On the Geometry of Grassmannians and the Symplectic Group: the Maslov Index and Its Applications.

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Introduction

It has become evident through many mathematical theories of our century that Geometry and Topology offer very powerful tools in the study of qualitative and also quantitative properties of differential equations. The main idea behind these theories is that some equations, or better, some classes of equations can be studied by means of their *symmetries*, where by symmetry we mean generically any algebraic or geometric structure which is preserved by their flow. Once such invariant structures are determined for a class of differential equations, many properties of the solutions of the class can be read off from the geometry of the curve obtained by the flow, taking values in the space (typically a Lie group) of all structure-preserving morphisms.

A simple, but instructive, example is given by the Sturmian theory for second order ordinary differential equations in $\mathbb{R}$. The Sturm oscillation theorem deals with equations of the form $-(px')'+rx=\lambda x$, where p and r are functions, $p>0$, and λ is a real parameter. The theorem states that, denoting by $C^1_o[\alpha,\beta]$ the space of C^1-functions on $[\alpha,\beta]$ vanishing at α and β, the index of the symmetric bilinear form $B(x,y)=\int_a^b[px'y'+rxy]\,\mathrm{d}t$ in $C^1_o[a,b]$ is equal to the sum over $t\in\,]a,b[$ of the dimension of the kernel of the bilinear form $\int_a^t[px'y'+rxy]\,\mathrm{d}t$ in $C^1_o[a,t]$.

The classical proof of the Sturm oscillation theorem (see for instance [**4**, Chapter 8]) is obtained by showing that the two quantities involved in the thesis can be obtained as the *winding number* of two homotopic closed curves in the real projective line.

The class of differential equations that we are interested in consists in the so called "symplectic differential systems"; these are linear systems in $\mathbb{R}^n\oplus\mathbb{R}^{n*}$ whose flow preserve the canonical *symplectic form*, given by $\omega\big((v,\alpha),(w,\beta)\big)=\beta(v)-\alpha(w)$. Recall that a symplectic form is a nondegenerate skew-symmetric bilinear form on a (necessarily even dimensional) vector space. These differential systems appear naturally in a great variety of fields of pure and applied mathematics, and many areas of mathematics and physics, like Calculus of Variations, Hamiltonian systems, (Pseudo-)Riemannian Geometry, Symplectic Geometry, Mechanics and Optimal Control Theory produce examples of symplectic systems as basic objects of investigation. For instance, Morse–Sturm systems are special cases of symplectic systems; such systems are obtained from the Jacobi equation along any pseudo-Riemannian geodesic by means of a parallel trivialization of the tangent bundle of the pseudo-Riemannian manifold along the geodesic. More in general, symplectic systems are obtained by considering the linearized Hamilton equations along any solution of a (possibly time-dependent) Hamiltonian problem, using a symplectic trivialization along the solution of the tangent bundle of the underlying symplectic manifold. Another large class of examples where the theory leads naturally to the study of symplectic systems is provided by Lagrangian variational theories in manifolds, possibly time-dependent, even in the case of *constrained* variational problems. Indeed, under a suitable *invertibility* assumption called *hyper-regularity*, the solutions to such problems correspond, via the *Legendre transform*, to the solutions of an associated Hamiltonian problem in the cotangent bundle.

The fundamental matrix of a symplectic system is a curve in the *symplectic group*, denoted by $\mathrm{Sp}(2n, \mathbb{R})$, which is a closed subgroup of the general linear group $\mathrm{GL}(2n, \mathbb{R})$, hence it has a Lie group structure. This structure is extremely rich, due to the fact that symplectic forms on a vector space are intimately related to its complex structures, and such relation produces other invariant geometric and algebraic structures, such as inner products and Hermitian products.

Many interesting questions can be answered by studying solutions of symplectic systems whose initial data belong to a fixed *Lagrangian subspace* of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$. Recall that a Lagrangian subspace of a symplectic space is a maximal subspace on which the symplectic form vanishes. Such initial conditions are obtained, for instance, in Riemannian or pseudo-Riemannian geometry when one considers Jacobi fields along a geodesic that are variations made of geodesics starting orthogonally at a given submanifold. Since symplectic maps preserve Lagrangian subspaces, the image of the initial Lagrangian by the flow of a symplectic system is a curve in the set Λ of all Lagrangian subspaces of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$. The set Λ is a smooth (indeed, real-analytic) submanifold of the *Grassmannian* $G_n(\mathbb{R}^n \oplus \mathbb{R}^{n*})$ of all n-dimensional subspaces of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$; Λ is called the *Lagrangian Grassmannian* of the symplectic space $\mathbb{R}^n \oplus \mathbb{R}^{n*}$.

The original interest of the authors was the study of conjugate points along geodesics in a semi-Riemannian manifold and their stability (see [**29, 34**]), with the aim of developing an infinite dimensional Morse Theory (see [**30, 12, 26**]) for semi-Riemannian geodesics. A few decades ago a new integer valued homological invariant, called the *Maslov index*, was introduced by the Russian school (see for instance [**1**] and the references therein) for closed curves in a Lagrangian submanifold M of the space $\mathbb{R}^{2n}$ endowed with its canonical symplectic structure. The notion of Maslov index has been immediately recognized as an important tool in the study of conjugate points, and it has has been thoroughly investigated and extended in several directions by mathematical-physicists, geometers and analysts. There is nowadays a very extensive literature about the subject, and it is almost impossible to acknowledge the work of all the many authors who have given significant contributions to the field. Our list of references ([**6, 9, 11, 14, 15, 16, 24, 28, 36, 42**]) is far from being exhaustive.

Periodic or non periodic solutions of Hamiltonian systems, like for instance geodesics in a semi-Riemannian manifold, define a curve in the symplectic group, or in the Lagrangian Grassmannian, hence they define a Maslov index. Roughly speaking, the Maslov index gives a sort of *algebraic count* of the conjugate points along a solution; here are some of the main properties of this invariant:

- it is always finite (even when the number of conjugate points is infinite);
- it is *stable* by "small" perturbations of the data;
- it coincides with the *geometric index* in the case of a causal (timelike or lightlike) Lorentzian geodesic;
- it is related to the *analytic index* (or, more in general, to the relative index) of the solution, which is the index of the second variation of an associated Lagrangian action functional;
- it is related to the spectral properties of the associated Hamiltonian second order differential operator.

Conjugate and focal points appear naturally in Optics, both classical and relativistic, and the Maslov index provides a new topological invariant. For instance, the optics of light rays in a general-relativistic medium, like vacuum, dust, or plasma, magnetized or not, etc., can be described using a Hamiltonian formalism. As the underlying spacetime model one assumes an arbitrary 4-dimensional Lorentzian manifold (M, g), and the trajectories of light rays are projections onto M of solutions in the cotangent bundle TM^* of some Hamiltonian function $H : TM^* \to \mathbb{R}$. Typically, the explicit form of the function H involves the spacetime metric and a number of tensor fields by which the medium is characterized. For a comprehensive discussion of this subject we refer to Perlick [**33**].

The aim of this booklet is to provide a complete, self-contained study of the geometry of the Grassmannian manifolds, the symplectic group and the Lagrangian Grassmannian. This study will lead us naturally to the notion of Maslov index, that will be introduced in the context of symplectic differential systems.

These notes are organized as follows. In Chapter 1 we describe the algebraic setup; we will study complex structures and symplectic structures on finite dimensional vector spaces. Special attention is given to the Lagrangian subspaces and to the Lagrangian decompositions of a symplectic space.

Chapter 2 is entirely dedicated to Differential Geometry; we will study at the differentiable structure of the Grassmannian manifolds and of the Lagrangian Grassmannians. We will develop briefly the theory of Lie groups and their actions on differentiable manifolds, so that we will be able to describe the Grassmannians as homogeneous spaces.

In Chapter 3 we will develop the algebraic topological framework of the theory, including the basics of homotopy theory and of singular homology theory for topological spaces. Using the long exact sequences in homotopy and in homology, and using the Hurewicz homomorphism we will compute the first homology and relative homology group of the Grassmannians.

In Chapter 4 we will introduce the notion of Maslov index for curves in the Lagrangian Grassmannian, and we will present some methods to compute it in terms of the change of signature of curves of bilinear forms.

In Chapter 5 we will show how symplectic differential systems are produced in several geometrical problems; more precisely, we will consider the case of the Jacobi equation along a semi-Riemannian geodesic, and the case of the linearized

Hamilton equation along the solution of a Hamiltonian problem in a symplectic manifold.

At the end of each Chapter we have given a list of exercises whose solution is based on the material presented in the chapter. The reader should be able to solve the problems as he/she goes along; the solution or a hint for the solution of (almost) all the exercises is given in Appendix A.

Ognuno sta solo sul cuor della terra
trafitto da un raggio di sole:
ed è subito sera.

CHAPTER 1

Symplectic Spaces

1.1. A Short Review of Linear Algebra

In this section we will briefly review some well known facts concerning the identification of bilinear forms and linear operators on vector spaces. These identifications will be used repeatedly during the exposition of the material, and, to avoid confusion, the reader is encouraged to take a few minutes to go through the pain of reading this section.

The results presented are valid for vector spaces over an arbitrary field K, however we will mainly be interested in the case that $K = I\!R$ or $K = \mathbb{C}$. Moreover, we emphasize that even though a few results presented are also valid in the case of infinite dimensional vector spaces, in this chapter *we will always assume* that the vector spaces involved are *finite dimensional*.

Let V and W be vector spaces. We denote by $\mathrm{Lin}(V,W)$ and by $\mathrm{B}(V,W)$ respectively the vector spaces of all the *linear operators* $T : V \to W$ and of *bilinear operators*, called also *bilinear forms*, $B : V \times W \to K$; by V^* we mean the *dual* space $\mathrm{Lin}(V,K)$ of V. Shortly, we set $\mathrm{Lin}(V) = \mathrm{Lin}(V,V)$ and $\mathrm{B}(V) = \mathrm{B}(V,V)$.

There is a *natural* isomorphism:

$$\mathrm{Lin}(V,W^*) \longrightarrow \mathrm{B}(V,W), \tag{1.1.1}$$

which is obtained by associating to each linear operator $T : V \to W^*$ the bilinear form $B_T \in \mathrm{B}(V,W)$ given by $B_T(v,w) = T(v)\big(w\big)$.

1.1.1. REMARK. Given vector spaces V, W, V_1, W_1 and a pair (L, M) of linear operators, with $L \in \mathrm{Lin}(V_1, V)$ and $M \in \mathrm{Lin}(W, W_1)$, one defines another linear operator:

$$\mathrm{Lin}(L,M) : \mathrm{Lin}(V,W) \longrightarrow \mathrm{Lin}(V_1,W_1) \tag{1.1.2}$$

by:

$$\mathrm{Lin}(L,M)\cdot T = M \circ T \circ L. \tag{1.1.3}$$

In this way, $\mathrm{Lin}(\cdot,\cdot)$ becomes a functor, contravariant in the first variable and covariant in the second, from the category of pairs of vector spaces to the category of vector spaces. Similarly, given linear operators $L \in \mathrm{Lin}(V_1, V)$ and $M \in \mathrm{Lin}(W_1, W)$, we can define a linear operator $\mathrm{B}(L,M) : \mathrm{B}(V,W) \to \mathrm{B}(V_1,W_1)$ by setting $\mathrm{B}(L,M)\cdot B = B(L\cdot, M\cdot)$. In this way, $\mathrm{B}(\cdot,\cdot)$ turns into a functor, contravariant in both variables, from the category of pairs of vector spaces to the category of vector spaces. This abstract formalism will infact be useful later (see Section 2.3).

The naturality of the isomorphism (1.1.1) may be meant in the technical sense of *natural isomorphism between the functors* $\mathrm{Lin}(\cdot,\cdot)$ *and* $\mathrm{B}(\cdot,\cdot)$ (see Exercise 1.1).

To avoid confusion, in this Section we will distinguish between the symbols of a bilinear form B and of the associated linear operator T_B, or between a linear operator T and the associated bilinear form B_T. However, in the rest of the book we will implicitly assume the isomorphism (1.1.1), and we will not continue with this distinction.

Given another pair of vector spaces V_1 and W_1 and operators $L_1 \in \mathrm{Lin}(V_1, V)$, $L_2 \in \mathrm{Lin}(W_1, W)$, the bilinear forms $B_T(L_1\cdot,\cdot)$ and $B_T(\cdot, L_2\cdot)$ correspond via (1.1.1) to the linear operators $T\circ L_1$ and $L_2^*\circ T$ respectively. Here, $L_2^* : W^* \to W_1^*$ denotes the *transpose linear operator* of L_2 given by:

$$L_2^*(\alpha) = \alpha \circ L_2, \quad \forall\, \alpha \in W^*.$$

We will identify every vector space V with its *bidual* V^{**} and every linear operator T with its *bitranspose* T^{**}. Given $T \in \mathrm{Lin}(V, W^*)$ we will therefore look at T^* as an element in $\mathrm{Lin}(W, V^*)$; if B_T is the bilinear operator associated to T, then the bilinear operator B_{T^*} associated to T^* is the *transpose bilinear operator* $B_T^* \in \mathrm{B}(W, V)$ defined by $B_T^*(w, v) = B_T(v, w)$.

Given $B \in \mathrm{B}(V)$, we say that B is *symmetric* if $B(v, w) = B(w, v)$ for all $v, w \in V$; we say that B is *anti-symmetric* if $B(v, w) = -B(w, v)$ for all $v, w \in V$ (see Exercise 1.2). The sets of symmetric bilinear forms and of anti-symmetric bilinear forms are subspaces of $\mathrm{B}(V)$, denoted respectively by $\mathrm{B}_{\mathrm{sym}}(V)$ and $\mathrm{B}_{\mathrm{a-sym}}(V)$.

The reader is warned that, unfortunately, the identification (1.1.1) does *not* behave well in terms of matrices, with the usual convention for the matrix representations of linear and bilinear operators.

If $(v_i)_{i=1}^n$ and $(w_i)_{i=1}^m$ are bases of V and W respectively, we denote by $(v_i^*)_{i=1}^n$ and $(w_i^*)_{i=1}^m$ the corresponding dual bases of V^* and W^*. For $T \in \mathrm{Lin}(V, W^*)$, the matrix representation (T_{ij}) of T satisfies:

$$T(v_j) = \sum_{i=1}^m T_{ij}\, w_i^*.$$

On the other hand, if $B \in \mathrm{B}(V, W)$, the matrix representation (B_{ij}) of B is defined by:

$$B_{ij} = B(v_i, w_j);$$

hence, for all $T \in \mathrm{Lin}(V, W^*)$ we have:

$$T_{ij} = T(v_j)(w_i) = B_T(v_j, w_i) = [B_T]_{ji}.$$

Thus, the matrix of a linear operator is the *transpose* of the matrix of the corresponding bilinear operator; in some cases we will be considering *symmetric* operators, and there will be no risk of confusion. However, when we deal with *symplectic forms* (see Section 1.4) one must be careful not to make sign errors.

1.1.2. DEFINITION. Given $T \in \mathrm{Lin}(V, W)$, we define the *pull-back* associated to T to be map:

$$T^{\#} : \mathrm{B}(W) \longrightarrow \mathrm{B}(V)$$

given by $T^{\#}(B) = B(T\cdot, T\cdot)$. When T is an *isomorphism*, we can also define the *push-forward* associated to T, which is the map:

$$T_{\#} : \mathrm{B}(V) \longrightarrow \mathrm{B}(W)$$

defined by $T_{\#}(B) = B(T^{-1}\cdot, T^{-1}\cdot)$.

1.1.3. EXAMPLE. Using (1.1.1) to identify linear and bilinear operators, we have the following formulas for the pull-back and the push-forward:

$$T^{\#}(B) = T^* \circ T_B \circ T, \quad T_{\#}(B) = (T^{-1})^* \circ T_B \circ T^{-1}. \tag{1.1.4}$$

The identities (1.1.4) can be interpreted as equalities involving the matrix representations, in which case one must use the matrices that represent B, $T^{\#}(B)$ and $T_{\#}(B)$ *as linear operators*.

For $B \in \mathrm{B}(V)$, the *kernel* of B is the subspace of V defined by:

$$\mathrm{Ker}(B) = \Big\{ v \in V : B(v, w) = 0, \ \forall\, w \in V \Big\}. \tag{1.1.5}$$

The kernel of B coincides with the kernel of the associated linear operator $T : V \to V^*$. The bilinear form B is said to be *nondegenerate* if $\mathrm{Ker}(B) = \{0\}$; this is equivalent to requiring that its associated linear operator T is injective, or equivalently, an isomorphism.

1.1.4. EXAMPLE. If $B \in \mathrm{B}(V)$ is nondegenerate, then B defines an isomorphism T_B between V and V^* and therefore we can define a bilinear form $[T_B]_{\#}(B)$ in V^* by taking the push-forward of B by T_B. By (1.1.4), such bilinear form is associated to the linear operator $(T_B^{-1})^*$; if B is symmetric, then $[T_B]_{\#}(B)$ is the bilinear form associated to the linear map T_B^{-1}.

1.1.5. DEFINITION. Let $B \in \mathrm{B}_{\mathrm{sym}}(V)$ be a symmetric bilinear form in V. We say that a linear operator $T : V \to V$ is *B-symmetric* (respectively, *B-anti-symmetric*) if the bilinear form $B(T\cdot, \cdot)$ is symmetric (respectively, anti-symmetric). We say that T is *B-orthogonal* if $T^{\#}[B] = B$, i.e., if $B(T\cdot, T\cdot) = B$.

1.1.6. EXAMPLE. Given $B \in \mathrm{B}_{\mathrm{sym}}$ and $T \in \mathrm{Lin}(V)$, the B-symmetry of T is equivalent to:

$$T_B \circ T = (T_B \circ T)^*; \tag{1.1.6}$$

clearly, the B-anti-symmetry is equivalent to $T_B \circ T = -(T_B \circ T)^*$.

When B is nondegenerate, we can also define the *transpose of T relatively to B*, which is the operator $\hat{T} \in \mathrm{Lin}(V)$ such that $B(Tv, w) = B(v, \hat{T}w)$ for all $v, w \in V$. Explicitly, we have

$$\hat{T} = T_B^{-1} \circ T^* \circ T_B. \tag{1.1.7}$$

Then, T is B-symmetric (resp., B-anti-symmetric) iff $\hat{T} = T$ (resp., iff $\hat{T} = -T$), and it is B-orthogonal iff $\hat{T} = T^{-1}$.

We also say that T is B-*normal* if T commutes with $\hat{T}$.

Given a subspace $S \subset V$ and a bilinear form $B \in \mathrm{B}(V)$, the *orthogonal complement* $S^\perp$ of S with respect to B is defined by:

$$S^\perp = \Big\{ v \in V : B(v, w) = 0,\ \forall\, w \in S \Big\}. \tag{1.1.8}$$

In particular, $\mathrm{Ker}(B) = V^\perp$. The *annihilator* S^o of S is the subspace of V^* defined as:

$$S^o = \Big\{ \alpha \in V^* : \alpha(w) = 0,\ \forall\, w \in S \Big\}.$$

Observe that $S^\perp = T_B^{-1}(S^o)$.

1.1.7. EXAMPLE. Assume that $B \in \mathrm{B}_{\mathrm{sym}}(V)$ is nondegenerate and let $T \in \mathrm{Lin}(V)$; denote by $\hat{T}$ the B-transpose of T. If $S \subset V$ is an *invariant subspace* for T, i.e., if $T(S) \subset S$, then the B-orthogonal complement $S^\perp$ of S is invariant for $\hat{T}$. This follows from (1.1.7) and from the identity $S^\perp = T_B^{-1}(S^o)$, observing that the annihilator S^o of S is invariant for T^*.

1.1.8. PROPOSITION. *If $B \in \mathrm{B}(V)$ is nondegenerate and $S \subset V$ is a subspace, then* $\dim(V) = \dim(S) + \dim(S^\perp)$.

PROOF. Simply note that $\dim(V) = \dim(S) + \dim(S^o)$ and that $\dim(S^\perp) = \dim(S^o)$, and $S^\perp = T_B^{-1}(S^o)$, with T_B an isomorphism, because B is nondegenerate. □

If B is either symmetric or anti-symmetric, then it is easy to see that $S \subset (S^\perp)^\perp$; the equality does *not* hold in general, but only if B is nondegenerate.

1.1.9. COROLLARY. *Suppose that $B \in \mathrm{B}(V)$ is either symmetric or anti-symmetric; if B is nondegenerate, then $S = (S^\perp)^\perp$.*

PROOF. It is $S \subset (S^\perp)^\perp$; by Proposition 1.1.8 $\dim(S) = \dim\big((S^\perp)^\perp\big)$. □

If $B \in \mathrm{B}(V)$ is nondegenerate and $S \subset V$ is a subspace, then the restriction of B to $S \times S$ may be degenerate. We have the following:

1.1.10. PROPOSITION. *The restriction $B|_{S\times S}$ is nondegenerate if and only if $V = S \oplus S^\perp$.*

PROOF. The kernel of the restriction $B|_{S\times S}$ is $S \cap S^\perp$; hence, if $V = S \oplus S^\perp$, it follows that B is nondegenerate on S. Conversely, if B is nondegenerate on S, then $S \cap S^\perp = \{0\}$. It remains to show that $V = S + S^\perp$. For, observe that the map:

$$S \ni x \longmapsto B(x, \cdot)|_S \in S^* \tag{1.1.9}$$

is an isomorphism. Hence, given $v \in V$, there exists $x \in S$ such that $B(x, \cdot)$ and $B(v, \cdot)$ coincide in S, thus $x - v \in S^\perp$. This concludes the proof. □

1.1.11. COROLLARY. *Suppose that $B \in \mathrm{B}(V)$ is either symmetric or anti-symmetric; if B is nondegenerate, then the following are equivalent:*

- *B is nondegenerate on S;*
- *B is nondegenerate on $S^\perp$.*

PROOF. Assume that B is nondegenerate on S. By Proposition 1.1.10 it is $V = S \oplus S^\perp$; by Corollary 1.1.9 we have $V = S^\perp \oplus (S^\perp)^\perp$, from which it follows that B is nondegenerate on $S^\perp$ by Proposition 1.1.10. The converse is analogous, since $(S^\perp)^\perp = S$. □

1.1.12. EXAMPLE. Proposition 1.1.10 actually does *not* hold if V is not finite dimensional. For instance, if V is the space of *square summable* sequences $x = (x_i)_{i\in\mathbb{N}}$ of real numbers, i.e., $\sum_{i\in\mathbb{N}} x_i^2 < +\infty$, B is the *standard Hilbert product in* V given by $B(x,y) = \sum_{i\in\mathbb{N}} x_i y_i$ and $S \subset V$ is the subspace consisting of all *almost null* sequences, i.e., $x_i \neq 0$ only for a finite number of indices $i \in \mathbb{N}$, then it is easy to see that $S^\perp = \{0\}$. What happens here is that the map (1.1.9) is injective, but not surjective.

1.1.13. REMARK. Observe that Proposition 1.1.10 is indeed true if we assume only that S is finite dimensional; for, in the proof presented, only the finiteness of $\dim(S)$ was used to conclude that the map (1.1.9) is an isomorphism.

As an application of Proposition 1.1.10 we can now prove that every symmetric bilinear form is diagonalizable. We say that a basis $(v_i)_{i=1}^n$ of V *diagonalizes* the bilinear form B if $B(v_i, v_j) = 0$ for all $i \neq j$, i.e., if B is represented by a diagonal matrix in the basis $(v_i)_{i=1}^n$.

1.1.14. THEOREM. *Suppose that K is a field of characteristic different from 2. Given $B \in \mathrm{B}_{\mathrm{sym}}(V)$, there exists a basis $(v_i)_{i=1}^n$ of V that diagonalizes B.*

PROOF. We prove the result by induction on $\dim(V)$. If $\dim(V) = 1$ the result is trivial; assume $\dim(V) = n$ and that the result holds true for every vector space of dimension less than n. If $B(v,v) = 0$ for all $v \in V$, then $B = 0$. For,

$$0 = B(v+w, v+w) = 2\,B(v,w),$$

and the filed K has characteristic different from 2. Since the result in the case that $B = 0$ is trivial, we can assume the existence of $v_1 \in V$ such that $B(v_1, v_1) \neq 0$. It follows that B is nondegenerate on the one-dimensional subspace $K\,v_1$ generated by v_1; by Proposition 1.1.10 we get:

$$V = K\,v_1 \oplus (K\,v_1)^\perp.$$

By the induction hypothesis, there exists a basis $(v_i)_{i=1}^n$ of $(K\,v_1)^\perp$ that diagonalizes the restriction of B; it is then easy to check that the basis $(v_i)_{i=2}^n$ diagonalizes B. □

1.2. Complex Structures

In this section we will study the procedure of changing the scalar field of a real vector space, making it into a complex vector space. Of course, given a complex vector space, one can always reduce the scalars to the real field: such operation will be called *reduction of the scalars*.

Passing from the real to the complex field requires the introduction of an additional structure, that will be called a *complex structure*. Many of the proofs in this section are elementary, so they will be omitted and left as an exercise for the reader.

For clarity, in this section we will refer to linear operators as $\mathbb{R}$-linear or $\mathbb{C}$-linear, and similarly we will talk about $\mathbb{R}$-bases or $\mathbb{C}$-bases, real or complex dimension, etc.

Let $\mathcal{V}$ be a complex vector space; we will denote by $\mathcal{V}_{\mathbb{R}}$ the real vector space obtained by restriction of the multiplication by scalars $\mathbb{C}\times\mathcal{V}\to\mathcal{V}$ to $\mathbb{R}\times\mathcal{V}\to\mathcal{V}$. Observe that the underlying set of vectors, as well as the operation of sum, coincides in $\mathcal{V}$ and $\mathcal{V}_{\mathbb{R}}$. We say that $\mathcal{V}_{\mathbb{R}}$ is a *realification* of $\mathcal{V}$, or that $\mathcal{V}_{\mathbb{R}}$ is obtained by a reduction of scalars from $\mathcal{V}$.

The endomorphism $v\mapsto iv$ of $\mathcal{V}$ given by the multiplication by the imaginary unit $i=\sqrt{-1}$ is $\mathbb{C}$-linear, hence also $\mathbb{R}$-linear. The square of this endomorphism is given by minus the identity of $\mathcal{V}$. This suggests the following definition:

1.2.1. DEFINITION. Let V be a real vector space. A *complex structure* in V is a linear operator $J:V\to V$ such that $J^2=J\circ J=-\mathrm{Id}$.

Clearly, a complex structure J is an isomorphism, since $J^{-1}=-J$.

Given a complex structure J on V it is easy to see that there exists a unique way of extending the multiplication by scalars $\mathbb{R}\times V\to V$ of V to a multiplication by scalar $\mathbb{C}\times V\to V$ in such a way that $J(v)=iv$. Explicitly, we define:

$$(a+b\,i)v=av+b\,J(v),\quad a,b\in\mathbb{R},\ v\in V. \tag{1.2.1}$$

Conversely, as we had already observed, every complex extension of multiplication by scalars for V defines a complex structure on V by $J(v)=iv$.

We will henceforth identify every pair (V,J), where V is a real vector space and J is a complex structure of V, with the complex vector space $\mathcal{V}$ obtained from (V,J) by (1.2.1). Observe that V is the realification $\mathcal{V}_{\mathbb{R}}$ of $\mathcal{V}$.

1.2.2. EXAMPLE. For every $n\in\mathbb{N}$, the space $\mathbb{R}^{2n}$ has a *canonical complex structure* defined by $J(x,y)=(-y,x)$, for $x,y\in\mathbb{R}^n$. We can identify $(\mathbb{R}^{2n},J)$ with the complex vector space $\mathbb{C}^n$ by $(x,y)\mapsto x+iy$. In terms of matrix representations, we have:

$$J=\begin{pmatrix}0 & -\mathrm{I}\\ \mathrm{I} & 0\end{pmatrix}, \tag{1.2.2}$$

where 0 and I denote respectively the 0 and the identity $n\times n$ matrices.

We have the following simple Lemma:

1.2.3. LEMMA. *Let (V_1,J_1) and (V_2,J_2) be real vector spaces endowed with complex structures. A $\mathbb{R}$-linear operator $T:V_1\to V_2$ is $\mathbb{C}$-linear if and only if $T\circ J_1=J_2\circ T$. In particular, the $\mathbb{C}$-linear endomorphisms of a vector space with complex structure (V,J) are the $\mathbb{R}$-linear endomorphisms of V that commute with J.*

PROOF. Left to the reader in Exercise 1.3. □

1.2.4. REMARK. Observe that if J is a complex structure on V, then also $-J$ is a complex structure, that will be called the *conjugate complex structure*. For $\lambda \in \mathbb{C}$ and $v \in V$, the product of λ and v in the complex space $(V, -J)$ is given by the product of $\bar{\lambda}$ and v in the complex space (V, J), where $\bar{\lambda}$ is the complex conjugate of λ. The set of complex bases of (V, J) and $(V, -J)$ coincide; observe however that the components of a vector in a fixed basis are conjugated when replacing J by $-J$.

A $\mathbb{C}$-linear operator T between complex spaces is still $\mathbb{C}$-linear when replacing the complex structures by their conjugates in both the domain and the counterdomain. The representations of T with respect to fixed bases in the complex structures and the same bases in the conjugate complex structures are given by conjugate matrices.

1.2.5. DEFINITION. A map T between complex vector spaces is said to be *anti-linear*, or *conjugate linear*, if it is additive and if $T(\lambda v) = \bar{\lambda} T(v)$ for all $\lambda \in \mathbb{C}$ and all v in the domain of T.

An anti-linear map is always $\mathbb{R}$-linear when we see it as a map between the realifications of the domain and the counterdomain. Moreover, a map is anti-linear if and only if it is $\mathbb{C}$-linear when the complex structure of its domain (or of its counter domain) is replaced by the complex conjugate. In particular, the anti-linear endomorphisms of (V, J) are the $\mathbb{R}$-linear endomorphisms of V that *anti-commute* with J.

We have the following relation between the bases of (V, J) and of V:

1.2.6. PROPOSITION. *Let V be a (possibly infinite dimensional) real vector space and J a complex structure on V. If $(b_j)_{j \in \mathcal{J}}$ is a $\mathbb{C}$-basis of (V, J), then the union of $(b_j) j \in \mathcal{J}$ and $\big(J(b_j)\big)_{j \in \mathcal{J}}$ is an $\mathbb{R}$-basis of V.*

PROOF. Left to the reader in Exercise 1.4. □

1.2.7. COROLLARY. *The real dimension of V is twice the complex dimension of (V, J); in particular, a (finite dimensional) vector space admits a complex structure if and only if its dimension is an even number.*

PROOF. We only need to show that every real vector space of even dimension admits a complex structure. This is easily established by choosing an isomorphism with $\mathbb{R}^{2n}$ and using the canonical complex structure given in Example 1.2.2. □

1.2.8. EXAMPLE. If (V, J) is a real vector space with complex structure, then the dual complex space of (V, J) is given by the set of $\mathbb{R}$-linear maps $\alpha : V \to \mathbb{C}$ such that:

$$\alpha \circ J(v) = i\,\alpha(v), \quad v \in V. \tag{1.2.3}$$

It is easy to see that (1.2.8) determines the imaginary part of α when it is known its real part; hence we have an $\mathbb{R}$-linear isomorphism:

$$(V, J)^* \ni \alpha \longmapsto \Re \circ \alpha \in V^*, \tag{1.2.4}$$

where $\Re : \mathbb{C} \to \mathbb{R}$ denotes the real part operator. The isomorphism (1.2.4) therefore induces a unique complex structure of V^* that makes (1.2.4) into a $\mathbb{C}$-*linear* isomorphism. Such complex structure is called the *dual complex structure*, and it is easy to see that it is given simply by the transpose operator J^*.

We conclude this section with a relation between the matrix representations of vectors and operators in real and complex bases. Let (V, J) be a $2n$-dimensional vector space with complex structure; a basis of V *adapted to* J , shortly a J-*basis*, is a basis of the form

$$(b_1, \ldots, b_n, J(b_1), \ldots, J(b_n)); \tag{1.2.5}$$

in this case, $(b_j)_{j=1}^n$ is a complex basis of (V, J). For instance, the canonical basis of $\mathbb{R}^{2n}$, endowed with the canonical complex structure, is a J-basis corresponding to the canonical basis of $\mathbb{C}^n$. In other words, the J-bases of a vector space are precisely those with respect to which the matrix representations of J is that given by (1.2.2). The existence of J-bases is given by Proposition 1.2.6.

Let a J-basis of V be fixed, corresponding to a complex basis $\mathcal{B} = (b_j)_{j=1}^n$ of (V, J). Given $v \in V$ with coordinates $(z_1, \ldots, z_n)$ in the basis $\mathcal{B}$, then its coordinates in the (real) J-basis of V are:

$$v \sim (x_1, \ldots, x_n, y_1, \ldots, y_n),$$

where $z_j = x_j + i\, y_j$, $x_j, y_j \in \mathbb{R}$. If T is a $\mathbb{C}$-linear operator represented in the complex basis by the matrix $Z = A + i\, B$ (A and B real), then its representation in the corresponding J-basis is:

$$T \sim \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{1.2.6}$$

1.2.9. REMARK. Formula (1.2.6) tells us that the map

$$Z = A + B\, i \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

is an *injective homomorphism* of the algebra of complex $n \times n$ matrices into the algebra of real $2n \times 2n$ matrices.

1.3. Complexification and Real Forms

In this section we show that any real vector space can be "extended" in a canonical way to a complex vector space, by mimicking the relation between $\mathbb{R}^n$ and $\mathbb{C}^n$; such an extension will be called a complexification of the space. We also show that, given a complex space, it can be seen as the complexification of several of its real subspaces, that will be called the real forms of the complex space. We will only consider the case of finite dimensional spaces, even though many of the results presented will hold in the case of infinite dimensional spaces, up to minor modifications. Some of the proofs are elementary, and they will be omitted and left to the reader as Exercises.

1.3.1. DEFINITION. Let $\mathcal{V}$ be a complex vector space; a *real form* in $\mathcal{V}$ is a real subspace $\mathcal{V}_0$ of $\mathcal{V}$ (or, more precisely, a subspace of the realification $\mathcal{V}_{\mathbb{R}}$ of $\mathcal{V}$) such that:

$$\mathcal{V}_{\mathbb{R}} = \mathcal{V}_0 \oplus i\,\mathcal{V}_0.$$

In other words, a real form $\mathcal{V}_0$ in $\mathcal{V}$ is a real subspace such that every $v \in \mathcal{V}$ can be written uniquely in the form $v = v_1 + i\,v_2$, with $v_1, v_2 \in \mathcal{V}_0$.

To a real form $\mathcal{V}_0$ we associate maps:

$$\Re : \mathcal{V} \longrightarrow \mathcal{V}_0, \quad \Im : \mathcal{V} \longrightarrow \mathcal{V}_0, \quad \mathfrak{c} : \mathcal{V} \longrightarrow \mathcal{V}, \tag{1.3.1}$$

given by $\Re(v_1 + i\,v_2) = v_1$, $\Im(v_1 + i\,v_2) = v_2$ and $\mathfrak{c}(v_1 + i\,v_2) = v_1 - i\,v_2$, for all $v_1, v_2 \in \mathcal{V}$. We call $\Re$, $\Im$ and $\mathfrak{c}$ respectively the *real part, imaginary part*, and *conjugation* operators associated to the real form $\mathcal{V}_0$. All these operators are $\mathbb{R}$-linear; the operator $\mathfrak{c}$ is also anti-linear. For $v \in \mathcal{V}$, we also say that $\mathfrak{c}(v)$ is the *conjugate* of v relatively to the real form $\mathcal{V}_0$, and we also write:

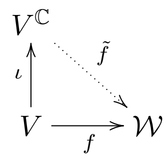
$$\mathfrak{c}(v) = \bar{v}.$$

1.3.2. DEFINITION. Let V be a real vector space. A *complexification* of V is a pair $(V^{\mathbb{C}}, \iota)$, where $V^{\mathbb{C}}$ is a complex vector space and $\iota : V \to V^{\mathbb{C}}$ is an injective $\mathbb{R}$-linear map such that $\iota(V)$ is a real form in $V^{\mathbb{C}}$.

The result of the following Proposition is usually known as the *universal property of the complexification*:

1.3.3. PROPOSITION. *Let V be a real vector space, $(V^{\mathbb{C}}, \iota)$ a complexification of V and $\mathcal{W}$ a complex vector space. Then, given an $\mathbb{R}$-linear map $f : V \to \mathcal{W}_{\mathbb{R}}$, there exists a unique $\mathbb{C}$-linear map $\tilde{f} : V^{\mathbb{C}} \to \mathcal{W}$ such that the following diagram commutes:*

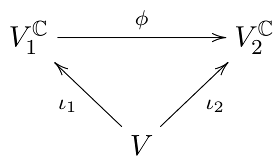
(1.3.2)



PROOF. Left to the reader in Exercise 1.5. □

As corollary, we obtain the uniqueness of a complexification, up to isomorphisms:

1.3.4. COROLLARY. *Suppose that $(V_1^{\mathbb{C}}, \iota_1)$ and $(V_2^{\mathbb{C}}, \iota_2)$ are complexifications of V. Then, there exists a unique $\mathbb{C}$-linear isomorphism $\phi : V_1^{\mathbb{C}} \to V_2^{\mathbb{C}}$ such that the following diagram commutes:*



PROOF. Left to the reader in Exercise 1.6. □

If V is a real vector space, we can make the direct sum $V \oplus V$ into a complex space by taking the complex structure $J(v,w) = (-w,v)$. Setting $\iota(v) = (v,0)$, it is easy to see that $(V \oplus V, \iota)$ is a complexification of V, that will be called the *canonical complexification* of the real vector space V.

By Corollary 1.3.4, we do not need to distinguish between complexifications of a vector space; so, from now on, we will denote by $V^{\mathbb{C}}$ the canonical complexification of V, or, depending on the context, we may use the symbol $V^{\mathbb{C}}$ to denote some other complexification of V, which is necessarily isomorphic to the canonical one.

The original space V will then be identified with $\iota(V)$, so that we will always think of an inclusion $V \subset V^{\mathbb{C}}$; since $\iota(V)$ is a real form in $V^{\mathbb{C}}$, then $V^{\mathbb{C}}$ is a direct sum of V and $i\,V$:

$$V^{\mathbb{C}} = V \oplus i\,V.$$

1.3.5. EXAMPLE. The subspace $\mathbb{R}^n \subset \mathbb{C}^n$ is a real form in $\mathbb{C}^n$, hence $\mathbb{C}^n$ is a complexification of $\mathbb{R}^n$.

1.3.6. EXAMPLE. The space $M_n(\mathbb{R})$ of real $n \times n$ matrices is a real form in the space $M_n(\mathbb{C})$ of complex $n \times n$ matrices.

A less trivial example of a real form in $M_n(\mathbb{C})$ is given by $\mathrm{u}(n)$, which is the space of *anti-Hermitian matrices*, i.e., matrices A such that $A^* = -A$, where A^* denotes the conjugate transpose matrix of A. In this example, $i\,\mathrm{u}(n)$ is the space of *Hermitian matrices*, i.e., the space of those matrices A such that $A^* = A$. It is easy to see that $M_n(\mathbb{C}) = \mathrm{u}(n) \oplus i\,\mathrm{u}(n)$, and so $\mathrm{u}(n)$ is a real form in $M_n(\mathbb{C})$ and $M_n(\mathbb{C})$ is a complexification of $\mathrm{u}(n)$.

1.3.7. EXAMPLE. If $\mathcal{V}$ is a complex vector space and if $(b_j)_{j=1}^n$ is a complex basis of $\mathcal{V}$, then the real subspace $\mathcal{V}_0$ of $\mathcal{V}_{\mathbb{R}}$ given by:

$$\mathcal{V}_0 = \Big\{ \sum_{j=1}^{n} \lambda_j\, b_j : \lambda_j \in \mathbb{R},\ \ \forall\, j \Big\}$$

is a real form in $\mathcal{V}$.

Actually, every real form of $\mathcal{V}$ can be obtained in this way; for, if $\mathcal{V}_0 \subset \mathcal{V}$ is a real form, then an $\mathbb{R}$-basis $(b_j)_{j=1}^n$ of $\mathcal{V}_0$ is also a $\mathbb{C}$-basis of $\mathcal{V}$. It follows in particular that the real dimension of a real form $\mathcal{V}_0$ is equal to the complex dimension of $\mathcal{V}$.

Example 1.3.7 tells us that every complex space admits infinitely many real forms; in the following proposition we give a characterization of the real forms in a complex space. We recall that a bijection ϕ of a set is said to be an *involution* if $\phi^2 = \phi \circ \phi = \mathrm{Id}$.

1.3.8. PROPOSITION. *Let $\mathcal{V}$ be a complex space. Then there exists a bijection between the set of real forms in $\mathcal{V}$ and the set of the anti-linear involutive automorphisms of $\mathcal{V}$. Such bijection is obtained by:*

- *associating to each real form $\mathcal{V}_0 \subset \mathcal{V}$ its conjugation operator $\mathfrak{c}$ (see* (1.3.1)*);*

- *associating to each anti-linear involutive automorphism* $\mathfrak{c}$ *of* $\mathcal{V}$ *the set of its fixed points* $\mathcal{V}_0 = \{v \in \mathcal{V} : \mathfrak{c}(v) = v\}$. □

The above result suggests an interesting comparison between the operation of realification of a complex space and the operation of complexification of a real space. In Section 1.2 we saw that, roughly speaking, the operations of realification and of addition of a complex structure are mutually inverse; the realification is a canonical procedure, while the addition of a complex structure employs an additional information, which is an automorphism J with $J^2 = -\mathrm{Id}$. In this section we have the opposite situation. The complexification is a canonical operation, while its "inverse" operation, which is the passage to a real form, involves an additional information, which is an anti-linear involutive automorphism.

Let us look now at the complexification as a *functorial* construction. Let V_1 and V_2 be real spaces; from the universal property of the complexification (Proposition 1.3.3) it follows that each linear operator $T : V_1 \to V_2$ admits a unique $\mathbb{C}$-linear extension $T^{\mathbb{C}} : V_1^{\mathbb{C}} \to V_2^{\mathbb{C}}$. We have the following commutative diagram:

$$\begin{array}{ccc} V_1^{\mathbb{C}} & \xrightarrow{T^{\mathbb{C}}} & V_2^{\mathbb{C}} \\ {\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle\iota} \\ V_1 & \xrightarrow[T]{} & V_2 \end{array}$$

The operator $T^{\mathbb{C}}$ is called the *complexification* of T; more concretely, we have that $T^{\mathbb{C}}$ is given by:

$$T^{\mathbb{C}}(v + i\,w) = T(v) + i\,T(w), \quad v, w \in V_1.$$

It is immediate that:

$$(T_1 \circ T_2)^{\mathbb{C}} = T_1^{\mathbb{C}} \circ T_2^{\mathbb{C}}, \quad \mathrm{Id}^{\mathbb{C}} = \mathrm{Id}, \tag{1.3.3}$$

and, when T is invertible

$$(T^{\mathbb{C}})^{-1} = (T^{-1})^{\mathbb{C}}. \tag{1.3.4}$$

The identities (1.3.3) imply that the complexification $V \to V^{\mathbb{C}}$, $T \mapsto T^{\mathbb{C}}$ is a *functor* from the category of real vector spaces with morphisms the $\mathbb{R}$-linear operators to the category of complex vector spaces with morphisms the $\mathbb{C}$-linear operators.

Given a linear operator $T : V_1 \to V_2$, it is easy to see that:

$$\mathrm{Ker}(T^{\mathbb{C}}) = (\mathrm{Ker}(T))^{\mathbb{C}}, \quad \mathrm{Im}(T^{\mathbb{C}}) = \mathrm{Im}(T)^{\mathbb{C}}; \tag{1.3.5}$$

in the context of categories, the identities (1.3.5) say that the complexification is an *exact functor*, i.e., it takes short exact sequences into short exact sequences.

If $U \subset V$ is a subspace, it is easy to see that the complexification $\mathfrak{i}^{\mathbb{C}}$ of the inclusion $\mathfrak{i} : U \to V$ is injective, and it therefore gives an identification of $U^{\mathbb{C}}$ with a subspace of $V^{\mathbb{C}}$. More concretely, the subspace $U^{\mathbb{C}}$ of $V^{\mathbb{C}}$ is the direct sum of the two real subspaces U and $i\,U$ of $V^{\mathbb{C}}$; equivalently, $U^{\mathbb{C}}$ is the complex subspace of $V^{\mathbb{C}}$ generated by the set $U \subset V^{\mathbb{C}}$. However, not every subspace of $V^{\mathbb{C}}$ is the complexification of some subspace of V. We have the following characterization:

1.3.9. LEMMA. *Let V be a real vector space and let $\mathcal{Z} \subset V^{\mathbb{C}}$ be a complex subspace of its complexification. Then, there exists a real subspace $U \subset V$ with $\mathcal{Z} = U^{\mathbb{C}}$ if and only if $\mathcal{Z}$ is* invariant by conjugation, *i.e.,*

$$\mathfrak{c}(\mathcal{Z}) \subset \mathcal{Z},$$

where $\mathfrak{c}$ is the conjugation relative to the real form $V \subset V^{\mathbb{C}}$. If $\mathcal{Z} = U^{\mathbb{C}}$, then such U is uniquely determined, and it is given explicitly by $U = \mathcal{Z} \cap V$.

