Bifurcation of periodic solutions to the singular Yamabe problem on spheres

International Conference of Mathematicians 2014
Seoul, South Korea

Paolo Piccione

Universidade de São Paulo, Brazil

August 16th, 2014

My co-authors

This is a joint work with:

Renato Bettiol
University of Notre Dame
USA

Bianca Santoro
City College of New York, CUNY
USA

The singular Yamabe problem (SYP)

The singular Yamabe problem (SYP)

Yamabe problem: (M, g) compact Riemannian manifold, find $\widetilde{g} \in [g]$ with $\mathrm{scal}_{\widetilde{g}} = \mathrm{const.}$

The singular Yamabe problem (SYP)

Yamabe problem: (M, g) compact Riemannian manifold, find $\widetilde{g} \in [g]$ with $\mathrm{scal}_{\widetilde{g}} = \mathrm{const.}$

Natural generalizations:

- $\boxed{\partial M \neq \emptyset}$ Find $\widetilde{g} \in [g]$ with constant scalar curvature + conditions on the second fundamental form of ∂M (tot. geodesic, minimal, tot. umbilical, CMC, etc.)

The singular Yamabe problem (SYP)

Yamabe problem: (M, g) compact Riemannian manifold, find $\widetilde{g} \in [g]$ with $\mathrm{scal}_{\widetilde{g}} = \mathrm{const.}$

Natural generalizations:

- $\boxed{\partial M \neq \emptyset}$ Find $\widetilde{g} \in [g]$ with constant scalar curvature + conditions on the second fundamental form of ∂M (tot. geodesic, minimal, tot. umbilical, CMC, etc.)
- $\boxed{M \text{ non-compact}}$ Find $\widetilde{g} \in [g]$ with constant scalar curvature + *conditions at infinity*

The singular Yamabe problem (SYP)

Yamabe problem: (M, g) compact Riemannian manifold, find $\widetilde{g} \in [g]$ with $\mathrm{scal}_{\widetilde{g}} = \mathrm{const.}$

Natural generalizations:

- $\boxed{\partial M \neq \emptyset}$ Find $\widetilde{g} \in [g]$ with constant scalar curvature + conditions on the second fundamental form of ∂M (tot. geodesic, minimal, tot. umbilical, CMC, etc.)
- $\boxed{M \text{ non-compact}}$ Find $\widetilde{g} \in [g]$ with constant scalar curvature + *conditions at infinity*

Warning. Need a control on the topology of M at infinity!

The singular Yamabe problem (SYP)

Yamabe problem: (M, g) compact Riemannian manifold, find $\tilde{g} \in [g]$ with $\mathrm{scal}_{\tilde{g}} = \mathrm{const.}$

Natural generalizations:

- $\boxed{\partial M \neq \emptyset}$ Find $\tilde{g} \in [g]$ with constant scalar curvature + conditions on the second fundamental form of ∂M (tot. geodesic, minimal, tot. umbilical, CMC, etc.)
- $\boxed{M \text{ non-compact}}$ Find $\tilde{g} \in [g]$ with constant scalar curvature + *conditions at infinity*

Warning. Need a control on the topology of M at infinity!

Formulation of the SYP

Given a compact (M, g), $\Lambda \subset M$ closed, find $\tilde{g}$ conformal to g in $M \setminus \Lambda$ satisfying:

- constant scalar curvature
- *complete* on $M \setminus \Lambda$

Case $\mathrm{scal} \leq 0$

- (1974) **Loewner–Niremberg**: solutions if $\dim_{\mathrm{H}}(\Lambda) \geq \frac{m-2}{2}$
- (1988) **Aviles–Mc Owen**: for general M and Λ submanifold, solutions exist iff $\dim(\Lambda) > \frac{m-2}{2}$.
- (1991) **Mazzeo**: uniqueness and regularity.

Case $\mathrm{scal} \le 0$

- (1974) **Loewner–Niremberg**: solutions if $\dim_{\mathrm{H}}(\Lambda) \ge \frac{m-2}{2}$
- (1988) **Aviles–Mc Owen**: for general M and Λ submanifold, solutions exist iff $\dim(\Lambda) > \frac{m-2}{2}$.
- (1991) **Mazzeo**: uniqueness and regularity.

