Bifurcation and symmetry breaking in geometric variational problems

Joint work with M. Koiso and B. Palmer

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Generalities on variational bifurcation

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Bifurcation at $\lambda_0 \in\,]a, b[$ if $\exists\, \lambda_n \to \lambda_0$ and $x_n \to x_{\lambda_0}$ as $n \to \infty$, with:

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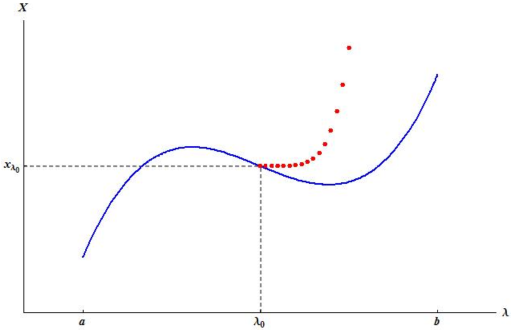

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Bifurcation occurs at *degenerate* critical points with *jumps of* the Morse index. In the equivariant case, bifurcation occurs at degenerate critical orbits where jumps of the *critical groups*.

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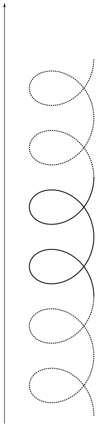
Variational principle

x has *constant* mean curvature (CMC) iff x is a stationary point for the *area functional* restricted to embeddings of fixed *volume*.

The nodary



The nodary and the symmetry axis.

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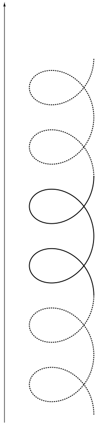

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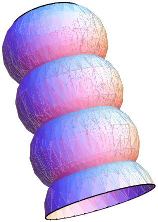
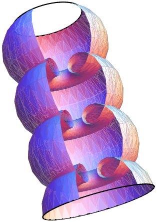
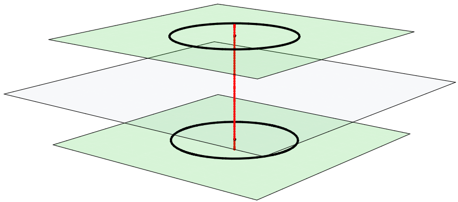




A portion of nodoid, with boundary on parallel planes orthogonal to the axis.

Fixed boundary problem



Circles for the CMC fixed boundary problem, lying on the planes Π_0 and Π_1. In the middle, the symmetry plane Π.

The 1-parameter family of nodoids

Σ_{a,H,t_0} surface of revolution around the x_3-axis with generatrix the *nodary*:

$$x_1(t) = \frac{\cos t + \sqrt{\cos^2 t + a}}{2|H|}, \qquad \boxed{t \in [-t_0, t_0]}$$

$$x_3(t) = \frac{1}{2|H|} \int_0^t \frac{\cos \tau + \sqrt{\cos^2 \tau + a}}{\sqrt{\cos^2 \tau + a}} \cos \tau \, \mathrm{d}\tau$$

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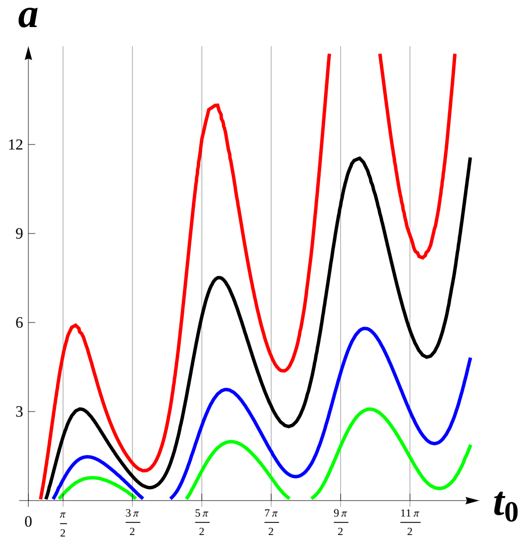
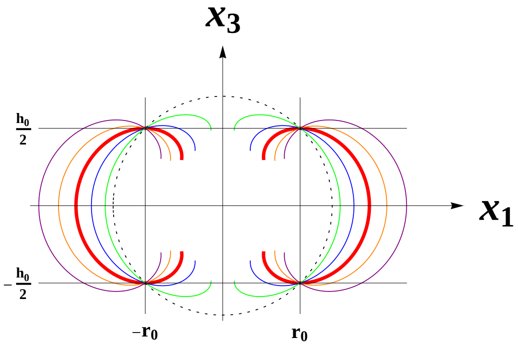
proposition

There exist *real-analytic* functions $a = a(t_0)$ and $H = H(t_0)$ such that $\Sigma_{t_0} = \Sigma_{a(t_0),H(t_0),t_0}$ satisfies the boundary condition.

$a = a(t_0)$



Nodaries through two circles



Nodary curves that generate nodoids which pass through 2 circles. The bifurcation point is in the middle (thicker/red), it has horizontal tangent at the point of intersection with the circles. The inner circle is a limit of the family when $a \to 0$.