PROOF. Left to the reader in Exercise 1.7. □

Given real vector spaces V_1 and V_2, observe that the map:

$$\mathrm{Lin}(V_1, V_2) \ni T \longmapsto T^{\mathbb{C}} \in \mathrm{Lin}(V_1^{\mathbb{C}}, V_2^{\mathbb{C}}) \tag{1.3.6}$$

is $I\!R$-linear; we actually have the following:

1.3.10. LEMMA. *The map* (1.3.6) *takes* $\mathrm{Lin}(V_1, V_2)$ *isomorphically onto a real form in* $\mathrm{Lin}(V_1^{\mathbb{C}}, V_2^{\mathbb{C}})$.

PROOF. Since $(V_2^{\mathbb{C}})_{I\!R} = V_2 \oplus i\, V_2$, it is easy to see that:

$$\mathrm{Lin}\Big(V_1, \big(V_2^{\mathbb{C}}\big)_{I\!R}\Big) = \mathrm{Lin}(V_1, V_2) \oplus i\mathrm{Lin}(V_1, V_2). \tag{1.3.7}$$

From the universal property of the complexification, it follows that the *restriction operator*

$$\mathrm{Lin}\big(V_1^{\mathbb{C}}, V_2^{\mathbb{C}}\big) \ni \mathcal{S} \xrightarrow{\cong} \mathcal{S}|_{V_1} \in \mathrm{Lin}\Big(V_1, \big(V_2^{\mathbb{C}}\big)_{I\!R}\Big) \tag{1.3.8}$$

is an isomorphism. From (1.3.7) and (1.3.8) it follows:

$$\mathrm{Lin}\big(V_1^{\mathbb{C}}, V_2^{\mathbb{C}}\big) \cong \mathrm{Lin}(V_1, V_2) \oplus \mathrm{Lin}(V_1, V_2), \tag{1.3.9}$$

where the two summands on the right of (1.3.9) are identified respectively with the image of (1.3.6) and with the same image multiplied by i. □

From Lemma 1.3.8 it follows in particular that the dual $V^* = \mathrm{Lin}(V, I\!R)$ can be identified with a real form of the dual of the complexification $(V^{\mathbb{C}})^* = \mathrm{Lin}(V^{\mathbb{C}}, \mathbb{C})$ (compare with Example 1.2.8).

Along the same lines of Lemma 1.3.9, in the next lemma we characterize the image of (1.3.6):

1.3.11. LEMMA. *Let V_1, V_2 be real vector spaces. Given a $\mathbb{C}$-linear map $\mathcal{S} : V_1^{\mathbb{C}} \to V_2^{\mathbb{C}}$, the following statements are equivalent:*

- *there exists an $I\!R$-linear map $T : V_1 \to V_2$ such that $\mathcal{S} = T^{\mathbb{C}}$;*
- $\mathcal{S}$ preserves real forms, *i.e., $\mathcal{S}(V_1) \subset V_2$;*
- $\mathcal{S}$ commutes with conjugation, *i.e.,* $\mathfrak{c} \circ \mathcal{S} = \mathcal{S} \circ \mathfrak{c}$, *where $\mathfrak{c}$ denotes the conjugation operators in $V_1^{\mathbb{C}}$ and$V_2^{\mathbb{C}}$ with respect to the real forms V_1 and V_2 respectively.*

When one (hence all) of the above conditions is satisfied, there exists a unique $T \in Lin(V_1, V_2)$ such that $\mathcal{S} = T^{\mathbb{C}}$, which is given by the restriction of $\mathcal{S}$. □

1.3.12. EXAMPLE. Let V_1, V_2 be real vector spaces; choosing bases for V_1 and V_2, the same will be bases for the complexifications $V_1^{\mathbb{C}}$ and $V_2^{\mathbb{C}}$ (see Example 1.3.7). With respect to these bases, the matrix representation of a linear operator $T : V_1 \to V_2$ is *equal* to the matrix representation of its complexification $T^{\mathbb{C}} : V_1^{\mathbb{C}} \to V_2^{\mathbb{C}}$ (compare with the result of Section 1.2, and more in particular with formula (1.2.6)). In terms of matrix representations, the map (1.3.6) is simply the inclusion of the real matrices into the complex matrices.

1.3.13. EXAMPLE. The real form in $\mathrm{Lin}(V_1^{\mathbb{C}}, V_2^{\mathbb{C}})$ defined in the statement of Lemma 1.3.10 corresponds to a conjugation operator in $\mathrm{Lin}(V_1^{\mathbb{C}}, V_2^{\mathbb{C}})$; given $\mathcal{S} \in \mathrm{Lin}(V_1^{\mathbb{C}}, V_2^{\mathbb{C}})$, we denote by $\overline{\mathcal{S}}$ its *conjugate operator*. Explicitly, $\overline{\mathcal{S}}$ is given by:

$$\overline{\mathcal{S}} = \mathfrak{c} \circ \mathcal{S} \circ \mathfrak{c}.$$

For, using Proposition 1.3.8 and Lemma 1.3.11, it suffices to observe that $\mathcal{S} \mapsto \mathfrak{c} \circ \mathcal{S} \circ \mathfrak{c}$ defines an anti-linear involutive automorphism of $\mathrm{Lin}(V_1^{\mathbb{C}}, V_2^{\mathbb{C}})$ whose fixed point set is the image of (1.3.6). Observe that we have the identity:

$$\overline{\mathcal{S}(v)} = \overline{\mathcal{S}}(\bar{v}), \quad v \in V_1^{\mathbb{C}}.$$

In terms of bases, the matrix representation of $\overline{\mathcal{S}}$ is the *complex conjugate* of the matrix representation of $\mathcal{S}$.

The theory presented in this section can be easily generalized to the case of *multi-linear* operators, anti-linear operators and operators with “mixed” multi-linearity, like sesquilinear operators. The latter case has special importance:

1.3.14. DEFINITION. Given complex vector spaces $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}$, we say that a map $\mathcal{B} : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathcal{V}$ is *sesquilinear* if for all $v_1 \in \mathcal{V}_1$ the map $\mathcal{B}(v_1, \cdot)$ is anti-linear and for all $v_2 \in \mathcal{V}_2$ the map $\mathcal{B}(\cdot, v_2)$ is $\mathbb{C}$-linear.

If $\mathcal{V}_1 = \mathcal{V}_2$ and if a real form is fixed in $\mathcal{V}$, we say that a sesquilinear map $\mathcal{B}$ is *Hermitian* (respectively, *anti-Hermitian*) if $\mathcal{B}(v_1, v_2) = \overline{\mathcal{B}(v_2, v_1)}$ (respectively, $\mathcal{B}(v_1, v_2) = -\overline{\mathcal{B}(v_2, v_1)}$) for all $v_1, v_2 \in \mathcal{V}_1$.

A *Hermitian form* in a complex space $\mathcal{V}$ is a sesquilinear Hermitian map $\mathcal{B} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$; if $\mathcal{B}$ is *positive definite*, i.e., $\mathcal{B}(v, v) > 0$ for all $v \neq 0$, we also say that $\mathcal{B}$ is a *positive Hermitian product*, or simply an *Hermitian product*, in $\mathcal{V}$.

In the same way that we define the complexification $T^{\mathbb{C}}$ for an $I\!R$-linear operator, we can define the complexification $B^{\mathbb{C}}$ of an $I\!R$-multilinear operator $B : V_1 \times \cdots \times V_p \to V$ as its *unique* extension to a $\mathbb{C}$-multi-linear operator $B^{\mathbb{C}} : V_1^{\mathbb{C}} \times \cdots \times V_p^{\mathbb{C}} \to V^{\mathbb{C}}$. Similarly, we can associate to an $I\!R$-linear operator its unique extension $T^{\underline{\mathbb{C}}} : V_1^{\mathbb{C}} \to V_2^{\mathbb{C}}$ to an *anti-linear operator*, and to an $I\!R$-bilinear operator $B : V_1 \times V_2 \to V$ its unique *sesquilinear extension* $B^{\mathbb{C}_s} : V_1^{\mathbb{C}} \times V_2^{\mathbb{C}} \to V^{\mathbb{C}}$.

In Exercise 1.8 the reader is asked to generalize the results of this section, in particular Proposition 1.3.3, Lemma 1.3.10 and Lemma 1.3.11, to the case of multi-linear, conjugate linear or sesquilinear operators.

1.3.15. EXAMPLE. If V is a real vector space and $B \in \mathrm{B_{sym}}(V)$ is a symmetric bilinear form on V, then the bilinear extension $B^{\mathbb{C}}$ of B to $V^{\mathbb{C}}$ is symmetric; on the other hand, the sesquilinear extension $B^{\mathbb{C}_s}$ of B is a Hermitian form on $V^{\mathbb{C}}$. Similarly, the bilinear extension of an anti-symmetric bilinear form is anti-symmetric, while the sesquilinear extension of an anti-symmetric form is anti-Hermitian.

The notions of kernel (see (1.1.5)), nondegeneracy and orthogonal complement (see (1.1.8)) for symmetric and anti-symmetric bilinear forms generalize in an obvious way to sesquilinear Hermitian and anti-Hermitian forms. If B is symmetric (or anti-symmetric), it is easy to see that the condition of nondegeneracy of B is equivalent to the nondegeneracy of either $B^{\mathbb{C}}$ or $B^{\mathbb{C}_s}$. Moreover, if $B \in \mathrm{B_{sym}}(V)$ is *positive definite*, i.e., $B(v, v) > 0$ for all $v \neq 0$, then its sesquilinear extension $B^{\mathbb{C}_s}$ is also positive definite. Observe that the $\mathbb{C}$-bilinear extension $B^{\mathbb{C}}$ will be nondegenerate, *but it is not positive definite* (see Exercise 1.9).

For instance, the *canonical inner product of* $\mathbb{R}^n$ is given by:

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j.$$

Its sesquilinear extension defines the *canonical Hermitian product in* $\mathbb{C}^n$, given by

$$\langle z, w \rangle^{\mathbb{C}_s} = \sum_{j=1}^{n} z_j \bar{w}_j, \tag{1.3.10}$$

while its $\mathbb{C}$-*bilinear extension* is given by:

$$\langle z, w \rangle^{\mathbb{C}} = \sum_{j=1}^{n} z_j w_j.$$

1.3.16. REMARK. In the spirit of Definition 1.1.5, given a complex space $\mathcal{V}$ and a Hermitian form $\mathcal{B}$ in $\mathcal{V}$, we say that a $\mathbb{C}$-linear operator $\mathcal{T} \in \mathrm{Lin}(\mathcal{V})$ is $\mathcal{B}$-*Hermitian* (respectively, $\mathcal{B}$-*anti-Hermitian*) if $\mathcal{B}(\mathcal{T}\cdot, \cdot)$ is a Hermitian (respectively, anti-Hermitian) form. We also say that $\mathcal{T}$ is $\mathcal{B}$-*unitary* if $\mathcal{B}(\mathcal{T}\cdot, \mathcal{T}\cdot) = \mathcal{B}$.

Given a real vector space V, $B \in \mathrm{B_{sym}}(V)$ and if $T \in \mathrm{Lin}(V)$ is a B-symmetric (respectively, B-anti-symmetric) operator, then its complexification $T^{\mathbb{C}}$ in $\mathrm{Lin}(V^{\mathbb{C}})$ is a $B^{\mathbb{C}_s}$-Hermitian (respectively, $B^{\mathbb{C}_s}$-anti-Hermitian) operator.

If T is B-orthogonal, then $T^{\mathbb{C}}$ is $B^{\mathbb{C}_s}$-unitary.

1.3.1. Complex structures and complexifications. The aim of this subsection is to show that there exists a natural correspondence between the complex structures of a real space V and certain direct sum decompositions of its complexification $V^{\mathbb{C}}$.

Let V be a real vector space and let $J : V \to V$ be a complex structure in V; we have that $J^{\mathbb{C}}$ is a $\mathbb{C}$-linear automorphism of the complexification $V^{\mathbb{C}}$ that satisfies $(J^{\mathbb{C}})^2 = -\mathrm{Id}$. It is then easy to see that $V^{\mathbb{C}}$ decomposes as the direct sum of the two eigenspaces of $J^{\mathbb{C}}$ corresponding to the eigenvalues i and $-i$ respectively;

more explicitly, we define:

$$V^{\mathfrak{h}} = \big\{ v \in V^{\mathbb{C}} : J^{\mathbb{C}}(v) = iv \big\},$$
$$V^{\mathfrak{a}} = \big\{ v \in V^{\mathbb{C}} : J^{\mathbb{C}}(v) = -iv \big\}.$$

Then, $V^{\mathfrak{h}}$ and $V^{\mathfrak{a}}$ are complex subspaces of $V^{\mathbb{C}}$, and $V^{\mathbb{C}} = V^{\mathfrak{h}} \oplus V^{\mathfrak{a}}$; the projections onto the subspaces $V^{\mathfrak{h}}$ and $V^{\mathfrak{a}}$ are given by:

$$\pi^{\mathfrak{h}}(v) = \frac{v - iJ^{\mathbb{C}}(v)}{2}, \quad \pi^{\mathfrak{a}}(v) = \frac{v + iJ^{\mathbb{C}}(v)}{2}, \quad v \in V^{\mathbb{C}}. \tag{1.3.11}$$

We call the spaces $V^{\mathfrak{h}}$ and $V^{\mathfrak{a}}$ respectively the *holomorphic* and the *anti-holomorphic* subspaces of $V^{\mathbb{C}}$. Next proposition justifies the names of these spaces (see also Example 1.3.18 below):

1.3.17. PROPOSITION. *Let V be a real vector space and J a complex structure in V. Then, the projections $\pi^{\mathfrak{h}}$ and $\pi^{\mathfrak{a}}$ given in* (1.3.11) *restricted to V define respectively a $\mathbb{C}$-linear isomorphism of (V, J) onto $V^{\mathfrak{h}}$ and a $\mathbb{C}$-anti-linear isomorphism of (V, J) onto $V^{\mathfrak{a}}$.* □

Proposition 1.3.17 tells us that, if we complexify a space V that already possesses a complex structure J, we obtain a complex space $V^{\mathbb{C}}$ that contains a copy of the original space (V, J) (the holomorphic subspace) and a copy of $(V, -J)$ (the anti-holomorphic subspace). Observe also that the holomorphic and the anti-holomorphic subspaces of $V^{\mathbb{C}}$ are *mutually conjugate*:

$$V^{\mathfrak{a}} = \mathfrak{c}\big(V^{\mathfrak{h}}\big), \quad V^{\mathfrak{h}} = \mathfrak{c}\big(V^{\mathfrak{a}}\big),$$

where $\mathfrak{c}$ denotes the conjugation of $V^{\mathbb{C}}$ relative to the real form V.

In our next example we make a short digression to show how the theory of this subsection appears naturally in the context of calculus with functions of several complex variables.

1.3.18. EXAMPLE. The construction of the holomorphic and the anti-holomorphic subspaces appears naturally when one studies calculus of several complex variables, or, more generally, when one studies the geometry of complex manifolds.

In this example we consider the space $\mathbb{C}^n$, that will be thought as the real space $\mathbb{R}^{2n}$ endowed with the canonical complex structure. The real canonical basis of $\mathbb{C}^n \simeq (\mathbb{R}^{2n}, J)$ will be denoted by:

$$\Big(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\Big);$$

this is a basis of $\mathbb{R}^{2n}$ adapted to J, and the corresponding complex basis of $\mathbb{C}^n$ is given by:

$$\big(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\big).$$

We now consider another complex space, given by the complexification $(\mathbb{R}^{2n})^{\mathbb{C}} \simeq \mathbb{C}^{2n}$. We denote by J the multiplication by the scalar i in $\mathbb{C}^n$, while in $\mathbb{C}^{2n}$ such multiplication will be denoted in the usual way $v \mapsto i\,v$. Let $J^{\mathbb{C}} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the complexification of J, which defines the holomorphic and the anti-holomorphic subspaces of $\mathbb{C}^{2n}$.

By Proposition 1.3.17, the projections $\pi^{\mathfrak{h}}$ and $\pi^{\mathfrak{a}}$ defined in (1.3.11) map the canonical complex basis of $\mathbb{C}^n$ respectively into a basis of the holomorphic subspace and a basis of the anti-holomorphic subspace of $\mathbb{C}^{2n}$. These bases are usually denoted by $\left(\frac{\partial}{\partial z^j}\right)_{j=1}^n$ and $\left(\frac{\partial}{\partial \bar{z}^j}\right)_{j=1}^n$; using (1.3.11) we compute explicitly:

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \quad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right).$$

Observe that the vector $\frac{\partial}{\partial \bar{z}^j}$ is conjugate to the vector $\frac{\partial}{\partial z^j}$.

The notation $\frac{\partial}{\partial x^j}$, $\frac{\partial}{\partial y^j}$ for the canonical basis of $I\!R^{2n}$ is justified by the identification of vectors in $I\!R^{2n}$ with the *partial derivative operators* on differentiable functions $f : I\!R^{2n} \to I\!R$. The complexification of $I\!R^{2n}$ is therefore identified with the space of partial derivative operators acting on complex differentiable functions $f : I\!R^{2n} \to \mathbb{C}$; in this notation, the *Cauchy–Riemann equations*, that characterize the *holomorphic functions*, are given by setting equal to 0 the derivatives in the directions of the anti-holomorphic subspace:

$$\frac{\partial}{\partial \bar{z}^j} f = 0, \quad j = 1, \ldots, n. \tag{1.3.12}$$

Observe that f satisfies (1.3.12) if and only if its differential at each point is a $\mathbb{C}$-linear operator from $\mathbb{C}^n \simeq (I\!R^{2n}, J)$ to $\mathbb{C}$.

We now show that the decomposition into holomorphic and anti-holomorphic subspace determines the complex structure:

1.3.19. PROPOSITION. *Let V be a real vector space and consider a direct sum of the complexification $V^{\mathbb{C}} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$, where $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are mutually conjugate subspaces of $V^{\mathbb{C}}$. Then, there exists a unique complex structure J on V such that $\mathcal{Z}_1 = V^{\mathfrak{h}}$; moreover, for such J, it is also $\mathcal{Z}_2 = V^{\mathfrak{a}}$.*

PROOF. The uniqueness follows from the fact that $V^{\mathfrak{h}}$ is the graph of $-J$ when we use the isomorphism $V^{\mathbb{C}} \simeq V \oplus V$. For the existence, consider the unique $\mathbb{C}$-linear operator in $V^{\mathbb{C}}$ that has $\mathcal{Z}_1$ and $\mathcal{Z}_2$ as eigenspaces corresponding to the eigenvalues i and $-i$ respectively. Clearly, such operator commutes with the conjugation and its square equals $-\mathrm{Id}$. From Lemma 1.3.11 it follows that it is of the form $J^{\mathbb{C}}$ for some complex structure $J : V \to V$. □

Let now T be a $\mathbb{C}$-linear endomorphism of (V, J), i.e., an $I\!R$-linear endomorphism of V such that $T \circ J = J \circ T$; let $T^{\mathbb{C}}$ be its complexification. It is easy to see that the the holomorphic and the anti-holomorphic subspaces of $V^{\mathbb{C}}$ are invariant by $T^{\mathbb{C}}$; moreover, we have the following commutative diagrams:

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ {\scriptstyle \pi^{\mathfrak{h}}|_V}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle \pi^{\mathfrak{h}}|_V} \\ V^{\mathfrak{h}} & \xrightarrow[T^{\mathbb{C}}]{} & V^{\mathfrak{h}} \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{T} & V \\ {\scriptstyle \pi^{\mathfrak{a}}|_V}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle \pi^{\mathfrak{a}}|_V} \\ V^{\mathfrak{a}} & \xrightarrow[T^{\mathbb{C}}]{} & V^{\mathfrak{a}} \end{array} \tag{1.3.13}$$

It follows from Proposition 1.3.17 that the vertical arrows in the diagram on the left are $\mathbb{C}$-linear isomorphisms of (V, J) with $V^{\mathfrak{h}}$ and the vertical arrows in the diagram on the right are $\mathbb{C}$-linear isomorphisms of $(V, -J)$ in $V^{\mathfrak{a}}$.

Let now $(b_j)_{j=1}^n$ be a complex basis of (V, J) and let $(b_j, J(b_j))_{j=1}^n$ be the corresponding real basis of V adapted to J. The latter is also a complex basis for $V^{\mathbb{C}}$ (see Example 1.3.7). By Proposition 1.3.17, the vectors u_j, $\bar{u}_j$ defined by:

$$u_j = \frac{b_j - iJ(b_j)}{2} \in V^{\mathfrak{h}}, \quad \bar{u}_j = \frac{b_j + iJ(b_j)}{2} \in V^{\mathfrak{a}}, \quad j = 1, \dots, n \tag{1.3.14}$$

form a complex basis of (V, J). If T is represented by the matrix $Z = A + B\,i$, with A, B real matrices, in the basis $(b_j)_{j=1}^n$ of $V^{\mathbb{C}}$ (hence it is represented by the matrix (1.2.6) with respect to the real basis of V), then it follows from (1.3.13) that the matrix representation of $T^{\mathbb{C}}$ with respect to the basis $(u_j, \bar{u}_j)_{j=1}^n$ of $V^{\mathbb{C}}$ is given by:

$$T^{\mathbb{C}} \sim \begin{pmatrix} Z & 0 \\ 0 & \overline{Z} \end{pmatrix}. \tag{1.3.15}$$

On the other hand, the matrix representation of $T^{\mathbb{C}}$ with respect to the basis $(b_j, J(b_j))_{j=1}^n$ is again (1.2.6) (see Example 1.3.12). This shows in particular that the matrices in (1.2.6) and in (1.3.15) are *equivalent* (or *conjugate*, i.e., representing the same operator in different bases).

We summarize the above observations into the following:

1.3.20. PROPOSITION. *Let V be a real vector space and J a complex structure in V. If T is a $\mathbb{C}$-linear endomorphism of (V, J), then:*

- *the* trace *of T as an operator on V is twice the real part of the trace of T as an operator on (V, J);*
- *the* determinant *of T as an operator on V is equal to the square of the absolute value of the determinant of T as an operator on (V, J).*

More explicitly, if A, B and real $n \times n$ matrices, $Z = A + B\,i$ and C is the matrix given in (1.2.6)*, then we have the following identities:*

$$\mathrm{tr}(C) = 2\Re\big(\mathrm{tr}(Z)\big), \quad \det(C) = |\det(Z)|^2,$$

where $\mathrm{tr}(U)$, $\det(U)$ denote respectively the trace and the determinant of the matrix U, and $\Re(\lambda)$, $|\lambda|$ denote respectively the real part and the absolute value of the complex number λ. □

1.3.21. REMARK. Suppose that V is endowed with a positive definite inner product g and that $J : V \to V$ is a complex structure which is g-anti-symmetric, Then, we have $J^{\#}g = g$, i.e., J is g-orthogonal. The operator $J^{\mathbb{C}}$ on $V^{\mathbb{C}}$ will then be anti-Hermitian (and unitary) with respect to the Hermitian product $g^{\mathbb{C}_s}$ in $V^{\mathbb{C}}$ (see Remark 1.3.16). It is easy to see that the holomorphic and the anti-holomorphic subspaces of J are *orthogonal* with respect to $g^{\mathbb{C}_s}$:

$$g^{\mathbb{C}_s}(v, w) = 0, \quad v \in V^{\mathfrak{h}},\ w \in V^{\mathfrak{a}}.$$

Using g and J, we can also define a Hermitian product $g_{\rm s}$ in V by setting:

$$g_{\rm s}(v,w)=g(v,w)+ig(v,Jw),\quad v,w\in V.$$

Actually, this is the *unique* Hermitian form in (V,J) that has g as its real part.

We have the following relations:

$$g^{\mathbb{C}_{\rm s}}\big(\pi^{\mathfrak{h}}(v),\pi^{\mathfrak{h}}(w)\big)=\frac{g_{\rm s}(v,w)}{2},\quad g^{\mathbb{C}_{\rm s}}\big(\pi^{\mathfrak{a}}(v),\pi^{\mathfrak{a}}(w)\big)=\frac{\overline{g_{\rm s}(v,w)}}{2},\quad v,w\in V;$$

they imply, in particular, that if $(b_j)_{j=1}^n$ is an orthonormal complex basis of (V,J) with respect to $g_{\rm s}$, then the vectors $\sqrt{2}\,u_j$, $\sqrt{2}\bar{u}_j$, $j=1,\ldots,n$, (see (1.3.14)) form an orthonormal real basis of $V^{\mathbb{C}}$ with respect to $g^{\mathbb{C}_{\rm s}}$. Also the vectors b_j and $J(b_j)$, $j=1,\ldots,n$, form an orthonormal real basis of V with respect to g, and therefore they form a complex orthonormal basis of $V^{\mathbb{C}}$ with respect to $g^{\mathbb{C}_{\rm s}}$. We conclude then that if $Z=A+B\,i$ (A,B real matrices), then the matrices in formulas (1.2.6) and (1.3.15) are *unitarily equivalent*, i.e., they represent the same complex operator in different orthonormal bases.

1.4. Symplectic Forms

In this section we will study the symplectic vector spaces. We define the notion of symplectomorphism, which is the equivalence in the category of symplectic vector spaces, and we show that symplectic vector spaces of the same dimension are equivalent.

1.4.1. DEFINITION. Let V be a real vector space; a *symplectic form* on V is an anti-symmetric nondegenerate bilinear form $\omega:V\times V\to\mathbb{R}$. We say that (V,ω) is a *symplectic vector space*.

1.4.2. REMARK. If $\omega\in{\rm B}_{\rm a-sym}(V)$ is a possibly degenerate anti-symmetric bilinear form on V, then ω defines an anti-symmetric bilinear form $\overline{\omega}$ on the quotient $V/{\rm Ker}(\omega)$; it is easy to see that $\overline{\omega}$ is nondegenerate, hence $(V/{\rm Ker}(\omega),\overline{\omega})$ is a symplectic space.

We start by giving a canonical form for the anti-symmetric bilinear forms; the proof is similar to the proof of Theorem 1.1.14.

1.4.3. THEOREM. *Let V be a p-dimensional vector space and $\omega\in{\rm B}_{\rm a-sym}(V)$ an anti-symmetric bilinear form on V. Then, there exists a basis of V with respect to which the matrix of ω (as a bilinear form) is given by:*

$$\omega\sim\begin{pmatrix}0_n & {\rm I}_n & 0_{n\times r}\\ -{\rm I}_n & 0_n & 0_{n\times r}\\ 0_{r\times n} & 0_{r\times n} & 0_r\end{pmatrix},\tag{1.4.1}$$

where $r=\dim({\rm Ker}(\omega))$, $p=2n+r$, and $0_{\alpha\times\beta}$, 0_α and ${\rm I}_\alpha$ denote respectively the zero $\alpha\times\beta$ matrix, the zero square matrix $\alpha\times\alpha$ and the identity $\alpha\times\alpha$ matrix.

PROOF. In first place, it is clear that, if a basis as in the thesis is found, then the last r vectors of this basis will be a basis for ${\rm Ker}(\omega)$, from which we get $r=\dim({\rm Ker}(\omega))$ and $p=2n+r$.

For the proof, we need to exhibit a basis $(b_i)_{i=1}^p$ of V such that:

$$\omega(b_i, b_{n+i}) = -\omega(b_{n+i}, b_i) = 1, \quad i = 1, \ldots, n, \tag{1.4.2}$$

and $\omega(b_i, b_j) = 0$ otherwise. We use induction on p; if $p \leq 1$ then necessarily $\omega = 0$ and the result is trivial.

Let's assume $p > 1$ and that the result is true for all vector spaces of dimension less than p. If $\omega = 0$ the result is trivial; let's assume then that $v, w \in V$ are chosen in such a way that $\omega(v, w) \neq 0$, for instance $\omega(v, w) = 1$. Then, it is easy to see that ω is nondegenerate when restricted to the two-dimensional plane generated by v and w; from Proposition 1.1.10 it follows that:

$$V = (\mathbb{R}v + \mathbb{R}w) \oplus (\mathbb{R}v + \mathbb{R}w)^{\perp}.$$

We now use the induction hypothesis to the restriction of ω to the $(p-2)$-dimensional vector space $(\mathbb{R}v + \mathbb{R}w)^{\perp}$, and we obtain a basis $(b_2, \ldots, b_n, b_{n+2}, \ldots, b_p)$ of $(\mathbb{R}v + \mathbb{R}w)^{\perp}$ in which ω takes the canonical form. This means that equality (1.4.2) holds for $i = 2, \ldots, n$, and $\omega(b_i, b_j) = 0$ otherwise. The desired basis for V is then obtained by setting $b_1 = v$ and $b_{n+1} = w$. □

1.4.4. COROLLARY. *If (V, ω) is a symplectic space, then V is even dimensional, and there exists a basis $(b_i)_{i=1}^{2n}$ of V with respect to which the matrix of ω as a bilinear form is given by:*

$$\omega \sim \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}, \tag{1.4.3}$$

where 0 and I denote respectively the zero and the identity $n \times n$ matrices.

1.4.5. DEFINITION. We say that $(b_i)_{i=1}^{2n}$ is a *symplectic basis* of (V, ω) if the matrix of ω as a bilinear form in this basis is given by (1.4.3).

Observe that the matrix of the linear operator $\omega : V \to V^*$ is given by the *transpose* of (1.4.3), i.e., it coincides with the matrix given in (1.2.2).

Corollary 1.4.4 tells us that every symplectic space admits a symplectic basis. We now define *sub-objects* and *morphisms* in the category of symplectic spaces.

1.4.6. DEFINITION. Let (V, ω) be a symplectic space; We say that S is a *symplectic subspace* if $S \subset V$ is a subspace and the restriction $\omega|_{S \times S}$ is nondegenerate. Hence, $(S, \omega|_{S \times S})$ is a symplectic space.

Let (V_1, ω_1) and (V_2, ω_2) be symplectic spaces; a linear operator $T : V_1 \to V_2$ is a *symplectic map* if $T^{\#}(\omega_2) = \omega_1$, i.e., if

$$\omega_2(T(v), T(w)) = \omega_1(v, w), \quad \forall\, v, w \in V_1.$$

We say that T is a *symplectomorphism* if T is an isomorphism and a symplectic map.

A symplectomorphism takes symplectic bases in symplectic bases; conversely, if $T : V_1 \to V_2$ is a linear map that takes some symplectic basis of V_1 into some symplectic basis of V_2, then T is a symplectomorphism.

In terms of the linear operators $\omega_1 \in \mathrm{Lin}(V_1, V_1^*)$ and $\omega_2 \in \mathrm{Lin}(V_2, V_2^*)$, a map $T \in \mathrm{Lin}(V_1, V_2)$ is symplectic if and only if:

$$T^* \circ \omega_2 \circ T = \omega_1. \tag{1.4.4}$$

1.4.7. REMARK. Observe that the right hand side of equality (1.4.4) is an isomorphism, from which it follows that *every symplectomorphism T is an injective map*. In particular, the image $T(V_1)$ is always a symplectic subspace of V_2.

1.4.8. EXAMPLE. We define a symplectic form in $\mathbb{R}^{2n}$ by setting:

$$\omega\big((v_1, w_1), (v_2, w_2)\big) = \langle v_1, w_2\rangle - \langle w_1, v_2\rangle, \tag{1.4.5}$$

for $v_1, v_2, w_1, w_2 \in \mathbb{R}^n$, where $\langle\cdot,\cdot\rangle$ denotes the canonical inner product of $\mathbb{R}^n$. We say that (1.4.5) is the *canonical symplectic form of $\mathbb{R}^{2n}$*; the canonical basis of $\mathbb{R}^{2n}$ is a symplectic basis for ω, hence the matrix of ω (as a bilinear map) with respect to the canonical basis of $\mathbb{R}^{2n}$ is (1.4.3).

The existence of a symplectic basis for a symplectic space (Corollary 1.4.4) implies that every symplectic space admits a symplectomorphism with $(\mathbb{R}^{2n}, \omega)$, hence the proof of every theorem concerning symplectic spaces can be reduced to the case of $(\mathbb{R}^{2n}, \omega)$.

We can also define a canonical symplectic form in $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ by setting:

$$\omega\big((v_1, \alpha_1), (v_2, \alpha_2)\big) = \alpha_2(v_1) - \alpha_1(v_2),$$

where $v_1, v_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}^{n*}$. Again, the canonical basis of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ is a symplectic basis for the canonical symplectic form of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$.

1.4.9. REMARK. Denoting by $(\mathrm{d}q_1, \ldots, \mathrm{d}q_n, \mathrm{d}p_1, \ldots, \mathrm{d}p_n)$ the canonical basis of $\mathbb{R}^{2n*}$ (dual of the canonical basis of $\mathbb{R}^{2n}$), the canonical symplectic form of $\mathbb{R}^{2n}$ is given by:

$$\omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

It follows easily:

$$\omega^n = \omega \wedge \ldots \wedge \omega = (-1)^{\frac{n(n-1)}{2}} \mathrm{d}q_1 \wedge \ldots \wedge \mathrm{d}q_n \wedge \mathrm{d}p_1 \wedge \ldots \wedge \mathrm{d}p_n.$$

Hence, ω^n is a *volume form* in $\mathbb{R}^{2n}$; for all symplectomorphism T of $(\mathbb{R}^{2n}, \omega)$ we therefore have:

$$T^{\#}(\omega^n) = \omega^n = \det(T)\,\omega^n,$$

from which it follows $\det(T) = 1$. In general, not every linear map T with $\det(T) = 1$ is a symplectomorphism of $(\mathbb{R}^{2n}, \omega)$; when $n = 1$ the symplectic form ω is a volume form, hence T is a symplectomorphism if and only if $\det(T) = 1$.

The symplectomorphisms of a symplectic space (V, ω) form a group by composition.

1.4.10. DEFINITION. Let (V, ω) be a symplectic space; the *symplectic group of (V, ω)* is the group of all symplectomorphisms of (V, ω), denoted by $\mathrm{Sp}(V, \omega)$. We denote by $\mathrm{Sp}(2n, \mathbb{R})$ the symplectic group of $\mathbb{R}^{2n}$ endowed with the canonical symplectic form.

Using a symplectic basis of (V, ω), a map $T \in \mathrm{Lin}(V)$ is a symplectomorphism if and only if the matrix M that represents T in such basis satisfies:

$$M^* \omega M = \omega, \tag{1.4.6}$$

where ω is the matrix given in (1.4.3). Writing

$$T \sim \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.4.7}$$

then (1.4.6) is equivalent to the following relations:

$$D^*A - B^*C = \mathrm{I}, \quad A^*C \text{ and } B^*D \text{ are symmetric}, \tag{1.4.8}$$

where A, B, C, D are $n \times n$ matrices, I is the $n \times n$ identity matrix, and * means transpose (see Exercise 1.10). A matrix of the form (1.4.7) satisfying (1.4.8) will be called a *symplectic matrix*.

We define direct sum of symplectic spaces.

1.4.11. DEFINITION. Given symplectic spaces (V_1, ω_1) and (V_2, ω_2), we define a symplectic form $\omega = \omega_1 \oplus \omega_2$ on $V_1 \oplus V_2$ by setting:

$$\omega((v_1, v_2), (w_1, w_2)) = \omega_1(v_1, w_1) + \omega_2(v_2, w_2), \ \ v_1, w_1 \in V_1, \ v_2, w_2 \in V_2.$$

The space $(V_1 \oplus V_2, \omega_1 \oplus \omega_2)$ is called the *direct sum of the symplectic spaces* (V_1, ω_1), (V_2, ω_2).

If (V, ω) is a symplectic space, two subspaces $S_1, S_2 \subset V$ are said to be ω-*orthogonal* if $\omega(v_1, v_2) = 0$ for all $v_i \in S_i$, $i = 1, 2$. If $V = S_1 \oplus S_2$ with S_1 and S_2 ω-orthogonal, then it is easy to see that both S_1 and S_2 are symplectic subspaces of (V, ω); in this case we say that V is the symplectic direct sum of the subspaces S_1 and S_2.

Observe that the notion of direct sum for symplectic spaces is *not* meant as a sum in a *categorical sense*, i.e., it is not true that a symplectic map on a direct sum $V_1 \oplus V_2$ is determined by its restriction to V_1 and V_2 (see Exercise 1.15).

1.4.12. EXAMPLE. If $T_i : V_i \to V_i'$, $i = 1, 2$, are symplectic maps, then the map $T = T_1 \oplus T_2 : V_1 \oplus V_2 \to V_1' \oplus V_2'$ defined by:

$$T(v_1, v_2) = (T_1(v_1), T_2(v_2)), \quad v_i \in V_i, \ i = 1, 2,$$

is also symplectic. If both T_1 and T_2 are symplectomorphisms, then also T is a symplectomorphism.

One needs to be careful with the notion of direct sum of symplectic spaces when working with symplectic bases; more explicitly, the concatenation of a symplectic basis $(b_i)_{i=1}^{2n}$ of V_1 and a symplectic basis $(b_j')_{j=1}^{2m}$ of V_2 is *not* a symplectic basis of $V_1 \oplus V_2$. In order to obtain a symplectic basis of $V_1 \oplus V_2$ we need to rearrange the vectors as follows:

$$(b_1, \ldots, b_n, b_1', \ldots, b_m', b_{n+1}, \ldots, b_{2n}, b_{m+1}', \ldots, b_{2m}').$$

Similar problems are encountered when dealing with symplectic matrices: the simple juxtaposition of along the diagonal of an element of $\mathrm{Sp}(2n, I\!R)$ and an element of $\mathrm{Sp}(2m, I\!R)$ *does not* produce an element of $\mathrm{Sp}(2(n+m), I\!R)$; in order to obtain a symplectic matrix it is necessary to perform a suitable permutation of the rows and the columns of such juxtaposition.

1.4.1. Isotropic and Lagrangian subspaces. In this subsection we consider a fixed symplectic space (V, ω), with $\dim(V) = 2n$.

1.4.13. DEFINITION. A subspace $S \subset V$ is said to be *isotropic* if $\omega|_{S\times S} = 0$.

Observe that S is isotropic if and only if it is contained in its orthogonal $S^\perp$ with respect to ω; from Proposition 1.1.8 we have:

$$\dim(S) + \dim(S^\perp) = 2n, \tag{1.4.9}$$

from which it follows that the dimension of an isotropic subspace is at most n. Observe that the notion of isotropic subspace is, roughly speaking, opposite to the notion of symplectic subspace; for, by Proposition 1.1.10, S is a symplectic subspace iff $S \cap S^\perp = \{0\}$.

We have the following:

1.4.14. LEMMA. *Let $L \subset V$ be a subspace; the following statements are equivalent:*

- *L is* maximal isotropic, *i.e., L is isotropic and it is not properly contained in any other isotropic subspace of V;*
- *$L = L^\perp$;*
- *L is isotropic and $\dim(L) = n$.*

PROOF. If L is maximal isotropic, then $L \subset L^\perp$ and for $v \in L^\perp$ the subspace $L + I\!R v$ is isotropic and it contains L. It follows that $L = L + I\!R v$, hence $v \in L$ and $L = L^\perp$. If $L = L^\perp$, then L is isotropic, and from (1.4.9) it follows that $\dim(L) = n$. Finally, if L is isotropic and $\dim(L) = n$, then L is maximal isotropic, because the dimension of an isotropic subspace is at most n. ☐

1.4.15. DEFINITION. A subspace $L \subset V$ is said to be *Lagrangian subspace* if it satisfies one (hence all) of the statements in Lemma 1.4.14.

1.4.16. EXAMPLE. The subspaces $\{0\} \oplus I\!R^n$ and $I\!R^n \oplus \{0\}$ are Lagrangian subspaces of $I\!R^{2n}$ endowed with the canonical symplectic structure. Given a linear map $T \in \mathrm{Lin}(I\!R^n)$, then its *graph* $\mathrm{Graph}(T) = \{v + T(v) : v \in I\!R^n\}$ is a Lagrangian subspace of $I\!R^{2n}$ endowed with the canonical symplectic structure if and only if T is symmetric with respect to the canonical inner product of $I\!R^n$.

1.4.17. EXAMPLE. If $S \subset V$ is an isotropic subspace, then the kernel of the restriction of ω to $S^\perp$ is the subspace $(S^\perp)^\perp \cap S^\perp = S$ (see Corollary 1.1.9). It follows that ω defines by passing to the quotient a symplectic form $\overline{\omega}$ in $S^\perp/S$ (Remark 1.4.2).

In the following definition we relate symplectic forms and complex structures on V:

1.4.18. DEFINITION. A complex structure $J : V \to V$ is said to be *compatible* with the symplectic form ω if $\omega(\cdot, J\cdot)$ is an *inner product* in V, i.e., a symmetric, positive definite bilinear form on V. More explicitly, J is compatible with ω if:

$$-\omega(Jv, w) = \omega(v, Jw), \quad \forall\, v, w \in V,$$

and if $\omega(Jv, v) > 0$ for all $v \neq 0$.

1.4.19. EXAMPLE. The canonical complex structure of $\mathbb{R}^{2n}$ (Example 1.2.2) is compatible with the canonical symplectic structure of $\mathbb{R}^{2n}$. The inner product $\omega(\cdot, J\cdot)$ is simply the canonical inner product of $\mathbb{R}^{2n}$. It follows that every symplectic space admits a complex structure compatible with the symplectic form: it is enough to define J by the matrix (1.2.2) with respect to any fixed symplectic basis. Such basis will then be an *orthonormal basis* with respect to the inner product $\omega(\cdot, J\cdot)$.

Let's assume that J is a given complex structure on V which is compatible with ω, and let's denote by g the inner product $\omega(\cdot, J\cdot)$; J is a symplectomorphism of (V, ω) (see Exercise 1.16) and the following identity holds:

$$g(J\cdot, \cdot) = \omega.$$

A compatible complex structure J can be used to construct a Lagrangian which is complementary to a given Lagrangian:

1.4.20. LEMMA. *If $L \subset V$ is a Lagrangian subspace and J is a complex structure compatible with ω, then $V = L \oplus J(L)$.*

PROOF. It suffices to observe that L and $J(L)$ are orthogonal subspaces with respect to the inner product g. □

1.4.21. COROLLARY. *Every Lagrangian subspace admits a complementary Lagrangian subspace.*

PROOF. It follows from Lemma 1.4.20, observing that $J(L)$ is Lagrangian, since J is a symplectomorphism (Exercise 1.16). □

We can define a complex valued sesquilinear form g_{s} (see Definition 1.3.14) in the complex space (V, J) by setting:

$$g_{\mathrm{s}}(v, w) = g(v, w) - i\,\omega(v, w). \tag{1.4.10}$$

It is easy to see that g_{s} is a positive Hermitian product in (V, J).

Recall from Remark 1.3.16 that a $\mathbb{C}$-linear endomorphism is g_{s}-unitary when $g_{\mathrm{s}}(T\cdot, T\cdot) = g_{\mathrm{s}}$; in this situation we also say that T *preserves* g_{s}. We have the following:

1.4.22. PROPOSITION. *Let $T \in \mathrm{Lin}(V)$ be an $\mathbb{R}$-linear map; the following statements are equivalent:*

- *T is $\mathbb{C}$-linear in (V, J) and g_{s}-unitary;*
- *T is orthogonal with respect to g and $T \in \mathrm{Sp}(V, \omega)$.*

PROOF. If T is $\mathbb{C}$-linear and g_s-unitary, then T preserves g_s, hence it preserves separately its real part, which is g, and its imaginary part, which is $-\omega$. Hence T is an orthogonal symplectomorphism.

Conversely, if T is an orthogonal symplectomorphism, then the following identities hold:

$$T^* \circ g \circ T = g, \quad T^* \circ \omega \circ T = \omega, \quad \omega = g \circ J,$$

considering g and ω as linear operators in $\mathrm{Lin}(V, V^*)$ (see Example 1.1.3)). It follows easily that $J \circ T = T \circ J$, i.e., T is $\mathbb{C}$-linear. Since T preserves both the real and the imaginary part of g_s, we conclude that T is g_s-unitary. □

1.4.23. EXAMPLE. The canonical complex structure J of $\mathbb{R}^{2n}$ (see Example 1.4.8) is compatible with its canonical symplectic structure (Example 1.2.2), and the inner product g corresponds to the canonical inner product of $\mathbb{R}^{2n}$. If we identify $(\mathbb{R}^{2n}, J)$ with $\mathbb{C}^n$ (Example 1.2.2), the Hermitian product g_s coincides with the canonical Hermitian product of $\mathbb{C}^n$ given in (1.3.10).