Case $\mathrm{scal} > 0$ (more involved)

- (1988) **Schoen, Schoen–Yau**: $\mathbb{S}^m \setminus \Lambda$ *admits a complete metric with* $\mathrm{scal} \ge c_0 > 0$ *only if* $\dim(\Lambda) \le \frac{m-2}{2}$
  + Examples with $\Lambda = \{P_1, \ldots, P_N\}$, $N \ge 2$.
- (1996, 1999) **Mazzeo–Pacard**: more examples with Λ disjoint union of submanifolds of dimension $\le \frac{m-2}{2}$.

An interesting example

An interesting example

- $M = \mathbb{S}^m$, $\Lambda = \mathbb{S}^k$, $k < m$

An interesting example

- $M = \mathbb{S}^m$, $\Lambda = \mathbb{S}^k$, $k < m$
- $\boxed{\mathbb{S}^m \setminus \mathbb{S}^k \cong_{\mathrm{conf}} \mathbb{R}^m \setminus \mathbb{R}^k}$

An interesting example

- $M = \mathbb{S}^m$, $\Lambda = \mathbb{S}^k$, $k < m$
- $\boxed{\mathbb{S}^m \setminus \mathbb{S}^k \cong_{\mathrm{conf}} \mathbb{R}^m \setminus \mathbb{R}^k}$
    - *cylindrical coordinates* $\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2$ on $\mathbb{R}^{m-k}$
    - $\mathrm{d}\theta^2$ round metric of $\mathbb{S}^{m-k-1}$
    - Euclidean coordinates $\mathrm{d}y^2$ on $\mathbb{R}^k$

An interesting example

- $M = \mathbb{S}^m$, $\Lambda = \mathbb{S}^k$, $k < m$
- $\boxed{\mathbb{S}^m \setminus \mathbb{S}^k \cong_{\mathrm{conf}} \mathbb{R}^m \setminus \mathbb{R}^k}$
  - *cylindrical coordinates* $\mathrm{d}r^2 + r^2\mathrm{d}\theta^2$ on $\mathbb{R}^{m-k}$
  - $\mathrm{d}\theta^2$ round metric of $\mathbb{S}^{m-k-1}$
  - Euclidean coordinates $\mathrm{d}y^2$ on $\mathbb{R}^k$
- $g_{\mathrm{round}} \cong_{\mathrm{conf}} \mathrm{d}r^2 + \mathrm{d}y^2 + r^2\mathrm{d}\theta^2 \cong_{\mathrm{conf}} \dfrac{\mathrm{d}r^2 + \mathrm{d}y^2}{r^2} + \mathrm{d}\theta^2$

An interesting example

- $M = \mathbb{S}^m$, $\Lambda = \mathbb{S}^k$, $k < m$
- $\boxed{\mathbb{S}^m \setminus \mathbb{S}^k \cong_{\mathrm{conf}} \mathbb{R}^m \setminus \mathbb{R}^k}$
  - *cylindrical coordinates* $\mathrm{d}r^2 + r^2\mathrm{d}\theta^2$ on $\mathbb{R}^{m-k}$
  - $\mathrm{d}\theta^2$ round metric of $\mathbb{S}^{m-k-1}$
  - Euclidean coordinates $\mathrm{d}y^2$ on $\mathbb{R}^k$
- $g_{\mathrm{round}} \cong_{\mathrm{conf}} \mathrm{d}r^2 + \mathrm{d}y^2 + r^2\mathrm{d}\theta^2 \cong_{\mathrm{conf}} \dfrac{\mathrm{d}r^2 + \mathrm{d}y^2}{r^2} + \mathrm{d}\theta^2$

Conclusion: $\boxed{(\mathbb{S}^m \setminus \mathbb{S}^k, g_{\mathrm{round}}) \cong_{\mathrm{conf}} \mathbb{H}^{k+1} \times \mathbb{S}^{m-k-1}}$
constant scalar curvature + complete.

