Chapter 5
Paracompact spaces

Paracompact spaces simultaneously generalize both compact spaces and metrizable spaces; although defined much later than the two latter classes, paracompact spaces quickly became popular among topologists and analysts, and now are considered to be one of the most important classes of topological spaces. Due to the introduction of paracompactness, many theorems in topology and analysis were generalized and many proofs were simplified. It also turned out that the notion of a locally finite family and notions related to it are very efficient and natural tools for studying topological spaces.

Section 5.1 is devoted to paracompact spaces. We start with three theorems containing various characterizations of paracompactness (the characterizations in terms of partitions of unity is particularly important for analysis). Then we prove that paracompact spaces have the property of collectionwise normality, which is much stronger than just normality, and we give a few examples. In the second part of the section we study operations on paracompact spaces and the behaviour of this class of spaces under mappings. The section concludes with the Tamano theorem establishing an interesting external characterization of paracompactness.

In Section 5.2 we study the class of countably paracompact spaces. The theorems in that section contain various characterizations of countable paracompactness.

Section 5.3 is devoted to weakly and strongly paracompact spaces. Like the class of countably paracompact spaces, those two classes are of much less importance than the class of paracompact spaces; however, they do play a role in dimension theory and in algebraic topology. Among the theorems in that section, the most important are the Nagami-Michael theorem stating that every collectionwise normal weakly paracompact space is paracompact, and the Worrell theorem establishing invariance of weak paracompactness under closed mappings.

The last section is a continuation of Section 4.4; five further metrization theorems are given there.

5.1 Paracompact spaces

The notion of a locally finite family of sets, introduced in Chapter 1, leads to the definition of an important class of topological spaces, the paracompact spaces. A topological space X is called a *paracompact space* if X is a Hausdorff space and every open cover of X has a locally finite open refinement.

Let us observe that, in contrast to the definition of compactness, in the definition of paracompactness the term "refinement" cannot be replaced by the term "subcover". In fact, one readily sees that every discrete space is paracompact – the cover consisting of all one-point sets is open and locally finite and refines any other cover of the space – and yet the open cover $\{N \cap [1,i]\}_{i=1}^{\infty}$ of the space of natural numbers N has no locally finite subcover (cf. Exercise 5.1.A(d)).

The definition of paracompactness yields

5.1.1. THEOREM. *Every compact space is paracompact.* ■

Using the notion of paracompactness, Theorems 3.8.11 and 4.4.1 can be stated as follows:

5.1.2. THEOREM. *Every Lindelöf space is paracompact.* ■

5.1.3. THEOREM. *Every metrizable space is paracompact.* ■

The reader can easily deduce from Theorem 5.1.12 and Remark 5.1.7 that the existence of open refinements which are both locally finite and σ-discrete, established for metrizable spaces in Theorem 4.4.1, is only formally stronger than paracompactness.

5.1.4. LEMMA. *Let X be a paracompact space and A, B a pair of closed subsets of X. If for every $x \in B$ there exist open sets U_x, V_x such that $A \subset U_x$, $x \in V_x$ and $U_x \cap V_x = \emptyset$, then there also exist open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

PROOF. The family $\{V_x\}_{x\in B} \cup \{X \setminus B\}$ is an open cover of the space X, so that it has a locally finite open refinement $\{W_s\}_{s\in S}$. Letting $S_0 = \{s \in S : W_s \cap B \neq \emptyset\}$ we have

$$A \cap \overline{W}_s = \emptyset \text{ for every } s \in S_0 \quad \text{and} \quad B \subset \bigcup_{s\in S_0} W_s.$$

By virtue of Theorem 1.1.11 the set $U = X \setminus \bigcup_{s\in S_0} \overline{W}_s$ is open; one readily sees that U and $V = \bigcup_{s\in S_0} W_s$ have all the required properties. ■

5.1.5. THEOREM. *Every paracompact space is normal.*

PROOF. Substituting one-point sets for A in the above lemma, we see that every paracompact space is regular; using this fact and applying the lemma again we obtain the theorem. ■

Let us observe that the last theorem is a common generalization of Theorems 1.5.16, 3.1.9 and 3.8.2.