The Jacobi operator

$$Jf = -\Delta f - (k_1^2 + k_2^2)f$$

Eigenvalues $\lambda_1 < \lambda_2 < \ldots \to +\infty$

Courant's nodal domain theorem

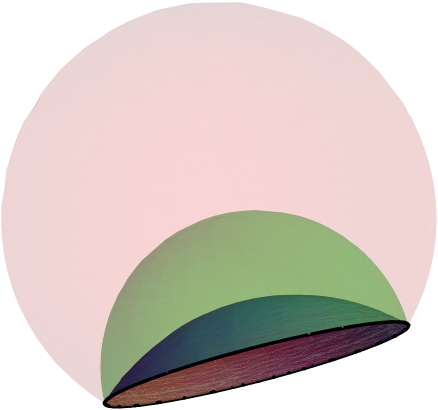
$Jf = \lambda_k f \quad \Longrightarrow \quad f$ has at least k *nodal domains*

Separation of variables: $f = T(\theta) \cdot S(s)$

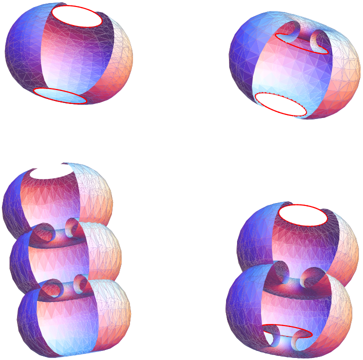
$$T'' + \kappa T = 0, \quad T(0) = T(2\pi), \quad T'(0) = T'(2\pi),$$

$$-(xS')' + \left(\frac{\kappa}{x} - x(k_1^2 + k_2^2)\right) S = \lambda x S, \quad S\big(-\tfrac{L}{2}\big) = S\big(\tfrac{L}{2}\big) = 0.$$

Spherical caps with the same boundary

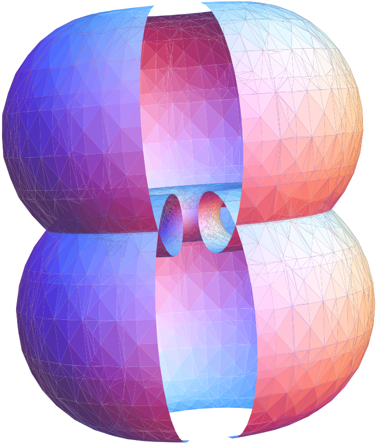


Degenerate nodoids

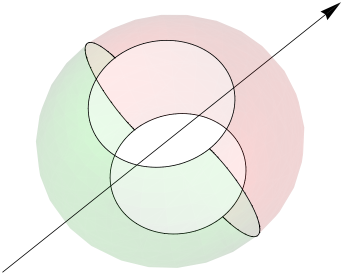


Degenerate nodoids are tangent to the planes containing their boundary. On the left, nodoids from the family Σ, on the right nodoids that are not symmetric with respect to the reflection around the plane Π.

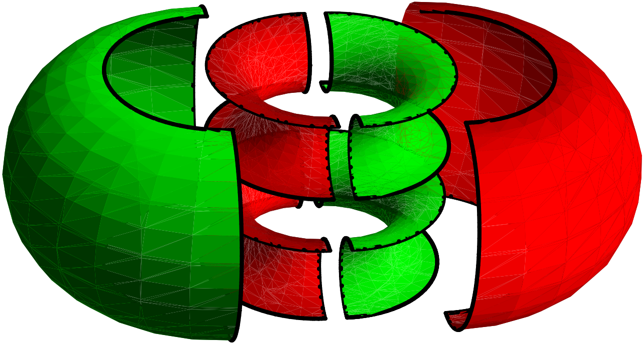
Degenerate nodoid with one bulge



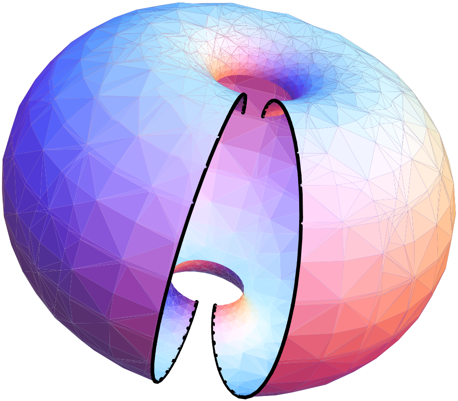
Two nodal domains



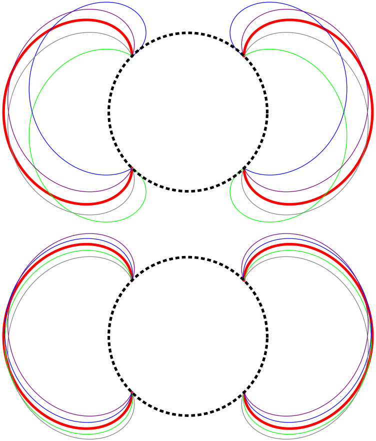
Six nodal domains



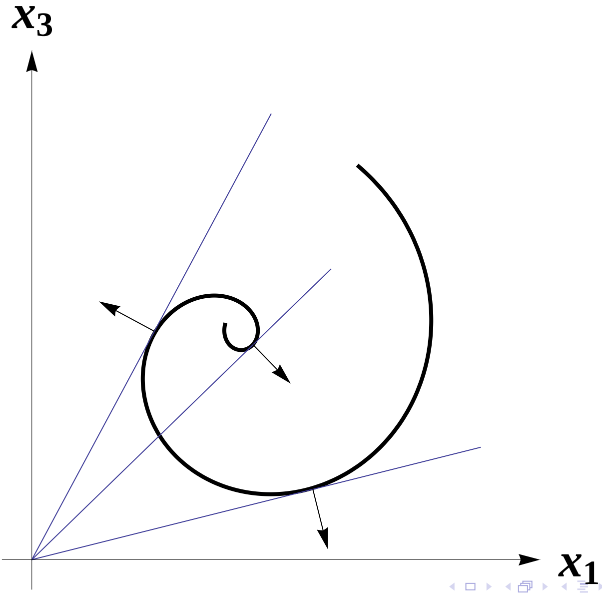
The degenerate nodoid Σ_π



Bifurcating branch of nodoids at the instant $t_0 = \pi$



Position vector



A picture of Miyuki and Bennett

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