An interesting example

- $M = \mathbb{S}^m$, $\Lambda = \mathbb{S}^k$, $k < m$
- $\boxed{\mathbb{S}^m \setminus \mathbb{S}^k \cong_{\text{conf}} \mathbb{R}^m \setminus \mathbb{R}^k}$
  - *cylindrical coordinates* $\mathrm{d}r^2 + r^2\mathrm{d}\theta^2$ on $\mathbb{R}^{m-k}$
  - $\mathrm{d}\theta^2$ round metric of $\mathbb{S}^{m-k-1}$
  - Euclidean coordinates $\mathrm{d}y^2$ on $\mathbb{R}^k$
- $g_{\text{round}} \cong_{\text{conf}} \mathrm{d}r^2 + \mathrm{d}y^2 + r^2\mathrm{d}\theta^2 \cong_{\text{conf}} \dfrac{\mathrm{d}r^2 + \mathrm{d}y^2}{r^2} + \mathrm{d}\theta^2$

Conclusion: $\boxed{(\mathbb{S}^m \setminus \mathbb{S}^k, g_{\text{round}}) \cong_{\text{conf}} \mathbb{H}^{k+1} \times \mathbb{S}^{m-k-1}}$
constant scalar curvature + complete.

$$\mathrm{scal}_{m,k} = (m-k+1)(m-k-2) - k(k+1)$$

An interesting example

- $M = \mathbb{S}^m$, $\Lambda = \mathbb{S}^k$, $k < m$
- $\boxed{\mathbb{S}^m \setminus \mathbb{S}^k \cong_{\mathrm{conf}} \mathbb{R}^m \setminus \mathbb{R}^k}$
  - *cylindrical coordinates* $\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2$ on $\mathbb{R}^{m-k}$
  - $\mathrm{d}\theta^2$ round metric of $\mathbb{S}^{m-k-1}$
  - Euclidean coordinates $\mathrm{d}y^2$ on $\mathbb{R}^k$
- $g_{\mathrm{round}} \cong_{\mathrm{conf}} \mathrm{d}r^2 + \mathrm{d}y^2 + r^2 \mathrm{d}\theta^2 \cong_{\mathrm{conf}} \dfrac{\mathrm{d}r^2 + \mathrm{d}y^2}{r^2} + \mathrm{d}\theta^2$

Conclusion: $\boxed{(\mathbb{S}^m \setminus \mathbb{S}^k, g_{\mathrm{round}}) \cong_{\mathrm{conf}} \mathbb{H}^{k+1} \times \mathbb{S}^{m-k-1}}$
constant scalar curvature + complete.

$$\mathrm{scal}_{m,k} = (m-k+1)(m-k-2) - k(k+1) > 0 \quad \Longleftrightarrow \quad k < \frac{m-2}{2}$$

An interesting example

- $M = \mathbb{S}^m$, $\Lambda = \mathbb{S}^k$, $k < m$
- $\boxed{\mathbb{S}^m \setminus \mathbb{S}^k \cong_{\mathrm{conf}} \mathbb{R}^m \setminus \mathbb{R}^k}$
  - *cylindrical coordinates* $\mathrm{d}r^2 + r^2\mathrm{d}\theta^2$ on $\mathbb{R}^{m-k}$
  - $\mathrm{d}\theta^2$ round metric of $\mathbb{S}^{m-k-1}$
  - Euclidean coordinates $\mathrm{d}y^2$ on $\mathbb{R}^k$
- $g_{\mathrm{round}} \cong_{\mathrm{conf}} \mathrm{d}r^2 + \mathrm{d}y^2 + r^2\mathrm{d}\theta^2 \cong_{\mathrm{conf}} \dfrac{\mathrm{d}r^2 + \mathrm{d}y^2}{r^2} + \mathrm{d}\theta^2$

Conclusion: $\boxed{(\mathbb{S}^m \setminus \mathbb{S}^k, g_{\mathrm{round}}) \cong_{\mathrm{conf}} \mathbb{H}^{k+1} \times \mathbb{S}^{m-k-1}}$
constant scalar curvature + complete.