1.4.24. REMARK. Observe that if (V, J) is a complex space endowed with a Hermitian product g_s, then the real part of g_s is a positive inner product g on V and the imaginary part of g_s is a symplectic form on V; moreover, defining ω as minus the imaginary part of g_s, it follows that J is compatible with ω and $g = \omega(\cdot, J\cdot)$.

1.4.25. REMARK. If V is a real vector space, g is a positive inner product on V and J is a complex structure which is g-anti-symmetric (or, equivalently, g-orthogonal), then we get a symplectic form on V by setting $\omega = g(J\cdot, \cdot)$. The complex structure J will then be compatible with ω, and $g = \omega(\cdot, J\cdot)$. Again, we also get a Hermitian product g_s in (V, J) defined by (1.4.10).

We have the following relation between Lagrangian subspaces and the Hermitian product g_s:

1.4.26. LEMMA. *A subspace $L \subset V$ is Lagrangian if and only if it is a real form which is* preserved by g_s, *i.e., $V = L \oplus J(L)$ and $g_s(L \times L) \subset \mathbb{R}$.*

PROOF. It follows from Lemma 1.4.20 and the observation that the imaginary part of g_s equals $-\omega$. □

As a corollary, we now prove that the group of g_s-unitary isomorphisms of (V, J) act *transitively* on the set of Lagrangian subspaces of (V, ω):

1.4.27. COROLLARY. *Given any pair of Lagrangian subspaces L_1, L_2 of V, there exists a $\mathbb{C}$-linear isomorphism T of (V, J) which is g_s-unitary and such that $T(L_1) = L_2$.*

PROOF. Let $(b_j)_{j=1}^n$ be an orthonormal basis of L_1 with respect to the inner product g; since L_1 is a real form of (V, J), it follows that $(b_j)_{j=1}^n$ is a complex basis of (V, J) (see Example 1.3.7). Moreover, since g_s is real on L_1, it follows that $(b_j)_{j=1}^n$ is an orthonormal basis of (V, J) with respect to g_s. Similarly, we consider a basis $(b'_j)_{j=1}^n$ of L_2 which is orthonormal with respect to g, and we obtain that $(b'_j)_{j=1}^n$ is a g_s-orthonormal basis of (V, J). It follows that the $\mathbb{C}$-linear isomorphism T defined by $T(b_j) = b'_j$, for all $j = 1, \ldots, n$, is unitary and satisfies $T(L_1) = L_2$. □

It follows that also the symplectic group acts transitively on the set of Lagrangian subspaces:

1.4.28. COROLLARY. *Given any pair L_1, L_2 of Lagrangian subspaces of the symplectic space (V, ω), there exists a symplectomorphism $T \in \mathrm{Sp}(V, \omega)$ such that $T(L_1) = L_2$.*

PROOF. It follows from Corollary 1.4.27, observing that every g_s-unitary map is a symplectomorphism (Proposition 1.4.22). □

1.4.29. REMARK. For later use, we will mention a mild refinement of the result of Corollary 1.4.27. Given Lagrangian subspaces $L_1, L_2 \subset V$ and chosen *orientations* $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively on the spaces L_1 and L_2, it is possible to find a $\mathbb{C}$-linear and g_s-unitary endomorphism T of (V, J) such that $T(L_1) = L_2$ and such that $T|_{L_1} : L_1 \to L_2$ is positively oriented. To see this, it suffices to choose in the proof of Corollary 1.4.27 the g-orthonormal bases $(b_j)_{j=1}^n$ and $(b'_j)_{j=1}^n$ of L_1 and L_2 respectively in such a way that they are positively oriented.

1.4.30. REMARK. Given a Lagrangian subspace $L_0 \subset V$, then it is always possible to find a basis $(b_j)_{j=1}^{2n}$ of V which is at the same time symplectic, adapted to J, and such that $(b_j)_{j=1}^n$ is a basis of L_0. For, if $(b_j)_{j=1}^n$ is a g-orthonormal basis of L_0, then the basis defined in (1.2.5) satisfies the required properties; moreover, such basis is g-orthonormal and the complex basis $(b_j)_{j=1}^n$ of (V, J) is g_s-orthonormal. We therefore obtain a basis that puts simultaneously all the objects $(V, \omega, J, g, g_\mathrm{s}, L_0)$ in their canonical forms.

In the spirit of Remark 1.4.24 and Remark 1.4.25, one can ask himself whether given a real space V endowed with a symplectic form ω and a positive inner product g, it is possible to construct a complex structure J and a Hermitian product g_s which are naturally associated to g and ω. If one requires the condition $\omega = g(J\cdot, \cdot)$, then this is clearly impossible in general, because there exists a unique operator $H \in \mathrm{Lin}(V)$ such that $\omega = g(H\cdot, \cdot)$, and such H does not in general satisfy $H^2 = -\mathrm{Id}$.

We conclude the subsection with a result in this direction:

1.4.31. PROPOSITION. *Let (V, ω) be a symplectic space and g a positive inner product in V. Then there exists a unique complex structure J in V which is g-anti-symmetric (or, equivalently, g-orthogonal) and compatible with ω.*

PROOF. The uniqueness is the hard part of the thesis, which we now prove. Suppose that J is a given g-anti-symmetric complex structure in V which is compatible with ω, and let $H \in \mathrm{Lin}(V)$ be the unique operator such that $\omega = g(H\cdot, \cdot)$. Then, H is a g-anti-symmetric isomorphism of V.

The compatibility of J with ω is equivalent to the condition that $g(HJ\cdot, \cdot)$ be a symmetric bilinear form on V which is *negative definite*. By the usual identification of linear and bilinear maps, we see that the g-anti-symmetry property of H and J, together with the g-symmetry of HJ are expressed by the following relations:

$$g\circ J=-J^*\circ g,\quad g\circ H=-H^*\circ g,\quad g\circ H\circ J=J^*\circ H^*\circ g,$$

from which it follows easily that $H\circ J=J\circ H$.

We now consider the complexifications $J^{\mathbb{C}},H^{\mathbb{C}}\in\mathrm{Lin}(V^{\mathbb{C}})$ and the unique sesquilinear extension $g^{\mathbb{C}_s}$ of g to $V^{\mathbb{C}}$; clearly, $g^{\mathbb{C}_s}$ is a positive Hermitian product in $V^{\mathbb{C}}$, with respect to which $H^{\mathbb{C}}$ and $J^{\mathbb{C}}$ are anti-Hermitian operators (see Example 1.3.15 and Remark 1.3.16); moreover, $H^{\mathbb{C}}\circ J^{\mathbb{C}}=J^{\mathbb{C}}\circ H^{\mathbb{C}}$ and $(J^{\mathbb{C}})^2=-\mathrm{Id}$.

Since $H^{\mathbb{C}}$ is $g^{\mathbb{C}_s}$-anti-Hermitian, then $H^{\mathbb{C}}$ can be diagonalized in a $g^{\mathbb{C}_s}$-orthonormal basis of $V^{\mathbb{C}}$ (see Exercise 1.18); its eigenvalues are pure imaginary (non zero, because $H^{\mathbb{C}}$ is invertible), and since $H^{\mathbb{C}}$ commutes with the conjugation, it follows that eigenspaces of $H^{\mathbb{C}}$ corresponding to two conjugate eigenvalues are mutually conjugate (see Lemma 1.3.11). We can then write a $g^{\mathbb{C}_s}$-orthogonal decomposition:

$$V^{\mathbb{C}}=\bigoplus_{j=1}^{r}\mathcal{Z}_{i\,\lambda_j}\oplus\bigoplus_{j=1}^{r}\mathcal{Z}_{-i\,\lambda_j},$$

where $\lambda_j>0$ for all j, $\mathcal{Z}_{i\,\lambda}$ the eigenspace of $H^{\mathbb{C}}$ corresponding to the eigenvalue $i\lambda$; also, $\mathcal{Z}_{-i\lambda}$ is the conjugate of $\mathcal{Z}_{i\lambda}$.

Since $J^{\mathbb{C}}$ commutes with $H^{\mathbb{C}}$, it follows that the eigenspaces of $H^{\mathbb{C}}$ are invariant by $J^{\mathbb{C}}$. The restriction of $J^{\mathbb{C}}$ to each $\mathcal{Z}_{i\lambda_j}$ is an anti-Hermitian operator whose square is $-\mathrm{Id}$, from which it follows that such restriction is diagonalizable, and its possible eigenvalues are i and $-i$. The restriction of $g^{\mathbb{C}_s}(J^{\mathbb{C}}\circ H^{\mathbb{C}}\,\cdot,\cdot)$ to $\mathcal{Z}_{i\lambda_j}$, that coincides with the restriction of $i\lambda_j\,g^{\mathbb{C}_s}(J^{\mathbb{C}}\,\cdot,\cdot))$ must be Hermitian and negative definite, from which it follows that the unique eigenvalue of the restriction of $J^{\mathbb{C}}$ to $\mathcal{Z}_{i\lambda_j}$ must be equal to i.

We conclude that the restriction of $J^{\mathbb{C}}$ to $\mathcal{Z}_{i\lambda_j}$ is the operator of multiplication by i, and the restriction of $J^{\mathbb{C}}$ to $\mathcal{Z}_{-i\lambda_j}$ is the operator of multiplication by $-i$; such conditions determine $J^{\mathbb{C}}$, which shows the uniqueness of J.

For the existence, simply consider the unique complex structure J on V whose holomorphic space coincides with $\bigoplus_j\mathcal{Z}_{i\lambda_j}$ (see Proposition 1.3.19). □

1.4.2. Lagrangian decompositions of a symplectic space. In this subsection we study the properties of Lagrangian decompositions of a symplectic space, that will be fundamental in the study of the Lagrangian Grassmannian in Section 2.5. Throughout this subsection we will fix a symplectic space (V,ω), with $\dim(V)=2n$. We start with a definition:

1.4.32. DEFINITION. A *Lagrangian decomposition* of (V,ω) is a pair (L_0,L_1) of Lagrangian subspaces of V with $V=L_0\oplus L_1$.

1.4.33. EXAMPLE. The pair $(\mathbb{R}^n\oplus\{0\},\{0\}\oplus\mathbb{R}^n)$ is a Lagrangian decomposition of $\mathbb{R}^{2n}$ endowed with the canonical symplectic structure. More generally, if $L\subset V$ is a Lagrangian subspace and J is a complex structure on V compatible with ω, then $(L, J(L))$ is a Lagrangian decomposition of (V, ω) (see Lemma 1.4.20 and the proof of Corollary 1.4.21).

Given a Lagrangian decomposition (L_0, L_1) of (V, ω), we define a a map:

$$\rho_{L_0,L_1} : L_1 \longrightarrow L_0^*$$

by setting

$$\rho_{L_0,L_1}(v) = \omega(v, \cdot)|_{L_0} \tag{1.4.11}$$

for all $v \in L_1$; it is easy to see that ρ_{L_0,L_1} is an isomorphism (see Exercise 1.19).

1.4.34. REMARK. The isomorphism ρ_{L_0,L_1} gives us an identification of L_1 with the dual space L_0^*, but the reader should be careful when using this identification for the following reason. The isomorphism ρ_{L_0,L_1} induces an isomorphism $(\rho_{L_0,L_1})^* : L_0^{**} \simeq L_0 \to L_1^*$; however, $(\rho_{L_0,L_1})^*$ does *not* coincide with ρ_{L_1,L_0}, but with its opposite:

$$(\rho_{L_0,L_1})^* = -\rho_{L_1,L_0}. \tag{1.4.12}$$

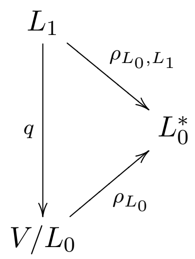
If $L \subset V$ is a Lagrangian subspace, we also define an isomorphism:

$$\rho_L : V/L \longrightarrow L^*,$$

by setting $\rho_L(v + L) = \omega(v, \cdot)|_L$.

Given a Lagrangian decomposition (L_0, L_1) of (V, ω), we have the following commutative diagram of isomorphisms:

(1.4.13)



where q is the restriction to L_1 of the quotient map $V \to V/L_0$.

An application of the isomorphism ρ_{L_0,L_1} is given in the following:

1.4.35. LEMMA. *If $L_0 \subset V$ is a Lagrangian subspace, then every basis $(b_i)_{i=1}^n$ of L_0 extends to a symplectic basis $(b_i)_{i=1}^{2n}$ of V; moreover, given any Lagrangian L_1 which is complementary to L_0, one can choose the basis $(b_i)_{i=1}^{2n}$ in such a way that $(b_i)_{i=n+1}^{2n}$ is a basis of L_1.*

PROOF. Observe first that the Lagrangian L_0 admits a complementary Lagrangian L_1 (see Corollary 1.4.21); given one such Lagrangian L_1, we define:

$$b_{n+i} = -\rho_{L_0,L_1}^{-1}(b_i^*), \quad i = 1, \ldots, n,$$

where $(b_i^*)_{i=1}^n$ is the basis of L_0^* which is dual to $(b_i)_{i=1}^n$. □

1.4.36. COROLLARY. *Given any Lagrangian decompositions (L_0, L_1) and (L_0', L_1') of V then every isomorphism from L_0 to L_0' extends to a symplectomorphism $T\colon V \to V$ such that $T(L_1) = L_1'$.*

PROOF. Let $(b_i)_{i=1}^n$ be a basis of L_0 and let $(b'_i)_{i=1}^n$ be the basis of L'_0 which corresponds to $(b_i)_{i=1}^n$ by the given isomorphism. Using Lemma 1.4.35 we can find symplectic bases $(b_i)_{i=1}^{2n}$ and $(b'_i)_{i=1}^{2n}$ of V in such a way that $(b_i)_{i=n+1}^{2n}$ and $(b'_i)_{i=n+1}^{2n}$ are bases of L_1 and L'_1 respectively; to conclude the proof one simply chooses T such that $T(b_i) = b'_i$, $i = 1, \ldots, 2n$. □

1.4.37. COROLLARY. *If $L_0 \subset V$ is a Lagrangian subspace, then every isomorphism of L_0 extends to a symplectomorphism of V.*

PROOF. Choose a Lagrangian L_1 complementary to L_0 (see Corollary 1.4.21) and apply Corollary 1.4.36. □

The technique of extending bases of Lagrangians to symplectic bases of the symplectic space may be used to give an alternative proof of Corollary 1.4.28. Roughly speaking, Corollary 1.4.28 tells us that Lagrangian subspaces are "indistinguishable" from the viewpoint of the symplectic structure; our next Proposition tells us that the only invariant of a *pair* (L_0, L_1) of Lagrangian subspaces is the dimension of their intersection $L_0 \cap L_1$:

1.4.38. PROPOSITION. *Given three Lagrangian subspaces $L_0, L, L' \subset V$ with $\dim(L_0 \cap L) = \dim(L_0 \cap L')$, there exists a symplectomorphism T of (V, ω) such that $T(L_0) = L_0$ and $T(L) = L'$.*

PROOF. By Corollary 1.4.37, there exists a symplectomorphism of (V, ω) that takes L_0 into itself and $L_0 \cap L$ onto $L_0 \cap L'$; we can therefore assume without loss of generality that $L_0 \cap L = L_0 \cap L'$.

Set $S = L_0 \cap L = L_0 \cap L'$; clearly S is isotropic and $L_0, L, L' \subset S^\perp$. We have a symplectic form $\overline{\omega}$ in $S^\perp/S$ obtained from ω by passing to the quotient (see Example 1.4.17).

Denote by $q : S^\perp \to S^\perp/S$ the quotient map; it is easy to see that $q(L_0)$, $q(L)$ and $q(L')$ are Lagrangian subspaces of $(S^\perp/S, \overline{\omega})$; moreover, $\big(q(L_0), q(L)\big)$ and $\big(q(L_0), q(L')\big)$ are both Lagrangian decompositions of $S^\perp/S$ and hence there exists a symplectomorphism $\overline{T}$ of $(S^\perp/S, \overline{\omega})$ such that $\overline{T}\big(q(L_0)\big) = q(L_0)$ and $\overline{T}\big(q(L)\big) = q(L')$ (see Corollary 1.4.28). The required symplectomorphism $T \in \mathrm{Sp}(V, \omega)$ is obtained from the following Lemma. □

1.4.39. LEMMA. *Let $L_0 \subset V$ be a Lagrangian subspace and let $S \subset L_0$ be any subspace. Consider the quotient symplectic form $\overline{\omega}$ on $S^\perp/S$; then, given any symplectomorphism $\overline{T}$ of $(S^\perp/S, \overline{\omega})$ with $\overline{T}(q(L_0)) = q(L_0)$, there exists a symplectomorphism T of (V, ω) such that $T(S) = S$ (hence also $T(S^\perp) = S^\perp$), $T(L_0) = L_0$, and such that the following diagram commutes*

$$\begin{array}{ccc} S^\perp & \xrightarrow{\;T|_{S^\perp}\;} & S^\perp \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ S^\perp/S & \xrightarrow[\;\overline{T}\;]{} & S^\perp/S \end{array}$$

where $q : S^\perp \to S^\perp/S$ denotes the quotient map.

PROOF. Write $L_0 = S \oplus R$; hence $L_0^* = S^o \oplus R^o$, where S^o and R^o are the annihilators of S and R respectively. Let L_1 be any complementary Lagrangian to L_0 in V (Corollary 1.4.21). We have:

$$L_1 = \rho_{L_0,L_1}^{-1}(S^o) \oplus \rho_{L_0,L_1}^{-1}(R^o).$$

We obtain a direct sum decomposition $V = V_1 \oplus V_2$ into ω-orthogonal subspaces given by:

$$V_1 = S \oplus \rho_{L_0,L_1}^{-1}(R^o), \quad V_2 = R \oplus \rho_{L_0,L_1}^{-1}(S^o),$$

from which it follows that V is direct sum of the symplectic spaces V_1 and V_2.

Observe that $S^\perp = V_2 \oplus S$, hence the quotient map q restricts to a symplectomorphism of V_2 into $S^\perp/S$; therefore, we have a unique symplectomorphism T' of V_2 such that the diagram:

$$\begin{array}{ccc} V_2 & \xrightarrow{\;T'\;} & V_2 \\ {\scriptstyle q|_{V_2}}\downarrow & & \downarrow{\scriptstyle q|_{V_2}} \\ S^\perp/S & \xrightarrow[\overline{T}]{} & S^\perp/S \end{array}$$

commutes. Since $\overline{T}$ preserves $q(L_0)$ it follows that T' preserves R; we then define T by setting $T|_{V_1} = \mathrm{Id}$ and $T|_{V_2} = T'$ (see Example 1.4.12). □

1.4.40. REMARK. We claim that one can actually choose the symplectomorphism T in the thesis of Proposition 1.4.38 in such a way that T restricts to a *positively oriented* isomorphism of L_0; namely, if $\dim(L_0 \cap L) = \dim(L_0 \cap L') = 0$ then this claim follows directly from Corollary 1.4.36. For the general case, we observe that in the last part of the proof of Lemma 1.4.39 one can define $T|_{V_1}$ to be any symplectomorphism of V_1 which preserves S (while $T|_{V_2} = T'$ is kept unchanged); since S is Lagrangian in V_1, using Corollary 1.4.37, we get that $T|_S$ can be choosen to be any isomorphism A of S given *a priori* (and $T|_R$ does not depend on A). Since $\dim(S) \geq 1$, this freedom in the choice of A can be used to *adjust* the orientation of $T|_{L_0}$.

Exercises for Chapter 1

EXERCISE 1.1. Show that the isomorphism between the spaces $\mathrm{Lin}(V, W^*)$ and $\mathrm{B}(V, W)$ given in (1.1.1) is natural in the sense that it gives a *natural isomorphism of the functors* $\mathrm{Lin}(\cdot, \cdot)$ and $\mathrm{B}(\cdot, \cdot)$ from the category of pairs of vector spaces to the category of vector spaces.

EXERCISE 1.2. Prove that $\mathrm{B}(V) = \mathrm{B}_{\mathrm{sym}}(V) \oplus \mathrm{B}_{\mathrm{a-sym}}(V)$.

EXERCISE 1.3. Prove Lemma 1.2.3.

EXERCISE 1.4. Prove Proposition 1.2.6.

EXERCISE 1.5. Prove Proposition 1.3.3.

EXERCISE 1.6. Prove Corollary 1.3.4.

EXERCISE 1.7. Prove Lemma 1.3.9.

EXERCISE 1.8. Generalize the results of Section 1.3, in particular Proposition 1.3.3, Lemma 1.3.10 and Lemma 1.3.11, to the case of anti-linear, multi-linear and sesquilinear operators.

EXERCISE 1.9. Prove that if $\mathcal{V}$ is a non trivial complex vector space, then there exists no $\mathbb{C}$-bilinear form on $\mathcal{V}$ which is positive definite.

EXERCISE 1.10. Prove that $T \in \mathrm{Lin}(V)$ is a symplectomorphism of (V, ω) if and only if its matrix representation with respect to a symplectic basis of (V, ω) satisfies the relations (1.4.8).

EXERCISE 1.11. Consider the symplectic space $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ endowed with its canonical symplectic structure. Prove that to each Lagrangian subspace L there corresponds a unique pair (P, S), where $P \subset \mathbb{R}^n$ is a subspace and $S : P \times P \to \mathbb{R}$ is a symmetric bilinear form on P, such that:

$$L = \big\{(v, \alpha) \in \mathbb{R}^n \oplus \mathbb{R}^{n*} : v \in P,\ \alpha|_P + S(v, \cdot) = 0\big\}.$$

More generally, if (L_0, L_1) is a Lagrangian decomposition of the symplectic space (V, ω), there exists a bijection between the Lagrangian subspaces $L \subset V$ and the pairs (P, S), where $P \subset L_1$ is any subspace and $S \in \mathrm{B}_{\mathrm{sym}}(P)$ is a symmetric bilinear form on P, so that (recall formula (1.4.11)):

$$L = \big\{v + w : v \in P,\ w \in L_0,\ \rho_{L_1,L_0}(w)|_P + S(v, \cdot) = 0\big\}. \tag{1.4.14}$$

EXERCISE 1.12. Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an element in $\mathrm{Sp}(2n, \mathbb{R})$ (recall formula (1.4.8)) and let $L_0 = \{0\} \oplus \mathbb{R}^{n*}$. Prove that the following two statements are equivalent:

(a) $T(L_0)$ is transverse to L_0;
(b) B is invertible.

Prove also that, in this case, the $n \times n$ matrices DB^{-1}, $B^{-1}A$ and $C - DB^{-1}A - B^{-1}$ are symmetric.

EXERCISE 1.13. Prove that the transpose of a symplectic matrix in $\mathrm{Sp}(2n, \mathbb{R})$ is again symplectic.

EXERCISE 1.14. Every invertible matrix M can be written in *polar form*:

$$M = PO, \quad P = (MM^*)^{\frac{1}{2}}, \quad O = P^{-1}M,$$

where P is symmetric and positive definite and O is orthogonal. Such decomposition is unique and it depends continuously on M.

Prove that $M \in \mathrm{Sp}(2n, \mathbb{R})$ if and only if both P and O are in $\mathrm{Sp}(2n, \mathbb{R})$.

EXERCISE 1.15. Prove that the direct sum of symplectic spaces is not categorical, i.e., it is not true in general that if a linear map $T : V_1 \oplus V_2 \to W$ is such that its restrictions $T|_{V_1}$ and $T|_{V_2}$ are symplectic, then T is symplectic.

EXERCISE 1.16. Prove that a complex structure on a symplectic space which is compatible with the symplectic form is a symplectomorphism.

EXERCISE 1.17. Let V be a real vector space and g a positive inner product. Prove that a complex structure J in V is g-anti-symmetric iff it is g-orthogonal.

EXERCISE 1.18. Let $\mathcal{V}$ be a complex space, g_s a positive Hermitian product in $\mathcal{V}$ and $\mathcal{T} \in \mathrm{Lin}(\mathcal{V})$ a g_s-normal operator. Show that $\mathcal{T}$ is diagonalizable in a g_s-orthonormal basis of $\mathcal{V}$.

EXERCISE 1.19. Given a Lagrangian decomposition (L_0, L_1) of a symplectic space (V, ω), prove that the map $\rho_{L_0,L_1} : L_1 \to L_0^*$ defined in (1.4.11) page 27 is an isomorphism.

EXERCISE 1.20. Let (V, ω) be a symplectic space, $S \subset V$ an isotropic subspace, and consider the quotient symplectic space $(S^\perp/S, \overline{\omega})$ defined in Example 1.4.17. Prove that if $L \subset V$ is a Lagrangian subspace of (V, ω), then $\pi(L)$ is Lagrangian in $(S^\perp/S, \overline{\omega})$, where $\pi : S^\perp \to S^\perp/S$ is the projection.

CHAPTER 2

The Geometry of Grassmannians

2.1. Differentiable Manifolds and Lie Groups

In this section we give the basic definitions and we fix some notations concerning calculus on manifolds. In this text, the term "manifold" will always mean a real, finite dimensional differentiable manifold whose topology satisfies the Hausdorff property and the second countability axiom, i.e., it admits a countable basis of open sets. The term "differentiable" will always mean "of class C^∞"; we will describe below the terminology used in the construction of a differentiable manifold structure.

Let M be a set; a *chart* in M is a bijection:

$$\phi : U \to \widetilde{U},$$

where $U \subset M$ is any subset and $\widetilde{U}$ is an open set in some Euclidean space $\mathbb{R}^n$; in some situation, with a slight abuse of terminology, we will allow that $\widetilde{U}$ be an open subset of some arbitrary real finite dimensional vector space.

We say that two charts $\phi : U \to \widetilde{U}$ and $\psi : V \to \widetilde{V}$ in M are *compatible* if $U \cap V = \emptyset$ or if $\phi(U \cap V)$ and $\psi(U \cap V)$ are both open sets and the *transition function*:

$$\psi \circ \phi^{-1} \colon \phi(U \cap V) \longrightarrow \psi(U \cap V)$$

is a differentiable diffeomorphism. A *differentiable atlas* $\mathcal{A}$ in M is a set of charts in M that are pairwise compatible and whose domains form a covering of M. A chart is said to be *compatible with a differentiable atlas* if it is compatible with all the charts of the atlas; it is easy to see that two charts that are compatible with an atlas are compatible with each other. Hence, every differentiable atlas $\mathcal{A}$ is contained in a *unique maximal differentiable atlas* which is obtained as the collection of all the charts in M that are compatible with $\mathcal{A}$.

A differentiable atlas $\mathcal{A}$ induces on M a unique topology τ such that each chart of $\mathcal{A}$ is a *homeomorphism* defined in an open subset of (M, τ); such topology τ is defined as the set of parts $A \subset M$ such that $\phi(A \cap U)$ is an open subset of $\widetilde{U}$ for every chart $\phi : U \to \widetilde{U}$ in $\mathcal{A}$.

A *(differentiable) manifold* is then defined as a pair $(M, \mathcal{A})$, where M is a set and $\mathcal{A}$ is a maximal differentiable atlas in M whose corresponding topology τ is Hausdorff and second countable; a *chart*, or a *coordinate system*, in a differentiable manifold $(M, \mathcal{A})$ is a chart that belongs to $\mathcal{A}$.

2.1.1. REMARK. Observe that some authors replace the assumption of second countability for a differentiable manifold with the assumption of *paracompactness*. In Exercise 2.1 the reader is asked to show that such assumption is "weaker", but indeed "not so much weaker".

Let M be a manifold and $N \subset M$ be a subset; we say that a chart $\phi : U \to \widetilde{U} \subset \mathbb{R}^n$ is a *submanifold chart* for N if $\phi(U \cap N)$ is equal to the intersection of $\widetilde{U}$ with a vector subspace S of $\mathbb{R}^n$. We then say that:

$$\phi|_{U \cap N} : U \cap N \longrightarrow \widetilde{U} \cap S$$

is the *chart in N induced by ϕ*. The subset N is said to be an *embedded submanifold* of M if for all $x \in N$ there exists a submanifold chart for N whose domain contains x. The inclusion $\mathfrak{i} : N \to M$ will then be an *embedding* of N in M, i.e., a differentiable immersion which is a homeomorphism onto its image endowed with the relative topology.

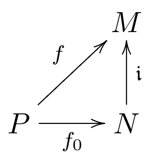
An *immersed submanifold* N in M is a manifold N such that N is a subset of M and such that the inclusion $\mathfrak{i} : N \to M$ is a differentiable immersion. Observe that a subset $N \subset M$ may admit *several* differentiable structures that make it into an immersed submanifold; however, if we fix a topology in N, then there exists at most one differentiable structure in N compatible with such topology and for which N is an immersed submanifold of M (see Exercise 2.3).

In general, if N and M are any two manifolds, and if $f : N \to M$ is an *injective* differentiable immersion, then there exists a unique differentiable structure on $f(N)$ that makes f into a differentiable diffeomorphism onto $f(N)$; hence, $f(N)$ is an immersed submanifold of M. If f is an embedding, then it follows from the local form of immersions that $f(N)$ is an embedded submanifold of M.

From now on, unless otherwise stated, by "submanifold" we will always mean "embedded submanifold".

2.1.2. REMARK. If P and M are two manifolds, $N \subset M$ is an embedded submanifold and $f : P \to M$ is a differentiable map such that $f(P) \subset N$, then there exists a unique map $f_0 : P \to N$ such that the following diagram commutes:

(2.1.1)



where $\mathfrak{i}$ denotes the inclusion. We say that f_0 is obtained from f by *change of counterdomain*, and we will often use the same symbol f for f_0; the map f_0 is differentiable. The same results *does not* hold in general if N is only an immersed submanifold; it holds under the assumption of continuity for f_0 (see Exercise 2.2).

Immersed submanifolds $N \subset M$ for which the differentiability of f in (2.1.1) implies the differentiability of f_0 are known as *almost embedded submanifolds* of M; examples of such submanifolds are *integral submanifolds of involutive distributions*, or immersed submanifolds that are *subgroups of Lie groups*.

2.1.3. REMARK. If $f : M \to N$ is a differentiable submersion, then it follows from the local form of the submersions that for all $y \in \mathrm{Im}(f)$ and for all $x \in f^{-1}(y) \subset M$ there exists a *local differentiable section* of f that takes y into x, i.e., there exists a differentiable map $s : U \to M$ defined in an open neighborhood U of y in N such that $s(y) = x$ and such that $f(s(z)) = z$ for all $z \in U$.

The existence of local differentiable sections allows to prove that differentiable submersions that are surjective have the *quotient property*; this means that if $f : M \to N$ is a surjective submersion and $g : M \to P$ is a differentiable map, and if there exists a map $\bar{g} : N \to P$ such that the following diagram commutes:

$$\begin{array}{ccc} M & & \\ {\scriptstyle f}\downarrow & \searrow^{g} & \\ N & \xrightarrow[\bar{g}]{} & P \end{array}$$

then also $\bar{g}$ is differentiable.

In particular, if M is a manifold and $f : M \to N$ is a surjective map, then there exists at most one differentiable structure on N that makes f into a differentiable submersion; such structure is called a *quotient differentiable structure induced by* f.

2.1.1. Classical Lie Groups and Lie Algebras. In this subsection we give a short description and we introduce the notations for the classical Lie groups and Lie algebras that will be used in the text.

A *Lie group* is a group G endowed with a differentiable structure such that the map $G \times G \ni (x, y) \mapsto xy^{-1} \in G$ is differentiable; the unit of G will be denoted by $1 \in G$.

A *Lie group homomorphism* will always means a group homomorphism which is also continuous; then, it will be automatically differentiable (see for instance [**47**, Theorem 2.11.2] and [**48**, Theorem 3.39]).

For $g \in G$, we denote by l_g and r_g respectively the diffeomorphisms of G given by the *left-translation* $l_g(x) = gx$ and by the *right-translation* $r_g(x) = xg$; by $\mathcal{I}_g = l_g \circ r_g^{-1}$ we denote the *inner automorphism* of G associated to g. If $g \in G$ and $v \in T_xG$ is a tangent vector to G, we write:

$$gv = \mathrm{d}l_g(x) \cdot v, \quad vg = \mathrm{d}r_g(x) \cdot v;$$

for all $X \in T_1G$ we define vector fields X^L and X^R in G by setting:

$$X^L(g) = gX, \quad X^R(g) = Xg, \tag{2.1.2}$$

for all $g \in G$. We say that X^L (respectively, X^R) is the *left-invariant* (respectively, the *right-invariant*) vector field in X associated to $X \in T_1G$.

The *Lie algebra* corresponding to G, denoted by $\mathfrak{g}$, is defined as the tangent space at 1 of the manifold G: $\mathfrak{g} = T_1G$; the *Lie bracket*, or *commutator*, in $\mathfrak{g}$ is obtained as the restriction of the Lie brackets of vector fields in G where we identify each $X \in \mathfrak{g}$ with the left-invariant vector field X^L.

We denote by $\exp : \mathfrak{g} \to G$ the *exponential map* of G, defined in such a way that, for each $X \in \mathfrak{g}$, the map:

$$\mathbb{R} \ni t \longmapsto \exp(tX) \in G \tag{2.1.3}$$

is a Lie group homomorphism whose derivative at $t = 0$ is equal to X. Then, the curve (2.1.3) is an integral curve of the vector fields X^L and X^R, that is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(tX) = X^L(\exp(tX)) = X^R(\exp(tX)), \tag{2.1.4}$$

for all $t \in \mathbb{R}$ (see [**48**, Theorem 3.31]).

A *Lie subgroup* of G is an immersed submanifold which is also a subgroup of G; then, H is also a Lie group with the group and the differentiable structure inherited from those of G (see Remark 2.1.2). A Lie subgroup $H \subset G$ will be an embedded submanifold if and only if H is closed in G (see [**47**, Theorem 2.5.4] and [**48**, Theorem 3.21]); moreover, every closed subgroup of a Lie group is a Lie subgroup of G (see [**47**, Theorem 2.12.6] and [**48**, Theorem 3.42]).

If $H \subset G$ is a Lie subgroup, then the differential of the inclusion map allows to identify the Lie algebra $\mathfrak{h}$ of H with a Lie subalgebra of $\mathfrak{g}$ (see [**48**, Proposition 3.33]); explicitly, we have:

$$\mathfrak{h} = \big\{X \in \mathfrak{g} : \exp(tX) \in H,\ \forall t \in \mathbb{R}\big\}. \tag{2.1.5}$$

Observe that every *discrete* subgroup $H \subset G$ is an embedded (and closed) Lie subgroup of G with $\dim(H) = 0$; in this case $\mathfrak{h} = \{0\}$.

If G^o denotes the connected component of G containing the identity (which is also an arc-connected component), then it is easy to see that G^o is a normal subgroup of G which is closed and open. Actually, every open subgroup of G is also closed, as its complementary is union of cosets of this subgroup, that are open. It follows that every open subgroup of G is the union of some connected components of G, and the Lie algebra of an open subgroup of G is identified with the Lie algebra of G.

2.1.4. REMARK. If G is a Lie group and $\mathfrak{h}$ is a subspace of $\mathfrak{g}$, then there exists a unique *left-invariant distribution* $\mathcal{D}^L$ and a unique *right-invariant distribution* $\mathcal{D}^R$ in G such that $\mathcal{D}^L(1) = \mathcal{D}^R(1) = \mathfrak{h}$. We have that $\mathcal{D}^L$, or $\mathcal{D}^R$, is *involutive* if and only if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. In this case, the maximal connected integral submanifold of $\mathcal{D}^L$, or of $\mathcal{D}^R$, passing through $1 \in G$ is a (connected) Lie subgroup of G whose Lie algebra is $\mathfrak{h}$; moreover, if $H \subset G$ is any Lie subgroup whose Lie algebra is $\mathfrak{h}$, then H^o is the maximal connected integral submanifold of $\mathcal{D}^L$, or of $\mathcal{D}^R$ passing through $1 \in G$. The other maximal connected integral submanifolds of $\mathcal{D}^L$ (respectively, of $\mathcal{D}^R$) are the left cosets gH (respectively, the right cosets Hg) of H. A proof of these facts can be found in [**47**, Theorem 2.5.2] and [**48**, Corollary (b), Theorem 3.19]; for the basic notions of involutive distributions, integral submanifolds and the *Frobenius Theorem* the reader may use, for instance, [**47**, Section 1.3] or [**48**, pages 41–49].

From the above observations we obtain that a curve $t \mapsto \gamma(t) \in G$ of class C^1 has image contained in some left coset of H if and only if

$$\gamma(t)^{-1}\gamma'(t) \in \mathfrak{h},$$

for all t; similarly, it has image in some right coset of H if and only if:

$$\gamma'(t)\gamma(t)^{-1} \in \mathfrak{h},$$

for all t.

We will now present a short list of the classical Lie groups that will be encountered in this text, and we will describe their Lie algebras. All these groups and algebras are formed by real or complex matrices, or by linear operators on real or complex vector spaces. The group multiplication will always be the multiplication of matrices, or the operator composition, and the Lie bracket will always be given by:

$$\big[X, Y\big] = XY - YX;$$

finally, the exponential map will always be:

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Typically, we will use capital letters to denote Lie groups and the corresponding small letters to denote their Lie algebras; all the vector spaces below will be meant to be finite dimensional.

- *The general linear group.* Let V be a real or a complex vector space; we denote by $\mathrm{GL}(V)$ the group of all linear automorphisms of V; its Lie algebra $\mathrm{gl}(V)$ coincides with the space of all linear endomorphisms $\mathrm{Lin}(V)$ of V. We call $\mathrm{GL}(V)$ the *general linear group of* V.

  We write $\mathrm{GL}(\mathbb{R}^n) = \mathrm{GL}(n, \mathbb{R})$, $\mathrm{gl}(\mathbb{R}^n) = \mathrm{gl}(n, \mathbb{R})$, $\mathrm{GL}(\mathbb{C}^n) = \mathrm{GL}(n, \mathbb{C})$ and $\mathrm{gl}(\mathbb{C}^n) = \mathrm{gl}(n, \mathbb{C})$; obviously, we can identify $\mathrm{GL}(n, \mathbb{R})$ (respectively, $\mathrm{GL}(n, \mathbb{C})$) with the group of invertible real (respectively, complex) $n \times n$ matrices, and $\mathrm{gl}(n, \mathbb{R})$ (resp., $\mathrm{gl}(n, \mathbb{C})$) with the algebra of all real (resp., complex) $n \times n$ matrices.

  Observe that if V is a real space and J is a complex structure on V, so that (V, J) is identified with a complex space, then $\mathrm{GL}(V, J)$ (resp., $\mathrm{gl}(V, J)$) can be seen as the subgroup (resp., the subalgebra) of $\mathrm{GL}(V)$ (resp., of $\mathrm{gl}(V)$) consisting of those operators that commute with J (see Lemma 1.2.3).

  In this way we obtain an inclusion of $\mathrm{GL}(n, \mathbb{C})$ into $\mathrm{GL}(2n, \mathbb{R})$ and of $\mathrm{gl}(n, \mathbb{C})$ into $\mathrm{gl}(2n, \mathbb{R})$ (see Example 1.2.2 and Remark 1.2.9).

- *The special linear group.*

  If V is a real or complex vector space, we denote by $\mathrm{SL}(V)$ the *special linear group of* V, given by the closed subgroup of $\mathrm{GL}(V)$ consisting of those endomorphisms with determinant equal to 1. Its Lie algebra $\mathrm{sl}(V)$ is given by the set of endomorphisms of V with null trace. We also write $\mathrm{SL}(\mathbb{R}^n) = \mathrm{SL}(n, \mathbb{R})$, $\mathrm{SL}(\mathbb{C}^n) = \mathrm{SL}(n, \mathbb{C})$, $\mathrm{sl}(\mathbb{R}^n) = \mathrm{sl}(n, \mathbb{R})$ and $\mathrm{sl}(\mathbb{C}^n) = \mathrm{sl}(n, \mathbb{C})$.

We identify $\mathrm{SL}(n, I\!R)$ (resp., $\mathrm{SL}(n, \mathbb{C})$) with the group of real (resp., complex) $n \times n$ matrices with determinant equal to 1, and $\mathrm{sl}(n, I\!R)$ (resp., $\mathrm{sl}(n, \mathbb{C})$) with the algebra of real (resp., complex) $n \times n$ matrices with null trace.

As in the case of the general linear group, we have inclusions: $\mathrm{SL}(n, \mathbb{C}) \subset \mathrm{SL}(2n, I\!R)$ and $\mathrm{sl}(n, \mathbb{C}) \subset \mathrm{sl}(2n, I\!R)$.

- *The orthogonal and the special orthogonal groups.*

If V is a real vector space endowed with a positive inner product g, we denote by $\mathrm{O}(V, g)$ the *orthogonal group of* (V, g), which is the closed subgroup of $\mathrm{GL}(V)$ consisting of the g-orthogonal operators. The *special orthogonal group of* (V, g) is defined by:

$$\mathrm{SO}(V, g) = \mathrm{O}(V, g) \cap \mathrm{SL}(V).$$

The Lie algebras of $\mathrm{O}(V, g)$ and of $\mathrm{SO}(V, g)$ coincide, and they are both denoted by $\mathrm{so}(V, g)$; this is the subalgebra of $\mathrm{gl}(V)$ consisting of g-anti-symmetric operators.

If $V = I\!R^n$ and g is the canonical inner product, then we write $\mathrm{O}(I\!R^n, g) = \mathrm{O}(n)$, $\mathrm{SO}(I\!R^n, g) = \mathrm{SO}(n)$ and $\mathrm{so}(I\!R^n, g) = \mathrm{so}(n)$; $\mathrm{O}(n)$ is identified with the group of $n \times n$ *orthogonal matrices* (a matrix is orthogonal if its transpose coincides with its inverse), $\mathrm{SO}(n)$ is the subgroup of $\mathrm{O}(n)$ consisting of those matrices with determinant equal to 1, and $\mathrm{so}(n)$ is the Lie algebra of real $n \times n$ anti-symmetric matrices.

- *The unitary and the special unitary groups.* Let $\mathcal{V}$ be a complex vector space endowed with a positive Hermitian product g_{s}. The *unitary group* of $(\mathcal{V}, g_{\mathrm{s}})$, denoted by $\mathrm{U}(\mathcal{V}, g_{\mathrm{s}})$, is the closed subgroup of $\mathrm{GL}(\mathcal{V})$ consisting of the g_{s}-unitary operators on $\mathcal{V}$; the *special unitary group* of $(\mathcal{V}, g_{\mathrm{s}})$ is defined by:

$$\mathrm{SU}(\mathcal{V}, g_{\mathrm{s}}) = \mathrm{U}(\mathcal{V}, g_{\mathrm{s}}) \cap \mathrm{SL}(\mathcal{V}).$$

The Lie algebra $\mathrm{u}(\mathcal{V}, g_{\mathrm{s}})$ of $\mathrm{U}(\mathcal{V}, gs)$ is the subalgebra of $\mathrm{gl}(\mathcal{V})$ consisting of the g_{s}-anti-Hermitian operators, and the Lie algebra $\mathrm{su}(\mathcal{V}, g_{\mathrm{s}})$ of $\mathrm{SU}(\mathcal{V}, g_{\mathrm{s}})$ is the subalgebra of $\mathrm{u}(\mathcal{V}, g_{\mathrm{s}})$ consisting of operators with null trace.

If V is a real space and J is a complex structure in V in such a way that (V, J) is identified with a complex vector space $\mathcal{V}$, then given a Hermitian product g_{s} in (V, J) we also write $\mathrm{U}(\mathcal{V}, g_{\mathrm{s}}) = \mathrm{U}(V, J, g_{\mathrm{s}})$, $\mathrm{SU}(\mathcal{V}, g_{\mathrm{s}}) = \mathrm{SU}(V, J, g_{\mathrm{s}})$, $\mathrm{u}(\mathcal{V}, g_{\mathrm{s}}) = \mathrm{u}(V, J, g_{\mathrm{s}})$ and $\mathrm{su}(\mathcal{V}, g_{\mathrm{s}}) = \mathrm{su}(V, J, g_{\mathrm{s}})$.

If $\mathcal{V} = \mathbb{C}^n$ and g_{s} is the canonical Hermitian product in $\mathbb{C}^n$, then we write $\mathrm{U}(\mathbb{C}^n, g_{\mathrm{s}}) = \mathrm{U}(n)$, $\mathrm{SU}(\mathbb{C}^n, g_{\mathrm{s}}) = \mathrm{SU}(n)$, $\mathrm{u}(\mathbb{C}^n, g_{\mathrm{s}}) = \mathrm{u}(n)$ and $\mathrm{su}(\mathbb{C}^n, g_{\mathrm{s}}) = \mathrm{su}(n)$; then $\mathrm{U}(n)$ is the group of complex $n \times n$ unitary matrices (a matrix is unitary if its conjugate transpose is equal to its inverse), $\mathrm{SU}(n)$ is the subgroup of $\mathrm{U}(n)$ consisting of matrices with determinant equal to 1, $\mathrm{u}(n)$ is the Lie algebra of all complex $n \times n$ anti-Hermitian matrices (a matrix is anti-Hermitian if its conjugate transpose equals its opposite), and $\mathrm{su}(n)$ is the subalgebra of $\mathrm{u}(n)$ consisting of matrices with null trace.

