On a Gromoll–Meyer type theorem in globally hyperbolic stationary Lorentzian manifolds

Joint work with L. Biliotti and F. Mercuri

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Santiago de Compostela, February 2007

Outline.

1. The celebrated result of Gromoll and Meyer
2. Some literature
3. On the Lorentzian result
4. Variational framework
5. Equivariant Morse theory

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- Bumpy metrics are generic (Abraham 1970, B. White Indiana J. Math. 1991)

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- McCleary & Ziller (Amer. J. Math., 1987, 1991) $\sup_k \beta_k(\Lambda M, \mathbb{Z}_2) = +\infty$ if M is homotopically equivalent to a compact simply connected *homogeneous space* not diffeomorphic to a symmetric space of rank 1.

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- Matthias (1980): closed geodesics in *Finsler* manifolds
- **Guruprasad, Haefliger (Topology 2006): closed geodesics in *orbifolds***

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- **Antonacci–Sampalmieri (Proc. Roy. Soc. Edinburgh, 1998) One closed geodesic in compact manifolds of *splitting type*.**

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Assumptions:

- M simply connected
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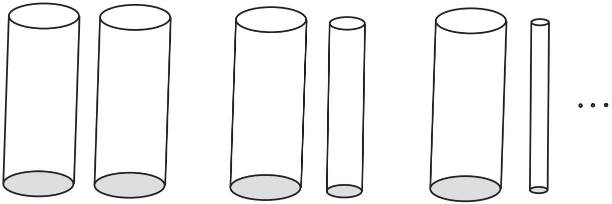

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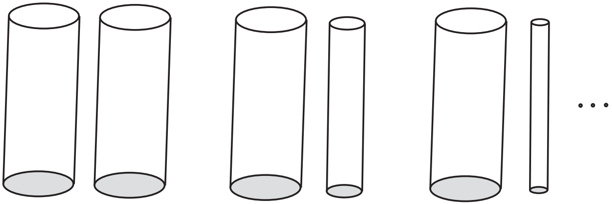

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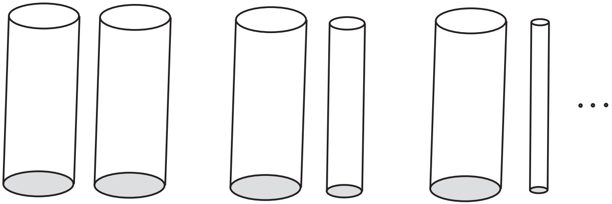

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Outline

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$\mathcal{N}$ is a smooth Hilbert submanifold of ΛM. If Y is *complete* then the inclusion $\mathcal{N} \hookrightarrow \Lambda M$ is a homotopy equivalence.

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$\exists\, \alpha \in \mathbb{R}^+,\ \beta \in \mathbb{R}$ such that, given any closed geodesic γ, either $\mu(\gamma^N)$ is *bounded*, or for s large enough:

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Iteration formulas for the Morse index

Problem. Estimate $\mu(\gamma^N)$ in terms of $\mu(\gamma)$.

- Since $\mu(\gamma^N) = \mathrm{i}_{\mathrm{Maslov}}(\gamma^N) +$ bounded terms, it suffices to estimate $\mathrm{i}_{\mathrm{Maslov}}(\gamma^N)$
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Assume there is only a finite number of distinct prime closed geodesics in M. Then, there exists a finite number of closed geodesics (not necessarily geometrically distinct) $\gamma_1, \ldots, \gamma_s$ in M such that:

- every closed geodesic γ is the iterate of some γ_i
- $\mathrm{nul}(\gamma) = \mathrm{nul}(\gamma_i)$.

Proof. Purely arithmetical.

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(work in progress with M. A. Javaloyes, L. L. de Lima)

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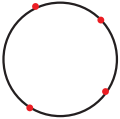

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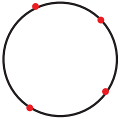

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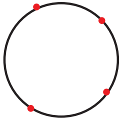

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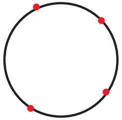

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Example of applications. (Ballmann, Thorbergsson, Ziller) *If* $\pi_1(M)$ *has a non trivial element of finite order a such that every closed geodesic freely homotopic to some power* a^q *is hyperbolic, then there are infinitely many distinct closed geodesics.*

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Outline

Homological invariants at isolated critical points

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Up to a change of coordinates, around $p = (0, 0)$:

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Closed sublevel: $\mathfrak{f}^c = \big\{x \in \mathcal{M} : \mathfrak{f}(x) \le c\big\}$.

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Shifting theorem (G & M, Topology 1969)

$\mu(p)$ = Morse index of $\mathfrak{f}$ at p $\Longrightarrow$ $\boxed{\mathfrak{H}_{k+\mu(p)}(\mathfrak{f}, p; \mathbb{F}) \cong \mathfrak{H}_k^0(\mathfrak{f}, p; \mathbb{F})}$

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Theorem

If $\mathfrak{f}^{-1}(c)$ contains a finite number of critical orbits $Gp_1, \ldots, Gp_r$:

$$H_*(\mathfrak{f}^{c+\varepsilon}, \mathfrak{f}^{c-\varepsilon}; \mathbb{F}) \cong \bigoplus_{i=1}^{r} \mathfrak{H}_*(\mathfrak{f}, Gp_i; \mathbb{F})$$

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$$\boxed{\mathrm{Geo}(\mathbb{S}^1, M_0 \times \mathbb{R}) \times \mathrm{Met}(M_0) \times \mathfrak{X}(M_0) \times C^\infty(M_0) \ni [\gamma, g_0, \delta, \beta] \mapsto [g_0, \delta, \beta]}$$

is a Fredholm nonlinear map with null index? If yes, apply Sard–Smale.

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- Apply the Morse inequalities to the filtration $\Lambda M = \bigcup_{n\geq 1} f^{c_n}$ to get a uniform upper bound on the Betti numbers of ΛM, getting a contradiction.

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This point does not work in the degenerate case

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$$\frac{\boxed{\textbf{Tricky Lemma}}}{+\ \boxed{\textbf{Even trickier Lemma}}} \Longrightarrow \boxed{\textbf{desired uniform bounds}}$$

QED

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THANKS FOR YOUR ATTENTION!!

These notes will be available on my web page:

`http://www.ime.usp.br/~piccione`