$$\mathrm{scal}_{m,k} = (m-k+1)(m-k-2) - k(k+1) > 0 \iff k < \frac{m-2}{2}$$

$$\boxed{\mathrm{scal}_{m,1} = m^2 - 5m + 4 = (m-1)(m-4) > 0 \iff m > 4}$$

$\boxed{\left(\mathbb{S}^m \setminus \mathbb{S}^1, g_{\mathrm{round}}\right) \;\cong_{\mathrm{conf}}\; \mathbb{H}^2 \times \mathbb{S}^{m-2}}$ (product metric)

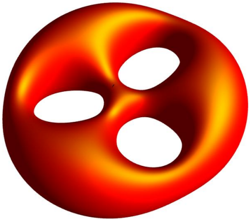
Periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$

$$\boxed{\left(\mathbb{S}^m \setminus \mathbb{S}^1, g_{\mathrm{round}}\right) \ \cong_{\mathrm{conf}} \ \mathbb{H}^2 \times \mathbb{S}^{m-2}} \text{ (product metric)}$$

- $\Gamma \subset \mathbb{H}^2$ *co-compact lattice*

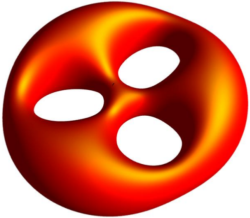
Periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$

$\boxed{\left(\mathbb{S}^m \setminus \mathbb{S}^1, g_{\mathrm{round}}\right) \;\cong_{\mathrm{conf}}\; \mathbb{H}^2 \times \mathbb{S}^{m-2}}$ (product metric)



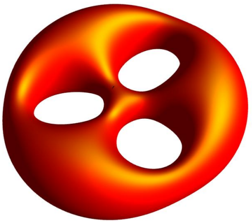
- $\Gamma \subset \mathbb{H}^2$ *co-compact lattice*
- $\Sigma = \mathbb{H}^2/\Gamma$ *hyperbolic surface*, $\mathrm{gen}(\Sigma) \geq 2$

$\boxed{\left(\mathbb{S}^m \setminus \mathbb{S}^1, g_{\mathrm{round}}\right) \ \cong_{\mathrm{conf}} \ \mathbb{H}^2 \times \mathbb{S}^{m-2}}$ (product metric)



- $\Gamma \subset \mathbb{H}^2$ *co-compact lattice*
- $\Sigma = \mathbb{H}^2/\Gamma$ *hyperbolic surface*, $\mathrm{gen}(\Sigma) \geq 2$
- $\pi : \mathbb{H}^2 \times \mathbb{S}^{m-2} \to \Sigma \times \mathbb{S}^{m-2}$ projection

$$\boxed{\left(\mathbb{S}^m \setminus \mathbb{S}^1, g_{\text{round}}\right) \cong_{\text{conf}} \mathbb{H}^2 \times \mathbb{S}^{m-2}}$$ (product metric)



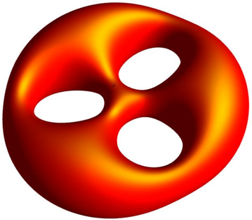
- $\Gamma \subset \mathbb{H}^2$ *co-compact lattice*
- $\Sigma = \mathbb{H}^2/\Gamma$ *hyperbolic surface*, $\mathrm{gen}(\Sigma) \geq 2$
- $\pi : \mathbb{H}^2 \times \mathbb{S}^{m-2} \to \Sigma \times \mathbb{S}^{m-2}$ projection

Definition

A solution $\widetilde{g}$ of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$ is periodic if $\widetilde{g} = \pi^*(g_0)$ for some g_0 metric with CSC on $\Sigma \times \mathbb{S}^{m-2}$.

Periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$

$\boxed{\left(\mathbb{S}^m \setminus \mathbb{S}^1, g_{\mathrm{round}}\right) \ \cong_{\mathrm{conf}} \ \mathbb{H}^2 \times \mathbb{S}^{m-2}}$ (product metric)



- $\Gamma \subset \mathbb{H}^2$ *co-compact lattice*
- $\Sigma = \mathbb{H}^2/\Gamma$ *hyperbolic surface*, $\mathrm{gen}(\Sigma) \geq 2$
- $\pi : \mathbb{H}^2 \times \mathbb{S}^{m-2} \to \Sigma \times \mathbb{S}^{m-2}$ projection

Definition

A solution $\widetilde{g}$ of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$ is periodic if $\widetilde{g} = \pi^*(g_0)$ for some g_0 metric with CSC on $\Sigma \times \mathbb{S}^{m-2}$. Automatically complete!