- *The symplectic group.*

Let (V, ω) be a symplectic space; in Definition 1.4.10 we have introduced the symplectic group $\mathrm{Sp}(V, \omega)$. We have that $\mathrm{Sp}(V, \omega)$ is a closed subgroup of $\mathrm{GL}(V)$; its Lie algebra consists of those linear endomorphisms X of V such that $\omega(X\cdot, \cdot)$ is a symmetric bilinear form, that is:

$$\omega(X(v), w) = \omega(X(w), v), \quad v, w \in V. \tag{2.1.6}$$

In terms of the linear operator $\omega\colon V \to V^*$, formula (2.1.6) is equivalent to the identity:

$$\omega \circ X = -X^* \circ \omega. \tag{2.1.7}$$

If ω is the canonical symplectic form of $\mathbb{R}^{2n}$, then we write $\mathrm{Sp}(\mathbb{R}^{2n}, \omega) = \mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{sp}(\mathbb{R}^{2n}, \omega) = \mathrm{sp}(2n, \mathbb{R})$. The matrix representations of elements of $\mathrm{Sp}(V, \omega)$ with respect to a symplectic basis are described in formulas (1.4.7) and (1.4.8). Using (2.1.7) it is easy to see that the matrix representation of elements of $\mathrm{sp}(V, \omega)$ in a symplectic basis is of the form:

$$\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}, \quad B, C \text{ symmetric},$$

where A^* denotes the transpose of A.

2.1.2. Actions of Lie Groups and Homogeneous Manifolds. In this subsection we state some results concerning actions of Lie groups on manifolds and we study the *homogeneous manifolds*, that are manifolds obtained as quotients of Lie groups.

If G is a group and M is a set, a *(left) action* of G on M is a map:

$$G \times M \ni (g, m) \longmapsto g \cdot m \in M \tag{2.1.8}$$

such that $g_1 \cdot (g_2 \cdot m) = (g_1 g_2) \cdot m$ and $1 \cdot m = m$ for all $g_1, g_2 \in G$ and for all $m \in M$, where 1 is the unit of G. Given an action of G on M, we get a map

$$\beta_m : G \longrightarrow M \tag{2.1.9}$$

given by $\beta_m(g) = g \cdot m$, and for all $g \in G$ we get a bijection:

$$\gamma_g : M \to M$$

of M given by $\gamma_g(m) = g \cdot m$; the map $g \mapsto \gamma_g$ is a group homomorphism from G to the group of bijections of M.

For all $m \in M$, we define the *orbit* of m relative to the action of G by:

$$G(m) = \{g \cdot m : g \in G\};$$

the orbits of the action of G form a partition of M; we also define the *isotropy group* of the element $m \in M$ by:

$$G_m = \{g \in G : g \cdot m = m\}.$$

It is easy to see that G_m is a subgroup of G.

We say that the action of G on M is *transitive* if $G(m) = M$ for some, hence for all, $m \in M$; we say that the action is *free*, or *without fixed points*, if $G_m = \{1\}$

for all $m \in M$. The action is *effective* if the homomorphism $g \mapsto \gamma_g$ is injective, i.e., if $\bigcap_{m \in M} G_m = \{1\}$.

If H is a subgroup of G, we will denote by G/H the set of *left cosets* of H in G:

$$G/H = \big\{gH : g \in G\big\},$$

where $gH = \{gh : h \in H\}$ is the left coset of $g \in G$. We have a natural action of G on G/H given by:

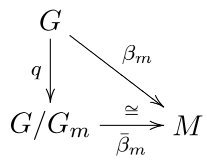
$$G \times G/H \ni (g_1, g_2H) \longmapsto (g_1 g_2)H \in G/H; \tag{2.1.10}$$

this action is called *action by left translation* of G in the left cosets of H. The action (2.1.10) is always transitive.

If G acts on M and G_m is the isotropy group of the element $m \in M$, then the map β_m of (2.1.9) passes to the quotient and defines a bijection:

$$\bar{\beta}_m : G/G_m \longrightarrow G(m) \tag{2.1.11}$$

given by $\bar{\beta}_m(gG_m) = g \cdot m$. We therefore have the following commutative diagram:



where $q\colon G \to G/G_m$ denotes the quotient map.

2.1.5. DEFINITION. Given actions of the group G on sets M and N, we say that a map $\phi : M \to N$ is *G-equivariant* if the following identity holds:

$$\phi(g \cdot m) = g \cdot \phi(m),$$

for all $g \in G$ and all $m \in M$. If ϕ is an equivariant bijection, we say that ϕ is an *equivariant isomorphism*; in this case ϕ^{-1} is automatically equivariant.

The bijection (2.1.11) is an equivariant isomorphism when we consider the action of G on G/G_m by left translation and the action of G on $G(m)$ obtained by the restriction of the action of G on M.

2.1.6. REMARK. It is possible to define also a *right action* of a group G on a set M as a map:

$$M \times G \ni (m, g) \longmapsto m \cdot g \in M \tag{2.1.12}$$

that satisfies $(m \cdot g_1) \cdot g_2 = m \cdot (g_1 g_2)$ and $m \cdot 1 = m$ for all $g_1, g_2 \in G$ and all $m \in M$. A theory totally analogous to the theory of left actions can be developed for right actions; as a matter of facts, every right action (2.1.12) defines a left action by $(g, m) \mapsto m \cdot g^{-1}$. Observe that in the theory of right actions, in order to define properly the bijection $\bar{\beta}_m$ in formula (2.1.11), the symbol G/H has to be meant as the set of *right cosets* of H.

Let's assume now that G is a Lie group and that M is a manifold; in this context we will always assume that the map (2.1.8) is differentiable, and we will say that G *acts differentiably on* M. If H is a closed subgroup of G, then there exists a unique differentiable structure in the set G/H such that the quotient map:

$$q : G \longrightarrow G/H$$

is a differentiable submersion (see Remark 2.1.3). The kernel of the differential $\mathrm{d}q(1)$ is precisely the Lie algebra $\mathfrak{h}$ of H, so that the tangent space to G/H at the point $1H$ may be identified with the quotient space $\mathfrak{g}/\mathfrak{h}$. Observe that, since q is open and surjective, it follows that G/H has the *quotient topology* induced by q from the topology of G.

By continuity, for all $m \in M$, the isotropy group G_m is a closed subgroup of G, hence we get a differentiable structure on G/G_m; it can be shown that the map $gG_m \mapsto g \cdot m$ is a differentiable immersion, from which we obtain the following:

2.1.7. PROPOSITION. *If G is a Lie group that acts differentiably on the manifold M, then for all $m \in M$ the orbit $G(m)$ has a unique differentiable structure that makes* (2.1.11) *into a differentiable diffeomorphism; with such structure $G(m)$ is an immersed submanifold of M, and the tangent space $T_mG(m)$ coincides with the image of the map:*

$$\mathrm{d}\beta_m(1) : \mathfrak{g} \longrightarrow T_mM,$$

where β_m is the map defined in (2.1.9). □

2.1.8. REMARK. If we choose a different point $m' \in G(m)$, so that $G(m') = G(m)$, then it is easy to see that the differentiable structure induced on $G(m)$ by $\bar{\beta}_{m'}$ coincides with that induced by $\bar{\beta}_m$.

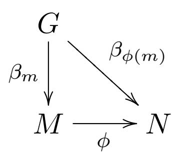
We also have the following:

2.1.9. COROLLARY. *If G acts transitively on M, then for all $m \in M$ the map* (2.1.11) *is a differentiable diffeomorphism of G/G_m onto M; in particular, the map β_m of* (2.1.9) *is a surjective submersion.* □

In the case of transitive actions, when we identify G/G_m with M by the diffeomorphism (2.1.11), we will say that m *is the base point* for such identification; we then say that M (or G/G_m) is a *homogeneous manifold*.

2.1.10. COROLLARY. *Let M, N be manifolds and let G be a Lie group that acts differentiably on both M and N. If the action of G on M is transitive, then every equivariant map $\phi : M \to N$ is differentiable.*

PROOF. Choose $m \in M$; the equivariance property of ϕ gives us the following commutative diagram:



and the conclusion follows from Corollary 2.1.9 and Remark 2.1.3. □

In some situations we will need to know if a given orbit of the action of a Lie group is an embedded submanifold. Let us give the following definition:

2.1.11. DEFINITION. Let X be a topological space; a subset $S \subset X$ is said to be *locally closed* if S is given by the intersection of an open and a closed subset of X. Equivalently, S is locally closed when it is open in the relative topology of its closure $\overline{S}$.

Exercise 2.4 is dedicated to the notion of locally closed subsets.

We have the following:

2.1.12. THEOREM. *Let G be a Lie group acting differentiably on the manifold M. Given $m \in M$, the orbit $G(m)$ is an embedded submanifold of M if and only if $G(m)$ is locally closed in M.*

PROOF. See [**47**, Theorem 2.9.7]. □

We conclude the subsection with a result that relates the notions of *fibration* and homogeneous manifold.

2.1.13. DEFINITION. Given manifolds F, E and B and a differentiable map $p : E \to B$, we say that p is a *differentiable fibration with typical fiber F* if for all $b \in B$ there exists a diffeomorphism:

$$\alpha : p^{-1}(U) \longrightarrow U \times F$$

such that $\pi_1 \circ \alpha = p|_{p^{-1}(U)}$, where $U \subset B$ is an open neighborhood of b in B and $\pi_1 : U \times F \to U$ is the projection onto the first factor. In this case, we say that α is a *local trivialization* of p around b.

2.1.14. THEOREM. *Let G be a Lie group and H, K closed subgroups of G with $K \subset H$; then the map:*

$$p : G/K \longrightarrow G/H$$

defined by $p(gK) = gH$ is a differentiable fibration with typical fiber H/K.

PROOF. It follows from Remark 2.1.3 that p is differentiable. Given $gH \in G/H$, let $s : U \to G$ be a local section of the submersion $q : G \to G/H$ defined in an open neighborhood $U \subset G/H$ of gH; it follows that $q \circ s$ is the inclusion of U in G/H. We define a local trivialization of p:

$$\alpha : p^{-1}(U) \longrightarrow U \times H/K$$

by setting $\alpha(xK) = (xH, s(xH)^{-1}\, xK)$. The conclusion follows. □

2.1.15. COROLLARY. *Under the assumptions of Corollary 2.1.9, the map β_m given in* (2.1.9) *is a differentiable fibration with typical fiber G_m.* □

2.1.16. COROLLARY. *Let $f\colon G \to G'$ be a Lie group homomorphism and let $H \subset G$, $H' \subset G'$ be closed subgroups such that $f(H) \subset H'$; consider the map:*

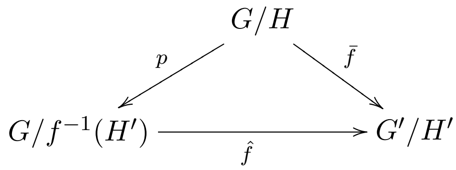
$$\bar{f}\colon G/H \longrightarrow G'/H'$$

induced from f by passage to the quotient, i.e., $\bar{f}(gH) = f(g)H'$ for all $g \in G$. If $\bar{f}$ is surjective, then $\bar{f}$ is a differentiable fibration with typical fiber $f^{-1}(H')/H$.

PROOF. Consider the action of G on G'/H' given by

$$G \times G'/H' \ni (g, g'H') \longmapsto (f(g)g')H' \in G'/H'.$$

The orbit of the element $1H' \in G'/H'$ is the image of $\bar{f}$, and its isotropy group is $f^{-1}(H')$; since $\bar{f}$ is surjective, it follows from Corollary 2.1.9 that the map $\hat{f} \colon G/f^{-1}(H') \to G'/H'$ induced from f by passage to the quotient is a diffeomorphism. We have the following commutative diagram:

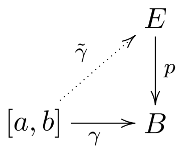


where p is induced from the identity of G by passage to the quotient; it follows from Theorem 2.1.14 that p is a differentiable fibration with typical fiber $f^{-1}(H')/H$. This concludes the proof. □

A *differentiable covering* is a differentiable fibering whose fiber is a *discrete* manifold (i.e., zero dimensional). We have the following:

2.1.17. COROLLARY. *Under the assumptions of Corollary 2.1.16, if H and $f^{-1}(H')$ have the same dimension, then $\bar{f}$ is a differentiable covering.* □

2.1.18. REMARK. Given a differentiable fibration $p : E \to B$ with typical fiber F, then every curve $\gamma : [a, b] \to B$ of class C^k, $0 \le k \le +\infty$, admits a *lift* $\overline{\gamma} : [a, b] \to E$ (i.e., $p \circ \overline{\gamma} = \gamma$) which is of class C^k:



The proof of this fact is left to the reader in Exercise 2.9.

2.1.3. Linearization of the Action of a Lie Group on a Manifold. In this subsection we will consider a Lie group G with a differentiable (left) action on the manifold M; we show that such action defines a anti-homomorphism of the Lie algebra $\mathfrak{g}$ of G to the Lie algebra of the differentiable vector fields on M.

Given $X \in \mathfrak{g}$, we define a differentiable vector field X^* on M by setting:

$$X^*(m) = \mathrm{d}\beta_m(1) \cdot X, \quad m \in M,$$

where β_m is the map defined in (2.1.9).

Recall that if $f : N_1 \to N_2$ is a differentiable map, the vector fields Y_1 and Y_2 on N_1 and N_2 respectively are said to be *f-related* if:

$$Y_2(f(n)) = \mathrm{d}f_n(Y_1(n)), \quad \forall\, n \in N_1.$$

2.1.19. REMARK. If Y_1, Z_1 are differentiable vector fields on the manifold N_1 that are f-related respectively with the fields Y_2, Z_2 on the manifold N_2, then the Lie bracket $[Y_1, Z_1]$ is f-related to the Lie bracket $[Y_2, Z_2]$.

Observe that, for all $g \in G$ and all $m \in M$, we have

$$\beta_{gm} = \beta_m \circ r_g,$$

hence

$$\mathrm{d}\beta_{gm}(1) = \mathrm{d}\beta_m(g) \circ \mathrm{d}r_g(1). \tag{2.1.13}$$

If X^R denotes the right invariant vector field on G corresponding to the element $X \in \mathfrak{g}$, then, using (2.1.13), we have:

$$X^*(g \cdot m) = \mathrm{d}\beta_m(g) \cdot X^R(g), \quad \forall m \in M. \tag{2.1.14}$$

The identity (2.1.14) tells us that, for all $m \in M$, the field X^* in M is β_m-related with the field X^R in G.

2.1.20. REMARK. Let us denote by X^L the left invariant vector field on G corresponding to $X \in \mathfrak{g}$; if G acts on the left on M, then in general it is not possible to construct a vector field in M which is β_m-related to X^L. Observe also that, in general, the field X^* is *not* invariant by the action of G in M; actually, it is not possible in general to construct a vector field on M which is invariant by the action of G and whose value at a given point is given.

As a corollary of (2.1.14) we get the following:

2.1.21. PROPOSITION. *Given $X, Y \in \mathfrak{g}$, then we have:*

$$[X, Y]^* = -[X^*, Y^*],$$

where the bracket on the left of the equality is the Lie product in $\mathfrak{g}$ and the bracket on the right denotes the Lie bracket of vector fields in M.

PROOF. Choose $m \in M$; since the vector fields X^* and Y^* are β_m-related respectively to the right invariant vector fields X^R and Y^R, it follows from Remark 2.1.19 that $[X^*, Y^*]$ is β_m-related to $[X^R, Y^R]$. To conclude the proof, we will show that:

$$[X^R, Y^R] = -[X, Y]^R; \tag{2.1.15}$$

observe now that from (2.1.15) it will follow that both $[X^*, Y^*]$ and $-[X, Y]^*$ are β_m-related to $[X^R, Y^R]$, hence they must coincide on $\mathrm{Im}(\beta_m) = G(m)$. Since m is arbitrary, the proof of Proposition 2.1.21 will follow.

In order to show (2.1.15), consider the inversion map $\mathrm{inv} : G \to G$ given by $\mathrm{inv}(g) = g^{-1}$; we have that $\mathrm{d}(\mathrm{inv})(1) = -\mathrm{Id}$. Then, it is easy to see that X^R is inv-related to the left invariant field $-X^L$, and, by Remark 2.1.19, $[X^R, Y^R]$ is inv-related to $[X^L, Y^L] = [X, Y]^L$; also, $-[X, Y]^R$ is inv-related to $[X, Y]^L$. The conclusion now follows from the fact that inv is surjective. □

The map $X \mapsto X^*$ is called the *linearization of the action of G in M*; Proposition 2.1.21 tells us that this map is a *anti-homomorphism* of the Lie algebra $\mathfrak{g}$ into the Lie algebra of differentiable vector fields on M.

2.1.22. REMARK. From (2.1.14) it follows easily that, for all $m \in M$, the map $t \mapsto \exp(tX) \cdot m$ is an integral curve of X^*.

More generally, given any map $I \ni t \mapsto X(t) \in \mathfrak{g}$ defined in an interval $I \subset \mathbb{R}$, we obtain a *time-dependent right invariant vector field* in G given by:

$$I \times G \ni (t, g) \longmapsto X(t)^R(g) = X(t)g \in T_gG; \tag{2.1.16}$$

we also have a time-dependent vector field in M by setting:

$$I \times M \ni (t, m) \longmapsto X(t)^*(m) \in T_mM. \tag{2.1.17}$$

From (2.1.14) it follows also that, for any $m \in M$, the map β_m takes integral curves of (2.1.16) into integral curves of (2.1.17); more explicitly, if $t \mapsto \gamma(t) \in G$ satisfies

$$\gamma'(t) = X(t)\gamma(t),$$

for all t then:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t) \cdot m) = X(t)^*(\gamma(t) \cdot m).$$

2.2. Grassmannians and Their Differentiable Structure

In this section we will study the geometry of the set of all k-dimensional subspaces of a Euclidean space.

Let n, k be fixed integers, with $n \geq 0$ and $0 \leq k \leq n$; we will denote by $G_k(n)$ the set of all k-dimensional vector subspaces of $\mathbb{R}^n$; $G_k(n)$ is called the *Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$*.

Our goal is to describe a differentiable atlas for $G_k(n)$, and the main idea is to view the points of $G_k(n)$ as *graphs* of linear maps defined on a fixed k-dimensional subspace of $\mathbb{R}^n$ and taking values in another fixed $(n-k)$-dimensional subspace of $\mathbb{R}^n$, where these two fixed subspaces are transversal.

To this aim, we consider a direct sum decomposition $\mathbb{R}^n = W_0 \oplus W_1$, where $\dim(W_0) = k$ (and obviously $\dim(W_1) = n - k$). For every linear operator $T : W_0 \to W_1$, the *graph* of T given by:

$$\mathrm{Gr}(T) = \{\, v + T(v) : v \in W_0 \,\}$$

is an element in $G_k(n)$. Moreover, an element $W \in G_k(n)$ is of the form $\mathrm{Gr}(T)$ if and only if it is transversal to W_1, i.e., iff it belongs to the set:

$$G_k^0(n, W_1) = \big\{\, W \in G_k(n) : W \cap W_1 = \{0\} \,\big\} \subset G_k(n).$$

In this situation, the operator T is uniquely determined by W. We can therefore define a bijection:

$$\phi_{W_0,W_1} \colon G_k^0(n, W_1) \longrightarrow \mathrm{Lin}(W_0, W_1), \tag{2.2.1}$$

by setting $\phi_{W_0,W_1}(W) = T$ when $W = \mathrm{Gr}(T)$.

More concretely, if π_0 and π_1 denote respectively the projections onto W_0 and W_1 in the decomposition $\mathbb{R}^n = W_0 \oplus W_1$, then the operator $T = \phi_{W_0,W_1}(W)$ is given by:

$$T = (\pi_1|_W) \circ (\pi_0|_W)^{-1}.$$

Observe that the condition that W be transversal to W_1 is equivalent to the condition that the restriction $\pi_0|_W$ be an isomorphism onto W_0.

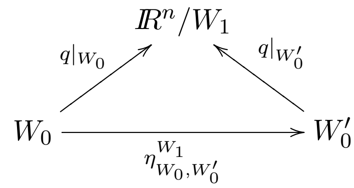
We will now show that the collection of the charts ϕ_{W_0,W_1}, when (W_0, W_1) run over the set of all direct sum decomposition of $\mathbb{R}^n$ with $\dim(W_0) = k$, is a differentiable atlas for $G_k(n)$. To this aim, we need to study the transition functions between these charts. Let us give the following:

2.2.1. DEFINITION. Given subspaces $W_0, W_0' \subset \mathbb{R}^n$ and given a common complementary subspace $W_1 \subset \mathbb{R}^n$ of theirs, i.e., $\mathbb{R}^n = W_0 \oplus W_1 = W_0' \oplus W_1$, then we have an isomorphism:

$$\eta = \eta^{W_1}_{W_0,W_0'} \colon W_0 \longrightarrow W_0',$$

obtained by the restriction to W_0 of the projection onto W_0' relative to the decomposition $\mathbb{R}^n = W_0' \oplus W_1$. We say that $\eta^{W_1}_{W_0,W_0'}$ is the *isomorphism of* W_0 *and* W_0' *determined by the common complementary subspace* W_1.

The inverse of $\eta^{W_1}_{W_0,W_0'}$ is simply $\eta^{W_1}_{W_0',W_0}$; we have the following commutative diagram of isomorphisms:



where $q : \mathbb{R}^n \to \mathbb{R}^n/W_1$ is the quotient map.

Let us consider charts ϕ_{W_0,W_1} and ϕ_{W_0',W_1} in $G_k(n)$, with $k = \dim(W_0) = \dim(W_0')$; observe that they have *the same domain*. In this case it is easy to obtain the following formula for the transition function:

$$\phi_{W_0',W_1} \circ (\phi_{W_0,W_1})^{-1}(T) = (\pi_1'|_{W_0} + T) \circ \eta^{W_1}_{W_0',W_0}, \tag{2.2.2}$$

where π_1' denotes the projection onto W_1 relative to the decomposition $\mathbb{R}^n = W_0' \oplus W_1$.

Let us now consider decompositions $\mathbb{R}^n = W_0 \oplus W_1 = W_0 \oplus W_1'$, with $\dim(W_0) = k$, and let us look at the transition function $\phi_{W_0,W_1'} \circ (\phi_{W_0,W_1})^{-1}$. In first place, we observe that its domain consists of those operators $T \in \mathrm{Lin}(W_0, W_1)$ such that $\mathrm{Gr}(T) \in G^0_k(n, W_1')$; it is easy to see that this condition is equivalent to the *invertibility* of the map:

$$\mathrm{Id} + (\pi_0'|_{W_1}) \circ T,$$

where π_0' denotes the projection onto W_0 relative to the decomposition $\mathbb{R}^n = W_0 \oplus W_1'$ and Id is the identity operator on W_0. We have the following formula for $\phi_{W_0,W_1'} \circ (\phi_{W_0,W_1})^{-1}$:

$$\phi_{W_0,W_1'} \circ (\phi_{W_0,W_1})^{-1}(T) = \eta^{W_0}_{W_1,W_1'} \circ T \circ \big(\mathrm{Id} + (\pi_0'|_{W_1}) \circ T\big)^{-1}. \tag{2.2.3}$$

We have therefore proven the following:

2.2.2. PROPOSITION. *The set of all charts* ϕ_{W_0,W_1} *in* $G_k(n)$*, where the pair* (W_0, W_1) *run over the set of all direct sum decompositions of* $\mathbb{R}^n$ *with* $\dim(W_0) = k$*, is a differentiable atlas for* $G_k(n)$.

PROOF. Since every subspace of $\mathbb{R}^n$ admits one complementary subspace, it follows that the domains of the charts ϕ_{W_0,W_1} cover $G_k(n)$. The transition functions (2.2.2) and (2.2.3) are differentiable maps defined in open subsets of the vector space $\mathrm{Lin}(W_0, W_1)$. The general case of compatibility between charts ϕ_{W_0,W_1} and $\phi_{W_0',W_1'}$ follows from transitivity. □

2.2.3. REMARK. As to the argument of transitivity mentioned in the proof of Proposition 2.2.2, we observe that in general the property of the compatibility of charts is *not* transitive. However, the following weaker transitivity property holds, and that applies to the case of Proposition 2.2.2: if ψ_0, ψ_1 and ψ_2 are charts on a set such that ψ_0 is compatible with ψ_1, ψ_1 is compatible with ψ_2 and the domain of ψ_0 *coincides* with the domain of ψ_1, then ψ_0 is compatible with ψ_2.

2.2.4. REMARK. Formulas (2.2.2) and (2.2.3) show indeed that the charts ϕ_{W_0,W_1} form a *real analytic* atlas for $G_k(n)$.

2.2.5. REMARK. Given a finite collection $V_1, \ldots, V_r$ of k-dimensional subspaces of $\mathbb{R}^n$, it is possible to find a subspace W which is complementary to all of the V_i's. For, if $k < n$, we can choose a vector $v_1 \in \mathbb{R}^n \setminus \bigcup_{i=1}^r V_i$. Let us now consider the subspaces $V_i' = V_i \oplus \mathbb{R}\, v_1$ of dimension $k+1$; by repeating the construction to the V_i''s, we determine inductively vectors $v_1, \ldots, v_{n-k}$ that form a basis for a common complementary to the V_i's. This argument shows that every *finite* subset of $G_k(n)$ belongs to the domain of some chart ϕ_{W_0,W_1}. In Exercise 2.6 the reader is asked to show that the same holds for *countable* subsets of $G_k(n)$.

We finally prove that $G_k(n)$ is a manifold:

2.2.6. THEOREM. *The differentiable atlas in Proposition 2.2.2 makes* $G_k(n)$ *into a differentiable manifold of dimension* $k(n-k)$.

PROOF. If $\dim(W_0) = k$ and $\dim(W_1) = n - k$, then $\dim(\mathrm{Lin}(W_0, W_1)) = k(n-k)$. It remains to prove that the topology defined by the atlas is Hausdorff and second countable. The Hausdorff property follows from the fact that every pair of points of $G_k(n)$ belongs to the domain of a chart. The second countability property follows from the fact that, if we consider the finite set of chart ϕ_{W_0,W_1}, where both W_0 and W_1 are generated by elements of the canonical basis of $\mathbb{R}^n$, we obtain a finite differentiable atlas for $G_k(n)$. □

2.2.7. REMARK. It follows immediately from the definition of topology induced by a differentiable atlas that the subsets $G_k^0(n, W_1) \subset G_k(n)$ are open; moreover, since the charts ϕ_{W_0,W_1} are surjective, it follows that $G_k^0(n, W_1)$ is homeomorphic (and diffeomorphic) to the vector space $\mathrm{Lin}(W_0, W_1)$.

2.2.8. EXAMPLE. The Grassmannian $G_1(n)$ of all the lines through the origin in $\mathbb{R}^n$ is also known as the *real projective space* $\mathbb{R}P^{n-1}$. By taking $W_0 = \{0\}^{n-1} \oplus \mathbb{R}$ and $W_1 = \mathbb{R}^{n-1} \oplus \{0\}$, the chart ϕ_{W_0,W_1} gives us what is usually known in projective geometry as the *homogeneous coordinates*. The space $I\!RP^{n-1}$ can also be described as the quotient of the sphere S^{n-1} obtained by identifying the antipodal points.

The *real projective line* $I\!RP^1$ is diffeomorphic to the circle S^1; in fact, considering $S^1 \subset \mathbb{C}$, the map $z \mapsto z^2$ is a two-fold covering of S^1 over itself that identifies antipodal points.

2.2.9. REMARK. The theory of this section can be repeated *verbatim* to define a manifold structure in the Grassmannian of all k-dimensional complex subspaces of $\mathbb{C}^n$. Formulas (2.2.2) and (2.2.3) are *holomorphic*, which says that such Grassmannian is a *complex manifold*, whose complex dimension is $k(n-k)$.

2.3. The tangent Space to a Grassmannian

In this section we give a concrete description of the tangent space $T_W G_k(n)$ for $W \in G_k(n)$, by showing that it can be naturally identified with the space $\mathrm{Lin}(W, I\!R^n/W)$. This identification will allow to compute in a simple way the derivative of a curve in $G_k(n)$.

We start with an informal approach. Suppose that we are given a differentiable curve $t \mapsto W(t)$ in $G_k(n)$, i.e., for all instants t we have k-dimensional subspace $W(t)$ of $I\!R^n$. How can we think of the derivative $W'(t_0)$ in an intuitive way? Consider a curve of vectors $t \mapsto w(t) \in I\!R^n$, with $v(t) \in W(t)$ for all t; in some sense, the derivative $v'(t_0)$ must *encode* part of the information contained in the derivative $W'(t_0)$. We now try to formalize these ideas.

For all t, write $W(t) = \mathrm{Ker}\big(A(t)\big)$, where $A(t) \in \mathrm{Lin}(I\!R^n, I\!R^{n-k})$; differentiating the identity $A(t)w(t) = 0$ in $t = t_0$ we get:

$$A'(t_0)w(t_0) + A(t_0)w'(t_0) = 0.$$

This identity shows that the value of $w'(t_0)$ is totally determined by $w(t_0)$ *up to elements of* $W(t_0)$. More precisely, to all $w_0 \in W(t_0)$, we can associate a class $w_0' + W(t_0) \in I\!R^n/W(t_0)$ by setting $w_0' = w'(t_0)$, where $t \mapsto w(t)$ is any differentiable curve in $I\!R^n$ with $w(t) \in W(t)$ for all t and $w(0) = w_0$. Using the above identity it is easy to see that such map is well defined, i.e., it does not depend on the choice of the curve $w(t)$. The map $w_0 \mapsto w_0' + W(t_0)$ is a linear operator from $W(t_0)$ to $I\!R^n/W(t_0)$, and we can look at it as the *derivative of the curve of subspaces* $W(t)$ *in* $t = t_0$.

We can now prove the existence of a canonical isomorphism of the tangent space $T_W G_k(n)$ with $\mathrm{Lin}(W, I\!R^n/W)$; in the following proposition we will use the abstract formalism concerning the functor $\mathrm{Lin}(\cdot,\cdot)$ introduced in Remark 1.1.1.

2.3.1. PROPOSITION. *Let* $W \in G_k(n)$ *and* W_1 *be a complementary subspace of* W *in* $I\!R^n$. *Denote by* $q_1 : W_1 \to I\!R^n/W$ *the restriction of the quotient map onto* $I\!R^n/W$. *We have an isomorphism:*

$$\mathrm{Lin}(\mathrm{Id}, q_1) \circ \mathrm{d}\phi_{W,W_1}(W) : T_W G_k(n) \longrightarrow \mathrm{Lin}(W, I\!R^n/W), \tag{2.3.1}$$

where

$$\mathrm{Lin}(\mathrm{Id}, q_1)\colon \mathrm{Lin}(W, W_1) \longrightarrow \mathrm{Lin}(W, I\!R^n/W) \tag{2.3.2}$$

is the operator of composition on the left $T \mapsto q_1 \circ T$ *(recall formulas* (1.1.2) *and* (1.1.3)*).*

The isomorphism (2.3.1) does not *depend on the choice of the complementary subspace* W_1.

PROOF. Since q_1 is an isomorphism and ϕ_{W,W_1} is a chart around W, obviously (2.3.1) is an isomorphism. The only non trivial fact in the statement is the independence of (2.3.1) from the choice of the subspace W_1. To prove this fact, consider a different complementary subspace W_1' of W in $I\!R^n$; observe that $\phi_{W,W_1}(W) = \phi_{W,W_1'}(W) = 0$. By differentiating the transition function (2.2.3) in $T = 0$ we see that the following diagram commutes:

$$\begin{array}{ccc} & T_W G_k(n) & \\ {\scriptstyle \mathrm{d}\phi_{W,W_1}(W)} \swarrow & & \searrow {\scriptstyle \mathrm{d}\phi_{W,W_1'}(W)} \\ \mathrm{Lin}(W, W_1) & \xrightarrow[\mathrm{Lin}\left(\mathrm{Id},\, \eta^W_{W_1,W_1'}\right)]{} & \mathrm{Lin}(W, W_1'). \end{array}$$

The conclusion now follows easily from the observation that also the diagram

$$\begin{array}{ccc} W_1 & \xrightarrow{\eta^W_{W_1,W_1'}} & W_1' \\ {\scriptstyle q_1} \searrow & & \swarrow {\scriptstyle q_1'} \\ & I\!R^n/W & \end{array} \tag{2.3.3}$$

is commutative, where q_1' denotes the restriction to W_1' of the quotient map onto R^n/W. □

2.3.2. REMARK. Observe that, from a functorial point of view, the conclusion of Proposition 2.3.1 follows by applying the functor $\mathrm{Lin}(W, \cdot)$ to the diagram (2.3.3).

Keeping in mind Proposition 2.3.1, we will henceforth identify the spaces $T_W G_k(n)$ and $\mathrm{Lin}(W, I\!R^n/W)$. Our next proposition will provide a justification for the informal reasons of such identification given at the beginning of the section:

2.3.3. PROPOSITION. *Let* $W : I \to G_k(n)$ *and* $w : I \to I\!R^n$ *be curves defined in an interval* I *containing* t_0*, both differentiable at* $t = t_0$*. Suppose that* $w(t) \in W(t)$ *for all* $t \in I$*. Then, the following identity holds:*

$$W'(t_0) \cdot w(t_0) = w'(t_0) + W(t_0) \in I\!R^n/W(t_0),$$

where we identify $W'(t_0)$ *with an element in* $\mathrm{Lin}(W, I\!R^n/W(t_0))$ *using the isomorphism* (2.3.1).

PROOF. Set $W_0 = W(t_0)$ and choose a complementary subspace W_1 of W_0 in $I\!R^n$. Set $T = \phi_{W_0,W_1} \circ W$, so that, for all $t \in I$ sufficiently close to t_0, we have $W(t) = \mathrm{Gr}(T(t))$. Denoting by π_0 the projection onto W_0 relative to the decomposition $I\!R^n = W_0 \oplus W_1$, we set $u = \pi_0 \circ w$.

Since $w(t) \in W(t)$, we have:

$$w(t) = u(t) + T(t) \cdot u(t), \quad t \in I. \tag{2.3.4}$$

Using the isomorphism (2.3.1) we see that $W'(t_0) \in T_{W_0}G_k(n)$ is identified with:

$$\mathrm{Lin}(\mathrm{Id}, q_1) \circ \mathrm{d}\phi_{W_0,W_1}(W_0) \cdot W'(t_0) = q_1 \circ T'(t_0) \in \mathrm{Lin}(W_0, I\!R^n/W_0),$$

where q_1 and $\mathrm{Lin}(\mathrm{Id}, q_1)$ are defined as in the statement of Proposition 2.3.1.

Hence, it remains to show that:

$$q_1 \circ T'(t_0) \cdot w(t_0) = w'(t_0) + W_0 \in I\!R^n/W_0.$$

Differentiating (2.3.4) in $t = t_0$ and observing that $T(t_0) = 0$, $u(t_0) = w(t_0)$, we obtain:

$$w'(t_0) = u'(t_0) + T'(t_0) \cdot w(t_0),$$

where $u'(t_0) \in W_0$. The conclusion follows. □

2.3.4. REMARK. Given a curve $W : I \to G_k(n)$, $t_0 \in I$ and a vector $w_0 \in W_0 = W(t_0)$, we can always find a curve $t \mapsto w(t) \in I\!R^n$ defined in a neighborhood of t_0 in I, with $w(t) \in W(t)$ for all t, with $w(t_0) = w_0$ and such that w has the *same regularity as* W. Indeed, for t near t_0, we write W in the form $W(t) = \mathrm{Gr}(T(t))$ using a local chart ϕ_{W_0,W_1}; then we can define $w(t) = w_0 + T(t) \cdot w_0$.

This implies that Proposition 2.3.3 can *always* be used to compute differentials of functions defined on, or taking values in, Grassmannian manifolds. Indeed, the computation of differentials may always be reduced to the computation of tangent vectors to curves, and to this aim we can always use Proposition 2.3.3 (see for instance the proofs of Lemma 2.3.5, Proposition 2.4.11 and Proposition 2.4.12).

We now compute the differential of a chart ϕ_{W_0,W_1} at a point W of its domain using the identification $T_W G_k(n) \simeq \mathrm{Lin}(W, I\!R^n/W)$:

2.3.5. LEMMA. *Consider a direct sum decomposition $I\!R^n = W_0 \oplus W_1$, with $\dim(W_0) = k$, and let $W \in G_k^0(n, W_1)$; then the differential of the chart ϕ_{W_0,W_1} at W is the operator:*

$$\mathrm{Lin}\big(\eta_{W_0,W}^{W_1}, q_1^{-1}\big) : \mathrm{Lin}(W, I\!R^n/W) \longrightarrow \mathrm{Lin}(W_0, W_1),$$

that is:

$$\mathrm{d}\phi_{W_0,W_1}(W) \cdot Z = q_1^{-1} \circ Z \circ \eta_{W_0,W}^{W_1}, \quad Z \in \mathrm{Lin}(W, I\!R^n/W) \cong T_W G_k(n),$$

where q_1 denotes the restriction to W_1 of the quotient map onto R^n/W and $\eta_{W_0,W}^{W_1}$ is the isomorphism of W_0 onto W determined by the common complementary W_1 (cf. Definition 2.2.1).

PROOF. It is a direct application of the technique described in Remark 2.3.4.

Let $t \mapsto \mathfrak{W}(t)$ be a differentiable curve in $G_k(n)$ with $\mathfrak{W}(0) = W$, $\mathfrak{W}'(0) = Z$; write $T(t) = \phi_{W_0,W_1}(\mathfrak{W}(t))$, so that $\mathfrak{W}(t) = \mathrm{Gr}(T(t))$ for all t; observe that $T'(0) = \mathrm{d}\phi_{W_0,W_1}(W) \cdot Z$.

Let $w \in W$; since $W = \mathrm{Gr}(T(0))$, we can write $w = w_0 + T(0) \cdot w_0$ with $w_0 \in W_0$. Then, $t \mapsto w(t) = w_0 + T(t) \cdot w_0$ is a curve in $I\!R^n$ with $w(t) \in \mathfrak{W}(t)$ for all t and $w(0) = w$. By Proposition 2.3.3 we have:

$$\mathfrak{W}'(0) \cdot w = Z \cdot w = w'(0) + W = T'(0) \cdot w_0 + W \in I\!R^n/W.$$

Observing that $w_0 = \eta^{W_1}_{W,W_0}(w)$, we conclude that

$$Z = q_1 \circ T'(0) \circ \eta^{W_1}_{W,W_0}.$$

The conclusion follows. □

2.4. The Grassmannian as a Homogeneous Space

In this section we will show that the natural action of the general linear group of $I\!R^n$ on $G_k(n)$ is differentiable. This action is transitive, even when restricted to the special orthogonal group; it will follow that the Grassmannian is a quotient of this group, and therefore it is a *compact and connected* manifold.

Each linear isomorphism $A \in \mathrm{GL}(n, I\!R)$ defines a bijection of $G_k(n)$ that associates to each $W \in G_k(n)$ its image $A(W)$; with a slight abuse of notation, this bijection will be denoted by the same symbol A. We therefore have a (left) action of $\mathrm{GL}(n, I\!R)$ on $G_k(n)$, that will be called the *natural action* of $\mathrm{GL}(n, I\!R)$ on $G_k(n)$.

We start by proving the differentiability of this action:

2.4.1. PROPOSITION. *The natural action* $\mathrm{GL}(n, I\!R) \times G_k(n) \to G_k(n)$ *is differentiable.*

PROOF. We simply compute the representation of this action in local charts.

Let $A \in \mathrm{GL}(n, I\!R)$ and $W_0 \in G_k(n)$ be fixed. Let W_1 be a common complementary for W_0 and $A(W_0)$; hence, ϕ_{W_0,W_1} is a chart whose domain contains both W_0 and $A(W_0)$. We compute $\phi_{W_0,W_1}(B(W))$ for B in a neighborhood of A and W in a neighborhood of W_0; writing $T = \phi_{W_0,W_1}(W)$ we have:

$$\phi_{W_0,W_1}(B(W)) = (B_{10} + B_{11} \circ T) \circ (B_{00} + B_{01} \circ T)^{-1}, \tag{2.4.1}$$

where B_{ij} denotes the component $\pi_i \circ (B|_{W_j})$ of B and π_i, $i = 0, 1$, denotes the projection onto W_i relative to the decomposition $I\!R^n = W_0 \oplus W_1$. Obviously, (2.4.1) is a differentiable function of the pair (B, T). □

The action of $\mathrm{GL}(n, I\!R)$ on $G_k(n)$ is transitive; actually, we have the following stronger result:

2.4.2. PROPOSITION. *The natural action of* $\mathrm{SO}(n)$ *in* $G_k(n)$*, obtained by restriction of the natural action of* $\mathrm{GL}(n, I\!R)$*, is transitive.*

PROOF. Let $W, W' \in G_k(n)$ be fixed; we can find orthonormal bases $(b_j)_{j=1}^n$ and $(b'_j)_{j=1}^n$ of $I\!R^n$ such that $(b_j)_{j=1}^k$ is a basis of W and $(b'_j)_{j=1}^k$ is a basis of W'. By possibly replacing b_1 with $-b_1$, we can assume that the two bases define the same orientation of $I\!R^n$. We can therefore find $A \in \mathrm{SO}(n)$ such that $A(b_j) = b'_j$ for all $j = 1, \ldots, n$, hence in particular $A(W) = W'$. □

2.4.3. COROLLARY. *The Grassmannian $G_k(n)$ is diffeomorphic to the quotients:*

$$\frac{\mathrm{O}(n)}{\mathrm{O}(k)\times\mathrm{O}(n-k)} \quad \textit{and} \quad \frac{\mathrm{SO}(n)}{\mathrm{S}\big(\mathrm{O}(k)\times\mathrm{O}(n-k)\big)}$$

where $\mathrm{S}\big(\mathrm{O}(k)\times\mathrm{O}(n-k)\big)$ denotes the intersection:

$$\mathrm{SO}(n)\cap\big(\mathrm{O}(k)\times\mathrm{O}(n-k)\big).$$

It follows in particular that $G_k(n)$ is a compact and connected manifold.

PROOF. The isotropy of the point $\mathbb{R}^k\oplus\{0\}^{n-k}$ by the action of $\mathrm{O}(n)$ is given by the group of orthogonal operators that leave the subspaces $\mathbb{R}^k\oplus\{0\}^{n-k}$ and $\{0\}^k\oplus\mathbb{R}^{n-k}$ invariant; this group is clearly isomorphic to $\mathrm{O}(k)\times\mathrm{O}(n-k)$. A similar argument applies to the case of the action of $\mathrm{SO}(n)$. The conclusion follows from Corollary 2.1.9 and Proposition 2.4.2. □

2.4.4. REMARK. Obviously, we could have added to the statement of Corollary 2.4.3 a representation of $G_k(n)$ as a quotient of $\mathrm{GL}(n,\mathbb{R})$. Observe that in this case the isotropy of $\mathbb{R}^k\oplus\{0\}^{n-k}$ is *not* $\mathrm{GL}(k)\times\mathrm{GL}(n-k)$ (see Exercise 2.7).

2.4.5. REMARK. As a matter of facts, formula (2.4.1) shows that the natural action of $\mathrm{GL}(n,\mathbb{R})$ on $G_k(n)$ is *real analytic*. In the case of a complex Grassmannian, the natural action of the linear group $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{C}^n$ is holomorphic. An obvious generalization of Proposition 2.4.2 shows that the action of the special unitary group $\mathrm{SU}(n)$ on the complex Grassmannian is transitive. Analogously to the result of Corollary 2.4.3, we conclude that the complex Grassmannian is compact, connected and isomorphic to the quotients $\mathrm{U}(n)/\big(\mathrm{U}(k)\times\mathrm{U}(n-k)\big)$ and $\mathrm{SU}(n)/\mathrm{S}\big(\mathrm{U}(k)\times\mathrm{U}(n-k)\big)$, where $\mathrm{S}\big(\mathrm{U}(k)\times\mathrm{U}(n-k)\big)$ denotes the intersection $\mathrm{SU}(n)\cap\big(\mathrm{U}(k)\times\mathrm{U}(n-k)\big)$.

We have two more interesting corollaries of the representation of $G_k(n)$ as the quotient of a Lie group.

2.4.6. PROPOSITION. *In an open neighborhood $\mathcal{U}$ of any point of $G_k(n)$ we can define a differentiable map $A:\mathcal{U}\to\mathrm{GL}(n,\mathbb{R})$ such that*

$$A(W)(\mathbb{R}^k\oplus\{0\}^{n-k})=W$$

for all $W\in\mathcal{U}$.