A family $\{f_s\}_{s\in S}$ of continuous functions from a space X to the closed unit interval I is called a *partition of unity* on the space X if $\sum_{s\in S} f_s(x) = 1$ for every $x \in X$. The last equality means that for each $x_0 \in X$ only countably many functions f_s do not vanish at x_0 and that the series $\sum_{i=1}^{\infty} f_{s_i}(x_0)$, where $\{s_1, s_2, \ldots\} = \{s \in S : f_s(x_0) \neq 0\}$, converges to 1; since the sequence is absolutely convergent, the arrangement of terms does not matter and convergence to 1 means that 1 is the least upper bound of the set consisting of all numbers of the form $f_{s_1}(x_0) + f_{s_2}(x_0) + \ldots + f_{s_k}(x_0)$, where $s_1, s_2, \ldots, s_k \in S$ and $k = 1, 2, \ldots$

We say that a partition of unity $\{f_s\}_{s\in S}$ on a space X is *locally finite* if the cover $\{f_s^{-1}((0,1])\}_{s\in S}$ of the space X is locally finite. This means that for each $x_0 \in X$ there exists a neighbourhood U_0 of the point x_0 and a finite set $S_0 = \{s_1, s_2, \ldots, s_k\} \subset S$ such that for every $x \in U_0$ we have $f_s(x) = 0$ whenever $s \in S \setminus S_0$, and $f_{s_1}(x) + f_{s_2}(x) + \ldots + f_{s_k}(x) = 1$.

A partition of unity $\{f_s\}_{s\in S}$ on a space X is *subordinated to a cover $\mathcal{A}$* of X if the cover $\{f_s^{-1}((0,1])\}_{s\in S}$ of the space X is a refinement of $\mathcal{A}$.

Our next theorem contains two characterizations of paracompactness in terms of partitions of unity; these characterizations are very useful not only in topology but also in analysis and differential geometry. The theorem will be preceded by two lemmas. The first one will also be applied later and is stated in a form slightly more general than needed here; in our theorem instead of using this lemma one could apply – less elementary – Theorem 1.5.18.

5.1.6. LEMMA. *If every open cover of a regular space X has a locally finite refinement (consisting of arbitrary sets), then for every open cover $\{U_s\}_{s\in S}$ of the space X there exists a closed locally finite cover $\{F_s\}_{s\in S}$ of X such that $F_s \subset U_s$ for every $s \in S$.*

PROOF. By regularity of X there exists an open cover $\mathcal{W}$ of the space X such that $\{\overline{W} : W \in \mathcal{W}\}$ is a refinement of $\{U_s\}_{s\in S}$. Take a locally finite refinement $\{A_t\}_{t\in T}$ of the cover $\mathcal{W}$, for every $t \in T$ choose an $s(t) \in S$ such that $\overline{A}_t \subset U_{s(t)}$, and let $F_s = \bigcup_{s(t)=s} \overline{A}_t$. From Theorems 1.1.11 and 1.1.13 it follows readily that $\{F_s\}_{s\in S}$ is a closed locally finite cover of X and the definition of the F_s's implies that $F_s \subset U_s$ for every $s \in S$. ■

5.1.7. REMARK. Let us note that if the cover $\{A_t\}_{t\in T}$ in the last proof is open, then the sets $V_s = \bigcup_{s(t)=s} A_t$ are open and $\overline{V}_s = F_s$. Hence, for every open cover $\{U_s\}_{s\in S}$ of a paracompact space there exists a locally finite open cover $\{V_s\}_{s\in S}$ such that $\overline{V}_s \subset U_s$ for every $s \in S$.

5.1.8. LEMMA. *If for an open cover $\mathcal{U}$ of a space X there exists a partition of unity $\{f_s\}_{s\in S}$ subordinated to it, then $\mathcal{U}$ has an open localy finite refinement.*

PROOF. To begin, let us observe that for every continuous function $g : X \to I$ and any point $x_0 \in X$ satisfying $g(x_0) > 0$ there exists a neighbourhood U_0 of the point x_0 and a finite set $S_0 \subset S$ such that

$$f_s(x) < g(x) \quad \text{for } x \in U_0 \text{ and } s \in S \setminus S_0. \tag{1}$$

Indeed, one easily verifies that any set $S_0 = \{s_1, s_2, \ldots, s_k\} \subset S$ such that

$$1 - \sum_{i=1}^{k} f_{s_i}(x_0) < g(x_0)$$

and the open set $U_0 = \{x \in X : 1 - \sum_{i=1}^{k} f_{s_i}(x) < g(x)\}$ satisfy (1).