Periodic solutions are non-trivial!

- For any hyperbolic metric $h_{\mathrm{hyperbolic}}$ on Σ, if:

$$\boxed{g_0 = h_{\mathrm{hyperbolic}} \times g_{\mathrm{round}}^{(m-2)}}$$

then $\widetilde{g} = \pi^*(g_0)$ is the product metric of $\mathbb{H}^2 \times \mathbb{S}^{m-2}$.

Periodic solutions are non-trivial!

- For any hyperbolic metric $h_{\text{hyperbolic}}$ on Σ, if:

$$\boxed{g_0 = h_{\text{hyperbolic}} \times g_{\text{round}}^{(m-2)}}$$

then $\widetilde{g} = \pi^*(g_0)$ is the product metric of $\mathbb{H}^2 \times \mathbb{S}^{m-2}$.

- On the other hand, if g_0 is conformal, but non-homothetic to a product $h_{\text{hyperbolic}} \times g_{\text{round}}^{(m-2)}$, then $\widetilde{g} = \pi^*(g_0)$ is a non-trivial solution of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$.

Periodic solutions are non-trivial!

- For any hyperbolic metric $h_{\mathrm{hyperbolic}}$ on Σ, if:

$$\boxed{g_0 = h_{\mathrm{hyperbolic}} \times g^{(m-2)}_{\mathrm{round}}}$$

then $\widetilde{g} = \pi^*(g_0)$ is the product metric of $\mathbb{H}^2 \times \mathbb{S}^{m-2}$.

- On the other hand, if g_0 is conformal, but non-homothetic to a product $h_{\mathrm{hyperbolic}} \times g^{(m-2)}_{\mathrm{round}}$, then $\widetilde{g} = \pi^*(g_0)$ is a non-trivial solution of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$.

Proposition

Non-trivial periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$ correspond to *fixed volume* metrics with CSC on $\Sigma \times \mathbb{S}^{m-2}$ conformal to products $h_{\mathrm{hyperbolic}} \times g^{(m-2)}_{\mathrm{round}}$.

Main result

Theorem

There exist uncountably many non-trivial periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$ when $m \geq 5$.

Main result

Theorem

There exist uncountably many non-trivial periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$ when $m \geq 5$. Their scalar curvature is arbitrarily close to $(m-1)(m-4) > 0$.

Main result

Theorem

There exist uncountably many non-trivial periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$ when $m \geq 5$. Their scalar curvature is arbitrarily close to $(m-1)(m-4) > 0$.

Proof.

Main result

Theorem

There exist uncountably many non-trivial periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$ when $m \geq 5$. Their scalar curvature is arbitrarily close to $(m-1)(m-4) > 0$.

Proof.

Bifurcation theory.

Main result

Theorem

There exist uncountably many non-trivial periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$ when $m \geq 5$. Their scalar curvature is arbitrarily close to $(m-1)(m-4) > 0$.

Proof.

Bifurcation theory.

- Fix Σ with $\mathrm{gen}(\Sigma) \geq 2$

Main result

Theorem

There exist uncountably many non-trivial periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$ when $m \geq 5$. Their scalar curvature is arbitrarily close to $(m-1)(m-4) > 0$.

Proof.

Bifurcation theory.

- Fix Σ with $\mathrm{gen}(\Sigma) \geq 2$
- Dynamics: paths $t \mapsto h_t$ of hyperbolic metrics on Σ

Main result

Theorem

There exist uncountably many non-trivial periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$ when $m \geq 5$. Their scalar curvature is arbitrarily close to $(m-1)(m-4) > 0$.

Proof.

Bifurcation theory.

