Details on proof of lemma 10 in chp. 4 of O'Neill's
Semi-Riemannian Geometry with Applications to Relativity

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M is a paracompact smooth manifold, hence metrizable.

Let d be a metric on M which induces the same topology M has as a smooth manifold to the metric space (M, d).

Let $\mathcal{D}$ be a locally finite open refinement of the open covering $\mathcal{C}^*$. We may suppose that given $D \in \mathcal{D}$, $\nexists \tilde{D} \in \mathcal{D}$ such that $\tilde{D} \subset D$. (one may obtain another locally finite open covering $\tilde{\mathcal{D}}$ by taking out the sets $D \in \mathcal{D}$ for which $\exists \tilde{D} \in \mathcal{D}$ such that $D \subset \tilde{D}$)

Let us construct an open covering $\mathcal{B}$ of M such that given $A, B \in \mathcal{B}$, if $A \cap B \neq \emptyset$ then there exists C in $\mathcal{C}^*$ such that $A \cup B \subset C$.

Given $p \in M$:

Let U_p be an open neighborhood of p which intersects a finite number of sets in $\mathcal{D}$, which we shall denote $D_{p,1}, ..., D_{p,n}$. $\mathcal{D}$ is a open covering of M, so we may suppose that there exists $D \in \mathcal{D}$ such that $U_p \subset D$. (indeed, as $\mathcal{D}$ is an open covering of M, there exists $D \in \mathcal{D}$ such that $p \in D$ so $\tilde{U_p} = U_p \cap D$ is an open neighborhood of p which intersects a finite number of elements in $\mathcal{D}$)

Fix $D^p \in \mathcal{D}$ such that $U_p \subset D^p$.

By renumbering the sets in $\mathcal{D}$ which intersect U_p, we may suppose $D_{p,1}, ..., D_{p,k}$ are all the sets $D_{p,i}$ such that $p \notin D_{p,i}$, where $0 \leq k \leq n$.

If $k = 0$, let $r_p > 0$ and $V_p = U_p \cap B_{r_p}(p) \cap D_{p,1} \cap ... \cap D_{p,n}$.

Otherwise, note that given $j = 1, ..., k$; $d(p, D_{p,j} - D^p) > 0$. (indeed, D^p is an open neighborhood of p and $p \notin D_{p,j}$ for $j = 1, ..., k$)

Let $r_p > 0$ be such that $r_p \leq \frac{1}{3} \min\{d(p, D_{p,j} - D^p) : j = 1, ..., k\}$.

Let $V_p = U_p \cap B_{r_p}(p) \cap D_{p,k+1} \cap ... \cap D_{p,n}$. (if $k = n$, let $V_p = U_p \cap B_{r_p}(p)$)

Finally, let $\mathcal{B} = \{V_p : p \in M\}$.

Let us prove that $\mathcal{B}$ has the desired properties.

Given $p, q \in M$, there are two possible cases:

1. $p \in D^q$ (or $q \in D^p$):

   In this case, $V_q \subset D^p$ ($V_p \subset D^q$) because D^p (D^q) is one of the $D_{q,i}$ ($D_{p,i}$) such that $q \in D_{q,i}$ ($p \in D_{p,i}$).

   Therefore, $V_p \cup V_q \subset D^p$ (D^q) and we have the desired property because $\mathcal{D}$ is an open refinement of $\mathcal{C}^*$.

2. $p \notin D^q$ and $q \notin D^p$:

   We shall show that $V_p \cap V_q = \emptyset$. Suppose otherwise.

   $V_p \subset B_{r_p}(p)$, $V_q \subset B_{r_q}(q)$ and $V_p \cap V_q \neq \emptyset$, so there exists $x \in B_{r_p}(p) \cap B_{r_q}(q)$.

   On one hand, $d(p,q) \leq d(p,x) + d(x,q) \leq r_p + r_q$.

   On the other hand, $V_p \cap V_q \neq \emptyset$ so D^p is one of the $D_{q,i}$ which intersects V_q but $q \notin D^p$ and conversely for D^q with respect to p. Therefore, $d(p,q) \geq d(p, D^q - D^p) \geq 3r_p$ and $d(p,q) \geq d(q, D^p - D^q) \geq 3r_q$ so $d(p,q) \geq \frac{3}{2}(r_p + r_q)$.

   Contradiction.