For every $x \in X$ there exists an $s(x) \in S$ such that $f_{s(x)}(x) > 0$. Letting $g = f_{s(x)}$ in the above observation we infer from 2.1.12 that the formula $f(x) = \sup_{s\in S} f_s(x)$ defines a continuous function $f : X \to (0, 1]$. For every $s \in S$ the set

$$V_s = \{x \in X : f_s(x) > \frac{1}{2} f(x)\}$$

is open, and the family $\mathcal{V} = \{V_s\}_{s\in S}$ is a refinement of $\mathcal{U}$. Letting $g = \frac{1}{2}f$ in our original observation we infer that $\mathcal{V}$ is a locally finite family. ■

5.1.9. THEOREM. *For every T_1-space X the following conditions are equivalent:*

(i) *The space X is paracompact.*

(ii) *Every open cover of the space X has a locally finite partition of unity subordinated to it.*

(iii) *Every open cover of the space X has a partition of unity subordinated to it.*

PROOF. Assume that X is paracompact and consider an open cover $\mathcal{A}$ of X. Let $\mathcal{U} = \{U_s\}_{s\in S}$ be a locally finite open refinement of $\mathcal{A}$. By virtue of Lemma 5.1.6 there exists a closed cover $\{F_s\}_{s\in S}$ of the space X such that $F_s \subset U_s$ for every $s \in S$. From Urysohn's lemma it follows that for every $s \in S$ one can find a continuous function $g_s : X \to I$ such that $g_s(x) = 0$ for $x \in X \setminus U_s$ and $g_s(x) = 1$ for $x \in F_s$. The family $\mathcal{U}$ being locally finite, by letting $g(x) = \sum_{s\in S} g_s(x)$ we define a continuous function $g : X \to R$. One easily sees that $\{f_s\}_{s\in S}$, where $f_s = g_s/g$, is a locally finite partition of unity subordinated to $\mathcal{A}$. Hence the implication (i)$\Rightarrow$(ii) is established.

Since the implication (ii)$\Rightarrow$(iii) is obvious, to conclude the proof it suffices to show that (iii)$\Rightarrow$(i), which – by virtue of Lemma 5.1.8 – reduces to showing that every T_1-space X satisfying (iii) is a Hausdorff space. Consider a pair of distinct poins $x_1, x_2 \in X$. The open cover $\mathcal{U} = \{X \setminus \{x_1\}, X \setminus \{x_2\}\}$ of the space X has a partition of unity $\{f_s\}_{s\in S}$ subordinated to it. Take an $s_0 \in S$ such that $f_{s_0}(x_1) = a > 0$; since the set $f_{s_0}^{-1}((0,1])$ is contained in $X \setminus \{x_2\}$, we have $f_{s_0}(x_2) = 0$. The open sets $U_1 = f_{s_0}^{-1}((a/2,1])$ and $U_2 = f_{s_0}^{-1}([0,a/2))$ are disjoint and contain x_1 and x_2 respectively.■

Three further characterizations of paracompactness are stated in the next theorem.

5.1.10. LEMMA. *Every open σ-locally finite cover $\mathcal{V}$ of a topological space X has a locally finite refinement.*

PROOF. Let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$, where $\mathcal{V}_i = \{V_s\}_{s\in S_i}$ is a locally finite family of open sets and $S_i \cap S_j = \emptyset$ whenever $i \neq j$. For every $s_0 \in S_i$ let

$$A_{s_0} = V_{s_0} \setminus \bigcup_{k<i} \bigcup_{s\in S_k} V_s;$$

the family $\mathcal{A} = \{A_s\}_{s\in S}$, where $S = \bigcup_{i=1}^{\infty} S_i$, covers X and is a refinement of $\mathcal{V}$. We shall show that $\mathcal{A}$ is locally finite. Consider a point $x \in X$, denote by k the smallest natural number such that $x \in \bigcup_{s\in S_k} V_s$, and take an $s_0 \in S_k$ satisfying $x \in V_{s_0}$; clearly V_{s_0} is a neighbourhood of x disjoint from all sets A_s with $s \in \bigcup_{i>k} S_i$. Since the families $\mathcal{V}_i$ are locally finite, for every $i \leq k$ there exists a neighbourhood U_i of x which meets only finitely many members of $\mathcal{V}_i$. The neighbourhood $U_1 \cap U_2 \cap \ldots \cap U_k \cap V_{s_0}$ of the point x meets only finitely many members of $\mathcal{A}$.■

5.1.11. THEOREM. *For every regular space X the following conditions are equivalent:*
(i) *The space X is paracompact.*
(ii) *Every open cover of the space X has an open σ-locally finite refinement.*
(iii) *Every open cover of the space X has a locally finite refinement.*
(iv) *Every open cover of the space X has a closed locally finite refinement.*

PROOF. The theorem follows from 5.1.10, 5.1.6 and 4.4.12.■

Let us note that the last theorem immediately implies that every Lindelöf space is paracompact. It should be also noted that in (ii) the second "open" cannot be replaced by "closed" (see the remark to Problem 5.5.3(a)).

