- Auslander–Kuranishi: $\exists$ Bieberbach N-group Γ' with $\mathrm{Hol}(\Gamma') = H$.
  Compact N-manifold $M_0 = \mathbb{R}^N/\Gamma' = \mathbb{T}^N/H$ (free action).
- Diagonal action of H on $\mathbb{T}^n \times \mathbb{T}^N$: this is free!
  Set $M = (\mathbb{T}^n \times \mathbb{T}^N)/H$, compact flat manifold.
- $\mathcal{O}$ is obtained by *collapsing* the factor $\mathbb{T}^N$ in M. $\square$

Boundary in the 3-D case

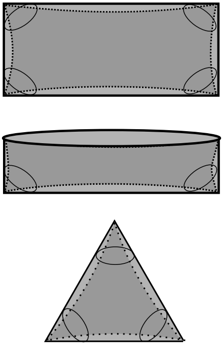
Theorem

The G-H limit of a sequence of compact flat 3 manifold belongs to one of these classes:

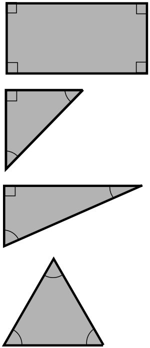
- *point*
- *closed interval, circle*
- *2-torus, Klein bottle, Möbius band, cylinder*
- *flat disk with two singularities:* $D^2(4;2)$, $D^2(3;3)$ *or* $D^2(2,2;)$
- *flat sphere with singularities:* $S^2(3,3,3;)$ *or* $S^2(2,2,2,2;)$ *(pillowcase)*
- *projective plane with 2 singularities* $\mathbb{R}P^2(2,2;)$.

A few more pictures

Collapse of flat 3-manifolds



Need higher dimensional collapse



Thanks for your attention!!

Paolo Piccione

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See you at ICM2018!