- Fix Σ with $\mathrm{gen}(\Sigma) \geq 2$
- Dynamics: paths $t \mapsto h_t$ of hyperbolic metrics on Σ
- Variational bifurcation for the metrics $h_t \times g_{\mathrm{round}}^{(m-2)}$ using the *Hilbert–Einstein* functional

Main result

Theorem

There exist uncountably many non-trivial periodic solutions of the SYP on $\mathbb{S}^m \setminus \mathbb{S}^1$ when $m \geq 5$. Their scalar curvature is arbitrarily close to $(m-1)(m-4) > 0$.

Proof.

Bifurcation theory.

- Fix Σ with $\mathrm{gen}(\Sigma) \geq 2$
- Dynamics: paths $t \mapsto h_t$ of hyperbolic metrics on Σ
- Variational bifurcation for the metrics $h_t \times g^{(m-2)}_{\mathrm{round}}$ using the *Hilbert–Einstein* functional
- Use spectral theory of hyperbolic metrics to determine uncountably many paths where bifurcation occurs:
  - jump of Morse index;
  - nondegeneracy at endpoints. $\square$

On the jump of Morse index condition

On the jump of Morse index condition

- $\mathcal{H}(\Sigma)$ space of hyperbolic metrics on Σ

On the jump of Morse index condition

- $\mathcal{H}(\Sigma)$ space of hyperbolic metrics on Σ
- Teichmüller space: $\mathcal{T}(\Sigma) = \mathcal{H}(\Sigma)/\mathrm{Diff}^+(\Sigma)$.

On the jump of Morse index condition

- $\mathcal{H}(\Sigma)$ space of hyperbolic metrics on Σ
- Teichmüller space: $\mathcal{T}(\Sigma) = \mathcal{H}(\Sigma)/\mathrm{Diff}^+(\Sigma)$.
  Real-analytic manifold of dimension $6\,\mathrm{gen}(\Sigma) - 6$.

On the jump of Morse index condition

- $\mathcal{H}(\Sigma)$ space of hyperbolic metrics on Σ
- Teichmüller space: $\mathcal{T}(\Sigma) = \mathcal{H}(\Sigma)/\mathrm{Diff}^+(\Sigma)$.
  Real-analytic manifold of dimension $6\,\mathrm{gen}(\Sigma) - 6$.
- Spectrum of the Jacobi operator related to the spectrum of the Laplacian of $h \in \mathcal{T}(\Sigma)$:

$$\boxed{\text{Jump of Morse index}} \iff \boxed{\text{Spectral flow of paths in } \mathcal{T}(\Sigma)}$$

On the jump of Morse index condition

- $\mathcal{H}(\Sigma)$ space of hyperbolic metrics on Σ
- Teichmüller space: $\mathcal{T}(\Sigma) = \mathcal{H}(\Sigma)/\mathrm{Diff}^+(\Sigma)$.
  Real-analytic manifold of dimension $6\,\mathrm{gen}(\Sigma) - 6$.
- Spectrum of the Jacobi operator related to the spectrum of the Laplacian of $h \in \mathcal{T}(\Sigma)$:

  $\boxed{\text{Jump of Morse index}} \iff \boxed{\text{Spectral flow of paths in } \mathcal{T}(\Sigma)}$

- **Small eigenvalues:** find hyperbolic metrics with arbitrarily many eigenvalues in $\left[0, \frac{1}{4} + \varepsilon\right]$ (Buser)

On the jump of Morse index condition

- $\mathcal{H}(\Sigma)$ space of hyperbolic metrics on Σ
- Teichmüller space: $\mathcal{T}(\Sigma) = \mathcal{H}(\Sigma)/\mathrm{Diff}^+(\Sigma)$.
  Real-analytic manifold of dimension $6\,\mathrm{gen}(\Sigma) - 6$.
- Spectrum of the Jacobi operator related to the spectrum of the Laplacian of $h \in \mathcal{T}(\Sigma)$:

  $\boxed{\text{Jump of Morse index}} \Longleftrightarrow \boxed{\text{Spectral flow of paths in } \mathcal{T}(\Sigma)}$

- **Small eigenvalues:** find hyperbolic metrics with arbitrarily many eigenvalues in $\left[0, \frac{1}{4} + \varepsilon\right]$ (Buser)
- **Large eigenvalues.** *Trivial*: $\lim_{k\to\infty} \lambda_k(h) \to +\infty$ for all h

On the nondegeneracy condition

On the nondegeneracy condition

Central question: For $\lambda = m - 4 \in \{1, 2, \dots\}$, is the set:

$$\boxed{\mathcal{T}_\lambda(\Sigma) := \big\{h \in \mathcal{T}(\Sigma) : \lambda \notin \mathrm{spec}(\Delta_h)\big\}}$$

dense in $\mathcal{T}(\Sigma)$?