PROOF. It follows from Propositions 2.4.1, 2.4.2 and from Corollary 2.1.9 that the map:

$$\mathrm{GL}(n,\mathbb{R})\ni B\longmapsto B\big(\mathbb{R}^k\oplus\{0\}^{n-k}\big)\in G_k(n)$$

is a submersion; the required map is simply a local differentiable section of this submersion (see Remark 2.1.3). □

2.4.7. COROLLARY. *In an open neighborhood $\mathcal{U}$ of any point of $G_k(n)$ there exist differentiable maps:*

$$Z_{\mathrm{ker}}:\mathcal{U}\longrightarrow\mathrm{Lin}(\mathbb{R}^n,\mathbb{R}^{n-k}) \quad \textit{and} \quad Z_{\mathrm{im}}:\mathcal{U}\longrightarrow\mathrm{Lin}(\mathbb{R}^k,\mathbb{R}^n)$$

such that $W=\mathrm{Ker}\big(Z_{\mathrm{ker}}(W)\big)=\mathrm{Im}\big(Z_{\mathrm{im}}(W)\big)$ for all $W\in\mathcal{U}$.

PROOF. Define A as in Proposition 2.4.6 and take $Z_{\rm ker} = \pi \circ A(W)^{-1}$ and $Z_{\rm im} = A(W) \circ i$, where $i : I\!R^k \to I\!R^n$ is the inclusion in the first k-coordinates and $\pi : I\!R^n \to I\!R^{n-k}$ is the projection onto the last $n-k$ coordinates. □

2.4.8. COROLLARY. *Let $S \subset I\!R^n$ be any subspace and let $r \in \mathbb{Z}$ be a non negative integer; then, the set of subspaces $W \in G_k(n)$ such that $\dim(W \cap S) \le r$ is open in $G_k(n)$.*

PROOF. Let $W_0 \in G_k(n)$ be fixed and let $Z_{\rm ker}$ be a map as in the statement of Corollary 2.4.7 defined in an open neighborhood $\mathcal{U}$ of W_0 in $G_k(n)$. For all $W \in \mathcal{U}$ we have:

$$W \cap S = \mathrm{Ker}\big(Z_{\rm ker}(W)|_S\big),$$

from which we get that $\dim(W \cap S) \le r$ if and only if the operator $Z_{\rm ker}(W)|_S \in \mathrm{Lin}(S, I\!R^{n-k})$ has rank greater or equal to $\dim(S) - r$; this condition defines an open subset of $\mathrm{Lin}(S, I\!R^{n-k})$, and the conclusion follows. □

We now consider the action of the product of Lie groups $\mathrm{GL}(n, I\!R) \times \mathrm{GL}(m, I\!R)$ on the vector space $\mathrm{Lin}(I\!R^n, I\!R^m)$ given by:

$$(A, B, T) \longmapsto B \circ T \circ A^{-1}, \tag{2.4.2}$$

for $A \in \mathrm{GL}(n, I\!R)$, $B \in \mathrm{GL}(m, I\!R)$ and $T \in \mathrm{Lin}(I\!R^n, I\!R^m)$. An elementary linear algebra argument shows that the orbits of the action (2.3.4) are the sets:

$$\mathrm{Lin}^r(I\!R^n, I\!R^m) = \Big\{ T \in \mathrm{Lin}(I\!R^n, I\!R^m) : T \text{ is a matrix of rank } r \Big\},$$

where $r = 1, \dots, \min\{n, m\}$. It is also easy to see that the sets:

$$\bigcup_{i \ge r} \mathrm{Lin}^i(I\!R^n, I\!R^m) \quad \text{and} \quad \bigcup_{i \le r} \mathrm{Lin}^i(I\!R^n, I\!R^m)$$

are respectively an open and a closed subset of $\mathrm{Lin}(I\!R^n, I\!R^m)$; it follows that each $\mathrm{Lin}^r(I\!R^n, I\!R^m)$ is locally closed in $\mathrm{Lin}(I\!R^n, I\!R^m)$.

Thus, we have the following:

2.4.9. LEMMA. *For each $r = 1, \dots, \min\{n, m\}$, the set $\mathrm{Lin}^r(I\!R^n, I\!R^m)$ is an embedded submanifold of $\mathrm{Lin}(I\!R^n, I\!R^m)$.*

PROOF. It follows from Theorem 2.1.12. □

We also obtain directly the following:

2.4.10. PROPOSITION. *Given non negative integers m, n and r, with $r \le \min\{n, m\}$, then the maps:*

$$\mathrm{Lin}^r(I\!R^n, I\!R^m) \ni T \longmapsto \mathrm{Im}(T) \in G_r(m) \tag{2.4.3}$$

$$\mathrm{Lin}^r(I\!R^n, I\!R^m) \ni T \longmapsto \mathrm{Ker}(T) \in G_{n-r}(n) \tag{2.4.4}$$

are differentiable.

PROOF. The product $\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(m,\mathbb{R})$ acts transitively on the orbit $\mathrm{Lin}^r(\mathbb{R}^n,\mathbb{R}^m)$, and it also acts transitively on $G_r(m)$, by considering the action for which $\mathrm{GL}(n,\mathbb{R})$ acts trivially and $\mathrm{GL}(m,\mathbb{R})$ acts on $G_r(m)$ with its natural action. The map (2.4.3) is equivariant, hence its differentiability follows from Corollary 2.1.10 and Proposition 2.4.1.

The differentiability of (2.4.4) follows similarly. □

In the next two propositions we compute the differential of the natural action of $\mathrm{GL}(n,\mathbb{R})$ on $G_k(n)$.

2.4.11. PROPOSITION. *For $A \in \mathrm{GL}(n,\mathbb{R})$, let us consider the diffeomorphism of $G_k(n)$, also denoted by A, given by $W \mapsto A(W)$. For $W \in G_k(n)$, the differential $\mathrm{d}A(W)$ of A at the point W is the operator:*

$$\mathrm{Lin}\big((A|_W)^{-1},\bar{A}\big)\colon \mathrm{Lin}(W,\mathbb{R}^n/W) \longrightarrow \mathrm{Lin}(A(W),\mathbb{R}^n/A(W))$$

given by $Z \mapsto \bar{A}\circ Z\circ (A|_W)^{-1}$, where

$$\bar{A}\colon \mathbb{R}^n/W \longrightarrow \mathbb{R}^n/A(W)$$

is induced from A by passing to the quotient.

PROOF. It is a direct application of the technique described in Remark 2.3.4.

Let $t \mapsto W(t)$ a differentiable curve in $G_k(n)$ with $W(0) = W$ and $W'(0) = Z$; let $t \mapsto w(t)$ be a differentiable curve in $\mathbb{R}^n$ with $w(t) \in W(t)$ for all t. It follows that $t \mapsto A(w(t))$ is a differentiable curve in $\mathbb{R}^n$ with $A(w(t)) \in A(W(t))$ for all t; by Proposition 2.3.3 we have:

$$(A\circ W)'(0)\cdot A(w(0)) = A(w'(0)) + A(W) \in \mathbb{R}^n/A(W). \tag{2.4.5}$$

Using again Proposition 2.3.3, we get:

$$W'(0)\cdot w(0) = w'(0) + W \in \mathbb{R}^n/W. \tag{2.4.6}$$

The conclusion follows from (2.4.5) and (2.4.6). □

2.4.12. PROPOSITION. *For $W \in G_k(n)$, the differential of the map:*

$$\beta_W\colon \mathrm{GL}(n,\mathbb{R}) \longrightarrow G_k(n)$$

given by $\beta_W(A) = A(W)$ is:

$$\mathrm{d}\beta_W(A)\cdot X = q\circ X\circ A^{-1}|_{A(W)},$$

for all $A \in \mathrm{GL}(n,\mathbb{R})$, $X \in \mathrm{Lin}(\mathbb{R}^n)$, where $q\colon \mathbb{R}^n \to \mathbb{R}^n/A(W)$ is the quotient map.

PROOF. We use again the technique described in Remark 2.3.4.

Let $t \mapsto A(t)$ be a differentiable curve in $\mathrm{GL}(n,\mathbb{R})$ with $A(0) = A$ and $A'(0) = X$; fix $w_0 \in W$. It follows that $t \mapsto A(t)(w_0)$ is a differentiable curve in $\mathbb{R}^n$ with $A(t)(w_0) \in \beta_W(A(t))$ for all t. Using Proposition 2.3.3 we get:

$$(\beta_W\circ A)'(0)\cdot A(w_0) = X(w_0) + A(W) \in \mathbb{R}^n/A(W).$$

The conclusion follows.

□

2.5. The Lagrangian Grassmannian

In this section we will show that the set Λ of all Lagrangian subspaces of a $2n$-dimensional symplectic space (V, ω) is a submanifold of the Grassmannian of all n-dimensional subspaces of V. We will call Λ the Lagrangian Grassmannian of (V, ω). We will study in detail the charts of Λ, its tangent space and the action of the symplectic group $\mathrm{Sp}(V, \omega)$ on Λ; we will show that, like the total Grassmannian, the Lagrangian Grassmannian is a homogeneous manifold.

We will make systematic use of the results concerning the Grassmannian manifolds presented in Sections 2.2, 2.3 and 2.4, as well as the results concerning the symplectic spaces presented in Section 1.4, and especially in Subsection 1.4.2.

We start with the observation that the theory of Grassmannians of subspaces of $\mathbb{R}^n$ developed in Sections 2.2, 2.3 and 2.4 can be generalized in an obvious way if we replace $\mathbb{R}^n$ with any other arbitrary finite dimensional real vector space V; let us briefly mention the changes in the notation that will be used in order to consider Grassmannians of subspaces of an arbitrary space V.

We will denote by $G_k(V)$ the set of all k-dimensional subspaces of V, with $0 \leq k \leq \dim(V)$; this set has a differentiable structure of dimension $k(\dim(V) - k)$, with charts described in Section 2.2. If $W_1 \subset V$ is a subspace of codimension k, we will denote by $G_k^0(V, W_1)$ (or more simply by $G_k^0(W_1)$ when the space V will be clear from the context) the subset of $G_k(V)$ consisting of those subspaces that are transversal to W_1:

$$G_k^0(V, W_1) = G_k^0(W_1) = \Big\{ W \in G_k(V) : V = W \oplus W_1 \Big\}.$$

If $W_0 \in G_k^0(W_1)$, then $G_k^0(W_1)$ is the domain of the chart ϕ_{W_0, W_1}.

For $W \in G_k(V)$, we will always consider the following identification of the tangent space $T_W G_k(V)$:

$$T_W G_k(V) \simeq \mathrm{Lin}(W, V/W),$$

that is constructed precisely as in Section 2.3. In Section 2.4 we must replace the general linear group $\mathrm{GL}(n, \mathbb{R})$ of $\mathbb{R}^n$ by the general linear group $\mathrm{GL}(V)$ of V; in Proposition 2.4.2 and in Corollary 2.4.3 the orthogonal and the special orthogonal group $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ of $\mathbb{R}^n$ must be replaced by the corresponding group $\mathrm{O}(V, g)$ and $\mathrm{SO}(V, g)$ associated to an arbitrary choice of an inner product g in V.

Let now be fixed for the rest of this section a symplectic space (V, ω) with $\dim(V) = 2n$. We denote by $\Lambda(V, \omega)$, or more simply by Λ, the set of all Lagrangian subspaces of (V, ω):

$$\Lambda(V, \omega) = \Lambda = \Big\{ L \in G_n(V) : L \text{ is Lagrangian} \Big\}.$$

We say that Λ is the *Lagrangian Grassmannian* of the symplectic space (V, ω).

We start with a description of submanifold charts for Λ:

2.5.1. LEMMA. *Let* (L_0, L_1) *be a Lagrangian decomposition of* V*; then a subspace* $L \in G_n^0(L_1)$ *is Lagrangian if and only if the bilinear form:*

$$\rho_{L_0,L_1} \circ \phi_{L_0,L_1}(L) \in \mathrm{Lin}(L_0, L_0^*) \simeq \mathrm{B}(L_0) \tag{2.5.1}$$

is symmetric.

PROOF. Since $\dim(L) = n$, then L is Lagrangian if and only if it is isotropic. Let $T = \phi_{L_0,L_1}(L)$, so that $T \in \mathrm{Lin}(L_0, L_1)$ and $L = \mathrm{Gr}(T)$; we have:

$$\omega\big(v + T(v), w + T(w)\big) = \omega(T(v), w) - \omega(T(w), v).$$

The conclusion follows by observing that the bilinear form (2.5.1) coincides with $\omega(T\cdot,\cdot)|_{L_0\times L_0}$. □

If $L_1 \subset V$ is a Lagrangian subspace, we denote by $\Lambda^0(L_1)$ the set of all Lagrangian subspaces of V that are transversal to L_1:

$$\Lambda^0(L_1) = \Lambda \cap G_n^0(L_1). \tag{2.5.2}$$

It follows from Lemma 2.5.1 that, associated to each Lagrangian decomposition (L_0, L_1) of V we have a bijection:

$$\varphi_{L_0,L_1} : \Lambda^0(L_1) \longrightarrow \mathrm{B_{sym}}(L_0) \tag{2.5.3}$$

given by $\varphi_{L_0,L_1}(L) = \rho_{L_0,L_1} \circ \phi_{L_0,L_1}(L)$. We therefore have the following:

2.5.2. COROLLARY. *The Grassmannian Lagrangian* Λ *is an embedded submanifold of* $G_n(V)$ *with dimension* $\dim(\Lambda) = \frac{1}{2}n(n+1)$*; the charts* φ_{L_0,L_1} *defined in* (2.5.3) *form a differentiable atlas for* Λ *as* (L_0, L_1) *runs over the set of all Lagrangian decompositions of* V.

PROOF. Given a Lagrangian decomposition (L_0, L_1) of V, it follows from Lemma 2.5.1 that the chart:

$$G_n^0(L_1) \ni W \longmapsto \rho_{L_0,L_1} \circ \phi_{L_0,L_1}(W) \in \mathrm{Lin}(L_0, L_0^*) \simeq \mathrm{B}(L_0) \tag{2.5.4}$$

of $G_n(V)$ is a submanifold chart for Λ, that induces the chart (2.5.3) of Λ. Moreover, $\dim(\mathrm{B_{sym}}(L_0)) = \frac{1}{2}n(n+1)$. The conclusion follows from the fact that, since every Lagrangian admits a complementary Lagrangian (Corollary 1.4.21), the domains of the charts (2.4.5) cover Λ as (L_0, L_1) runs over the set of all Lagrangian decompositions of V. □

2.5.3. REMARK. It follows from formula (2.5.2) and Remark 2.2.7 that the subset $\Lambda^0(L_1)$ is open in Λ; moreover, since the chart (2.5.3) is surjective, we have that $\Lambda^0(L_1)$ is homeomorphic (and diffeomorphic) to the Euclidean space $\mathrm{B_{sym}}(L_0)$.

It is sometimes useful to have an explicit formula for the transition functions between the charts (2.5.3) of the Lagrangian Grassmannian; we have the following:

2.5.4. LEMMA. *Given Lagrangian decompositions* (L_0, L_1) *and* (L_0', L_1) *of* V *then:*

$$\varphi_{L_0',L_1} \circ (\varphi_{L_0,L_1})^{-1}(B) = \varphi_{L_0',L_1}(L_0) + \big(\eta^{L_1}_{L_0',L_0}\big)^{\#}(B) \in \mathrm{B_{sym}}(L_0'), \tag{2.5.5}$$

for every $B \in \mathrm{B}_{\mathrm{sym}}(L_0)$, where $\eta^{L_1}_{L_0',L_0}$ denotes the isomorphism of L_0' onto L_0 determined by the common complementary L_1 (recall Definitions 1.1.2 and2.2.1); if (L_0, L_1') is also a Lagrangian decomposition of V then the following identity holds:

$$\varphi_{L_0,L_1'} \circ (\varphi_{L_0,L_1})^{-1}(B) = B \circ \big(\mathrm{Id} + (\pi_0'|_{L_1}) \circ \rho^{-1}_{L_0,L_1} \circ B\big)^{-1}, \tag{2.5.6}$$

for all $B \in \varphi_{L_0,L_1}(\Lambda^0(L_1')) \subset \mathrm{B}_{\mathrm{sym}}(L_0)$, where π_0' denotes the projection onto L_0 relative to the decomposition $V = L_0 \oplus L_1'$.

Observe that the following identity holds:

$$(\pi_0'|_{L_1}) \circ \rho^{-1}_{L_0,L_1} = (\rho_{L_0,L_1})_{\#}\big(\varphi_{L_1,L_0}(L_1')\big). \tag{2.5.7}$$

PROOF. Using (2.2.2) it is easy to see that:

$$\varphi_{L_0',L_1} \circ (\varphi_{L_0,L_1})^{-1}(B) = \rho_{L_0',L_1} \circ (\pi_1'|_{L_0} + \rho^{-1}_{L_0,L_1} \circ B) \circ \eta^{L_1}_{L_0',L_0}, \tag{2.5.8}$$

where π_1' denotes the projection onto L_1 relative to the decomposition $V = L_0' \oplus L_1$; it is also easy to prove that:

$$\rho_{L_0',L_1} \circ \rho^{-1}_{L_0,L_1} = \big(\eta^{L_1}_{L_0',L_0}\big)^* \colon L_0^* \longrightarrow L_0'^*$$

and substituting in (2.5.8) we obtain (see also (1.1.4)):

$$\varphi_{L_0',L_1} \circ (\varphi_{L_0,L_1})^{-1}(B) = \rho_{L_0',L_1} \circ (\pi_1'|_{L_0}) \circ \eta^{L_1}_{L_0',L_0} + \big(\eta^{L_1}_{L_0',L_0}\big)^{\#}(B). \tag{2.5.9}$$

Setting $B = 0$ in (2.5.9) we conclude that

$$\varphi_{L_0',L_1}(L_0) = \rho_{L_0',L_1} \circ (\pi_1'|_{L_0}) \circ \eta^{L_1}_{L_0',L_0},$$

which completes the proof of (2.5.5).

Now, using (2.2.3) it is easy to see that:

$$\varphi_{L_0,L_1'} \circ (\varphi_{L_0,L_1})^{-1}(B) = \rho_{L_0,L_1'} \circ \eta^{L_0}_{L_1,L_1'} \circ \rho^{-1}_{L_0,L_1} \circ B \circ \big(\mathrm{Id} + (\pi_0'|_{L_1}) \circ \rho^{-1}_{L_0,L_1} \circ B\big)^{-1};$$

and it is also easy to prove that:

$$\rho_{L_0,L_1'} \circ \eta^{L_0}_{L_1,L_1'} \circ \rho^{-1}_{L_0,L_1} = \mathrm{Id} \colon L_0^* \longrightarrow L_0^*,$$

and this concludes the proof. □

In our next Lemma we show an interesting formula that involves the charts (2.5.3).

2.5.5. LEMMA. *Let L_0, L_1 and L be Lagrangian subspaces of V that are pairwise complementary; the following identities hold:*

$$\varphi_{L_0,L_1}(L) = -\varphi_{L_0,L}(L_1), \tag{2.5.10}$$

$$\varphi_{L_0,L_1}(L) = -(\rho_{L_1,L_0})^{\#}\big(\varphi_{L_1,L_0}(L)^{-1}\big); \tag{2.5.11}$$

PROOF. Let $T = \phi_{L_0,L_1}(L)$; then $T \in \mathrm{Lin}(L_0, L_1)$ and $L = \mathrm{Gr}(T)$. Observe that $\mathrm{Ker}(T) = L_0 \cap L = \{0\}$ and so T is invertible; hence:

$$L_1 = \{v + (-v - T(v)) : v \in L_0\}$$

and therefore:

$$\phi_{L_0,L}(L_1)\colon L_0 \ni v \longmapsto -v - T(v) \in L.$$

For all $v, w \in L_0$, we now compute, :

$$\varphi_{L_0,L}(L_1) \cdot (v, w) = \omega(-v - T(v), w) = -\omega(T(v), w) = -\varphi_{L_0,L_1}(L) \cdot (v, w),$$

which completes the proof of (2.5.10). To show (2.5.11) observe that $\phi_{L_1,L_0}(L) = T^{-1}$; then:

$$\varphi_{L_1,L_0}(L) = \rho_{L_1,L_0} \circ T^{-1}, \quad \varphi_{L_0,L_1}(L) = \rho_{L_0,L_1} \circ T,$$

from which we get:

$$\varphi_{L_0,L_1}(L) = \rho_{L_0,L_1} \circ \varphi_{L_1,L_0}(L)^{-1} \circ \rho_{L_1,L_0}.$$

The conclusion follows from (1.4.12) and (1.1.4). □

We will now study the tangent space $T_L\Lambda$ of the Lagrangian Grassmannian.

2.5.6. PROPOSITION. *Let $L \in \Lambda$ be fixed; then the isomorphism:*

$$\mathrm{Lin}(\mathrm{Id}, \rho_L) : \mathrm{Lin}(L, V/L) \longrightarrow \mathrm{Lin}(L, L^*) \simeq \mathrm{B}(L) \tag{2.5.12}$$

given by $Z \mapsto \rho_L \circ Z$ takes $T_L\Lambda \subset T_L G_n(V) \simeq \mathrm{Lin}(L, V/L)$ onto the subspace $\mathrm{B}_{\mathrm{sym}}(L) \subset \mathrm{B}(L)$.

PROOF. Let L_1 be a Lagrangian complementary to L. As in the proof of Corollary 2.5.2, the chart:

$$G_n^0(L_1) \ni W \longmapsto \rho_{L,L_1} \circ \phi_{L,L_1}(W) \in \mathrm{B}(L) \tag{2.5.13}$$

of $G_n(V)$ is a submanifold chart for Λ that induces the chart φ_{L,L_1} of Λ; hence, the differential of (2.5.13) at the point L is an isomorphism that takes $T_L\Lambda$ onto $\mathrm{B}_{\mathrm{sym}}(L)$. By Lemma 2.3.5, the differential of ϕ_{L,L_1} at the point L is $\mathrm{Lin}(\mathrm{Id}, q_1^{-1})$, where q_1 denotes the restriction to L_1 of the quotient map onto V/L; it follows from the diagram (1.4.13) that the differential of (2.5.13) at L coincides with the isomorphism (2.5.12). □

Using the result of Proposition 2.5.6, *we will henceforth identify the tangent space $T_L\Lambda$ with* $\mathrm{B}_{\mathrm{sym}}(L)$. We will now prove versions of Lemma 2.3.5 and Propositions 2.4.11 and 2.4.12 for the Lagrangian Grassmannian; in these proofs we must keep in mind the isomorphism (2.5.12) that identifies $T_L\Lambda$ and $\mathrm{B}_{\mathrm{sym}}(L)$.

2.5.7. LEMMA. *Consider a Lagrangian decomposition (L_0, L_1) of V and let $L \in \Lambda^0(L_1)$ be fixed; then, the differential of the chart φ_{L_0,L_1} at the point L is the push-forward operator:*

$$\left(\eta_{L,L_0}^{L_1}\right)_{\#} : \mathrm{B}_{\mathrm{sym}}(L) \longrightarrow \mathrm{B}_{\mathrm{sym}}(L_0),$$

where $\eta_{L,L_0}^{L_1}$ denotes the isomorphism of L onto L_0 determined by the common complementary L_1 (see Definition 2.2.1).

PROOF. By differentiating the equality:

$$\varphi_{L_0,L_1} = \mathrm{Lin}(\mathrm{Id}, \rho_{L_0,L_1}) \circ \phi_{L_0,L_1}$$

at the point L and keeping in mind the identification $T_L\Lambda \simeq \mathrm{B}_{\mathrm{sym}}(L)$, we obtain:

$$\mathrm{d}\varphi_{L_0,L_1}(L) = \mathrm{Lin}\big(\eta^{L_1}_{L_0,L}, \rho_{L_0,L_1} \circ q_1^{-1} \circ \rho_L^{-1}\big)|_{\mathrm{B}_{\mathrm{sym}}(L)} : \mathrm{B}_{\mathrm{sym}}(L) \longrightarrow \mathrm{B}_{\mathrm{sym}}(L_0),$$

where q_1 denotes the restriction to L_1 of the quotient map onto V/L. On the other hand, it is easy to see that:

$$\rho_{L_0,L_1} \circ q_1^{-1} \circ \rho_L^{-1} = \big(\eta^{L_1}_{L_0,L}\big)^*.$$

This concludes the proof. □

Clearly, the natural action of $\mathrm{GL}(V)$ on the Grassmannian $G_n(V)$ restricts to an action of the symplectic group $\mathrm{Sp}(V,\omega)$ on the Lagrangian Grassmannian Λ; we have the following:

2.5.8. PROPOSITION. *The natural action of* $\mathrm{Sp}(V,\omega)$ *on* Λ *is differentiable.*

PROOF. It follows directly from Proposition 2.4.1. □

Let us now compute the differential of the action of $\mathrm{Sp}(V,\omega)$ on Λ:

2.5.9. PROPOSITION. *For* $A \in \mathrm{Sp}(V,\omega)$*, consider the diffeomorphism, also denoted by* A*, of* Λ *given by* $L \mapsto A(L)$*. For* $L \in \Lambda$*, the differential* $\mathrm{d}A(L)$ *is the push-forward operator:*

$$(A|_L)_\# : \mathrm{B}_{\mathrm{sym}}(L) \longrightarrow \mathrm{B}_{\mathrm{sym}}(A(L)).$$

PROOF. Using Proposition 2.4.11 and keeping in mind the identifications of the tangent spaces $T_L\Lambda \simeq \mathrm{B}_{\mathrm{sym}}(L)$ and $T_{A(L)}\Lambda \simeq \mathrm{B}_{\mathrm{sym}}(A(L))$, we see that the differential $\mathrm{d}A(L)$ is obtained by the restriction to $\mathrm{B}_{\mathrm{sym}}(L)$ of the map Φ defined by the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{B}(L) & \xrightarrow{\quad\Phi\quad} & \mathrm{B}(A(L)) \\
{\scriptstyle \mathrm{Lin}(\mathrm{Id},\rho_L)}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{Lin}(\mathrm{Id},\rho_{A(L)})} \\
\mathrm{Lin}(L, V/L) & \xrightarrow[\mathrm{Lin}((A|_L)^{-1},\bar{A})]{} & \mathrm{Lin}\big(A(L), V/A(L)\big)
\end{array}$$

where $\bar{A} : V/L \to V/A(L)$ is induced from A by passing to the quotient, hence:

$$\Phi = \mathrm{Lin}\big((A|_L)^{-1}, \rho_{A(L)} \circ \bar{A} \circ \rho_L^{-1}\big).$$

It is easy to see that:

$$\rho_{A(L)} \circ \bar{A} \circ \rho_L^{-1} = (A|_L)^{*-1}.$$

This concludes the proof. □

2.5.10. PROPOSITION. *For $L \in \Lambda$, the differential of the map:*

$$\beta_L\colon \mathrm{Sp}(V,\omega) \longrightarrow \Lambda$$

given by $\beta_L(A) = A(L)$ is:

$$\mathrm{d}\beta_L(A)\cdot X = \omega(X\circ A^{-1}\cdot,\cdot)|_{A(L)\times A(L)},$$

for all $A \in \mathrm{Sp}(V,\omega)$, $X \in T_A\mathrm{Sp}(V,\omega) = \mathrm{sp}(V,\omega)\cdot A$.

PROOF. It follows easily from Proposition 2.4.11, keeping in mind the identification $T_{A(L)}\Lambda \simeq \mathrm{B}_{\mathrm{sym}}(A(L))$ obtained by the restriction of the isomorphism $\mathrm{Lin}(\mathrm{Id},\rho_{A(L)})$. □

We will now show that the Lagrangian Grassmannian can be obtained as a quotient of the unitary group. Let J be a complex structure in V which is compatible with the symplectic form ω; consider the corresponding inner product $g = \omega(\cdot, J\cdot)$ on V and the Hermitian product g_{s} in (V,J) defined in (1.4.10). Using the notation introduced in Subsection 2.1.1, Proposition 1.4.22 tells us that

$$\mathrm{U}(V,J,g_{\mathrm{s}}) = \mathrm{O}(V,g)\cap \mathrm{Sp}(V,\omega).$$

Let us now fix a Lagrangian $L_0 \subset V$; by Lemma 1.4.26, L_0 is a real form in (V,J) where g_{s} is real. It follows that g_{s} is the unique sesquilinear extension of the inner product $g|_{L_0\times L_0}$ in L_0. Since L_0 is a real form in (V,J), we have that (V,J) is a complexification of L_0, from which it follows that every $\mathbb{R}$-linear endomorphism $T \in \mathrm{Lin}(L_0)$ extends uniquely to a $\mathbb{C}$-linear endomorphism of (V,J). From Remark 1.3.16 it follows that $T \in \mathrm{Lin}(L_0)$ is g-orthogonal if and only if $T^{\mathbb{C}}$ is g_{s}-unitary; we therefore have an injective homomorphism of Lie groups:

$$\mathrm{O}\big(L_0, g|_{L_0\times L_0}\big) \ni T \longmapsto T^{\mathbb{C}} \in \mathrm{U}(V,J,g_{\mathrm{s}}) \tag{2.5.14}$$

whose image consists precisely of the elements in $\mathrm{U}(V,J,g_{\mathrm{s}})$ that leave L_0 invariant (see Lemma 1.3.11). Corollary 1.4.27 tells us that the subgroup $\mathrm{U}(V,J,g_{\mathrm{s}})$ of $\mathrm{Sp}(V,\omega)$ acts transitively on Λ; from Corollary 2.1.9 we therefore obtain the following:

2.5.11. PROPOSITION. *Fix $L_0 \in \Lambda$ and a complex structure J on V which is compatible with ω; the map:*

$$\mathrm{U}(V,J,g_{\mathrm{s}}) \ni A \longmapsto A(L_0) \in \Lambda$$

induces a diffeomorphism

$$\mathrm{U}(V,J,g_{\mathrm{s}})/\mathrm{O}\big(L_0, g|_{L_0\times L_0}\big) \simeq \Lambda,$$

where $\mathrm{O}\big(L_0, g|_{L_0\times L_0}\big)$ is identified with a closed subgroup of $\mathrm{U}(V,J,g_{\mathrm{s}})$ through (2.5.14). □

Obviously, the choice of a symplectic basis in V induces an isomorphism between the Lagrangian Grassmannian of (V,ω) and the Lagrangian Grassmannian of $\mathbb{R}^{2n}$ endowed with the canonical symplectic structure. Hence we have the following:

2.5.12. COROLLARY. *The Lagrangian Grassmannian* Λ *is isomorphic to the quotient* $\mathrm{U}(n)/\mathrm{O}(n)$*; in particular,* Λ *is a compact and connect manifold.*

2.5.1. The submanifolds $\mathbf{\Lambda^k(L_0)}$. In this subsection we will consider a fixed symplectic space (V, ω), with $\dim(V) = 2n$, and a Lagrangian subspace $L_0 \subset V$. For $k = 0, \ldots, n$ we define the following subsets of Λ:

$$\Lambda^k(L_0) = \big\{L \in \Lambda : \dim(L \cap L_0) = k\big\}.$$

Observe that, for $k = 0$, the above definition is compatible with the definition of $\Lambda^0(L_0)$ given in (2.5.2). Our goal is to show that each $\Lambda^k(L_0)$ is a submanifold of Λ and to compute its tangent space; we will also show that $\Lambda^1(L_0)$ has codimension 1 in Λ, and that it admits a canonical transverse orientation in Λ.

Let us denote by $\mathrm{Sp}(V, \omega, L_0)$ the closed subgroup of $\mathrm{Sp}(V, \omega)$ consisting of those symplectomorphisms that preserve L_0:

$$\mathrm{Sp}(V, \omega, L_0) = \big\{A \in \mathrm{Sp}(V, \omega) : A(L_0) = L_0\big\}. \tag{2.5.15}$$

It is easy to see that the Lie algebra $\mathrm{sp}(V, \omega, L_0)$ of $\mathrm{Sp}(V, \omega, L_0)$ is given by (see formula (2.1.5)):

$$\mathrm{sp}(V, \omega, L_0) = \big\{X \in \mathrm{sp}(V, \omega) : X(L_0) \subset L_0\big\}.$$

In the next Lemma we compute more explicitly this algebra:

2.5.13. LEMMA. *The Lie algebra* $\mathrm{sp}(V, \omega, L_0)$ *consists of those linear endomorphisms* $X \in \mathrm{Lin}(V)$ *such that* $\omega(X\cdot, \cdot)$ *is a* symmetric *bilinear form that vanishes on* L_0.

PROOF. It follows from the characterization of the algebra $\mathrm{sp}(V, \omega)$ given in Subsection 2.1.1, observing that $\omega(X\cdot, \cdot)|_{L_0 \times L_0} = 0$ if and only if $X(L_0)$ is contained in the ω-orthogonal complement $L_0^{\perp}$ of L_0. But L_0 is Lagrangian, hence $L_0^{\perp} = L_0$. $\square$

It is clear that the action of $\mathrm{Sp}(V, \omega)$ on Λ leaves each subset $\Lambda^k(L_0)$ invariant; moreover, by Proposition 1.4.38, it follows that $\Lambda^k(L_0)$ is an orbit of the action of $\mathrm{Sp}(V, \omega, L_0)$. The strategy then is to use Theorem 2.1.12 to conclude that $\Lambda^k(L_0)$ is an embedded submanifold of Λ; to this aim, we need to show that $\Lambda^k(L_0)$ is locally closed in Λ.

For each $k = 0, \ldots, n$ we define:

$$\Lambda^{\geq k}(L_0) = \bigcup_{i=k}^{n} \Lambda^i(L_0), \quad \Lambda^{\leq k}(L_0) = \bigcup_{i=0}^{k} \Lambda^i(L_0).$$

We have the following:

2.5.14. LEMMA. *For all* $k = 0, \ldots, n$*, the subset* $\Lambda^{\leq k}(L_0)$ *is open and the subset* $\Lambda^{\geq k}(L_0)$ *is closed in* Λ.

PROOF. It follows from Corollary 2.4.8 that the set of spaces $W \in G_n(V)$ such that $\dim(W \cap L_0) \leq k$ is open in $G_n(V)$; since Λ has the topology induced by that of $G_n(V)$, it follows that $\Lambda^{\leq k}(L_0)$ is open in Λ. Since $\Lambda^{\geq k}(L_0)$ is the complementary of $\Lambda^{\leq k-1}(L_0)$, the conclusion follows. $\square$

2.5.15. COROLLARY. *For all $k = 0, \dots, n$, the subset $\Lambda^k(L_0)$ is locally closed in Λ.*

PROOF. Simply observe that $\Lambda^k(L_0) = \Lambda^{\geq k}(L_0) \cap \Lambda^{\leq k}(L_0)$. □

As a corollary, we obtain the main result of the subsection:

2.5.16. THEOREM. *For each $k = 0, \dots, n$, $\Lambda^k(L_0)$ is an embedded submanifold of Λ with codimension $\frac{1}{2}k(k+1)$; its tangent space is given by:*

$$T_L\Lambda^k(L_0) = \big\{B \in \mathrm{B}_{\mathrm{sym}}(L) : B|_{(L_0\cap L)\times(L_0\cap L)} = 0\big\}, \tag{2.5.16}$$

for all $L \in \Lambda^k(L_0)$.

PROOF. It follows from Proposition 1.4.38 that $\Lambda^k(L_0)$ is an orbit of the action of $\mathrm{Sp}(V, \omega, L_0)$ on Λ. From Theorem 2.1.12 and Corollary 2.5.15 it follows that $\Lambda^k(L_0)$ is an embedded submanifold of Λ. It remains to prove the identity in (2.5.16), because then it will follow that

$$T_L\Lambda \cong \mathrm{B}_{\mathrm{sym}}(L) \ni B \longmapsto B|_{(L_0\cap L)\times(L_0\cap L)} \in \mathrm{B}_{\mathrm{sym}}(L_0 \cap L) \tag{2.5.17}$$

is a surjective linear operator whose kernel is $T_L\Lambda^k(L_0)$, which implies the claim on the codimension of $\Lambda^k(L_0)$.

Using Propositions 2.1.7, 2.5.10 and Lemma 2.5.13, we have that:

$$T_L\Lambda^k(L_0) = \big\{B|_{L\times L} : B \in \mathrm{B}_{\mathrm{sym}}(V),\ B|_{L_0\times L_0} = 0\big\},$$

for all $L \in \Lambda^k(L_0)$. It remains to prove that every symmetric bilinear form $B \in \mathrm{B}_{\mathrm{sym}}(L)$ that vanishes on vectors in $L\cap L_0$ can be extended to a symmetric bilinear form on V that vanishes on L_0. This fact is left to the reader in Exercise 2.8. □

2.5.17. REMARK. One can actually prove that the manifolds $\Lambda^k(L_0)$ are connected; namely, Remark 1.4.40 implies that the group $\mathrm{Sp}_+(V, \omega, L_0)$ of symplectomorphisms of V which restrict to a *positive* isomorphism of L_0 acts transitively on $\Lambda^k(L_0)$. The connectedness of $\Lambda^k(L_0)$ then follows from the conectedness of $\mathrm{Sp}_+(V, \omega, L_0)$ (see Example 3.2.36).

2.5.18. REMARK. It follows from Theorem 2.5.16 that $\Lambda^0(L_0)$ is a dense open subset of Λ; indeed, its complement $\Lambda^{\geq 1}(L_0)$ is a finite union of positive codimension submanifolds, all of which have therefore *null measure*. It follows that given any sequence $(L_i)_{i\in\mathbb{N}}$ of Lagrangian subspaces of V, then the set

$$\bigcap_{i\in\mathbb{N}} \Lambda^0(L_i) = \big\{L \in \Lambda : L \cap L_i = \{0\},\ \forall i \in \mathbb{N}\big\}$$

is dense in Λ, because its complement is a countable union of sets of null measure. The same conclusion can be obtained by using Baire's Lemma instead of the "null measure argument".

We are now able to define a transverse orientation for $\Lambda^1(L_0)$ in Λ. Recall that if N is a submanifold of M, then a *transverse orientation* for N in M is an orientation for the *normal bundle* $\mathfrak{i}^*(TM)/TN$, where $\mathfrak{i} : N \to M$ denotes the inclusion; more explicitly, a transverse orientation for N in M is a choice of an orientation for the quotient space T_nM/T_nN that depends *continuously* on $n \in I\!N$. The *continuous dependence* of the choice of an orientation has to be meant in the following sense: given any $n_0 \in N$ there exists an open neighborhood $U \subset N$ of n_0 and there exist *continuous* functions $X_i : U \to TM$, $i = 1, \ldots, r$, such that $(X_i(n) + T_nN)_{i=1}^r$ is a positively oriented basis of T_nM/T_nN for all $n \in U$. It follows that, if such continuous maps X_i exist, then we can replace them with differentiable maps X_i that satisfy the same condition.

Observe that, for each $L \in \Lambda^k(L_0)$, the map (2.5.17) passes to the quotient and defines an isomorphism:

$$T_L\Lambda/T_L\Lambda^k(L_0) \xrightarrow{\cong} \mathrm{B}_{\mathrm{sym}}(L_0 \cap L). \tag{2.5.18}$$

2.5.19. DEFINITION. For each $L \in \Lambda^1(L_0)$ we define an orientation in the quotient $T_L\Lambda/T_L\Lambda^1(L_0)$ in the following way:

- we give an orientation to the unidimensional space $\mathrm{B}_{\mathrm{sym}}(L_0 \cap L)$ by requiring that an element $B \in \mathrm{B}_{\mathrm{sym}}(L_0 \cap L)$ is a positively oriented basis if $B(v, v) > 0$ for some (hence for all) $v \in L_0 \cap L$ with $v \neq 0$;
- we consider the unique orientation in $T_L\Lambda/T_L\Lambda^1(L_0)$ that makes the isomorphism (2.5.18) positively oriented.

2.5.20. PROPOSITION. *The orientation chosen in Definition 2.5.19 for the space $T_L\Lambda/T_L\Lambda^1(L_0)$ makes $\Lambda^1(L_0)$ into a transversally oriented submanifold of Λ; this transverse orientation is invariant by the action of $\mathrm{Sp}(V, \omega, L_0)$, i.e., for all $A \in \mathrm{Sp}(V, \omega, L_0)$ and for all $L \in \Lambda^1(L_0)$ the isomorphism:*

$$T_L\Lambda/T_L\Lambda^1(L_0) \longrightarrow T_{A(L)}\Lambda/T_{A(L)}\Lambda^1(L_0)$$

induced from $\mathrm{d}A(L)$ by passage to the quotient is positively oriented.

PROOF. It follows from Proposition 2.5.9 that the differential $\mathrm{d}A(L)$ coincides with the push-forward $A_\#$; hence we have the following commutative diagram:

$$\begin{array}{ccc} T_L\Lambda & \xrightarrow{\ \mathrm{d}A(L)\ } & T_{A(L)}\Lambda \\ \downarrow & & \downarrow \\ \mathrm{B}_{\mathrm{sym}}(L \cap L_0) & \xrightarrow[(A|_{L\cap L_0})_\#]{} & \mathrm{B}_{\mathrm{sym}}(A(L) \cap L_0) \end{array} \tag{2.5.19}$$

where the vertical arrows are the operators of restriction of bilinear forms. Then, the orientation given in Definition 2.5.19 is $\mathrm{Sp}(V, \omega, L_0)$-invariant.

The continuous dependence on L of such orientation now follows from the fact that the action of $\mathrm{Sp}(V, \omega, L_0)$ on $\Lambda^1(L_0)$ is transitive.[1] □

[1]The required transverse orientation can be seen as a section $\mathcal{O}$ of the ($\mathbb{Z}_2$-principal) fiber bundle over $\Lambda^1(L_0)$ whose fiber at the point $L \in \Lambda^1(L_0)$ is the set consisting of the two possible orientations of $T_L\Lambda/T_L\Lambda^1(L_0)$. Under this viewpoint, the $\mathrm{Sp}(V, \omega, L_0)$-invariance of this transverse orientation means that the map $\mathcal{O}$ is $\mathrm{Sp}(V, \omega, L_0)$-equivariant, and the differentiability of $\mathcal{O}$ follows then from Corollary 2.1.10.

2.5.21. REMARK. If $A \in \mathrm{Sp}(V,\omega)$ is a symplectomorphism with $A(L_0) = L_0'$, then, as in the proof of Proposition 2.5.20, it follows that the isomorphism:

$$T_L\Lambda/T_L\Lambda^1(L_0) \longrightarrow T_{A(L)}\Lambda/T_{A(L)}\Lambda^1(L_0')$$

induced by the differential $\mathrm{d}A(L)$ by passage to the quotient is positively oriented for all $L \in \Lambda^1(L_0)$. To see this, simply replace L_0 by L_0' in the right column of the diagram (2.5.19).

Exercises for Chapter 2

EXERCISE 2.1. Let X be a locally compact Hausdorff topological space. Show that if X is second countable then X is paracompact; conversely, show that if X is paracompact, connected and locally second countable then X is second countable.

EXERCISE 2.2. Suppose that P, M are manifolds, $N \subset M$ is an immersed submanifold and $f : P \to M$ is a differentiable map. Suppose that $f(P) \subset N$; prove that if $f_0 : P \to N$ is continuous (f_0 is defined by the diagram (2.1.1)) when N is endowed with the topology induced by its differentiable atlas, then $f_0 : P \to N$ is differentiable.

EXERCISE 2.3. Let M be a manifold, $N \subset M$ a subset and τ a topology for N. Prove that there exists at most one differentiable structure on N that induces the topology τ and that makes N an immersed submanifold of M.

EXERCISE 2.4. Prove that every locally compact subspace of a Hausdorff space is locally closed and, conversely, that in a locally compact Hausdorff space every locally closed subset is locally compact in the induced topology.

EXERCISE 2.5. Let G be a Lie group acting differentiably on the manifold M; let $X \in \mathfrak{g}$ and let X^* be the vector field given by (2.1.14). Prove that X^* is *complete* in M, i.e., its maximal integral lines are defined over the whole real line.

EXERCISE 2.6. Show that, given any countable family $\{V_i\}_{i=1}^{\infty}$ of k-dimensional subspaces of $\mathbb{R}^n$, with $k < n$, then there exist a $(n-k)$-dimensional subspace $W \subset \mathbb{R}^n$ which is complementary to all the V_i's.

EXERCISE 2.7. Determine the isotropy of the element $\mathbb{R}^k \oplus \{0\}^{n-k} \in G_k(n)$ with respect to the natural action of $\mathrm{GL}(n,\mathbb{R})$ on $G_k(n)$.

EXERCISE 2.8. Let (V,ω) be a (finite dimensional) symplectic space and L, L_0 be Lagrangian subspaces of V. Suppose that $B \in \mathrm{B}_{\mathrm{sym}}(L)$ is a symmetric bilinear form on L that vanishes in $L \cap L_0$. Prove that B extends to a symmetric bilinear form on V that vanishes in V_0.

EXERCISE 2.9. Prove that if $P : E \to B$ is a differentiable fibration, then every curve of class C^k, $\gamma : [a,b] \to B$, admits a lift $\overline{\gamma} : [a,b] \to B$ of class C^k, $0 \le k \le +\infty$ (see Remark 2.1.18).

EXERCISE 2.10. Show that the map

$$\mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m) \ni T \longmapsto \mathrm{Gr}(T) \in G_n(n+m)$$

is a diffeomorphism onto an open set and compute its differential.