We shall now introduce some notions related to the notion of a cover, which will be applied to establish further characterizations of paracompactness. Let $\mathcal{A} = \{A_s\}_{s\in S}$ be a cover of a set X; the *star of a set $M \subset X$ with respect to $\mathcal{A}$* is the set $\mathrm{St}(M,\mathcal{A}) = \bigcup\{A_s :$

$M \cap A_s \neq \emptyset\}$. The star of a one-point set $\{x\}$ with respect to a cover $\mathcal{A}$ is called the *star of the point x with respect to $\mathcal{A}$* and is denoted by $\mathrm{St}(x, \mathcal{A})$. We say that a cover $\mathcal{B} = \{B_t\}_{t\in T}$ of a set X is a *star refinement* of another cover $\mathcal{A} = \{A_s\}_{s\in S}$ of the same set X if for every $t \in T$ there exists an $s \in S$ such that $\mathrm{St}(B_t, \mathcal{B}) \subset A_s$; if for every $x \in X$ there exists an $s \in S$ such that $\mathrm{St}(x, \mathcal{B}) \subset A_s$, then we say that $\mathcal{B}$ is a *barycentric refinement* of $\mathcal{A}$. Clearly, every star refinement is a barycentric refinement and every barycentric refinement is a refinement.

The next theorem contains yet three characterizations of paracompactness; it will be deduced directly from Lemmas 5.1.13, 5.1.15, and 5.1.16 that are stated and proved below.

5.1.12. THEOREM. *For every T_1-space X the following conditions are equivalent:*
(i) *The space X is paracompact.*
(ii) *Every open cover of the space X has an open barycentric refinement.*
(iii) *Every open cover of the space X has an open star refinement.*
(iv) *The space X is regular and every open cover of X has an open σ-discrete refinement.*

5.1.13. LEMMA. *If an open cover $\mathcal{U}$ of a topological space X has a closed locally finite refinement, then $\mathcal{U}$ has also an open barycentric refinement.*

PROOF. Let $\mathcal{F} = \{F_t\}_{t\in T}$ be a closed locally finite refinement of $\mathcal{U} = \{U_s\}_{s\in S}$. For every $t \in T$ choose an $s(t) \in S$ such that $F_t \subset U_{s(t)}$. It follows from the local finiteness of $\mathcal{F}$ that the set $T(x) = \{t \in T : x \in F_t\}$ is finite for every $x \in X$, and this implies that the set

$$V_x = \bigcap_{t\in T(x)} U_{s(t)} \cap (X \setminus \bigcup_{t\notin T(x)} F_t) \tag{2}$$

is open for every $x \in X$. As $x \in V_x$, the family $\mathcal{V} = \{V_x\}_{x\in X}$ is an open cover of X. Let x_0 be a point of X and t_0 an element of $T(x_0)$; it follows from (2) that if $x_0 \in V_x$, then $t_0 \in T(x)$, and thus $V_x \subset U_{s(t_0)}$. Hence we have $\mathrm{St}(x_0, \mathcal{V}) \subset U_{s(t_0)}$ which shows that $\mathcal{V}$ is a barycentric refinement of $\mathcal{U}$. ■

5.1.14. REMARK. The same proof shows that if a locally finite open cover of a topological space has a closed locally finite refinement then it has also a locally finite open barycentric refinement; indeed, if the cover $\mathcal{U}$ is locally finite, then the family of all sets of the form (2) is a locally finite open barycentric refinement of $\mathcal{U}$.

5.1.15. LEMMA. *If a cover $\mathcal{A} = \{A_s\}_{s\in S}$ of a set X is a barycentric refinement of a cover $\mathcal{B} = \{B_t\}_{t\in T}$ of X, and $\mathcal{B}$ is a barycentric refinement of a cover $\mathcal{C} = \{C_z\}_{z\in Z}$ of the same set, then $\mathcal{A}$ is a star refinement of $\mathcal{C}$.*

PROOF. Let us take an $s_0 \in S$ and for every $x \in A_{s_0}$ let us choose a $t(x) \in T$ such that

$$\mathrm{St}(x, \mathcal{A}) \subset B_{t(x)}. \tag{3}$$

Thus we have

$$\mathrm{St}(A_{s_0}, \mathcal{A}) = \bigcup_{x\in A_{s_0}} \mathrm{St}(x, \mathcal{A}) \subset \bigcup_{x\in A_{s_0}} B_{t(x)}. \tag{4}$$

