MAT-3210 – CÁLCULO DIFERENCIAL E INTEGRAL II

Soluções 1.

Soluções de alguns exercícios.

$\boxed{\text{A.6}}$ $\displaystyle\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{4(x^2+x+\frac{1}{4})+1} = \int \frac{dx}{[2(x+\frac{1}{2})]^2+1},$ fazendo $u = 2(x+\dfrac{1}{2})$, $du = 2xdx$

$\displaystyle = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan(2x+1) + C$

$\boxed{\text{B.3}}$ $\displaystyle\int \frac{e^x-1}{e^x+1}\,dx = \int \frac{e^x}{e^x+1}\,dx - \int \frac{dx}{e^x+1} = \int \frac{e^x}{e^x+1}\,dx - \int \frac{e^{-x}}{e^{-x}(e^x+1)}\,dx$

$\displaystyle = \int \frac{e^x}{e^x+1}\,dx - \int \frac{e^{-x}}{e^{-x}+1}\,dx,$ fazendo $u = e^x + 1$ na primeira integral e $v = e^{-x} + 1$ na segunda

$du = e^x dx$; $dv = -e^{-x}dx$, temos

$\displaystyle = \int \frac{du}{u} + \int \frac{dv}{v} = \ln u + \ln v + C = \ln(e^x+1) + \ln(e^{-x}+1) + C$

$\boxed{\text{B.4}}$ $\displaystyle\int \frac{x^2+2x+2}{x+1}\,dx = \int \frac{(x^2+2x+1)+1}{x+1}\,dx = \int \frac{(x+1)^2+1}{x+1}\,dx = \int (x+1)\,dx + \int \frac{dx}{x+1}$

$= \int (x+1)\,dx + \int \dfrac{du}{u} = \dfrac{x^2}{2} + x + \ln(x+1) + C$, onde $u = x+1$, $du = dx$

$\boxed{\text{B.6}}$ $\displaystyle\int \frac{x^2-5x+6}{x^2+4}\,dx = \int \frac{(x^2+4)-5x+2}{x^2+4}\,dx = \int dx - 5\int \frac{x}{x^2+4}\,dx + 2\int \frac{dx}{x^2+4}$

$\displaystyle\int dx - 5\int \frac{x}{x^2+4}\,dx + \frac{1}{2}\int \frac{dx}{(\frac{x}{2})^2+1}$, fazendo $u = x^2+4$ na segunda integral e $v = \dfrac{x}{2}$ na terceira, $du = 2xdx$,

$dv = \dfrac{dx}{2}$, temos

$$\int dx - \frac{5}{2}\int \frac{du}{u} + \int \frac{dv}{v^2+1} = x - \frac{5}{2}\ln u + \arctan v + C = x - \ln(x^2+4) + \arctan(\frac{x}{2}) + C$$

$\boxed{\text{B.7}}$ $\int 4^{2-3x}\,dx = \int e^{(2-3x)\ln 4} = e^{2\ln 4}\int e^{(-3\ln 4)x}$, se $u = (-3\ln 4)x$, temos $du = -3\ln 4\,dx$

$$= \frac{-1}{3\ln 4}\int e^u du = -\frac{e^u}{3\ln 4} + C = -\frac{e^{(-3\ln 4)x}}{3\ln 4} + C$$

$\boxed{\text{C.3}}$ $\int e^{ax}\cos bx\,dx$

$dv = e^{ax}\,dx \Rightarrow v = \dfrac{e^{ax}}{a}$

$u = \cos bx \Rightarrow du = -b\sin bx.$

$\int e^{ax}\cos bx\,dx = \int udv = uv - \int vdu = \dfrac{e^{ax}\cos bx}{a} + \dfrac{b}{a}\int e^{ax}\sin bx\,dx\,(*)$

$dv = e^{ax}\,dx \Rightarrow v = \dfrac{e^{ax}}{a}$,

$u = \sin bx \Rightarrow du = b\cos bx$, em $\int e^{ax}\sin bx\,dx$

$(*) \;=\; \dfrac{e^{ax}\cos bx}{a} + \dfrac{be^{ax}\sin bx}{a^2} - \dfrac{b^2}{a^2}\int e^{ax}\cos bx\,dx$

Assim

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\cos bx}{a} + \frac{be^{ax}\sin bx}{a^2} - \frac{b^2}{a^2}\int e^{ax}\cos bx\,dx$$

$$\int e^{ax}\cos bx\,dx = \frac{\dfrac{e^{ax}\cos bx}{a} + \dfrac{be^{ax}\sin bx}{a^2}}{1+\dfrac{b^2}{a^2}} + C$$