EXERCISE 2.11. Prove that a map $\mathcal{D} : [a, b] \to G_k(n)$ is of class C^p if and only if there exist maps $Y_1, \ldots, Y_k : [a, b] \to \mathbb{R}^n$ of class C^p such that $(Y_i(t))_{i=1}^k$ is a basis of $\mathcal{D}(t)$ for all t.

EXERCISE 2.12. The *Grassmannian of oriented k-dimensional subspaces of* $\mathbb{R}^n$ is the set $G_k^+(n)$ of all pairs $(W, \mathcal{O})$ where $W \subset \mathbb{R}^n$ is a k-dimensional subspace and $\mathcal{O}$ is an orientation in W. Define an action of $\mathrm{GL}(n, \mathbb{R})$ in $G_k^+(n)$ and show that its restriction to $\mathrm{SO}(n)$ is transitive if $k < n$. Conclude that, if $k < n$, $G_k^+(n)$ has a natural structure of homogeneous manifold which is compact and connected.

EXERCISE 2.13. Given a Lagrangian L_0 of a symplectic space (V, ω), denote by Fix_{L_0} the subgroup of $\mathrm{Sp}(V, \omega)$ consisting of those symplectomorphisms T such that $T|_{L_0} = \mathrm{Id}$, i.e., such that $T(v) = v$ for all $v \in L_0$. Prove that Fix_{L_0} is a Lie subgroup of $\mathrm{Sp}(V, \omega)$, and that it acts freely and transitively on $\Lambda^0(L_0)$. Conclude that Fix_{L_0} is diffeomorphic to $\Lambda^0(L_0)$.

EXERCISE 2.14. In the notations of Exercise 2.13, prove that Fix_{L_0} is isomorphic as a Lie group to the additive group of $n \times n$ real symmetric matrices.

EXERCISE 2.15. Given $L_0, L \in \Lambda$ with $L \cap L_0 = \{0\}$ and $B \in \mathrm{B}_{\mathrm{sym}}(L_0)$ a nondegenerate symmetric bilinear form on L_0, prove that there exists $L_1 \in \Lambda$ with $L_1 \cap L_0 = \{0\}$ and such that $\varphi_{L_0,L_1}(L) = B$.

CHAPTER 3

Topics of Algebraic Topology

3.1. The Fundamental Groupoid and Group

In this section we will give a short summary of the definition and of the main properties of the fundamental groupoid and group of a topological space X. We will denote by I the unit closed interval $[0,1]$ and by $C^0(Y,Z)$ the set of continuous maps $f: Y \to Z$ between any two topological spaces Y and Z.

Let us begin with a general definition:

3.1.1. DEFINITION. If Y and Z are topological spaces, we say that two maps $f, g \in C^0(Y,Z)$ are *homotopic* when there exists a continuous function:

$$H : I \times Y \longrightarrow Z$$

such that $H(0,y) = f(y)$ and $H(1,y) = g(y)$ for every $y \in Y$. We then say that H is a *homotopy* between f and g and we write $H\colon f \cong g$. For $s \in I$, we denote by $H_s : Y \to Z$ the map $H_s(y) = H(s,y)$.

Intuitively, a homotopy $H\colon f \cong g$ is a *one-parameter family* $(H_s)_{s\in I}$ in $C^0(Y,Z)$ that *deforms continuously* $H_0 = f$ into $H_1 = g$.

In our context, the following notion of homotopy is more interesting:

3.1.2. DEFINITION. Let $\gamma, \mu : [a,b] \to X$ be continuous curves in a topological space X; we say that γ *is homotopic to* μ *with fixed endpoints* if there exists a homotopy $H\colon \gamma \cong \mu$ such that $H(s,a) = \gamma(a) = \mu(a)$ and $H(s,b) = \gamma(b) = \mu(b)$ for every $s \in I$. In this case, we say that H is a *homotopy with fixed endpoints* between γ and μ.

Clearly, two curves $\gamma, \mu : [a,b] \to X$ can only be homotopic with fixed endpoints if they have the same endpoints, i.e., if $\gamma(a) = \mu(a)$ and $\gamma(b) = \mu(b)$; given a homotopy with fixed endpoints H the stages H_s are curves with the same endpoints as γ and μ.

It is easy to see that the "homotopy" and the "homotopy with fixed endpoints" are equivalence relations in $C^0(Y,Z)$ and in $C^0([a,b],X)$ respectively.

For this section we will fix a topological space X and we will denote by $\Omega(X)$ the set of all continuous curves $\gamma : I \to X$:

$$\Omega(X) = C^0(I,X).$$

For $\gamma \in \Omega(X)$, we denote by $[\gamma]$ the equivalence class of all curves homotopic to γ with fixed endpoints; we also denote by $\overline{\Omega}(X)$ the set of such classes:

$$\overline{\Omega}(X) = \Big\{ [\gamma] : \gamma \in \Omega(X) \Big\}.$$

If $\gamma, \mu \in \Omega(X)$ are such that $\gamma(1) = \mu(0)$, we define the *concatenation of* γ *and* μ to be the curve $\gamma \cdot \mu$ in $\Omega(X)$ defined by:

$$(\gamma \cdot \mu)(t) = \begin{cases} \gamma(2t), & t \in [0, \frac{1}{2}], \\ \mu(2t-1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

In this way, the map $(\gamma, \mu) \mapsto \gamma \cdot \mu$ defines a *partial binary operation* in the set $\Omega(X)$. For $\gamma \in \Omega(X)$, we define $\gamma^{-1} \in \Omega(X)$ by setting:

$$\gamma^{-1}(t) = \gamma(1-t), \quad t \in I.$$

For each point $x \in X$ we denote by $\mathfrak{o}_x \in \Omega(X)$ the constant curve equal to x:

$$\mathfrak{o}_x(t) = x, \quad t \in I.$$

It is not hard to prove that, if $\gamma(1) = \mu(0)$, $[\gamma] = [\gamma_1]$ and $[\mu] = [\mu_1]$, then:

$$[\gamma \cdot \mu] = [\gamma_1 \cdot \mu_1], \quad \big[\gamma^{-1}\big] = \big[\gamma_1^{-1}\big].$$

These identities show that the operations $(\gamma, \mu) \mapsto \gamma \cdot \mu$ and $\gamma \mapsto \gamma^{-1}$ *pass to the quotient* and they define operations in the set $\overline{\Omega}(X)$; we then define:

$$[\gamma] \cdot [\mu] = [\gamma \cdot \mu], \quad [\gamma]^{-1} = \big[\gamma^{-1}\big].$$

The homotopy class $[\gamma]$ of a curve γ is invariant by reparameterizations:

3.1.3. LEMMA. *Let $\gamma \in \Omega(X)$ be a continuous curve and consider a* reparameterization $\gamma \circ \sigma$ *of* γ*, where $\sigma : I \to I$ is a continuous map. If $\sigma(0) = 0$ and $\sigma(1) = 1$, then $[\gamma] = [\gamma \circ \sigma]$; if $\sigma(0) = \sigma(1)$, then $\gamma \circ \sigma$ is homotopic with fixed endpoints to a constant curve, i.e., $[\gamma \circ \sigma] = [\mathfrak{o}_{\gamma(\sigma(0))}]$.*

PROOF. Define $H(s,t) = \gamma\big((1-s)t + s\,\sigma(t)\big)$ to prove the first statement and $H(s,t) = \gamma\big((1-s)\sigma(t) + s\,\sigma(0)\big)$ to prove the second statement. □

3.1.4. REMARK. In some cases we may need to consider homotopy classes of curves $\gamma : [a, b] \to X$ defined on an arbitrary closed interval $[a, b]$; in this case we will denote by $[\gamma]$ the homotopy class with fixed endpoints of the curve:

$$I \ni t \longmapsto \gamma\big((b-a)t + a\big) \in X; \tag{3.1.1}$$

it follows from Lemma 3.1.3 that (3.1.1) is homotopic with fixed endpoints to every reparameterization $\gamma \circ \sigma$ of γ, where $\sigma : I \to [a, b]$ is a continuous map with $\sigma(0) = a$ and $\sigma(1) = b$. Also the concatenation of curves defined on arbitrary closed intervals should be understood in the sense of the concatenation of their affine reparameterizations on the interval I.

3.1.5. COROLLARY. *Given $\gamma, \mu, \kappa \in \Omega(X)$ with $\gamma(1) = \mu(0)$ and $\mu(1) = \kappa(0)$, then:*

$$\big([\gamma] \cdot [\mu]\big) \cdot [\kappa] = [\gamma] \cdot \big([\mu] \cdot [\kappa]\big). \tag{3.1.2}$$

Moreover, for $\gamma \in \Omega(X)$ we have:

$$[\gamma] \cdot [\mathfrak{o}_{\gamma(1)}] = [\gamma], \quad [\mathfrak{o}_{\gamma(0)}] \cdot [\gamma] = [\gamma] \tag{3.1.3}$$

and also:

$$[\gamma]\cdot[\gamma]^{-1}=[\mathfrak{o}_{\gamma(0)}],\quad [\gamma]^{-1}\cdot[\gamma]=[\mathfrak{o}_{\gamma(1)}]. \tag{3.1.4}$$

PROOF. The identity (3.1.2) follows from the observation that $(\gamma\cdot\mu)\cdot\kappa$ is a reparameterization of $\gamma\cdot(\mu\cdot\kappa)$ by a continuous map $\sigma : I \to I$ with $\sigma(0)=0$ and $\sigma(1)=1$. Similarly, the identities in (3.1.3) are obtained by observing that $\gamma\cdot\mathfrak{o}_{\gamma(1)}$ and $\mathfrak{o}_{\gamma(0)}\cdot\gamma$ are reparameterizations of γ by a map σ with $\sigma(0)=0$ and $\sigma(1)=1$. The first identity in (3.1.4) follows from the fact that $\gamma\cdot\gamma^{-1}=\gamma\circ\sigma$ where $\sigma : I \to I$ satisfies $\sigma(0)=\sigma(1)=0$; the second identity in (3.1.4) is obtained similarly. □

The identity (3.1.2) tells us that the concatenation is *associative* in $\overline{\Omega}(X)$ when all the products involved are defined; the identities in (3.1.3), roughly speaking, say that the classes $[\mathfrak{o}_x]$, $x\in X$, act like *neutral elements* for the operation of concatenation, and the identities in (3.1.4) tell us that the class $[\gamma^{-1}]$ acts like the *inverse* of the class $[\gamma]$ with respect to the concatenation.

If we fix a point $x_0\in X$, we denote by $\Omega_{x_0}(X)$ the set of *loops in X with basepoint* x_0:

$$\Omega_{x_0}(X)=\big\{\gamma\in\Omega(X):\gamma(0)=\gamma(1)=x_0\big\}.$$

We also consider the image of $\Omega_{x_0}(X)$ in the quotient $\overline{\Omega}(X)$, that will be denoted by:

$$\pi_1(X,x_0)=\big\{[\gamma]:\gamma\in\Omega_{x_0}(X)\big\}.$$

The (partially defined) binary operation of concatenation in $\overline{\Omega}(X)$ restricts to a (totally defined) binary operation in $\pi_1(X,x_0)$; from Corollary 3.1.5 we obtain the following:

3.1.6. THEOREM. *The set $\pi_1(X,x_0)$ endowed with the concatenation operation is a group.* □

This is the main definition of the section:

3.1.7. DEFINITION. The set $\overline{\Omega}(X)$ endowed with the (partially defined) operation of concatenation is called the *fundamental groupoid* of the topological space X. For all $x_0\in X$, the group $\pi_1(X,x_0)$ (with respect to the concatenation operation) is called the *fundamental group of X with basepoint* x_0.

3.1.8. REMARK. A *groupoid* is normally defined as a *small category*, i.e., a category whose objects form a set, whose morphisms are all isomorphisms. In this context it will not be important to study this abstract notion of groupoid, nevertheless it is important to observe that Corollary 3.1.5 shows that the fundamental groupoid of a topological space is indeed a groupoid in this abstract sense.

3.1.9. REMARK. If $X_0\subset X$ is the arc-connected component of x_0 in X, then $\pi_1(X,x_0)=\pi_1(X_0,x_0)$, since every loop in X with basepoint in x_0 has image contained in X_0, as well as every homotopy between such loops has image in X_0.

In the following lemma we describe the functoriality properties of the fundamental groupoid and group:

3.1.10. LEMMA. *Let $f : X \to Y$ be a continuous map; for $\gamma \in \Omega(X)$, the homotopy class $[f \circ \gamma]$ depends only on the homotopy class $[\gamma]$ of γ; hence, we have a well defined map*

$$f_*\colon \overline{\Omega}(X) \longrightarrow \overline{\Omega}(Y)$$

given by $f_*([\gamma]) = [f \circ \gamma]$. For $\gamma, \mu \in \Omega(X)$ with $\gamma(1) = \mu(0)$ and for every $x_0 \in X$ the following identities hold:

$$f_*([\gamma]\cdot[\mu]) = f_*([\gamma])\cdot f_*([\mu]), \quad f_*\big([\gamma]^{-1}\big) = f_*([\gamma])^{-1}, \quad f_*([\mathfrak{o}_{x_0}]) = [\mathfrak{o}_{f(x_0)}].$$

In particular, if $f(x_0) = y_0$ then f_* restricts to a map

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

which is a group homomorphism. □

Clearly, given $f \in C^0(X, Y)$ and $g \in C^0(Y, Z)$ then:

$$(g \circ f)_* = g_* \circ f_*,$$

and that, if Id denotes the identity of X, then Id_* is the identity of $\overline{\Omega}(X)$; it follows that, if $f : X \to Y$ is a homeomorphism, then f_* is a bijection, and it induces an isomorphism of $\pi_1(X, x_0)$ onto $\pi_1(Y, f(x_0))$. The map f_* is said to be *induced* by f in the fundamental groupoid or in the fundamental group.

The following proposition relates the fundamental groups relative to different basepoints:

3.1.11. PROPOSITION. *Given $x_0, x_1 \in X$ and a continuous curve $\lambda : I \to X$ with $\lambda(0) = x_0$ and $\lambda(1) = x_1$, we have an isomorphism:*

$$\lambda_\#\colon \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

defined by $\lambda_\#([\gamma]) = [\lambda]^{-1}\cdot[\gamma]\cdot[\lambda]$, for every $\gamma \in \Omega_{x_0}(X)$. □

3.1.12. COROLLARY. *If x_0 and x_1 belong to the same arc-connected component of X, then the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.* □

The following commutative diagram relates the homomorphisms f_* and $\lambda_\#$:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\ {\scriptstyle \lambda_\#}\big\downarrow & & \big\downarrow{\scriptstyle (f\circ\lambda)_\#} \\ \pi_1(X, x_1) & \xrightarrow[f_*]{} & \pi_1(Y, y_1) \end{array}$$

where $f \in C^0(X, Y)$, $x_0, x_1 \in X$, $y_0 = f(x_0)$, $y_1 = f(x_1)$ and $\lambda \in \Omega(X)$ is a curve from x_0 to x_1.

3.1.13. REMARK. In spite of the fact that $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic if x_0 and x_1 are in the same arc-connected component of X, such isomorphism is *not* canonical; more explicitly, if $\lambda_0, \lambda_1 \in \Omega(X)$ are curves from x_0 to x_1, then:

$$(\lambda_1)_\#^{-1} \circ (\lambda_0)_\# = \mathcal{I}_{[\lambda]},$$

where $\lambda = \lambda_1 \cdot \lambda_0^{-1}$ and $\mathcal{I}_{[\lambda]}$ denotes the operator of conjugation by the element $[\lambda]$ in $\pi_1(X, x_0)$. If $\pi_1(X, x_0)$ is abelian it follows that $(\lambda_0)_\# = (\lambda_1)_\#$, and therefore the fundamental groups with basepoints in the same arc-connected components can be canonically identified (compare with Remark 3.3.34).

3.1.14. DEFINITION. We say that a topological space X is *simply connected* if it is arc-connected and if $\pi_1(X, x_0)$ is the trivial group $\{\mathfrak{o}_{x_0}\}$ for some (hence for all) $x_0 \in X$.

Observe that, if X is simply connected, then $[\gamma] = [\mu]$ for all continuous curves $\gamma, \mu : I \to X$ such that $\gamma(0) = \mu(0)$ and $\gamma(1) = \mu(1)$; for, in this case, $[\gamma]\cdot[\mu]^{-1} = [\mathfrak{o}_{x_0}]$.

3.1.15. EXAMPLE. A subset $X \subset \mathbb{R}^n$ is said to be *star-shaped* around the point $x_0 \in X$ if for every $x \in X$ the segment:

$$[x_0, x] = \big\{(1-t)x_0 + tx : t \in I\big\}$$

is contained in X; we say that X is *convex* if it is star-shaped at each one of its points. If X is star-shaped at x_0, then X is simply connected; indeed, X is clearly arc-connected, and, given a loop $\gamma \in \Omega_{x_0}(X)$, we can define a homotopy:

$$I \times I \ni (s,t) \mapsto (1-s)\gamma(t) + s\,x_0 \in X$$

between γ and $\mathfrak{o}_{x_0}$.

3.1.16. REMARK. Two loops $\gamma \in \Omega_{x_0}(X)$ and $\mu \in \Omega_{x_1}(X)$ are said to be *freely homotopic* if there exists a homotopy $H\colon \gamma \cong \mu$ such that, for every $s \in I$, the curve H_s is a loop in X, i.e., $H(s,0) = H(s,1)$ for every s. In this situation, if we set $\lambda(s) = H(s,0)$, we have the following identity:

$$\lambda_{\#}([\gamma]) = [\mu]. \tag{3.1.5}$$

The identity (3.1.5) follows from the fact that, since the square $I \times I$ is convex, the homotopy class in $\overline{\Omega}(I \times I)$ of the loop that is obtained by considering the boundary of $I \times I$ run counterclockwise is trivial, hence so is its image by H_*. Such image is precisely the difference of the terms on the two sides of the equality in (3.1.5). In Exercise 3.3 the reader is asked to show that, conversely, any loop γ is always freely homotopic to $\lambda^{-1} \cdot \gamma \cdot \lambda$, for any curve λ with $\lambda(0) = \gamma(0)$.

In particular, if $\gamma, \mu \in \Omega_{x_0}(X)$ are freely homotopic, then the classes $[\gamma]$ and $[\mu]$ are *conjugate* in $\pi_1(X, x_0)$; it follows that $\gamma \in \Omega_{x_0}(X)$ is such that $[\gamma] = [\mathfrak{o}_{x_0}]$ if and only if γ is freely homotopic to a constant loop. With this argument we have shown that *an arc-connected topological space X is simply connected if and only if every loop in X is freely homotopic to a constant loop*.

3.1.17. EXAMPLE. A topological space X is said to be *contractible* if the identity map of X is homotopic to a constant map, i.e., if there exists a continuous map $H : I \times X \to X$ and $x_0 \in X$ such that $H(0,x) = x$ and $H(1,x) = x_0$ for every $x \in X$. For instance, if $X \subset \mathbb{R}^n$ is star-shaped at x_0, then X is contractible: the required homotopy H is given by $H(s,x) = (1-s)x + s\,x_0$.

It is easy to see that every contractible space is arc-connected (see Exercise 3.1). Moreover, if X is contractible then X is simply connected; indeed, if $H\colon \mathrm{Id} \cong x_0$ is a homotopy and $\gamma \in \Omega(X)$ is a loop, then the map $(s,t) \mapsto H(s, \gamma(t))$ is a free homotopy between γ and the constant loop $\mathfrak{o}_{x_0}$ (see Remark 3.1.16).

3.1.1. Stability of the homotopy class of a curve. In this subsection we show that, under reasonable assumptions on the topology of the space X, two continuous curves in X that are *sufficiently close* belong to the same homotopy class. We begin with a definition of "proximity" for continuous maps:

3.1.18. DEFINITION. Let Y, Z be topological spaces; for $K \subset Y$ compact and $U \subset Z$ open, we define:

$$\mathcal{V}(K;U) = \big\{f \in C^0(Y,Z) : f(K) \subset U\big\}.$$

The *compact-open topology* in $C^0(Y,Z)$ is the topology generated by the sets $\mathcal{V}(K;U)$ with $K \subset Y$ compact and $U \subset Z$ open; more explicitly, an open set in the compact-open topology is union of intersections of the form:

$$\mathcal{V}(K_1;U_1) \cap \ldots \cap \mathcal{V}(K_n;U_n)$$

with each $K_i \subset Y$ compact and each $U_i \subset Z$ open, $i = 1, \ldots, n$.

3.1.19. REMARK. When the topology of the counterdomain Z is metrizable, i.e., it is induced by a metric d, the compact-open topology in $C^0(Y,Z)$ is also called the *topology of the uniform convergence on compacta*; in this case it is not too hard to prove that, for $f \in C^0(Y,Z)$, a fundamental systems of open neighborhood of f is obtained by considering the sets:

$$\mathcal{V}(f;K,\varepsilon) = \Big\{g \in C^0(Y,Z) : \sup_{y\in K} d(f(y), g(y)) < \varepsilon\Big\},$$

where $K \subset Y$ is an arbitrary compact set and $\varepsilon > 0$. In this topology, a sequence (or a net) f_n converges to f if and only if f_n converges uniformly to f on each compact subset of Y.

In the context of differential topology, if Y and Z are manifolds (possibly with boundary), the compact-open topology in $C^0(Y,Z)$ is also known as the C^0-*topology* or as the C^0*-weak Whitney topology*.

3.1.20. REMARK. To each map $f : X \times Y \to Z$ which is continuous in the second variable there corresponds a map:

$$\tilde{f} \colon X \longrightarrow C^0(Y,Z).$$

An interesting property of the compact-open topology in $C^0(Y,Z)$ is that, if Y is Hausdorff, the continuity of $\tilde{f}$ is equivalent to the continuity of $f|_{X\times K}$ for every compact $K \subset Y$ (see [**22**, Proposição 21, §8, Capítulo 9]). In particular, if Y is Hausdorff and locally compact, the continuity of f and the continuity of $\tilde{f}$ are equivalent.

We will now introduce suitable conditions on the topological space X that will allow to prove the stability of the homotopy class of curves.

3.1.21. DEFINITION. We say that the topological space X is *locally arc-connected* if every point of X has a fundamental system of open neighborhoods consisting of arc-connected subsets, i.e., if for every $x \in X$ and every neighborhood V of x in X there exists an open arc-connected subset $U \subset X$ with $x \in U \subset V$.

We say that X is *semi-locally simply connected* if every $x \in X$ has a neighborhood V such that every loop on V is contractible in X, i.e., given $\gamma \in \Omega(X)$ with $\gamma(0) = \gamma(1)$ and $\mathrm{Im}(\gamma) \subset V$, then γ is homotopic (in X) with fixed endpoints to a constant curve.

3.1.22. EXAMPLE. If every point of X has a simply connected neighborhood, then X is semi-locally simply connected; in particular, every differentiable (or even topological) manifold is locally arc-connected and semi-locally simply connected.

This is the main result of the subsection:

3.1.23. THEOREM. *Let X be a locally arc-connected and semi-locally simply connected topological space; given a curve $\gamma \in \Omega(X)$, there exists a neighborhood $\mathcal{U}$ of γ in the space $C^0(I, X)$ endowed with the compact-open topology such that for every $\mu \in \mathcal{U}$, if $\mu(0) = \gamma(0)$ and $\mu(1) = \gamma(1)$ then $[\mu] = [\gamma]$.*

PROOF. Write $X = \bigcup_{\alpha\in\mathcal{A}} U_\alpha$, where each $U_\alpha \subset X$ is open and such that every lace in U_α is contractible in X. Then, the inverse images $\gamma^{-1}(U_\alpha)$, $\alpha \in \mathcal{A}$, form an open covering of the compact space I, which has a *Lebesgue number* $\delta > 0$, i.e., every subset of I whose diameter is less than δ is contained in some $\gamma^{-1}(U_\alpha)$.

Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be a partition of I with $t_{r+1} - t_r < \delta$ and let $\alpha_r \in \mathcal{A}$ be such that $\gamma([t_r, t_{r+1}]) \subset U_{\alpha_r}$ for every $r = 0, \ldots, k-1$. For each r, the point $\gamma(t_r) \in U_{\alpha_{r-1}} \cap U_{\alpha_r}$ has an open arc-connected neighborhood V_r contained in the intersection $U_{\alpha_{r-1}} \cap U_{\alpha_r}$; define the neighborhood $\mathcal{U}$ of γ in $C^0(I, X)$ by:

$$\mathcal{U} = \bigcap_{r=0}^{k-1} \mathcal{V}([t_r, t_{r+1}]; U_{\alpha_r}) \ \cap \ \bigcap_{r=1}^{k-1} \mathcal{V}(\{t_r\}; V_r).$$

Clearly, $\gamma \in \mathcal{U}$. Let now $\mu \in \mathcal{U}$ be such that $\mu(0) = \gamma(0)$ and $\mu(1) = \gamma(1)$; we need to show that $[\gamma] = [\mu]$.

For each $r = 1, \ldots, k-1$ choose a curve $\lambda_r \in \Omega(V_r)$ with $\lambda_r(0) = \gamma(t_r)$ and $\lambda_r(1) = \mu(t_r)$; set $\lambda_0 = \mathfrak{o}_{\gamma(0)}$ and $\lambda_k = \mathfrak{o}_{\gamma(1)}$. For $r = 0, \ldots, k-1$, we have (see Remark 3.1.4):

$$[\mu|_{[t_r,t_{r+1}]}] = [\lambda_r]^{-1} \cdot [\gamma|_{[t_r,t_{r+1}]}] \cdot [\lambda_{r+1}], \tag{3.1.6}$$

because the curve on the right hand side of (3.1.6) concatenated with the inverse of the curve on the left hand side of (3.1.6) is the homotopy class of a loop in U_{α_r}, hence trivial in $\overline{\Omega}(X)$. Moreover,

$$\begin{aligned} [\mu] &= [\mu|_{[t_0,t_1]}] \cdot \cdots \cdot [\mu|_{[t_{k-1},t_k]}], \\ [\gamma] &= [\gamma|_{[t_0,t_1]}] \cdot \cdots \cdot [\gamma|_{[t_{k-1},t_k]}]. \end{aligned} \tag{3.1.7}$$

The conclusion now follows from (3.1.7) by concatenating the curves on both sides of the identities (3.1.6) for $r = 0, \ldots, k-1$. □

3.1.24. EXAMPLE. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit n-dimensional sphere. From the proof of Theorem 3.1.23 it follows that every curve $\gamma : I \to S^n$ is homotopic with fixed endpoints to a curve which is piecewise C^1. If $n \geq 2$, such curve cannot be surjective onto the sphere, because its image must have null measure in S^n. Hence, if $n \geq 2$ and $\gamma : I \to S^n$ is a piecewise C^1 loop, there exists $x \in S^n$ such that $\mathrm{Im}(\gamma) \subset S^n \setminus \{x\}$. Using the stereographic projection, we see that $S^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n$, therefore it is simply connected. From this argument it follows that the sphere S^n is simply connected for $n \geq 2$; the circle S^1 is *not* simply connected (see Example 3.2.24).

We will need also a version of Theorem 3.1.23 for the case of homotopies with free endpoints in a given set.

3.1.25. DEFINITION. Let $A \subset X$ be a subset and let $\gamma, \mu : [a, b] \to X$ be given curves with $\gamma(a), \mu(a), \gamma(b), \mu(b) \in A$; we say that γ and μ are *homotopic with endpoints free in* A if there exists a homotopy $H\colon \gamma \cong \mu$ such that $H_s(a), H_s(b) \in A$ for every $s \in I$; in this case we say that H *is a homotopy with free endpoints in* A between γ and μ.

The relation of "homotopy with free endpoints in A" is an equivalence relation in the set of curves $\gamma \in C^0([a, b], X)$ such that $\gamma(a), \gamma(b) \in A$; obviously, if two curves with endpoints in A are homotopic with fixed endpoints then they will be homotopic with free endpoints in A.

3.1.26. REMARK. If $\gamma \in \Omega(X)$ is a curve with endpoints in A and $\lambda \in \Omega(A)$ is such that $\gamma(1) = \lambda(0)$, then the concatenation $\gamma \cdot \lambda$ is homotopic to γ with free endpoints in A. Indeed, for each $s \in I$, denote by $\lambda_s \in \Omega(A)$ the curve $\lambda_s(t) = \lambda\big((1 - s)t\big)$. Then, $H_s = \gamma \cdot \lambda_s$ defines a homotopy with free endpoints in A between $\gamma \cdot \lambda$ and $\gamma \cdot \mathfrak{o}_{\lambda(0)}$; the conclusion follows from the fact that γ and $\gamma \cdot \mathfrak{o}_{\lambda(0)}$ are homotopic with fixed endpoints.

Similarly, one shows that if $\lambda \in \Omega(A)$ is such that $\lambda(1) = \gamma(0)$, then $\lambda \cdot \gamma$ is homotopic to γ with free endpoints in A.

We have the following version of Theorem 3.1.23 for homotopies with free endpoints in a set:

3.1.27. THEOREM. *Let X be a locally arc-connected and semi-locally simply connected topological space; let $A \subset X$ be a locally arc-connected subspace of X. Given a curve $\gamma : I \to X$ with endpoints in A, then there exists a neighborhood $\mathcal{U}$ of γ in $C^0(I, X)$ endowed with the compact-open topology such that, for every $\mu \in \mathcal{U}$ with endpoints in A, the curves γ and μ are homotopic with free endpoints in A.*

PROOF. We will only show how to adapt the proof of Theorem 3.1.23 to this case. Once the open sets U_{α_r} and V_r are constructed, we also choose open neighborhood V_0 and V_k of $\gamma(t_0)$ and $\gamma(t_k)$ respectively in such a way that $V_0 \cap A$ and $V_k \cap A$ are arc-connected and contained respectively in U_{α_0} and in $U_{\alpha_{k-1}}$. Then, we define $\mathcal{U}$ by setting:

$$\mathcal{U} = \bigcap_{r=0}^{k-1} \mathcal{V}([t_r, t_{r+1}]; U_{\alpha_r}) \ \cap \ \bigcap_{r=0}^{k} \mathcal{V}(\{t_r\}; V_r).$$

Let $\mu \in \mathcal{U}$ be a curve with endpoints in A; we must show that γ and μ are homotopic with free endpoints in A. The curves λ_0 and λ_k are now chosen in such a way that $\lambda_r(0) = \gamma(t_r)$, $\lambda_r(1) = \mu(t_r)$ and $\mathrm{Im}(\lambda_r) \subset V_r \cap A$ for $r = 0, k$. The identity (3.1.6) still holds for $r = 0, \ldots, k-1$. Using the same argument of that proof, we now obtain:

$$[\mu] = [\lambda_0]^{-1} \cdot [\gamma] \cdot [\lambda_k];$$

and the conclusion follows from Remark 3.1.26. □

3.2. The Homotopy Exact Sequence of a Fibration

In this section we will give a short exposition of the definition and the basic properties of the (absolute and relative) homotopy groups of a topological space; we will describe the exact sequence in homotopy of a pair (X, A), and as a corollary we will obtain the homotopy exact sequence of a fibration $p : E \to B$.

As in Section 3.1, we will denote by I the closed unit interval $[0, 1]$ and by $C^0(Y, Z)$ the set of continuous maps from Y to Z. We will denote by I^n the *unit n-dimensional cube*, and by ∂I^n its boundary, that is:

$$\partial I^n = \big\{ t \in I^n : t_i \in \{0, 1\} \text{ for some } i = 1, \ldots, n \big\}.$$

If $n = 0$, we define $I^0 = \{0\}$ and $\partial I^0 = \emptyset$.

Let $\mathbb{R}^\infty$ denote the space of all sequences $(t_i)_{i \geq 1}$ of real numbers; we identify I^n with the subset of $\mathbb{R}^\infty$:

$$I^n \cong \{(t_1, \ldots, t_n, 0, 0, \ldots) : 0 \leq t_i \leq 1,\ i = 1, \ldots, n\} \subset \mathbb{R}^\infty$$

in such a way that, for $n \geq 1$, the cube I^{n-1} will be identified with the face of I^n:

$$I^{n-1} \cong \{t \in I^n : t_n = 0\} \subset I^n;$$

we will call this face the *initial face* of I^n. We denote by J^{n-1} the union of the other faces of I^n:

$$J^{n-1} = \big\{ t \in I^n : t_n = 1 \text{ or } t_i \in \{0, 1\} \text{ for some } i = 1, \ldots, n-1 \big\}.$$

We will henceforth fix a topological space X; for every $x_0 \in X$ we denote by $\Omega^n_{x_0}(X)$ the set:

$$\Omega^n_{x_0}(X) = \big\{\ \phi \in C^0(I^n, X) : \phi(\partial I^n) \subset \{x_0\}\ \big\}.$$

If $n = 0$, we identify a map $\phi : I^0 \to X$ with the point $\phi(0) \in X$, so that $\Omega^0_{x_0}(X)$ is identified with the set X (observe that $\Omega^0_{x_0}(X)$ does not actually depend on x_0). The set $\Omega^1_{x_0}(X)$ is the loop space with basepoint x_0 introduced in Section 3.1.

We say that (X, A) is a *pair of topological spaces* if X is a topological space and $A \subset X$ is a subspace. If (X, A) is a pair of topological spaces, $x_0 \in A$ and $n \geq 1$ we denote by $\Omega^n_{x_0}(X, A)$ the set:

$$\Omega^n_{x_0}(X, A) = \big\{\ \phi \in C^0(I^n, X) : \phi(I^{n-1}) \subset A,\ \phi(J^{n-1}) \subset \{x_0\}\ \big\}.$$

Observe that, for $\phi \in \Omega^n_{x_0}(X, A)$, we have $\phi(\partial I^n) \subset A$; also:

$$\Omega^n_{x_0}(X) = \Omega^n_{x_0}(X, \{x_0\}), \quad n \geq 1. \tag{3.2.1}$$

If $n = 1$, the cube I^n is the interval I, the initial face I^{n-1} is the point $\{0\}$ and $J^{n-1} = \{1\}$; the set $\Omega^1_{x_0}(X, A)$ therefore is simply the set of continuous curves $\gamma : I \to X$ with $\gamma(0) \in A$ and $\gamma(1) = x_0$.

3.2.1. DEFINITION. If X is a topological space, $x_0 \in X$ and $n \geq 0$, we say that $\phi, \psi \in \Omega^n_{x_0}(X)$ are homotopic in $\Omega^n_{x_0}(X)$ if there exists a homotopy $H\colon \phi \cong \psi$ such that $H_s \in \Omega^n_{x_0}(X)$ for every $s \in I$; the "homotopy in $\Omega^n_{x_0}(X)$" is an equivalence relation, and for every $\phi \in \Omega^n_{x_0}(X)$ we denote by $[\phi]$ its equivalence class. The quotient set is denoted by:

$$\pi_n(X, x_0) = \big\{[\phi] : \phi \in \Omega^n_{x_0}(X)\big\}.$$

We say that $[\phi]$ is the *homotopy class defined by ϕ in $\pi_n(X, x_0)$*.

Similarly, if (X, A) is a pair of topological spaces, $x_0 \in A$ and $n \geq 1$, we say that $\phi, \psi \in \Omega^n_{x_0}(X, A)$ are *homotopic in $\Omega^n_{x_0}(X, A)$* when there exists a homotopy $H\colon \phi \cong \psi$ such that $H_s \in \Omega^n_{x_0}(X, A)$ for every $s \in I$; then we have an equivalence relation in $\Omega^n_{x_0}(X, A)$ and we also denote the equivalence classes by $[\phi]$. The quotient set is denoted by:

$$\pi_n(X, A, x_0) = \big\{[\phi] : \phi \in \Omega^n_{x_0}(X, A)\big\}.$$

We say that $[\phi]$ is the *homotopy class defined by ϕ in $\pi_n(X, A, x_0)$*.

Observe that the set $\pi_0(X, x_0)$ does not depend on the point x_0, and it is identified with the set of *arc-connected components* of X; for every $x \in X$, $[x]$ will denote then the arc-connected component of X that contains x.

From (3.2.1) it follows that:

$$\pi_n(X, \{x_0\}, x_0) = \pi_n(X, x_0), \quad n \geq 1. \tag{3.2.2}$$

Given $\phi, \psi \in \Omega^n_{x_0}(X)$ with $n \geq 1$, or given $\phi, \psi \in \Omega^n_{x_0}(X, A)$ with $n \geq 2$, we define the *concatenation* of ϕ with ψ as the map $\phi \cdot \psi : I^n \to X$ given by:

$$(\phi \cdot \psi)(t) = \begin{cases} \phi(2t_1, t_2, \ldots, t_n), & t_1 \in [0, \frac{1}{2}], \\ \psi(2t_1 - 1, t_2, \ldots, t_n), & t_1 \in [\frac{1}{2}, 1], \end{cases} \tag{3.2.3}$$

for every $t = (t_1, \ldots, t_n) \in I^n$. Observe that the definition (3.2.3) does *not* make sense in general for $\phi, \psi \in \Omega^0_{x_0}(X)$ or for $\phi, \psi \in \Omega^1_{x_0}(X, A)$.

The concatenation is a binary operation in $\Omega^n_{x_0}(X)$ for $n \geq 1$ and in $\Omega^n_{x_0}(X, A)$ for $n \geq 2$; it is easy to see that this binary operation passes to the quotient and it defines operations in the sets $\pi_n(X, x_0)$ and $\pi_n(X, A, x_0)$ of the homotopy classes, given by:

$$[\phi] \cdot [\psi] = [\phi \cdot \psi].$$

We generalize Theorem 3.1.6 as follows:

3.2.2. THEOREM. *For $n \geq 1$, the set $\pi_n(X, x_0)$ is a group (with respect to the concatenation operation) and for $n \geq 2$ also the set $\pi_n(X, A, x_0)$ is a group; in both cases, the neutral element is the class $\mathfrak{o}_{x_0}$ of the constant map $\mathfrak{o}_{x_0} : I^n \to X$:*

$$\mathfrak{o}_{x_0}(t) = x_0, \quad t \in I^n, \tag{3.2.4}$$

and the inverse of $[\phi]$ *is the homotopy class* $[\phi^{-1}]$ *of the map* $\phi^{-1} : I^n \to X$ *given by:*

$$\phi^{-1}(t) = \phi(1 - t_1, t_2, \dots, t_n), \quad t \in I^n.$$

□

3.2.3. DEFINITION. A *pointed set* is a pair (C, c_0) where C is an arbitrary set and $c_0 \in C$ is an element of C. We say that c_0 is the *distinguished element* of (C, c_0). A *map of pointed sets* $f : (C, c_0) \to (C', c_0')$ is an arbitrary map $f : C \to C'$ such that $f(c_0) = c_0'$; in this case we define the *kernel* of f by:

$$\mathrm{Ker}(f) = f^{-1}(c_0'), \tag{3.2.5}$$

If $\mathrm{Ker}(f) = C$ we say that f is the *null map* of (C, c_0) in (C', c_0'). A pointed set (C, c_0) with $C = \{c_0\}$ will be called the *null pointed set*. Both the null pointed set and the null map of pointed sets will be denoted by 0 when there is no danger of confusion.

Given a group G, we will always think of G as the pointed set $(G, 1)$, where 1 is the identity of G; with this convention, the group homomorphisms are maps of pointed sets, and the definition of kernel (3.2.5) coincides with the usual definition of kernel of a homomorphism.

3.2.4. DEFINITION. For $n \geq 1$, the group $\pi_n(X, x_0)$ is called the n*-th (absolute) homotopy group* of the space X with basepoint x_0; for $n \geq 2$, the group $\pi_n(X, A, x_0)$ is called the n*-th relative homotopy group* of the pair (X, A) with basepoint $x_0 \in A$. We call $\pi_0(X, x_0)$ and $\pi_1(X, A, x_0)$ respectively the *zero-th set of homotopy* of X with basepoint $x_0 \in X$ and the *first set of homotopy* of the pair (X, A) with basepoint $x_0 \in A$; all the sets and groups of homotopy (absolute or relative) will be seen as pointed sets, being the class $[\mathfrak{o}_{x_0}]$ their distinguished element.

3.2.5. REMARK. Arguing as in Example 3.1.9, one concludes that if X_0 is the arc-connected component of X containing x_0, then $\pi_n(X, x_0) = \pi_n(X_0, x_0)$ for every $n \geq 1$; if $x_0 \in A \subset X_0$, then also $\pi_n(X, A, x_0) = \pi_n(X_0, A, x_0)$ for every $n \geq 1$. If $x_0 \in A \subset X$ and if A_0 denotes the arc-connected component of A containing x_0, then $\pi_n(X, A, x_0) = \pi_n(X_0, A_0, x_0)$ for every $n \geq 2$.

3.2.6. EXAMPLE. If $X \subset \mathbb{R}^d$ is star-shaped around the point $x_0 \in X$, then $\pi_n(X, x_0) = 0$ for every $n \geq 0$; for, given $\phi \in \Omega^n_{x_0}(X)$ we define a homotopy $H \colon \phi \cong \mathfrak{o}_{x_0}$ by setting:

$$H(s, t) = (1 - s)\phi(t) + s\, x_0, \quad s \in I,\ t \in I^n.$$

3.2.7. EXAMPLE. For $n \geq 1$, if $\phi \in \Omega^n_{x_0}(X, A)$ is such that $\mathrm{Im}(\phi) \subset A$, then $[\phi] = [\mathfrak{o}_{x_0}]$ in $\pi_n(X, A, x_0)$; for, a homotopy $H \colon \phi \cong \mathfrak{o}_{x_0}$ in $\Omega^n_{x_0}(X, A)$ can be defined by:

$$H(s, t) = \phi\big(t_1, \dots, t_{n-1}, 1 - (1 - s)(1 - t_n)\big), \quad t \in I^n,\ s \in I.$$

In particular, we have $\pi_n(X, X, x_0) = 0$.

3.2.8. DEFINITION. Let X, Y be topological spaces and let $x_0 \in X$, $y_0 \in Y$ be given. If $f : X \to Y$ is a continuous map such that $f(x_0) = y_0$, we say that f *preserves basepoints*, and we write

$$f\colon (X, x_0) \longrightarrow (Y, y_0).$$

Then, for $n \geq 0$, f induces a map of pointed sets:

$$f_*\colon \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0) \tag{3.2.6}$$

defined by $f_*([\phi]) = [f \circ \phi]$.

Given pairs (X, A) and (Y, B) of topological spaces, then a *map of pairs*

$$f\colon (X, A) \longrightarrow (Y, B)$$

is a continuous map $f : X \to Y$ such that $f(A) \subset B$. If a choice of basepoints $x_0 \in A$ and $y_0 \in B$ is done, we say that f *preserves basepoints* if $f(x_0) = y_0$, in which case we write:

$$f\colon (X, A, x_0) \longrightarrow (Y, B, y_0).$$

For $n \geq 1$, such a map induces a map f_* of pointed sets:

$$f_*\colon \pi_n(X, A, x_0) \longrightarrow \pi_n(Y, B, y_0) \tag{3.2.7}$$

defined by $f_*([\phi]) = [f \circ \phi]$.

It is easy to see that the maps f_* are well defined, i.e., they do not depend on the choice of representatives in the homotopy classes. Given maps:

$$f\colon (X, A, x_0) \longrightarrow (Y, B, y_0), \quad g\colon (Y, B, y_0) \longrightarrow (Z, C, z_0)$$

then $(g \circ f)_* = g_* \circ f_*$; if Id denotes the identity of (X, A, x_0), then Id_* is the identity of $\pi_n(X, A, x_0)$. It follows that if $f : (X, A, x_0) \to (Y, B, y_0)$ is a *homeomorphism of triples*, i.e., $f : X \to Y$ is a homeomorphism, $f(A) = B$ and $f(x_0) = y_0$, then f_* is a bijection. Similar observations can be made for the absolute homotopy groups $\pi_n(X, x_0)$. We also have the following:

3.2.9. PROPOSITION. *Given $f : (X, x_0) \to (Y, y_0)$, then, for $n \geq 1$, the map f_* given in* (3.2.6) *is a* group homomorphism*; moreover, if*

$$f\colon (X, A, x_0) \to (Y, B, y_0),$$

then for $n \geq 2$ the map f_* given in (3.2.7) *is a group homomorphism.* □

3.2.10. EXAMPLE. If $X = X_1 \times X_2$, and $\mathrm{pr}_1 : X \to X_1$, $\mathrm{pr}_2 : X \to X_2$ denote the projections, then a continuous map $\phi : I^n \to X$ is completely determined by its coordinates:

$$\mathrm{pr}_1 \circ \phi = \phi_1\colon I^n \longrightarrow X^1, \quad \mathrm{pr}_2 \circ \phi = \phi_2\colon I^n \longrightarrow X^2,$$

from which it is easy to see that, given $x = (x_1, x_2) \in X$ and $n \geq 0$, we have a bijection:

$$\pi_n(X, x) \xrightarrow[\cong]{\big((\mathrm{pr}_1)_*, (\mathrm{pr}_2)_*\big)} \pi_n(X_1, x_1) \times \pi_n(X_2, x_2)$$

which is also a group homomorphism if $n \geq 1$. More generally, given $A_1 \subset X_1$, $A_2 \subset X_2$, $x \in A = A_1 \times A_2$, then for $n \geq 1$ we have a bijection:

$$\pi_n(X, A, x) \xrightarrow[\cong]{\big((\mathrm{pr}_1)_*, (\mathrm{pr}_2)_*\big)} \pi_n(X_1, A_1, x_1) \times \pi_n(X_2, A_2, x_2)$$

which is also a group homomorphism if $n \geq 2$. Similar observations can be made for products of an arbitrary number (possibly infinite) of topological spaces.