On the nondegeneracy condition

Central question: For $\lambda = m - 4 \in \{1, 2, \ldots\}$, is the set:

$$\boxed{\mathcal{T}_\lambda(\Sigma) := \{h \in \mathcal{T}(\Sigma) : \lambda \notin \mathrm{spec}(\Delta_h)\}}$$

dense in $\mathcal{T}(\Sigma)$?

Proposition

If $\mathcal{T}_\lambda(\Sigma) \neq \emptyset$, then $\mathcal{T}_\lambda(\Sigma)$ is dense in $\mathcal{T}(\Sigma)$.

$\mathcal{T}_\lambda(\Sigma) \neq \emptyset$ (**Middle C embarrassment**):

On the nondegeneracy condition

Central question: For $\lambda = m - 4 \in \{1, 2, \dots\}$, is the set:

$$\boxed{\mathcal{T}_\lambda(\Sigma) := \{h \in \mathcal{T}(\Sigma) : \lambda \notin \operatorname{spec}(\Delta_h)\}}$$

dense in $\mathcal{T}(\Sigma)$?

Proposition

If $\mathcal{T}_\lambda(\Sigma) \neq \emptyset$, then $\mathcal{T}_\lambda(\Sigma)$ is dense in $\mathcal{T}(\Sigma)$.

$\mathcal{T}_\lambda(\Sigma) \neq \emptyset$ (**Middle C embarrassment**):

- For $\lambda = 1, 2, 3$ (i.e., $m = 5, 6, 7$): (Σ, h) Bolza surface, $\boxed{\lambda_1 \cong 3.8}$

On the nondegeneracy condition

Central question: For $\lambda = m - 4 \in \{1, 2, \ldots\}$, is the set:

$$\boxed{\mathcal{T}_\lambda(\Sigma) := \{h \in \mathcal{T}(\Sigma) : \lambda \notin \mathrm{spec}(\Delta_h)\}}$$

dense in $\mathcal{T}(\Sigma)$?

Proposition

If $\mathcal{T}_\lambda(\Sigma) \neq \emptyset$, then $\mathcal{T}_\lambda(\Sigma)$ is dense in $\mathcal{T}(\Sigma)$.

$\mathcal{T}_\lambda(\Sigma) \neq \emptyset$ (**Middle C embarrassment**):

- For $\lambda = 1, 2, 3$ (i.e., $m = 5, 6, 7$): (Σ, h) Bolza surface, $\boxed{\lambda_1 \cong 3.8}$
- for $\lambda > 3$, *perturbative results* by Wolpert (Ann. of Math. 1994)

On the nondegeneracy condition

Central question: For $\lambda = m - 4 \in \{1, 2, \dots\}$, is the set:

$$\boxed{\mathcal{T}_\lambda(\Sigma) := \{h \in \mathcal{T}(\Sigma) : \lambda \notin \mathrm{spec}(\Delta_h)\}}$$

dense in $\mathcal{T}(\Sigma)$?

Proposition

If $\mathcal{T}_\lambda(\Sigma) \neq \emptyset$, then $\mathcal{T}_\lambda(\Sigma)$ is dense in $\mathcal{T}(\Sigma)$.

$\mathcal{T}_\lambda(\Sigma) \neq \emptyset$ (**Middle C embarrassment**):

- For $\lambda = 1, 2, 3$ (i.e., $m = 5, 6, 7$):
  (Σ, h) Bolza surface, $\boxed{\lambda_1 \cong 3.8}$
- for $\lambda > 3$, *perturbative results* by Wolpert (Ann. of Math. 1994)

THANK YOU!