$\boxed{\text{C.9}}$ $\int \sec^3 x\,dx = \int \sec^2 x \sec\ dx$

$dv = \sec^2 x \Rightarrow v = \tan x$,

$u = \sec x \Rightarrow du = \sec x \tan x$

$\int \sec^3 x\,dx = \int \sec^2 x \sec\ dx = \int udv = uv - \int vdu = \sec x\tan x - \int \tan x\,(\sec x\tan x)\,dx$

$= \sec x\tan x - \int \sec x\tan^2 x\,dx = \sec x\tan x - \int \sec x\,(\sec^2 x - 1)\,dx = \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$= \sec x\tan x - \int \sec^3 x\,dx + \ln(\tan x + \sec x)$

assim,

$\int \sec^3 x\,dx = \sec x\tan x - \int \sec^3 x\,dx + \ln(\tan x + \sec x)$, portanto

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln(\tan x + \sec x) + C$$

$\boxed{\text{D.3}}$ $\displaystyle\int \frac{3x^2+2x-2}{x^3-1}\,dx = \int \frac{3x^2+2x-2}{(x-1)(x^2+x+1)}\,dx = \int \frac{Adx}{x-1} + \int \frac{Bx+C}{x^2+x+1}\,dx$

$$= \int \frac{(A+B)x^2 + (A+C-B)x + (A-C)}{(x-1)(x^2+x+1)}\,dx$$

portanto

$A + B = 3$

$A + C - B = 2$

$A - C = -2$

de onde, $A = 1$, $B = 2$, $C = 3$

$$\int \frac{3x^2+2x-2}{x^3-1}\,dx = \int \frac{dx}{x-1} + \int \frac{2x+3}{x^2+x+1}\,dx = \int \frac{dx}{x-1} + 2\int \frac{x}{x^2+x+1}\,dx + 3\int \frac{dx}{x^2+x+1} (*)$$

Agora:

$$(\star) \int \frac{dx}{x-1} = \ln(x-1)$$

$$(\star\star) \int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \int \frac{du}{u^2+\frac{3}{4}} = \frac{4}{3}\int \frac{du}{(\frac{2u}{\sqrt{3}})^2+1} = \frac{2\sqrt{3}}{3}\int \frac{dw}{w^2+1}, \text{ onde } w = \frac{2u}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{3}\arctan w + C = \frac{2\sqrt{3}}{3}\arctan \frac{\sqrt{3}}{3}(2x+1) + C.$$

$(\star\star\star) \int \frac{x}{x^2+x+1}$ por integração por partes temos

$u = x \Rightarrow du = dx$

$$dv = \frac{1}{x^2+x+1} \Rightarrow v = \frac{2\sqrt{3}}{3}\arctan \frac{\sqrt{3}}{3}(2x+1)$$

$$\int \frac{x}{x^2+x+1}\,dx = \frac{2\sqrt{3}x}{3} \arctan \frac{\sqrt{3}}{3}(2x+1) - \frac{\sqrt{3}}{3} \int \arctan \frac{2\sqrt{3}}{3}(2x+1)$$

$$= \frac{2\sqrt{3}x}{3} \arctan \frac{\sqrt{3}}{3}(2x+1) - \int \arctan u\,du, \text{ onde } u = \frac{\sqrt{3}}{3}(2x+1)$$

$$= \frac{2\sqrt{3}x}{3} \arctan \frac{\sqrt{3}}{3}(2x+1) - (\frac{2\sqrt{3}x}{3} + \frac{\sqrt{3}}{3}) \arctan (\frac{2\sqrt{3}x}{3} + \frac{\sqrt{3}}{3}) + \frac{1}{2} \ln(1 + \frac{(2x+1)^2}{3}) + C$$

substituindo o valor de $(\star)$, $(\star\star)$ e $(\star\star\star)$ em $(*)$ temos o resultado final.

$\boxed{\text{E.5}}$ $\int \sqrt{a-x^2}\,dx$

se $x = \sqrt{a}\sin y \Rightarrow dx = \sqrt{a}\cos y\,dy$ temos

$$= \int \sqrt{a - (\sqrt{a}\sin y)^2}\,\sqrt{a}\cos y\,dy = \sqrt{a}\int \sqrt{a(1-\sin^2 y)}\cos y\,dy = a\int \sqrt{\cos^2 y}\,\cos y dy = a\int \cos^2 y\,dy$$

$$= \frac{a}{2}\int(\cos 2y + 1)\,dy = a\sin 2y + \frac{ay}{2} + C = 2a\sin y\cos y + \frac{ay}{2} + C$$

$$\sin y = \frac{x}{\sqrt{a}} \Rightarrow \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{x^2}{a}} = \frac{\sqrt{a-x^2}}{\sqrt{a}}$$

assim $2a\sin y\cos y + \dfrac{ay}{2} + C = 2x\sqrt{a-x^2} + \dfrac{a}{2}\arcsin(\dfrac{x}{\sqrt{a}}) + C$