Give a pair (X, A) and $x_0 \in A$, we have the following maps:

$$\mathfrak{i}\colon (A, x_0) \longrightarrow (X, x_0), \quad \mathfrak{q}\colon (X, \{x_0\}, x_0) \longrightarrow (X, A, x_0),$$

induced respectively by the inclusion of A into X and by the identity of X. Keeping in mind (3.2.2) and Definition 3.2.8, we therefore obtain maps of pointed sets:

$$\mathfrak{i}_*\colon \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0), \quad \mathfrak{q}_*\colon \pi_n(X, x_0) \longrightarrow \pi_n(X, A, x_0); \tag{3.2.8}$$

explicitly, we have $\mathfrak{i}_*([\phi]) = [\phi]$ and $\mathfrak{q}_*([\phi]) = [\phi]$. For $n \geq 1$ we define the *connection operator* relative to the triple (X, A, x_0):

$$\partial_*\colon \pi_n(X, A, x_0) \longrightarrow \pi_{n-1}(A, x_0) \tag{3.2.9}$$

by setting $\partial_*([\phi]) = [\phi|_{I^{n-1}}]$; it is easy to see that ∂_* is well defined, i.e., it does not depend on the choice of a representative of the homotopy class. Moreover, ∂_* is always a map of pointed sets, and it is a group homomorphism if $n \geq 2$.

3.2.11. DEFINITION. A sequence of pointed sets and maps of pointed sets of the form:

$$\cdots \xrightarrow{f_{i+2}} (C_{i+1}, c_{i+1}) \xrightarrow{f_{i+1}} (C_i, c_i) \xrightarrow{f_i} (C_{i-1}, c_{i-1}) \xrightarrow{f_{i-1}} \cdots$$

is said to be *exact at* (C_i, c_i) if $\mathrm{Ker}(f_i) = \mathrm{Im}(f_{i+1})$; the sequence is said to be *exact* if it is exact at each (C_i, c_i) for every i.

We can now prove one of the main results of this section:

3.2.12. THEOREM. *If (X, A) is a pair of topological spaces and $x_0 \in A$, then the sequence:*

(3.2.10)

$$\cdots \xrightarrow{\partial_*} \pi_n(A, x_0) \xrightarrow{\mathfrak{i}_*} \pi_n(X, x_0) \xrightarrow{\mathfrak{q}_*} \pi_n(X, A, x_0) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \xrightarrow{\mathfrak{i}_*}$$
$$\cdots \xrightarrow{\mathfrak{q}_*} \pi_1(X, A, x_0) \xrightarrow{\partial_*} \pi_0(A, x_0) \xrightarrow{\mathfrak{i}_*} \pi_0(X, x_0)$$

is exact, where for each n the pointed set maps $\mathfrak{i}_*$, $\mathfrak{q}_*$ and ∂_* are given in formulas (3.2.8) *and* (3.2.9)

PROOF. The proof is done by considering several cases in which the homotopies are explicitly exhibited.

- *Exactness at* $\pi_n(X, x_0)$. The fact that $\mathrm{Im}(\mathfrak{i}_*) \subset \mathrm{Ker}(\mathfrak{q}_*)$ follows from Example 3.2.7. Let $\phi \in \Omega_{x_0}^n(X)$ be such that there exists a homotopy $H\colon \phi \cong \mathfrak{o}_{x_0}$ in $\Omega_{x_0}^n(X, A)$. Define $K\colon I \times I^n \to X$ by setting:

$$K_s(t) = K(s,t) = \begin{cases} H_{2t_n}(t_1, \ldots, t_{n-1}, 0), & 0 \le 2t_n \le s, \\ H_s\Big(t_1, \ldots, t_{n-1}, \frac{2t_n - s}{2-s}\Big), & s \le 2t_n \le 2; \end{cases}$$

It is easy to see that $\psi = K_1 \in \Omega_{x_0}^n(A)$ and that $K\colon \phi \cong \psi$ is a homotopy in $\Omega_{x_0}^n(X)$. It follows $[\phi] = \mathfrak{i}_*([\psi])$.

- *Exactness at* $\pi_n(X, A, x_0)$.

  The inclusion $\mathrm{Im}(\mathfrak{q}_*) \subset \mathrm{Ker}(\partial_*)$ is trivial. Let $\phi \in \Omega_{x_0}^{n-1}(X, A)$ be such that there exists a homotopy $H\colon \phi|_{I^{n-1}} \cong \mathfrak{o}_{x_0}$ in $\Omega_{x_0}^n(A)$. Define K : $I \times I^n \to X$ by the following formula:

$$K_s(t) = K(s,t) = \begin{cases} H_{s-2t_n}(t_1, \ldots, t_{n-1}), & 0 \le 2t_n \le s, \\ \phi\Big(t_1, \ldots, t_{n-1}, \frac{2t_n - s}{2-s}\Big), & s \le 2t_n \le 2; \end{cases}$$

It is easy to see that $\psi = K_1 \in \Omega_{x_0}^n(X)$ and that $K\colon \phi \cong \psi$ is a homotopy in $\Omega_{x_0}^n(X, A)$. It follows that $[\phi] = \mathfrak{q}_*([\psi])$.

- *Exactness at* $\pi_n(A, x_0)$.

  We first show that $\mathrm{Im}(\partial_*) \subset \mathrm{Ker}(\mathfrak{i}_*)$. To this aim, let $\phi \in \Omega_{x_0}^{n+1}(X, A)$. Define $H\colon I \times I^n \to X$ by setting:

$$H_s(t) = H(s,t) = \phi(t,s), \quad s \in I, \ t \in I^n;$$

It is easy to see that $H\colon \phi|_{I^n} \cong \mathfrak{o}_{x_0}$ is a homotopy in $\Omega_{x_0}^n(X)$, so that

$$(\mathfrak{i}_* \circ \partial_*)([\phi]) = [\mathfrak{o}_{x_0}].$$

  Let now $\psi \in \Omega_{x_0}^n(A)$ be such that there exists a homotopy $K\colon \psi \cong \mathfrak{o}_{x_0}$ in $\Omega_{x_0}^n(X)$. Then, define:

$$\phi(t) = K_{t_{n+1}}(t_1, \ldots, t_n), \quad t \in I^{n+1};$$

it follows that $\phi \in \Omega_{x_0}^{n+1}(X, A)$ and $\partial_*([\phi]) = [\psi]$.

This concludes the proof. □

The exact sequence (3.2.10) is known as the the *long exact homotopy sequence of the pair* (X, A) relative to the basepoint x_0. The exactness property of (3.2.10) at $\pi_1(X, A, x_0)$ can be refined a bit as follows:

3.2.13. PROPOSITION. *The map*

$$\pi_1(X, A, x_0) \times \pi_1(X, x_0) \ni ([\gamma], [\mu]) \longmapsto [\gamma \cdot \mu] \in \pi_1(X, A, x_0) \tag{3.2.11}$$

defines a right action of the group $\pi_1(X, x_0)$ *on the set* $\pi_1(X, A, x_0)$*; the orbit of the distinguished element* $[\mathfrak{o}_{x_0}] \in \pi_1(X, A, x_0)$ *is the kernel of the connection operator*

$$\partial_*\colon \pi_1(X, A, x_0) \longrightarrow \pi_0(A, x_0),$$

and the isotropy group of $[\mathfrak{o}_{x_0}]$ *is the image of the homomorphism:*

$$\mathfrak{i}_*\colon \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0);$$

in particular, the map

$$\mathfrak{q}_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(X, A, x_0) \tag{3.2.12}$$

induces, by passage to the quotient, a bijection between the set of right cosets $\pi_1(X, x_0)/\mathrm{Im}(\mathfrak{i}_*)$ *and the set* $\mathrm{Ker}(\partial_*)$.

PROOF. It is easy to see that (3.2.11) does indeed define a right action (see Corollary 3.1.5). The other statements follow from the long exact sequence of the pair (X, A) and from the elementary theory of actions of groups on sets, by observing that the map of "action on the element $\mathfrak{o}_{x_0}$":

$$\beta_{[\mathfrak{o}_{x_0}]}\colon \pi_1(X, x_0) \longrightarrow \pi_1(X, A, x_0)$$

given by $\beta_{[\mathfrak{o}_{x_0}]}([\mu]) = [\mathfrak{o}_{x_0} \cdot \mu]$ coincides with (3.2.12). □

We now proceed with the study of fibrations.

3.2.14. DEFINITION. Let F, E, B be topological spaces; a continuous map $p : E \to B$ is said to be a *locally trivial fibration* with *typical fiber* F if for every $b \in B$ there exists an open neighborhood U of b in B and a homeomorphism:

$$\alpha\colon p^{-1}(U) \longrightarrow U \times F \tag{3.2.13}$$

such that $\mathrm{pr}_1 \circ \alpha = p|_{p^{-1}(U)}$, where pr_1 denotes the first projection of the product $U \times F$; we then say that α is a *local trivialization* of p around b, and we also say that the fibration p is *trivial* on the open set $U \subset B$. We call E the *total space* and B the *base* of the fibration p; for every $b \in B$ the subset $E_b = p^{-1}(b) \subset E$ will be called the *fiber* over b.

Clearly, any local trivialization of p around b induces a homeomorphism of the fiber E_b onto the typical fiber F.

We have the following:

3.2.15. LEMMA. *Let* $p : E \to B$ *a locally trivial fibration, with typical fiber* F*; then, given* $e_0 \in E$, $b_0 \in B$ *with* $p(e_0) = b_0$*, the map:*

$$p_*\colon \pi_n(E, E_{b_0}, e_0) \longrightarrow \pi_n(B, \{b_0\}, b_0) = \pi_n(B, b_0) \tag{3.2.14}$$

is a bijection for every $n \geq 1$.

The proof of Lemma 3.2.15 is based on the following technical Lemma:

3.2.16. LEMMA. *Let* $p : E \to B$ *be a locally trivial fibration with typical fiber* F*; then, for* $n \geq 1$*, given continuous maps* $\phi : I^n \to B$ *and* $\psi : J^{n-1} \to E$ *with* $p \circ \psi = \phi|_{J^{n-1}}$*, there exists a continuous map* $\widetilde{\phi}\colon I^n \to E$ *such that* $\widetilde{\phi}|_{J^{n-1}} = \psi$ *and such that the following diagram commutes:*

$$\begin{array}{ccc} & & E \\ & \overset{\widetilde{\phi}}{\nearrow} & \big\downarrow p \\ I^n & \xrightarrow[\phi]{} & B \end{array}$$

PROOF. The proof is split into several steps.

(1) *There exists a* retraction $r : I^n \to J^{n-1}$*, i.e.,* r *is continuous and* $r|_{J^{n-1}} = \mathrm{Id}$.

Fix $\bar{t} = \left(\frac{1}{2}, \dots, \frac{1}{2}, -1\right) \in \mathbb{R}^n$; for each $t \in I^n$ define $r(t)$ as the unique point of J^{n-1} that belongs to the straight line through $\bar{t}$ and t.

(2) *The Lemma holds if there exists a trivialization* (3.2.13) *of* p *with* $\mathrm{Im}(\phi) \subset U$.

Let $\psi_0 \colon J^{n-1} \to F$ be such that

$$\alpha(\psi(t)) = (\phi(t), \psi_0(t)), \quad t \in J^{n-1};$$

then, we consider:

$$\tilde{\phi}(t) = \alpha^{-1}\big(\phi(t), \psi_0(r(t))\big), \quad t \in I^n.$$

(3) *The Lemma holds if* $n = 1$.

Let $0 = u_0 < u_1 < \ldots < u_k = 1$ be a partition of I such that, for $i = 0, \dots, k-1$, $\phi([u_i, u_{i+1}])$ is contained in an open subset of B over which the fibration p is trivial (see the idea of the proof of Theorem 3.1.23); using step (2), define $\widetilde{\phi}$ on the interval $[u_i, u_{i+1}]$ starting with $i = k-1$ and proceeding inductively up to $i = 0$.

(4) *The Lemma holds in general*.

We prove the general case by induction on n; the base of induction is step (3). Suppose then that the Lemma holds for cubes of dimensions less than n. Consider a partition:

$$0 = u_0 < u_1 < \ldots < u_k = 1 \tag{3.2.15}$$

of the interval I; let $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_{n-1})$ be such that for each $i = 1, \dots, n-1$, the set $\mathfrak{a}_i$ is equal to one of the intervals $[u_j, u_{j+1}]$, $j = 0, \dots, k-1$ of the partition (3.2.15), or else $\mathfrak{a}_i$ is equal to one of the points $\{u_j\}$, $j = 1, \dots, k-1$; define:

$$I_{\mathfrak{a}} = I_{\mathfrak{a}_1} \times \cdots \times I_{\mathfrak{a}_{n-1}} \subset I^{n-1}.$$

If $r \in \{0, \dots, n-1\}$ is the number of indices i such that $\mathfrak{a}_i$ is an interval (containing more than one point), we will say that $I_{\mathfrak{a}}$ is a *block of dimension* r. The partition (3.2.15) could have been chosen in such a way that each $\phi\big(I_{\mathfrak{a}} \times [u_j, u_{j+1}]\big)$ is contained in an open subset of B over which the fibration is trivial (see the idea of the proof of Theorem 3.1.23).

Using the induction hypotheses (or step (3)) we define the map $\widetilde{\phi}$ on the subsets $I_{\mathfrak{a}} \times I$ where $I_{\mathfrak{a}}$ is a block of dimension one. We then proceed inductively until when $\widetilde{\phi}$ is defined on each $I_{\mathfrak{a}} \times I$ such that $I_{\mathfrak{a}}$ is a block of dimension $r \leq n-2$.

Fix now $\mathfrak{a}$ in such a way that $I_{\mathfrak{a}}$ is a block of dimension $n-1$; using step (2) we define $\widetilde{\phi}$ on $i_{\mathfrak{a}} \times [u_j, u_{j+1}]$ starting with $j = k-1$ and continuing inductively until $j = 0$. This concludes the proof.

□

The map $\widetilde{\phi}$ in the statement of Lemma 3.2.16 is called a *lifting* of ϕ relatively to p.

PROOF OF LEMMA 3.2.15. Given $[\phi] \in \pi_n(B, b_0)$, by Lemma 3.2.16 there exists a lifting $\widetilde{\phi} : I^n \to E$ of ϕ relatively to p, such that $\widetilde{\phi}$ is constant equal to e_0 on J^{n-1}; then $[\widetilde{\phi}] \in \pi_n(E, E_{b_0}, e_0)$ and $p_*\big([\widetilde{\phi}]\big) = [\phi]$. This shows that p_* is surjective; we now show that p_* is injective.

Let $[\psi_1], [\psi_2] \in \pi_n(E, E_{b_0}, e_0)$ be such that $p_*([\psi_1]) = p_*([\psi_2])$; then, there exists a homotopy

$$H\colon I \times I^n = I^{n+1} \longrightarrow B$$

such that $H_0 = p \circ \psi_1$, $H_1 = p \circ \psi_2$ and $H_s \in \Omega^n_{b_0}(B)$ for every $s \in I$. Observe that:

$$J^n = \big(I \times J^{n-1}\big) \cup (\{0\} \times I^n) \cup (\{1\} \times I^n);$$

we can therefore define a continuous map

$$\psi\colon J^n \longrightarrow E$$

by setting $\psi(0, t) = \psi_1(t)$, $\psi(1, t) = \psi_2(t)$ for $t \in I^n$, and $\psi(s, t) = e_0$ for $s \in I$, $t \in J^{n-1}$. It follows from Lemma 3.2.16 that there exists a continuous map:

$$\tilde{H}\colon I \times I^n = I^{n+1} \longrightarrow E$$

such that $p \circ \tilde{H} = H$ e $\tilde{H}|_{J^n} = \psi$; it is then easy to see that $\tilde{H}\colon \psi_1 \cong \psi_2$ is a homotopy in $\Omega^n_{e_0}(E, E_{b_0})$ and therefore $[\psi_1] = [\psi_2] \in \pi_n(E, E_{b_0}, e_0)$. This concludes the proof. □

The idea now is to "replace" $\pi_n(E, E_{b_0}, e_0)$ by $\pi_n(B, b_0)$ in the long exact homotopy sequence of the pair (E, E_{b_0}), obtaining a new exact sequence. Towards this goal, we consider a locally trivial fibration $p\colon E \to B$ with typical fiber F; choose $b_0 \in B$, $f_0 \in F$, a homeomorphism $\mathfrak{h}\colon E_{b_0} \to F$ and let $e_0 \in E_{b_0}$ be such that $\mathfrak{h}(e_0) = f_0$. We then define maps ι_* e δ_* in such a way that the following diagrams commute:

(3.2.16)

$$\begin{array}{ccc} & \pi_n\big(E_{b_0}, e_0\big) & \\ {\scriptstyle \mathfrak{h}_*\,\cong}\swarrow & & \searrow{\scriptstyle \mathfrak{i}_*} \\ \pi_n(F, f_0) & \xrightarrow[\iota_*]{} & \pi_n(E, e_0) \end{array}$$

(3.2.17)

$$\begin{array}{ccc} \pi_n\big(E, E_{b_0}, e_0\big) & \xrightarrow{\partial_*} & \pi_{n-1}\big(E_{b_0}, e_0\big) \\ {\scriptstyle p_*}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \mathfrak{h}_*} \\ \pi_n(B, b_0) & \xrightarrow[\delta_*]{} & \pi_{n-1}(F, f_0) \end{array}$$

where $\mathfrak{i}_*$ is induced by inclusion, and ∂_* is the connection operator corresponding to the triple (E, E_{b_0}, e_0).

We then obtain the following:

3.2.17. COROLLARY. *Let* $p\colon E\to B$ *be a locally trivial fibration with typical fiber* F*; choosing* $b_0\in B$*,* $f_0\in F$*, a homeomorphism* $\mathfrak{h}\colon E_{b_0}\to F$ *and taking* $e_0\in E_{b_0}$ *such that* $\mathfrak{h}(e_0)=f_0$ *we obtain an exact sequence*

(3.2.18)

$$\cdots\xrightarrow{\delta_*}\pi_n(F,f_0)\xrightarrow{\iota_*}\pi_n(E,e_0)\xrightarrow{p_*}\pi_n(B,b_0)\xrightarrow{\delta_*}\pi_{n-1}(F,f_0)\xrightarrow{\iota_*}$$
$$\cdots\xrightarrow{p_*}\pi_1(B,b_0)\xrightarrow{\delta_*}\pi_0(F,f_0)\xrightarrow{\iota_*}\pi_0(E,e_0)\xrightarrow{p_*}\pi_0(B,b_0)$$

where ι_* *e* δ_* *are defined respectively by the commutative diagrams* (3.2.16) *and* (3.2.17)*.*

PROOF. Everything except for the exactness at $\pi_0(E,e_0)$ follows directly from the long exact sequence of the pair (E,E_{b_0}) and from the definitions of ι_* and δ_*. The exactness at $\pi_0(E,e_0)$ is obtained easily from Lemma 3.2.16 with $n=1$. □

The exact sequence (3.2.18) is known as the *long exact homotopy sequence of the fibration* p*.*

3.2.18. DEFINITION. A map $p:E\to B$ is said to be a *covering* if p is a locally trivial fibration with typical fiber F that is a discrete space.

We have the following:

3.2.19. COROLLARY. *If* $p:E\to B$ *is a covering, then, given* $e_0\in E$ *and* $b_0\in B$ *with* $p(e_0)=b_0$*, the map:*

$$p_*\colon\pi_n(E,e_0)\longrightarrow\pi_n(B,b_0)$$

is an isomorphism for every $n\ge 2$*.*

PROOF. It follows directly from the long exact homotopy sequence of the fibration p, observing that, since F is discrete, it is $\pi_n(F,f_0)=0$ for every $n\ge 1$. □

3.2.20. REMARK. Let $p:E\to B$ be a locally trivial fibration with typical fiber F; choose $b_0\in B$ and a homeomorphism $\mathfrak{h}:E_{b_0}\to F$. Let us take a closer look at the operator δ_* defined by diagram (3.2.17), in the case $n=1$.

For each $f\in F$, we denote by δ_*^f the operator defined by diagram (3.2.17) taking $n=1$ and replacing f_0 by f and e_0 by $\mathfrak{h}^{-1}(f)$ in this diagram. We have the following explicit formula:

$$\delta_*^f([\gamma])=\big[\mathfrak{h}\big(\tilde\gamma(0)\big)\big]\in\pi_0(F,f),\quad \gamma\in\Omega^1_{b_0}(B),\tag{3.2.19}$$

where $\tilde\gamma:I\to E$ is any lifting of γ (i.e., $p\circ\tilde\gamma=\gamma$) with $\tilde\gamma(1)=\mathfrak{h}^{-1}(f)$. The existence of the lifting $\tilde\gamma$ follows from Lemma 3.2.16 with $n=1$.

Using (3.2.19), it is easy to see that δ_*^f only depends on the arc-connected component $[f]$ of F containing f; for, if $f_1,f_2\in F$ and $\lambda:I\to F$ is a continuous curve with $\lambda(0)=f_1$ and $\lambda(1)=f_2$, then, given a lifting $\tilde\gamma$ of γ with $\tilde\gamma(1)=\mathfrak{h}^{-1}(f_1)$, it follows that $\tilde\mu=\tilde\gamma\cdot(\mathfrak{h}^{-1}\circ\lambda)$ is a lifting of $\mu=\gamma\circ\mathfrak{o}_{b_0}$ with $\tilde\mu(1)=\mathfrak{h}^{-1}(f_2)$, and so

$$\delta_*^{f_1}([\gamma])=\big[\mathfrak{h}\big(\tilde\gamma(0)\big)\big]=\big[\mathfrak{h}\big(\tilde\mu(0)\big)\big]=\delta_*^{f_2}([\mu])=\delta_*^{f_2}([\gamma]).$$

Denoting by $\pi_0(F)$ the set of arc-connected components of F (disregarding the distinguished point) we obtain a map

$$\pi_1(B, b_0) \times \pi_0(F) \longrightarrow \pi_0(F) \tag{3.2.20}$$

given by $([\gamma], [f]) \mapsto \delta_*^f([\gamma])$. It follows easily from (3.2.19) that (3.2.20) defines a left action of the group $\pi_1(B, b_0)$ on the set $\pi_0(F)$.

Let us now fix $f_0 \in F$ and let us set $e_0 = \mathfrak{h}^{-1}(f_0)$; using the long exact sequence of the fibration p it follows that the sequence

$$\pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\delta_* = \delta_*^{f_0}} \pi_0(F, f_0) \xrightarrow{\iota_*} \pi_0(E, e_0)$$

is exact. This means that the orbit of the point $[f_0] \in \pi_0(F)$ relatively to the action (3.2.20) is equal to the kernel of ι_* and that the isotropy group of $[f_0]$ is equal to the image of p_*; hence the operator δ_* induces by passing to the quotient a bijection between the set of left cosets $\pi_1(B, b_0)/\mathrm{Im}(p_*)$ and the set $\mathrm{Ker}(\iota_*)$.

3.2.21. EXAMPLE. Let $p : E \to B$ be a locally trivial fibration with discrete typical fiber F, i.e., p is a covering. Choose $b_0 \in B$, $e_0 \in E_{b_0}$ and a homeomorphism $\mathfrak{h} : E_{b_0} \to F$ (actually, in the case of discrete fiber, every bijection $\mathfrak{h}$ will be a homeomorphism); set $f_0 = \mathfrak{h}(e_0)$.

Since $\pi_1(F, f_0) = 0$, it follows from the long exact sequence of the fibration that the map

$$p_* \colon \pi_1(E, e_0) \longrightarrow \pi_1(B, b_0)$$

is injective; we can therefore identify $\pi_1(E, e_0)$ with the image of p_*. Observe that the set $\pi_0(F, f_0)$ may be identified with F.

Under the assumption that E is arc-connected, we have $\pi_0(E, e_0) = 0$, and it follows from Remark 3.2.20 that the map δ_* induces a bijection:

$$\pi_1(B, b_0)/\pi_1(E, e_0) \xrightarrow{\cong} F. \tag{3.2.21}$$

Unfortunately, since F has no group structure, the bijection (3.2.21) does not give any information about the group structures of $\pi_1(E, e_0)$ and $\pi_1(B, b_0)$.

Let us now assume that the fiber F has a group structure and that there exists a continuous right action:

$$E \times F \ni (e, f) \longmapsto e \bullet f \in E \tag{3.2.22}$$

of F on E (since F is discrete, continuity of (3.2.22) in this context means continuity in the second variable); let us also assume that the action (3.2.22) is free, i.e., without fixed points, and that its orbits are the fibers of p. If $f_0 = 1$ is the unit of F and the homeomorphism $\mathfrak{h} : E_{b_0} \to F$ is the inverse of the bijection:

$$\beta_{e_0} \colon F \ni f \longmapsto e_0 \bullet f \in E_{b_0},$$

we will show that the map

$$\delta_* \colon \pi_1(B, b_0) \longrightarrow \pi_0(F, f_0) \cong F \tag{3.2.23}$$

is a group homomorphism; it will then follow that $\mathrm{Im}(p_*) \simeq \pi_1(E, e_0)$ is a normal subgroup of $\pi_1(B, b_0)$ and that the bijection (3.2.21) is an isomorphism of groups.

Let us show that (3.2.23) is a homomorphism. To this aim, let $\gamma, \mu \in \Omega^1_{b_0}(B)$ and let $\tilde{\gamma}, \tilde{\mu} : I \to E$ be lifts of γ and μ respectively, with $\tilde{\gamma}(1) = \tilde{\mu}(1) = e_0$; using (3.2.19) and identifying $\pi_0(F, f_0)$ with F we obtain:

$$\delta_*([\gamma]) = \mathfrak{h}\big(\tilde{\gamma}(0)\big), \quad \delta_*([\mu]) = \mathfrak{h}\big(\tilde{\mu}(0)\big).$$

Define $\hat{\gamma}\colon I \to E$ by setting:

$$\hat{\gamma}(t) = \tilde{\gamma}(t) \bullet \mathfrak{h}\big(\tilde{\mu}(0)\big), \quad t \in I;$$

then $\tilde{\kappa} = \hat{\gamma} \cdot \tilde{\mu}$ is a lifting of $\kappa = \gamma \cdot \mu$ with $\tilde{\kappa}(1) = e_0$ and, using again (3.2.19) we obtain:

$$\delta_*([\gamma] \cdot [\mu]) = \mathfrak{h}\big(\tilde{\kappa}(0)\big) = \mathfrak{h}\big(\hat{\gamma}(0)\big) = \mathfrak{h}(\tilde{\gamma}(0))\, \mathfrak{h}\big(\tilde{\mu}(0)\big) = \delta_*([\gamma])\delta_*([\mu]),$$

which concludes the argument.

3.2.22. REMARK. The groups $\pi_1(X, x_0)$ and $\pi_2(X, A, x_0)$ may not be abelian, in general; however, it can be shown that $\pi_n(X, x_0)$ is always abelian for ≥ 2 and $\pi_n(X, A, x_0)$ is always abelian for $n \geq 3$ (see for instance [**18**, Proposition 2.1, Proposition 3.1, Chapter 4]).

3.2.23. REMARK. Generalizing the result of Proposition 3.1.11, given $n \geq 1$ it is possible to associate to each curve $\lambda : I \to X$ with $\lambda(0) = x_0$ and $\lambda(1) = x_1$ an isomorphism:

$$\lambda_\#\colon \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1);$$

in particular, if x_0 e x_1 belong to the same arc-connected component of X then $\pi_n(X, x_0)$ is isomorphic to $\pi_n(X, x_1)$. The isomorphism $\lambda_\#$ is defined by setting:

$$\lambda_\#([\phi]) = [\psi],$$

where ψ is constructed using a homotopy $H\colon \phi \cong \psi$ such that $H_s(t) = \lambda(t)$ for every $t \in \partial I^n$ and all $s \in I$ (for the details, see [**18**, Theorem 14.1, Chapter 4]). Then, as in Example 3.1.17, it is possible to show that if X is contractible, then $\pi_n(X, x_0) = 0$ for every $n \geq 0$.

If $\mathrm{Im}(\lambda) \subset A \subset X$ then, given $n \geq 1$, we can also define a bijection of pointed sets:

$$\lambda_\#\colon \pi_n(X, A, x_0) \longrightarrow \pi_n(X, A, x_1),$$

which is a group isomorphism for $n \geq 2$ (see [**18**, Exercises of Chapter 4]).

3.2.1. Applications to the theory of classical Lie groups. In this subsection we will use the long exact homotopy sequence of a fibration to compute the fundamental group and the connected components of the classical Lie groups introduced in Subsection 2.1.1. All the spaces considered in this section are differentiable manifolds, hence the notions of connectedness and of arc-connectedness will always be equivalent (see Exercise 3.2).

We will assume familiarity with the concepts and the notions introduced in Subsections 2.1.1 and 2.1.2; in particular, without explicit mention, we will make systematic use of the results of Theorem 2.1.14 and of Corollaries 2.1.9, 2.1.15, 2.1.16 and 2.1.17.

The relative homotopy groups will not be used in this Section; from Section 3.2 the reader is required to keep in mind the Examples 3.2.6, 3.2.10 and 3.2.21, and, obviously, Corollary 3.2.17.

In order to simplify the notation, *we will henceforth omit the specification of the basepoint* x_0 when we refer to a homotopy group, or set, $\pi_n(X, x_0)$, provided that the choice of such basepoint is not relevant in the context (see Corollary 3.1.12 and Remark 3.2.23); therefore, we will write $\pi_n(X)$.

We start with an easy example:

3.2.24. EXAMPLE. Denote by $S^1 \subset \mathbb{C}$ the unit circle; then, the map $p : I\!R \to S^1$ given by $p(t) = e^{2\pi i t}$ is a surjective group homomorphism whose kernel is $\mathrm{Ker}(p) = \mathbb{Z}$. It follows that p is a covering map. Moreover, the action of $\mathbb{Z}$ on $I\!R$ by translation is free, and its orbits are the fibers of p; it follows from Example 3.2.21 that we have an isomorphism:

$$\delta_*\colon \pi_1(S^1, 1) \longrightarrow \mathbb{Z}$$

given by $\delta_*([\gamma]) = \tilde{\gamma}(0)$, where $\tilde{\gamma} : I \to I\!R$ is a lifting of γ such that $\tilde{\gamma}(1) = 0$. In particular, the homotopy class of the loop $\gamma : I \to S^1$ given by:

$$\gamma(t) = e^{2\pi i t}, \quad t \in I, \tag{3.2.24}$$

is a generator of $\pi_1(S^1, 1) \simeq \mathbb{Z}$.

3.2.25. EXAMPLE. Let us show that the special unitary group $\mathrm{SU}(n)$ is (connected and) simply connected. First, observe that the canonical action of the group $\mathrm{SU}(n+1)$ on $\mathbb{C}^{n+1}$ restricts to an action of $\mathrm{SU}(n+1)$ on the unit sphere S^{2n+1}; it is easy to see that this action is transitive, and that the isotropy group of the point $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{C}^{n+1}$ is identified with $\mathrm{SU}(n)$. It follows that the quotient $\mathrm{SU}(n+1)/\mathrm{SU}(n)$ is diffeomorphic to the sphere S^{2n+1}; we therefore have a fibration:

$$p : \mathrm{SU}(n+1) \longrightarrow S^{2n+1},$$

with typical fiber $\mathrm{SU}(n)$. Since the sphere S^{2n+1} is simply connected (see Example 3.1.24), the long exact homotopy sequence of the fibration p gives us:

$$\pi_0(\mathrm{SU}(n)) \longrightarrow \pi_0(\mathrm{SU}(n+1)) \longrightarrow 0 \tag{3.2.25}$$

$$\pi_1(\mathrm{SU}(n)) \longrightarrow \pi_1(\mathrm{SU}(n+1)) \longrightarrow 0. \tag{3.2.26}$$

Since $\mathrm{SU}(1) = \{1\}$ is clearly simply connected, from the exactness of (3.2.25) it follows by induction on n that $\mathrm{SU}(n)$ is connected. Moreover, from the exactness of (3.2.26) it follows by induction on n that $\mathrm{SU}(n)$ is simply connected.

3.2.26. EXAMPLE. Let us show now that the unitary group $\mathrm{U}(n)$ is connected, and that $\pi_1(\mathrm{U}(n)) \simeq \mathbb{Z}$ for every $n \geq 1$. Consider the *determinant map*

$$\det\colon \mathrm{U}(n) \longrightarrow S^1;$$

we have that $\det$ is a surjective homomorphism of Lie groups, and therefore it is a fibration with typical fiber $\mathrm{Ker}(\det) = \mathrm{SU}(n)$. Keeping in mind that $\mathrm{SU}(n)$

is simply connected (see Example 3.2.25), from the fibration $\det$ we obtain the following exact sequence:

$$0 \longrightarrow \pi_0(\mathrm{U}(n)) \longrightarrow 0 \tag{3.2.27}$$

$$0 \longrightarrow \pi_1(\mathrm{U}(n), 1) \xrightarrow{\det_*} \pi_1(S^1, 1) \longrightarrow 0 \tag{3.2.28}$$

From (3.2.27) we conclude that $\mathrm{U}(n)$ is connected, and from (3.2.28) we obtain that the map

$$\det_* \colon \pi_1(\mathrm{U}(n), 1) \xrightarrow{\cong} \pi_1(S^1, 1) \cong \mathbb{Z} \tag{3.2.29}$$

is an isomorphism.

3.2.27. EXAMPLE. We will now show that the special orthogonal group $\mathrm{SO}(n)$ is connected for $n \geq 1$. The canonical action of $\mathrm{SO}(n+1)$ on $\mathbb{R}^{n+1}$ restricts to an action of $\mathrm{SO}(n+1)$ on the unit sphere S^n; it is easy to see that this action is transitive, and that the isotropy group of the point $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ is identified with $\mathrm{SO}(n)$. It follows that the quotient $\mathrm{SO}(n+1)/SO(n)$ is diffeomorphic to the sphere S^n, and we obtain a fibration:

$$p \colon \mathrm{SO}(n+1) \to S^n, \tag{3.2.30}$$

with typical fiber $\mathrm{SO}(n)$; then, we have an exact sequence:

$$\pi_0(\mathrm{SO}(n)) \longrightarrow \pi_0(\mathrm{SO}(n+1)) \longrightarrow 0$$

from which it follows, by induction on n, that $\mathrm{SO}(n)$ is connected for every n (clearly, $\mathrm{SO}(1) = \{1\}$ is connected). The determinant map induces an isomorphism between the quotient $\mathrm{O}(n)/\mathrm{SO}(n)$ and the group $\{1, -1\} \simeq \mathbb{Z}_2$, from which it follows that $\mathrm{O}(n)$ has precisely two connected components: $\mathrm{SO}(n)$ and its complementary.

3.2.28. EXAMPLE. We now show that the group $\mathrm{GL}_+(n, \mathbb{R})$ is connected. If we choose any basis $(b_i)_{i=1}^n$ of $\mathbb{R}^n$, it is easy to see that there exists a unique orthonormal basis $(u_i)_{i=1}^n$ of $\mathbb{R}^n$ such that, for every $k = 1, \ldots, n$, the vectors $(b_i)_{i=1}^k$ and $(u_i)_{i=1}^k$ are a basis of the same k-dimensional subspace of $\mathbb{R}^n$ and define the *same orientation* of this subspace. The vectors $(u_i)_{i=1}^n$ can be written explicitly in terms of the $(b_i)_{i=1}^n$; such formula is known as the *Gram–Schmidt orthogonalization process*.

Given any invertible matrix $A \in \mathrm{GL}(n, \mathbb{R})$, we denote by $r(A)$ the matrix obtained from A by an application of the Gram-Schmidt orthogonalization process on its columns; the map r from $\mathrm{GL}(n, \mathbb{R})$ onto $\mathrm{O}(n)$ is differentiable (but it is not a homomorphism). Observe that if $A \in \mathrm{O}(n)$, then $r(A) = A$; for this we say that r is a *retraction*. Denote by T_+ the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of upper triangular matrices with positive entries on the diagonal, i.e.,

$$\mathrm{T}_+ = \{(a_{ij})_{n \times n} \in \mathrm{GL}(n, \mathbb{R}) : a_{ij} = 0 \text{ if } i > j,\ a_{ii} > 0,\ i, j = 1, \ldots, n\}.$$

Then, it is easy to see that we obtain a diffeomorphism:

$$\mathrm{GL}(n, \mathbb{R}) \ni A \longmapsto \big(r(A), r(A)^{-1}A\big) \in \mathrm{O}(n) \times \mathrm{T}_+. \tag{3.2.31}$$

We have that (3.2.31) restricts to a diffeomorphism of $\mathrm{GL}_+(n,\mathbb{R})$ onto $\mathrm{SO}(n) \times T_+$. It follows from Example 3.2.27 that $\mathrm{GL}_+(n,\mathbb{R})$ is connected, and that the general linear group $\mathrm{GL}(n,\mathbb{R})$ has two connected components: $\mathrm{GL}_+(n,\mathbb{R})$ and its complementary.

3.2.29. REMARK. Actually, it is possible to show that $\mathrm{GL}_+(n,\mathbb{R})$ is connected by an elementary argument, using the fact that every invertible matrix can be written as the product of matrices corresponding to *elementary row operations*. Then, the map $r : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{O}(n)$ defined as in Example 3.2.28 gives us an alternative proof of the connectedness of $\mathrm{SO}(n)$.

3.2.30. EXAMPLE. We will now show that the group $\mathrm{GL}(n,\mathbb{C})$ is connected and that:

$$\pi_1(\mathrm{GL}(n,\mathbb{C})) \cong \mathbb{Z}.$$

We use the same idea as in Example 3.2.28; observe that it is possible to define a Gram-Schmidt orthonormalization process also for bases of $\mathbb{C}^n$. Then, we obtain a diffeomorphism:

$$\mathrm{GL}(n,\mathbb{C}) \ni A \longmapsto \big(r(A), r(A)^{-1}A\big) \in \mathrm{U}(n) \times \mathrm{T}_+(\mathbb{C}),$$

where $T_+(\mathbb{C})$ denotes the subgroup of $\mathrm{GL}(n,\mathbb{C})$ consisting of those upper triangular matrices having positive real entries on the diagonal:

$$\begin{aligned}\mathrm{T}_+(\mathbb{C}) = \{(a_{ij})_{n\times n} \in \mathrm{GL}(n,\mathbb{C}) : a_{ij} = 0 \text{ if } i > j,\\ a_{ii} \in \mathbb{R} \text{ and } a_{ii} > 0,\ i,j = 1,\ldots,n\}.\end{aligned}$$

It follows from Example 3.2.26 that $\mathrm{GL}(n,\mathbb{C})$ is connected and that $\pi_1(\mathrm{GL}(n,\mathbb{C}))$ is isomorphic to $\mathbb{Z}$ for $n \geq 1$; more explicitly, we have that the inclusion $\mathfrak{i}\colon \mathrm{U}(n) \to \mathrm{GL}(n,\mathbb{C})$ induces an isomorphism:

$$\mathfrak{i}_*\colon \pi_1(\mathrm{U}(n),1) \xrightarrow{\cong} \pi_1(\mathrm{GL}(n,\mathbb{C}),1).$$

3.2.31. REMARK. Also the connectedness of $\mathrm{GL}(n,\mathbb{C})$ can be proven by a simpler method, using *elementary row reduction* of matrices. Then, the Gram-Schmidt orthonormalization process gives us an alternative proof of the connectedness of $\mathrm{U}(n)$ (see Remark 3.2.29).

3.2.32. EXAMPLE. We will now consider the groups $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})$. We have a Lie group isomorphism:

$$\mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^+ \ni (T,c) \longmapsto c\,T \in \mathrm{GL}_+(n,\mathbb{R}),$$

where $\mathbb{R}^+ = \,]0,+\infty[$ is seen as a multiplicative group; it follows from Example 3.2.28 that $\mathrm{SL}(n,\mathbb{R})$ is connected, and that the inclusion $\mathfrak{i}\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}_+(n,\mathbb{R})$ induces an isomorphism:

$$\mathfrak{i}_*\colon \pi_1(\mathrm{SL}(n,\mathbb{R}),1) \xrightarrow{\cong} \pi_1(\mathrm{GL}_+(n,\mathbb{R}),1).$$

The group $\pi_1(\mathrm{GL}_+(n,\mathbb{R}))$ will be computed in Example 3.2.35 ahead.

Let us look now at the complex case: for $z \in \mathbb{C} \setminus \{0\}$ we define the diagonal matrix:

$$\sigma(z) = \begin{pmatrix} z & & & \\ & 1 & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & 1 \end{pmatrix} \in \mathrm{GL}(n, \mathbb{C});$$

we then obtain a diffeomorphism (which is *not* an isomorphism):

$$\mathrm{SL}(n, \mathbb{C}) \times \mathbb{R}^{+} \times S^1 \ni (T, c, z) \longmapsto \sigma(cz)T \in \mathrm{GL}(n, \mathbb{C}).$$

Then, it follows from Example 3.2.30 that $\mathrm{SL}(n, \mathbb{C})$ is connected and that:

$$\pi_1(\mathrm{GL}(n, \mathbb{C})) \cong \mathbb{Z} \cong \mathbb{Z} \times \pi_1(\mathrm{SL}(n, \mathbb{C})),$$

from which we get that $\mathrm{SL}(n, \mathbb{C})$ is simply connected.

In order to compute the fundamental group of the special orthogonal group $\mathrm{SO}(n)$ we need the following result:

3.2.33. LEMMA. *If $S^n \subset \mathbb{R}^{n+1}$ denotes the unit sphere, then, for every $x_0 \in S^n$, we have $\pi_k(S^n, x_0) = 0$ for $0 \le k < n$.*

PROOF. Let $\phi \in \Omega^k_{x_0}(S^n)$. If ϕ is not surjective, then there exists $x \in S^n$ with $\mathrm{Im}(\phi) \subset S^n \setminus \{x\}$; but $S^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n$ by the stereographic projection, hence $[\phi] = [\mathfrak{o}_{x_0}]$. It remains to show that any $\phi \in \Omega^k_{x_0}(S^n)$ is homotopic in $\Omega^k_{x_0}(S^n)$ to a map which is not surjective.

Let $\varepsilon > 0$ be fixed; it is known that there exists a differentiable[1] map $\psi\colon I^k \to \mathbb{R}^{n+1}$ such that $\|\phi(t) - \psi(t)\| < \varepsilon$ for every $t \in I^k$ (see [**23**, Teorema 10, §5, Capítulo 7]). Let $\xi\colon \mathbb{R} \to [0, 1]$ be a differentiable map such that $\xi(s) = 0$ for $|s| \le \varepsilon$ and $\xi(s) = 1$ for $|s| \ge 2\varepsilon$. Define $\rho\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by setting

$$\rho(x) = \xi(\|x - x_0\|)(x - x_0) + x_0, \quad x \in \mathbb{R}^{n+1};$$

then, ρ is differentiable in $\mathbb{R}^{n+1}$, $\rho(x) = x_0$ for $\|x - x_0\| \le \varepsilon$ and $\|\rho(x) - x\| \le 2\varepsilon$ for every $x \in \mathbb{R}^{n+1}$. It follows that $\rho \circ \psi\colon I^k \to \mathbb{R}^{n+1}$ is a differentiable map $(\rho \circ \psi)(\partial I^k) \subset \{x_0\}$ and $\|(\rho \circ \psi)(t) - \phi(t)\| \le 3\varepsilon$ for every $t \in I^k$. Choosing $\varepsilon > 0$ with $3\varepsilon < 1$, then we can define a homotopy $H\colon \phi \cong \theta$ in $\Omega^k_{x_0}(S^n)$ by setting:

$$H_s(t) = \frac{(1 - s)\phi(t) + s(\rho \circ \psi)(t)}{\|(1 - s)\phi(t) + s(\rho \circ \psi)(t)\|}, \quad t \in I^k,\ s \in I,$$

where $\theta(t) = (\rho \circ \psi)(t)/\|(\rho \circ \psi)(t)\|$, $t \in I^k$, is a differentiable map; since $k < n$, it follows that θ cannot be surjective, because its image has null measure in S^n (see [**23**, §2, Capítulo 6]). This concludes the proof. □

[1] The differentiability of a map ψ defined in a non necessarily open subset of $\mathbb{R}^k$ means that the map ψ admits a differentiable extension to some open subset containing its domain.