Let x_0 be a fixed element of A_{s_0}; from (3) it follows that $x_0 \in B_{t(x)}$ for every $x \in A_{s_0}$, so that

$$\bigcup_{x\in A_{s_0}} B_{t(x)} \subset \mathrm{St}(x_0, \mathcal{B}).$$

Since $\operatorname{St}(x_0, \mathcal{B}) \subset C_z$ for a $z \in Z$, the last inclusion, along with (4), implies that $\mathcal{A}$ is a star refinement of $\mathcal{C}$. ■

5.1.16. LEMMA. *If every open cover of a topological space X has an open star refinement, then every open cover of X has also an open σ-discrete refinement.*

PROOF. Consider an open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of the space X. Let $\mathcal{U}_0 = \mathcal{U}$ and denote by $\mathcal{U}_1, \mathcal{U}_2, \ldots$ a sequence of open covers of X such that

$$\text{(5)} \qquad \mathcal{U}_{i+1} \text{ is a star refinement of } \mathcal{U}_i \text{ for } i = 0, 1, \ldots$$

For every $s \in S$ and $i = 1, 2, \ldots$ take the open set

$$U_{s,i} = \{x \in X : x \text{ has a neighbourhood } V \text{ such that } \operatorname{St}(V, \mathcal{U}_i) \subset U_s\}.$$

The family $\{U_{s,i}\}_{s\in S}$ is an open refinement of $\mathcal{U}$ for $i = 1, 2, \ldots$ Let us observe that

$$\text{(6)} \qquad \text{if } x \in U_{s,i} \text{ and } y \notin U_{s,i+1}, \text{ then there is no } U \in \mathcal{U}_{i+1} \text{ such that } x, y \in U.$$

Indeed, it follows from (5) that for every $U \in \mathcal{U}_{i+1}$ there exists a $W \in \mathcal{U}_i$ such that $\operatorname{St}(U, \mathcal{U}_{i+1}) \subset W$; therefore if $x \in U \cap U_{s,i}$, then $W \subset \operatorname{St}(x, \mathcal{U}_i) \subset U_s$ which implies that $\operatorname{St}(U, \mathcal{U}_{i+1}) \subset U_s$ and $U \subset U_{s,i+1}$.

Take a well-ordering relation $<$ on the set S and let

$$\text{(7)} \qquad V_{s_0,i} = U_{s_0,i} \setminus \overline{\bigcup_{s<s_0} U_{s,i+1}}.$$

For every pair s_1, s_2 of distinct elements of S we have either $s_1 < s_2$ or $s_2 < s_1$; depending on which part of the alternative holds, by virtue of (7) we have

$$\text{either} \quad V_{s_2,i} \subset X \setminus U_{s_1,i+1} \quad \text{or} \quad V_{s_1,i} \subset X \setminus U_{s_2,i+1}.$$

Hence, it follows from (6) that if $x \in V_{s_1,i}$ and $y \in V_{s_2,i}$, where $s_1 \neq s_2$, then there is no $U \in \mathcal{U}_{i+1}$ such that $x, y \in U$. Thus the family of open sets $\{V_{s,i}\}_{s\in S}$ is discrete for $i = 1, 2, \ldots$

To conclude the proof it suffices to show that the family $\{V_{s,i}\}_{i=1,s\in S}^{\infty}$ is a cover of X. Let x be a point of X; denote by $s(x)$ the smallest element in S such that $x \in U_{s(x),i}$ for some positive integer i – the existence of $s(x)$ follows from the fact that for $i = 1, 2, \ldots$ the family $\{U_{s,i}\}_{s\in S}$ is a cover of X. Since $x \notin U_{s,i+2}$ for $s < s(x)$, it follows from (6) that

$$\operatorname{St}(x, \mathcal{U}_{i+2}) \cap \bigcup_{s<s(x)} U_{s,i+1} = \emptyset,$$

and this shows that $x \in V_{s(x),i}$. ■

PROOF OF THEOREM 5.1.12. By virtue of the last three lemmas and by Theorem 5.1.11, it suffices to show that every T_1-space X satisfying (iii) is regular. Consider a point $x \in X$ and a closed set $F \subset X$ such that $x \notin F$ and take an open star refinement $\mathcal{U}$ of the open cover $\{X \setminus F, X \setminus \{x\}\}$ of the space X. Let U be a member of $\mathcal{U}$ that contains x. As $\operatorname{St}(U, \mathcal{U}) \subset X \setminus F$, we have $\overline{U} \cap F = \emptyset$, so that the space X is regular. ■