3.2.34. EXAMPLE. The group $\mathrm{SO}(1)$ is trivial, therefore it is simply connected; the group $\mathrm{SO}(2)$ is isomorphic to the unit circle S^1 (see Example 3.2.27, hence:

$$\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}.$$

For $n \geq 3$, Lemma 3.2.33 tells us that $\pi_2(S^n) = 0$, and so the long exact homotopy sequence of the fibration (3.2.30) becomes:

$$0 \longrightarrow \pi_1(\mathrm{SO}(n), 1) \xrightarrow[\cong]{\mathfrak{i}_*} \pi_1(SO(n+1), 1) \longrightarrow 0$$

where $\mathfrak{i}_*$ is induced by the inclusion $\mathfrak{i}\colon \mathrm{SO}(n) \to \mathrm{SO}(n+1)$; it follows that $\pi_1(\mathrm{SO}(n))$ is isomorphic to $\pi_1(\mathrm{SO}(n+1))$. We will show next that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$, from which it will then follow that

$$\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2, \quad n \geq 3.$$

Consider the inner product g in the Lie algebra $\mathrm{su}(2)$ defined by

$$g(X, Y) = \mathrm{tr}(XY^*), \quad X, Y \in \mathrm{su}(2),$$

where Y^* denotes here the conjugate transpose of the matrix Y and $\mathrm{tr}(U)$ denotes the trace of the matrix U; consider the *adjoint representation* of $\mathrm{SU}(2)$:

$$\mathrm{Ad}\colon \mathrm{SU}(2) \longrightarrow \mathrm{SO}(\mathrm{su}(2), g) \tag{3.2.32}$$

given by $\mathrm{Ad}(A) \cdot X = AXA^{-1}$ for $A \in \mathrm{SU}(2)$, $X \in \mathrm{su}(2)$; it is easy to see that the linear endomorphism $\mathrm{Ad}(A)$ of $\mathrm{su}(2)$ is actually g-orthogonal for every $A \in \mathrm{SU}(2)$ and that (3.2.32) is a Lie group homomorphism. Clearly, $\mathrm{SO}(\mathrm{su}(2), g)$ is isomorphic to $\mathrm{SO}(3)$.

An explicit calculation shows that $\mathrm{Ker}(\mathrm{Ad}) = \{\mathrm{Id}, -\mathrm{Id}\}$, and since the domain and the counterdomain of (3.2.32) have the same dimension, it follows that the image of (3.2.32) is an open subgroup of $\mathrm{SO}(\mathrm{su}(2), g)$; since $\mathrm{SO}(\mathrm{su}(2), g)$ is connected (Example 3.2.27), we conclude that (3.2.32) is surjective, and so it is a covering map. Since $\mathrm{SU}(2)$ is simply connected (Example 3.2.25), it follows from Example 3.2.21 that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$, keeping in mind the action of $\mathbb{Z}_2 \cong \{\mathrm{Id}, -\mathrm{Id}\}$ on $\mathrm{SU}(2)$ by translation. The non trivial element of $\pi_1(\mathrm{SO}(3), 1)$ coincides with the homotopy class of any loop of the form $\mathrm{Ad}\circ\gamma$, where $\gamma\colon I \to \mathrm{SU}(2)$ is a curve joining Id and $-\mathrm{Id}$.

3.2.35. EXAMPLE. The diffeomorphism (3.2.31) shows that the inclusion $\mathfrak{i}$ of $\mathrm{SO}(n)$ into $\mathrm{GL}_+(n, I\!R)$ induces an isomorphism:

$$\mathfrak{i}_*\colon \pi_1(\mathrm{SO}(n), 1) \xrightarrow{\cong} \pi_1(\mathrm{GL}_+(n, I\!R), 1). \tag{3.2.33}$$

It follows from Example 3.2.34 that $\pi_1(\mathrm{GL}_+(n, I\!R))$ is trivial for $n = 1$, it is isomorphic to $\mathbb{Z}$ for $n = 2$, and it is isomorphic to $\mathbb{Z}_2$ for $n \geq 3$.

3.2.36. EXAMPLE. We will now look at the symplectic group $\mathrm{Sp}(2n, I\!R)$ and we will show that it is connected for every $n \geq 1$. Let ω be the canonical symplectic form of $I\!R^{2n}$ and let Λ_+ be the *Grassmannian of oriented Lagrangians* of the symplectic space $(I\!R^{2n}, \omega)$, that is:

$$\Lambda_+ = \Big\{(L, \mathcal{O}) : L \subset I\!R^{2n} \text{ is Lagrangian, and } \mathcal{O} \text{ is an orientation of } L\Big\}.$$

We have an action of the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ on the set Λ_+ given by $T\circ(L,\mathcal{O})=(T(L),\mathcal{O}')$, where $\mathcal{O}'$ is the unique orientation of $T(L)$ that makes $T|_L : L\to T(L)$ positively oriented.

By Remark 1.4.29, we have that the restriction of this action to the unitary group $\mathrm{U}(n)$ is transitive. Consider the Lagrangian $L_0=\mathbb{R}^n\oplus\{0\}$ and let $\mathcal{O}$ be the orientation of L_0 corresponding to the canonical basis of $\mathbb{R}^n$; then, the isotropy group of $(L_0,\mathcal{O})$ relative to the action of $\mathrm{U}(n)$ is $\mathrm{SO}(n)$. The isotropy group of $(L_0,\mathcal{O})$ relative to the action of $\mathrm{Sp}(2n,\mathbb{R})$ will be denoted by $\mathrm{Sp}_+(2n,\mathbb{R},L_0)$. In formulas (1.4.7) and (1.4.8) we have given an explicit description of the matrix representations of the elements of $\mathrm{Sp}(2n,\mathbb{R})$; using these formulas it is easy to see that $\mathrm{Sp}_+(2n,\mathbb{R},L_0)$ consists of matrices of the form:

(3.2.34) $$T=\begin{pmatrix} A & AS\\ 0 & A^{*-1}\end{pmatrix},\ A\in\mathrm{GL}_+(n,\mathbb{R}),\ S\ n\times n \text{ symmetric matrix},$$

where A^* denotes the transpose of A. It follows that we have a diffeomorphism:

(3.2.35) $$\mathrm{Sp}_+(2n,\mathbb{R},L_0)\ni T\longmapsto(A,S)\in\mathrm{GL}_+(n,\mathbb{R})\times\mathrm{B}_{\mathrm{sym}}(\mathbb{R}^n)$$

where A and S are defined by (3.2.34). We have the following commutative diagrams of bijections:

$$\begin{array}{ccc} \mathrm{U}(n)/\mathrm{SO}(n) & \xrightarrow{\ \bar{\mathfrak{i}}\ } & \mathrm{Sp}(2n,\mathbb{R})/\mathrm{Sp}_+(2n,\mathbb{R},L_0)\\ & {\scriptstyle\beta_1}\searrow \quad \swarrow{\scriptstyle\beta_2} & \\ & \Lambda_+ & \end{array}$$

where the maps β_1 and β_2 are induced respectively by the actions of $\mathrm{U}(n)$ and of $\mathrm{Sp}(2n,\mathbb{R})$ on Λ_+ and $\bar{\mathfrak{i}}$ is induced by the inclusion $\mathfrak{i}\colon \mathrm{U}(n)\to\mathrm{Sp}(2n,\mathbb{R})$ by passage to the quotient; we have that $\bar{\mathfrak{i}}$ is a diffeomorphism. Hence, we have a fibration:

(3.2.36) $$p\colon \mathrm{Sp}(2n,\mathbb{R})\longrightarrow\mathrm{Sp}(2n,\mathbb{R})/\mathrm{Sp}_+(2n,\mathbb{R},L_0)\cong\mathrm{U}(n)/\mathrm{SO}(n)$$

whose typical fiber is $\mathrm{Sp}_+(2n,\mathbb{R},L_0)\cong\mathrm{GL}_+(n,\mathbb{R})\times\mathrm{B}_{\mathrm{sym}}(\mathbb{R}^n)$. By Example 3.2.28 this typical fiber is connected, and by Example 3.2.26 the base manifold $\mathrm{U}(n)/\mathrm{SO}(n)$ is connected. It follows now easily from the long exact homotopy sequence of the fibration (3.2.36) that the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is connected.

3.2.37. EXAMPLE. We will now show that the fundamental group of the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is isomorphic to $\mathbb{Z}$. Using the exact sequence of the fibration (3.2.36) and the diffeomorphism (3.2.35), we obtain an exact sequence:

(3.2.37) $$\pi_1(\mathrm{GL}_+(n,\mathbb{R}))\xrightarrow{\ \iota_*\ }\pi_1(\mathrm{Sp}(2n,\mathbb{R}))\xrightarrow{\ p_*\ }\pi_1\big(\mathrm{U}(n)/\mathrm{SO}(n)\big)\longrightarrow 0$$

where ι_* is induced by the map $\iota\colon \mathrm{GL}_+(n,\mathbb{R})\to\mathrm{Sp}(2n,\mathbb{R})$ given by:

$$\iota(A)=\begin{pmatrix} A & 0\\ 0 & A^{*-1}\end{pmatrix},\quad A\in\mathrm{GL}_+(n,\mathbb{R}).$$

We will show first that the map ι_* is the null map; we have the following commutative diagram (see (3.2.29) and (3.2.33)):

$$\begin{array}{ccccc} \pi_1(\mathrm{SO}(n)) & \longrightarrow & \pi_1(\mathrm{U}(n)) & \xrightarrow[\cong]{\det_*} & \pi_1(S^1) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow & & \\ \pi_1(\mathrm{GL}_+(n,\mathbb{R})) & \xrightarrow[\iota_*]{} & \pi_1(\mathrm{Sp}(2n,\mathbb{R})) & & \end{array} \qquad (\pi_1(\mathrm{SO}(n)) \xrightarrow{0} \pi_1(S^1)) \tag{3.2.38}$$

where the unlabeled arrows are induced by inclusion.[2] A simple analysis of the diagram (3.2.38) shows that $\iota_* = 0$.

Now, the exactness of the sequence (3.2.37) implies that p_* is an isomorphism of $\pi_1(\mathrm{Sp}(2n,\mathbb{R}))$ onto the group $\pi_1\big(\mathrm{U}(n)/\mathrm{SO}(n)\big)$; let us compute this group. Consider the quotient map:

$$q\colon \mathrm{U}(n) \longrightarrow \mathrm{U}(n)/\mathrm{SO}(n);$$

we have that q is a fibration. We obtain a commutative diagram:

$$\begin{array}{ccccccc} \pi_1(\mathrm{SO}(n)) & \longrightarrow & \pi_1(\mathrm{U}(n)) & \xrightarrow{q_*} & \pi_1\big(\mathrm{U}(n)/\mathrm{SO}(n)\big) & \longrightarrow & 0 \\ & {\scriptstyle 0}\searrow & {\scriptstyle\cong}\big\downarrow{\scriptstyle\det_*} & & & & \\ & & \pi_1(S^1) & & & & \end{array} \tag{3.2.39}$$

The upper horizontal line in (3.2.39) is a portion of the homotopy exact sequence of the fibration q; it follows that q_* is an isomorphism. Finally, denoting by $\mathfrak{i}$ the inclusion of $\mathrm{U}(n)$ in $\mathrm{Sp}(2n,\mathbb{R})$ we obtain a commutative diagram:

$$\begin{array}{ccccc} & & \pi_1(\mathrm{Sp}(2n,\mathbb{R})) & & \\ & {\scriptstyle\mathfrak{i}_*}\nearrow & & \searrow{\scriptstyle p_*,\ \cong} & \\ \pi_1(\mathrm{U}(n)) & & \xrightarrow[q_*]{\cong} & & \pi_1\big(\mathrm{U}(n)/\mathrm{SO}(n)\big) \end{array}$$

from which it follows that $\mathfrak{i}_*$ is an isomorphism:

$$\mathbb{Z} \cong \pi_1(\mathrm{U}(n),1) \xrightarrow[\cong]{\mathfrak{i}_*} \pi_1(\mathrm{Sp}(2n,\mathbb{R}),1).$$

3.3. Singular Homology Groups

In this section we will give a brief exposition of the definition and the basic properties of the group of (relative and absolute) singular homology of a topological space X; we will describe the homology exact sequence of a pair of topological spaces.

For all $p \geq 0$, we will denote by $(e_i)_{i=1}^p$ the canonical basis of $\mathbb{R}^p$ and by e_0 the zero vector of $\mathbb{R}^p$; by $\mathbb{R}^0$ we will mean the trivial space $\{0\}$. Observe that, with this notations, we will have a small ambiguity due to the fact that, if $q \geq p \geq i$,

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[2]The inclusion of $\mathrm{U}(n)$ into $\mathrm{Sp}(2n,\mathbb{R})$ depends on the identification of $n\times n$ complex matrices with $2n \times 2n$ real matrices; see Remark 1.2.9.

the symbol e_i will denote at the same time a vector of $\mathbb{R}^p$ and also a vector of $\mathbb{R}^q$; however, this ambiguity will be of a harmless sort and, if necessary, the reader may consider identifications $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^\infty$.

Given $p \geq 0$, the *p-th standard simplex* is defined as the convex hull Δ_p of the set $\{e_i\}_{i=0}^p$ in $\mathbb{R}^p$; more explicitly:

$$\Delta_p = \Big\{ \sum\nolimits_{i=0}^p t_i e_i : \sum\nolimits_{i=0}^p t_i = 1,\ t_i \geq 0,\ i = 0, \ldots, p \Big\}.$$

Observe that Δ_0 is simply the point $\{0\}$ and Δ_1 is the unit interval $I = [0,1]$. Let us fix some terminology concerning the concepts related to free abelian groups:

3.3.1. DEFINITION. If G is an abelian group, then a *basis*[3] of G is a family $(b_\alpha)_{\alpha \in \mathcal{A}}$ such that every $g \in G$ is written uniquely in the form $g = \sum_{\alpha \in \mathcal{A}} n_\alpha b_\alpha$, where each n_α is in $\mathbb{Z}$ and $n_\alpha = 0$ except for a finite number of indices $\alpha \in \mathcal{A}$. If G' is another abelian group, then a homomorphism $f : G \to G'$ is uniquely determined when we specify its values on the elements of some basis of G. An abelian group that admits a basis is said to be *free*.

If $\mathcal{A}$ is any set, the *free abelian group $G_{\mathcal{A}}$ generated by $\mathcal{A}$* is the group of all “almost zero” maps $N : \mathcal{A} \to \mathbb{Z}$, i.e., $N(\alpha) = 0$ except for a finite number of indices $\alpha \in \mathcal{A}$; the sum in $G_{\mathcal{A}}$ is defined in the obvious way: $(N_1 + N_2)(\alpha) = N_1(\alpha) + N_2(\alpha)$. We then identify each $\alpha \in A$ with the function $N_\alpha \in G_{\mathcal{A}}$ defined by $N_\alpha(\alpha) = 1$ and $N_\alpha(\beta) = 0$ for every $\beta \neq \alpha$. Then, G_α is indeed a free abelian group, and $\mathcal{A} \subset G_{\mathcal{A}}$ is a basis of $G_{\mathcal{A}}$.

3.3.2. DEFINITION. For $p \geq 0$, a *singular p-simplex* is an arbitrary continuous map:

$$T \colon \Delta_p \longrightarrow X.$$

We denote by $\mathfrak{S}_p(X)$ the free abelian group generated by the set of all singular p-simplexes in X; the elements in $\mathfrak{S}_p(X)$ are called *singular p-chains*.

If $p = 0$, we identify the singular p-simplexes in X with the points of X, and $\mathfrak{S}_0(X)$ is the free abelian group generated by X. If $p < 0$, our convention will be that $\mathfrak{S}_p(X) = \{0\}$.

Each singular p-chain can be written as:

$$c = \sum_{\substack{T \text{ singular} \\ p\text{-simplex}}} n_T \cdot T,$$

where $n_T \in \mathbb{Z}$ and $n_T = 0$ except for a finite number of singular p-simplexes; the coefficients n_T are uniquely determined by c.

Given a finite dimensional vector space V and given $v_0, \ldots, v_p \in V$, we will denote by $\ell(v_0, \ldots, v_p)$ the singular p-simplex in V defined by:

$$\ell(v_0, \ldots, v_p)\Big(\sum_{i=0}^p t_i e_i\Big) = \sum_{i=0}^p t_i v_i, \tag{3.3.1}$$

[3]An abelian group is a $\mathbb{Z}$-module, and our definition of basis for an abelian group coincides with the usual definition of basis for a module over a ring.

where each $t_i \geq 0$ and $\sum_{i=0}^{p} t_i = 1$; observe that $\ell(v_0, \ldots, v_p)$ is the *unique affine function that takes e_i into v_i for every $i = 0, \ldots, p$.*

For each $p \in \mathbb{Z}$, we will now define a homomorphism:

$$\partial_p \colon \mathfrak{S}_p(X) \longrightarrow \mathfrak{S}_{p-1}(X).$$

If $p \leq 0$ we set $\partial_p = 0$. For $p > 0$, since $\mathfrak{S}_p(X)$ is free, it suffices to define ∂_p on a basis of $\mathfrak{S}_p(X)$; we then define $\partial_p(T)$ when T is a singular p-simplex in X by setting:

$$\partial_p(T) = \sum_{i=0}^{p} (-1)^i \, T \circ \ell(e_0, \ldots, \widehat{e_i}, \ldots, e_p),$$

where $\widehat{e_i}$ means that the term e_i is omitted in the sequence.

For each $i = 0, \ldots, p$, the image of the singular $(p-1)$-simplex in $\mathbb{R}^p$ $\ell(e_0, \ldots, \widehat{e_i}, \ldots, e_p)$ can be visualized as the face of the standard simplex Δ_p which is *opposite* to the vertex e_i.

If $c \in \mathfrak{S}_p(X)$ is a singular p-chain, we say that $\partial_p(c)$ is its *boundary*; observe that if $T : [0,1] \to X$ is a singular 1-simplex, then $\partial_1(T) = T(1) - T(0)$.

We have thus obtained a sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{\partial_{p+1}} \mathfrak{S}_p(X) \xrightarrow{\partial_p} \mathfrak{S}_{p-1}(X) \xrightarrow{\partial_{p-1}} \cdots \tag{3.3.2}$$

The sequence (3.3.2) has the property that the composition of two consecutive arrows vanishes:

3.3.3. LEMMA. *For all $p \in \mathbb{Z}$, we have $\partial_{p-1} \circ \partial_p = 0$.*

PROOF. If $p \leq 1$ the result is trivial; for the case $p \geq 2$ it suffices to show that $\partial_{p-1}(\partial_p(T)) = 0$ for every singular p-simplex T. Observing that

$$\ell(v_0, \ldots, v_q) \circ \ell(e_0, \ldots, \widehat{e_i}, \ldots, e_q) = \ell(v_0, \ldots, \widehat{v_i}, \ldots, v_q)$$

we compute as follows:

$$\begin{aligned} \partial_{p-1}(\partial_p(T)) = & \sum_{j<i} (-1)^{j+i} \, T \circ \ell(e_0, \ldots, \widehat{e_j}, \ldots, \widehat{e_i}, \ldots, e_p) \\ & + \sum_{j>i} (-1)^{j+i-1} \, T \circ \ell(e_0, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, \ldots, e_p) = 0. \end{aligned}$$ □

Let us give the following general definition:

3.3.4. DEFINITION. A *chain complex* is a family $\mathfrak{C} = (\mathfrak{C}_p, \delta_p)_{p \in \mathbb{Z}}$ where each $\mathfrak{C}_p$ is an abelian group, and each $\delta_p \colon \mathfrak{C}_p \to \mathfrak{C}_{p-1}$ is a homomorphism such that $\delta_{p-1} \circ \delta_p = 0$ for every $p \in \mathbb{Z}$. For each $p \in \mathbb{Z}$ we define:

$$Z_p(\mathfrak{C}) = \mathrm{Ker}(\delta_p), \quad B_p(\mathfrak{C}) = \mathrm{Im}(\delta_{p+1}),$$

and we say that $Z_p(\mathfrak{C})$, $B_p(\mathfrak{C})$ are respectively the *group of p-cycles* and the *group of p-boundaries* of the complex $\mathfrak{C}$. Clearly, $B_p(\mathfrak{C}) \subset Z_p(\mathfrak{C})$, and we can therefore define:

$$H_p(\mathfrak{C}) = Z_p(\mathfrak{C}) / B_p(\mathfrak{C});$$

we say that $H_p(\mathfrak{C})$ is the *p-th homology group* of the complex $\mathfrak{C}$.

If $c \in Z_p(\mathfrak{C})$ is a p-cycle, we denote by $c + B_p(\mathfrak{C})$ its equivalence class in $H_p(\mathfrak{C})$; we say that $c + B_p(\mathfrak{C})$ is the *homology class* determined by c. If $c_1, c_2 \in Z_p(\mathfrak{C})$ determine the same homology class (that is, if $c_1 - c_2 \in B_p(\mathfrak{C})$) we say that c_1 and c_2 are *homologous* cycles.

Lemma 3.3.3 tells us that $\mathfrak{S}(X) = (\mathfrak{S}_p(X), \partial_p)_{p\in\mathbb{Z}}$ is a chain complex; we say that $\mathfrak{S}(X)$ is the *singular complex* of the topological space X. We write:

$$Z_p(\mathfrak{S}(X)) = Z_p(X), \quad B_p(\mathfrak{S}(X)) = B_p(X), \quad H_p(\mathfrak{S}(X)) = H_p(X);$$

and we call $Z_p(X)$, $B_p(X)$ and $H_p(X)$ respectively the *group of singular p-cycles*, the *group of singular p-boundaries* and the *p-th singular homology group* of the topological space X.

Clearly, $H_p(X) = 0$ for $p < 0$ and $H_0(X) = \mathfrak{S}_0(X)/B_0(X)$.

We define a homomorphism

$$\varepsilon\colon \mathfrak{S}_0(X) \longrightarrow \mathbb{Z} \tag{3.3.3}$$

by setting $\varepsilon(x) = 1$ for every singular 0-simplex $x \in X$. It is easy to see that $\varepsilon \circ \partial_1 = 0$; for, it suffices to see that $\varepsilon(\partial_1(T)) = 0$ for every singular 1-simplex T in X. We therefore obtain a chain complex:

$$\begin{aligned} &\cdots \xrightarrow{\partial_{p+1}} \mathfrak{S}_p(X) \xrightarrow{\partial_p} \mathfrak{S}_{p-1}(X) \xrightarrow{\partial_{p-1}} \\ &\cdots \xrightarrow{\partial_1} \mathfrak{S}_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \end{aligned} \tag{3.3.4}$$

3.3.5. DEFINITION. The homomorphism (3.3.3) is called the *augmentation map* of the singular complex $\mathfrak{S}(X)$; the chain complex in (3.3.4), denoted by $(\mathfrak{S}(X), \varepsilon)$ is called the *augmented singular complex* of the space X. The groups of p-cycles, of p-boundaries and the p-th homology group of $(\mathfrak{S}(X), \varepsilon)$ are denoted by $\tilde{Z}_p(X)$, $\tilde{B}_p(X)$ and $\tilde{H}_p(X)$ respectively; we say that $\tilde{H}_p(X)$ is the *p-th reduced singular homology group* of X.

Clearly, for $p \geq 1$ we have:

$$\tilde{Z}_p(X) = Z_p(X), \quad \tilde{B}_p(X) = B_p(X), \quad \tilde{H}_p(X) = H_p(X).$$

From now on we will no longer specify the index p in the operator ∂_p, and we will write more concisely:

$$\partial_p = \partial, \quad p \in \mathbb{Z}.$$

3.3.6. EXAMPLE. If $X = \emptyset$ is the empty set, then obviously $\mathfrak{S}_p(X) = 0$ for every $p \in \mathbb{Z}$, hence $H_p(X) = 0$ for every p, and $\tilde{H}_p(X) = 0$ for every $p \neq -1$; on the other hand, we have $\tilde{H}_{-1}(X) = \mathbb{Z}$.

If X is non empty, then any singular 0-simplex $x_0 \in X$ is such that $\varepsilon(x_0) = 1$, and so ε is surjective; it follows that $\tilde{H}_{-1}(X) = 0$. Concerning the relation between $H_0(X)$ and $\tilde{H}_0(X)$, it is easy to see that we can identify $\tilde{H}_0(X)$ with a subgroup of $H_0(X)$, and that

$$H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}\cdot(x_0 + B_0(X)) \cong \tilde{H}_0(X) \oplus \mathbb{Z},$$

where $\mathbb{Z}\cdot(x_0 + B_0(X))$ is the subgroup (infinite cyclic) generated by the homology class of x_0 in $H_0(X)$.

3.3.7. EXAMPLE. If X is arc-connected and not empty , then any two singular 0-simplexes $x_0, x_1 \in X$ are homologous; indeed, if $T : [0,1] \to X$ is a continuous curve from x_0 to x_1, then T is a singular 1-simplex and $\partial T = x_1 - x_0 \in B_0(X)$. It follows that the homology class of any $x_0 \in X$ generates $H_0(X)$, and since $\varepsilon(x_0) = 1$, it follows that no non zero multiple of x_0 is a boundary; therefore:

$$H_0(X) \cong \mathbb{Z}, \quad \tilde{H}_0(X) = 0.$$

3.3.8. EXAMPLE. If X is not arc-connected, we can write $X = \bigcup_{\alpha\in\mathcal{A}} X_\alpha$, where each X_α is an arc-connected component of X. Then, every singular simplex in X has image contained in some X_α and therefore:

$$\mathfrak{S}_p(X) = \bigoplus_{\alpha\in\mathcal{A}} \mathfrak{S}_p(X_\alpha),$$

from which it follows that:

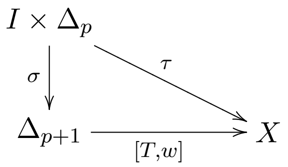
$$H_p(X) = \bigoplus_{\alpha\in\mathcal{A}} H_p(X_\alpha).$$

In particular, it follows from Example 3.3.7 that:

$$H_0(X) = \bigoplus_{\alpha\in\mathcal{A}} \mathbb{Z}.$$

The reader should compare this fact with Remark 3.2.5.

3.3.9. EXAMPLE. Suppose that $X \subset I\!R^n$ is a star-shaped subset around the point $w \in X$. For each singular p-simplex T in X we define a singular $(p+1)$-simplex $[T, w]$ in X in such a way that the following diagram commutes:



where σ and τ are defined by:

$$\sigma(s,t) = (1-s)t + s\, e_{p+1}, \quad \tau(s,t) = (1-s)T(t) + sw, \quad t \in \Delta_p,\ s \in I;$$

geometrically, the singular $(p+1)$-simplex $[T, w]$ coincides with T on the face $\Delta_p \subset \Delta_{p+1}$, it takes the vertex e_{p+1} on w and it is affine on the segment that joins t with e_{p+1} for every $t \in \Delta_p$.

The map $T \mapsto [T, w]$ extends to a homomorphism:

$$\mathfrak{S}_p(X) \ni c \longmapsto [c, w] \in \mathfrak{S}_{p+1}(X).$$

It is easy to see that for each singular p-chain $c \in \mathfrak{S}_p(X)$ we have:

$$\partial[c,w] = \begin{cases} [\partial c, w] + (-1)^{p+1} c, & p \geq 1 \\ \varepsilon(c) w - c, & p = 0; \end{cases} \tag{3.3.5}$$

for, it suffices to consider the case that $c = T$ is a singular p-simplex, in which case (3.3.5) follows from an elementary analysis of the definition of $[T, w]$ and of the definition of the boundary operator. In particular, we have $\partial[c, w] = (-1)^{p+1}c$ for every $c \in \tilde{Z}_p(X)$ and therefore $c \in \tilde{B}_p(X)$; we conclude that, if X is star shaped, then

$$\tilde{H}_p(X) = 0, \quad p \in \mathbb{Z}.$$

3.3.10. DEFINITION. Let $\mathfrak{C} = (\mathfrak{C}_p, \delta_p)$, $\mathfrak{C}' = (\mathfrak{C}'_p, \delta'_p)$ be chain complexes; a *chain map* $\phi\colon \mathfrak{C} \to \mathfrak{C}'$ is a sequence of homomorphisms $\phi_p\colon \mathfrak{C}_p \to \mathfrak{C}'_p$, $p \in \mathbb{Z}$, such that for every p the diagram

$$\begin{array}{ccc} \mathfrak{C}_p & \xrightarrow{\delta_p} & \mathfrak{C}_{p-1} \\ {\scriptstyle \phi_p}\downarrow & & \downarrow{\scriptstyle \phi_{p-1}} \\ \mathfrak{C}'_p & \xrightarrow[\delta'_p]{} & \mathfrak{C}'_{p-1} \end{array}$$

commutes; in general, we will write ϕ rather than ϕ_p. It is easy to see that if ϕ is a chain map, then $\phi(Z_p(\mathfrak{C})) \subset Z_p(\mathfrak{C}')$ and $\phi(B_p(\mathfrak{C})) \subset B_p(\mathfrak{C}')$, so that ϕ induces by passage to quotients a homomorphism

$$\phi_*\colon H_p(\mathfrak{C}) \longrightarrow H_p(\mathfrak{C}');$$

we say that ϕ_* is the *map induced in homology* by the chain map ϕ.

Clearly, if $\phi\colon \mathfrak{C} \to \mathfrak{C}'$ and $\psi\colon \mathfrak{C}' \to \mathfrak{C}''$ are chain maps, then also their composition $\psi \circ \phi$ is a chain map; moreover, $(\psi \circ \phi)_* = \psi_* \circ \phi_*$, and if Id is the *identity of the complex* $\mathfrak{C}$, i.e., Id_p is the identity of $\mathfrak{C}_p$ for every p, then Id_* is the identity of $H_p(\mathfrak{C})$ for every p. It follows that if ϕ is a *chain isomorphism,* i.e., ϕ_p is an isomorphism for every p, then ϕ_* is an isomorphism between the homology groups, and $(\phi^{-1})_* = (\phi_*)^{-1}$.

If X, Y are topological spaces and $f : X \to Y$ is a continuous map, then for each p we define a homomorphism:

$$f_\# : \mathfrak{S}_p(X) \longrightarrow \mathfrak{S}_p(Y)$$

by setting $f_\#(T) = f \circ T$ for every singular p-simplex T in X. It is easy to see that $f_\#$ is a chain map; we say that $f_\#$ is the *chain map induced by* f. It is clear that, given continuous maps $f\colon X \to Y$, $g\colon Y \to Z$ then $(g \circ f)_\# = g_\# \circ f_\#$, and that if Id is the identity map of X, then $\mathrm{Id}_\#$ is the identity of $\mathfrak{S}(X)$; in particular, if f is a homeomorphism, then $f_\#$ is a chain isomorphism, and $(f^{-1})_\# = (f_\#)^{-1}$. We have that the chain map $f_\#$ induces a homomorphism

$$f_*\colon H_p(X) \longrightarrow H_p(Y)$$

between the groups of singular homology of X and Y, that will be denoted simply by f_*.

3.3.11. REMARK. If A is a subspace of X, then we can identify the set of singular p-simplexes in A with a subset of the set of singular p-simplexes in X; then $\mathfrak{S}_p(A)$ is identified with a subgroup of $\mathfrak{S}_p(X)$. If $\mathfrak{i}\colon A \to X$ denotes the inclusion, then $\mathfrak{i}_\#$ is simply the inclusion of $\mathfrak{S}_p(A)$ into $\mathfrak{S}_p(X)$. However, observe that the induced map in homology $\mathfrak{i}_*$ is in general *not injective*, and there exists no identification of $H_p(A)$ with a subgroup of $H_p(X)$.

Recall that (X, A) is called a pair of topological spaces when X is a topological space and $A \subset X$ is a subspace. We define the *singular complex* $\mathfrak{S}(X, A)$ *of the pair* (X, A) by setting:

$$\mathfrak{S}_p(X, A) = \mathfrak{S}_p(X)/\mathfrak{S}_p(A);$$

the boundary operator of $\mathfrak{S}(X, A)$ is defined using the boundary operator of $\mathfrak{S}(X)$ by passage to the quotient. Clearly, $\mathfrak{S}(X, A)$ is a chain complex; we write

$$H_p(\mathfrak{S}(X, A)) = H_p(X, A).$$

We call $H_p(X, A)$ the p*-th group of relative homology of the pair* (X, A).

If $f\colon (X, A) \to (Y, B)$ is a map of pairs (Definition 3.2.8) then the chain map $f_\#$ passes to the quotient and it defines a chain map

$$f_\#\colon \mathfrak{S}(X, A) \longrightarrow \mathfrak{S}(Y, B)$$

that will also be denoted by $f_\#$; then $f_\#$ induces a homomorphism between the groups of relative homology, that will be denoted by f_*. Clearly, if $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (Z, C)$ are maps of pairs, then $(g \circ f)_\# = g_\# \circ f_\#$ and that if Id is the identity map of X, then $\mathrm{Id}_\#$ is the identity of $\mathfrak{S}(X, A)$; also, if $f\colon (X, A) \to (Y, B)$ is a *homeomorphism of pairs*, i.e., f is a homeomorphism of X onto Y with $f(A) = B$, then $f_\#$ is a chain isomorphism.

3.3.12. REMARK. An intuitive way of thinking of the groups of relative homology $H_p(X, A)$ is to consider them as the reduced homology groups $\tilde{H}_p(X/A)$ of the space X/A which is obtained from X by *collapsing* all the points of A to a single point. This idea is indeed a theorem that holds in the case that $A \subset X$ is closed and it is a *deformation retract* of some open subset of X. The proof of this theorem requires further development of the theory, and it will be omitted in these notes (see [**31**, Exercise 2, §39, Chapter 4]).

3.3.13. EXAMPLE. If A is the empty set, then $\mathfrak{S}(X, A) = \mathfrak{S}(X)$, and therefore $H_p(X, A) = H_p(X)$ for every $p \in \mathbb{Z}$; for this reason, we will not distinguish between the space X and the pair $(X, \emptyset)$.

3.3.14. EXAMPLE. The identity map of X induces a map of pairs:

$$\mathfrak{q}\colon (X, \emptyset) \longrightarrow (X, A); \tag{3.3.6}$$

then $\mathfrak{q}_\#\colon \mathfrak{S}(X) \to \mathfrak{S}(X, A)$ is simply the quotient map. We define

$$Z_p(X, A) = \mathfrak{q}_\#^{-1}\big(Z_p(\mathfrak{S}(X, A))\big), \quad B_p(X, A) = \mathfrak{q}_\#^{-1}\big(B_p(\mathfrak{S}(X, A))\big);$$

we call $Z_p(X, A)$ and $B_p(X, A)$ respectively the *group of relative* p*-cycles* and the *group of relative* p*-boundaries* of the pair (X, A). More explicitly, we have

$$Z_p(X, A) = \big\{c \in \mathfrak{S}_p(X) : \partial c \in \mathfrak{S}_{p-1}(A)\big\} = \partial^{-1}(\mathfrak{S}_{p-1}(A)),$$
$$B_p(X, A) = \big\{\partial c + d : c \in \mathfrak{S}_{p+1}(X),\ d \in \mathfrak{S}_p(A)\big\} = B_p(X) + \mathfrak{S}_p(A);$$

Observe that

$$Z_p(\mathfrak{S}(X, A)) = Z_p(X, A)/\mathfrak{S}_p(A), \quad B_p(\mathfrak{S}(X, A)) = B_p(X, A)/\mathfrak{S}_p(A);$$

it follows from elementary theory of quotient of groups that:

$$H_p(X,A) = H_p(\mathfrak{S}(X,A)) \cong Z_p(X,A)/B_p(X,A). \tag{3.3.7}$$

Given $c \in Z_p(X,A)$, the equivalence class $c + B_p(X,A) \in H_p(X,A)$ is called the *homology class determined by* c *in* $H_p(X,A)$; if $c_1, c_2 \in Z_p(X,A)$ determine the same homology class in $H_p(X,A)$, i.e., if $c_1 - c_2 \in B_p(X,A)$, we say that c_1 and c_2 are *homologous in* $\mathfrak{S}(X,A)$.

3.3.15. EXAMPLE. If X is arc-connected and $A \neq \emptyset$, then arguing as in Example 3.3.7 we conclude that any two 0-simplexes in X are homologous in $\mathfrak{S}(X,A)$; however, in this case every point of A is a singular 0-simplex which is homologous to 0 in $\mathfrak{S}(X,A)$, hence:

$$H_0(X,A) = 0.$$

If X is not arc-connected, then we write $X = \bigcup_{\alpha \in \mathcal{A}} X_\alpha$, where each X_α is an arc-connected component of X; writing $A_\alpha = A \cap X_\alpha$, as in Example 3.3.8 we obtain:

$$\mathfrak{S}_p(X,A) = \bigoplus_{\alpha \in \mathcal{A}} \mathfrak{S}_p(X_\alpha, A_\alpha);$$

and it follows directly that:

$$H_p(X,A) = \bigoplus_{\alpha \in \mathcal{A}} H_p(X_\alpha, A_\alpha).$$

In the case $p = 0$, we obtain in particular that:

$$H_0(X,A) = \bigoplus_{\alpha \in \mathcal{A}'} \mathbb{Z},$$

where $\mathcal{A}'$ is the subset of indices $\alpha \in \mathcal{A}$ such that $A_\alpha = \emptyset$.

Our goal now is to build an exact sequence that relates the homology groups $H_p(X)$ and $H_p(A)$ with the relative homology groups $H_p(X,A)$.

3.3.16. DEFINITION. Given chain complexes $\mathfrak{C}$, $\mathfrak{D}$, $\mathcal{E}$, we say that

$$0 \longrightarrow \mathfrak{C} \xrightarrow{\ \phi\ } \mathfrak{D} \xrightarrow{\ \psi\ } \mathcal{E} \longrightarrow 0 \tag{3.3.8}$$

is a *short exact sequence* of chain complexes if ϕ and ψ are chain maps and if for every $p \in \mathbb{Z}$ the sequence of abelian groups and homomorphisms

$$0 \longrightarrow \mathfrak{C}_p \xrightarrow{\ \phi\ } \mathfrak{D}_p \xrightarrow{\ \psi\ } \mathcal{E}_p \longrightarrow 0$$

is exact.

We have the following result of Homological Algebra:

3.3.17. LEMMA (The Zig-Zag Lemma). *Given a short exact sequence of chain complexes* (3.3.8)*, there exists an exact sequence of abelian groups and homomorphisms:*

$$\cdots \xrightarrow{\ \delta_*\ } H_p(\mathfrak{C}) \xrightarrow{\ \phi_*\ } H_p(\mathfrak{D}) \xrightarrow{\ \psi_*\ } H_p(\mathcal{E}) \xrightarrow{\ \delta_*\ } H_{p-1}(\mathfrak{C}) \xrightarrow{\ \phi_*\ } \cdots \tag{3.3.9}$$

where ϕ_* and ψ_* are induced by ϕ and ψ respectively, and the homomorphism δ_* is defined by:

$$\delta_*\big(e + B_p(\mathcal{E})\big) = c + B_{p-1}(\mathfrak{C}), \quad e \in Z_p(\mathcal{E}), \tag{3.3.10}$$

where $c \in \mathfrak{C}_{p-1}$ is chosen in such a way that $\phi(c) = \delta d$ and $d \in \mathfrak{D}_p$ is chosen in such a way that $\psi(d) = e$; the definition (3.3.10) *does not depend on the arbitrary choices involved.*

PROOF. The proof, based on an exhaustive analysis of all the cases, is elementary and it will be omitted. The details can be found in [**31**, §24, Chapter 3]. □

The exact sequence (3.3.9) is known as the *long exact homology sequence* corresponding to the short exact sequence of chain complexes (3.3.8)

Coming back to the topological considerations, if (X, A) is a pair of topological spaces, we have a short exact sequence of chain complexes:

$$0 \longrightarrow \mathfrak{S}(A) \xrightarrow{\mathfrak{i}_\#} \mathfrak{S}(X) \xrightarrow{\mathfrak{q}_\#} \mathfrak{S}(X, A) \longrightarrow 0 \tag{3.3.11}$$

where $\mathfrak{i}_\#$ is induced by the inclusion $\mathfrak{i}\colon A \to X$ and $\mathfrak{q}_\#$ is induced by (3.3.6).

Then, it follows directly from the Zig-Zag Lemma the following:

3.3.18. PROPOSITION. *Given a pair of topological spaces (X, A) then there exists an exact sequence*

$$\cdots \xrightarrow{\partial_*} H_p(A) \xrightarrow{\mathfrak{i}_*} H_p(X) \xrightarrow{\mathfrak{q}_*} H_p(X, A) \xrightarrow{\partial_*} H_{p-1}(A) \xrightarrow{\mathfrak{i}_*} \cdots \tag{3.3.12}$$

where $\mathfrak{i}_*$ is induced by the inclusion $\mathfrak{i}\colon A \to X$, $\mathfrak{q}_*$ is induced by (3.3.6) *and the homomorphism ∂_* is defined by:*

$$\partial_*\big(c + B_p(X, A)\big) = \partial c + B_{p-1}(A), \quad c \in Z_p(X, A);$$

such definition does not depend on the choices involved. If $A \neq \emptyset$ we also have an exact sequence

$$\cdots \xrightarrow{\partial_*} \tilde{H}_p(A) \xrightarrow{\mathfrak{i}_*} \tilde{H}_p(X) \xrightarrow{\mathfrak{q}_*} H_p(X, A) \xrightarrow{\partial_*} \tilde{H}_{p-1}(A) \xrightarrow{\mathfrak{i}_*} \cdots \tag{3.3.13}$$

whose arrows are obtained by restriction of the corresponding arrows in the sequence (3.3.12)*.*

PROOF. The sequence (3.3.12) is obtained by applying the Zig-Zag Lemma to the short exact sequence (3.3.11). If $A \neq \emptyset$, we replace $\mathfrak{S}(A)$ and $\mathfrak{S}(X)$ by the corresponding augmented complexes; we then apply the Zig-Zag Lemma and we obtain the sequence (3.3.13). □

The exact sequence (3.3.12) is known as the *long exact homology sequence of the pair (X, A)*; the sequence (3.3.13) is called the *long exact reduced homology sequence of the pair (X, A)*.

3.3.19. EXAMPLE. If $A \neq \emptyset$ is homeomorphic to a star-shaped subset of $I\!R^n$, then $\tilde{H}_p(A) = 0$ for every $p \in \mathbb{Z}$ (see Example 3.3.9); hence, the long exact reduced homology sequence of the pair (X, A) implies that the map:

$$\mathfrak{q}_*\colon \tilde{H}_p(X) \longrightarrow H_p(X, A)$$

is an isomorphism for every $p \in \mathbb{Z}$.

Now, we want to show the *homotopical invariance of the singular homology*; more precisely, we want to show that two homotopic continuous maps induce the same homomorphisms of the homology groups. We begin with an algebraic definition.

3.3.20. DEFINITION. Let $\mathfrak{C} = (\mathfrak{C}_p, \delta_p)$ and $\mathfrak{C}' = (\mathfrak{C}'_p, \delta'_p)$ be chain complexes. Given a chain map $\phi, \psi\colon \mathfrak{C} \to \mathfrak{C}'$ then a *chain homotopy* between ϕ and ψ is a sequence $(D_p)_{p\in\mathbb{Z}}$ of homomorphisms $D_p\colon \mathfrak{C}_p \to \mathfrak{C}'_{p+1}$ such that

$$\phi_p - \psi_p = \delta'_{p+1} \circ D_p + D_{p-1} \circ \delta_p, \tag{3.3.14}$$

for every $p \in \mathbb{Z}$; in this case we write $D\colon \phi \cong \psi$ and we say that ϕ and ψ are *chain-homotopic*.

The following Lemma is a trivial consequence of formula (3.3.14)

3.3.21. LEMMA. *If two chain maps ϕ and ψ are chain-homotopic, then ϕ and ψ induce the same homomorphisms in homology, i.e., $\phi_* = \psi_*$.* □

Our next goal is to prove that if f and g are two homotopic continuous maps, then the chain maps $f_\#$ and $g_\#$ are chain-homotopic. To this aim, we consider the maps:

$$i_X\colon X \to I \times X, \quad j_X\colon X \to I \times X \tag{3.3.15}$$

defined by $i_X(x) = (0, x)$ and $j_X(x) = (1, x)$ for every $x \in X$, where $I = [0, 1]$. We will show first that the chain maps $(i_X)_\#$ and $(j_X)_\#$ are chain-homotopic:

3.3.22. LEMMA. *For all topological space X there exists a chain homotopy $D_X\colon (i_X)_\# \cong (j_X)_\#$ where i_X and j_X are given in* (3.3.15)*; moreover, the association $X \mapsto D_X$ may be chosen in a* natural *way, i.e., in such a way that, given a continuous map $f\colon X \to Y$, then the diagram*

$$\begin{array}{ccc} \mathfrak{S}_p(X) & \xrightarrow{(D_X)_p} & \mathfrak{S}_{p+1}(I \times X) \\ {\scriptstyle (f_\#)_p}\downarrow & & \downarrow{\scriptstyle ((\mathrm{Id}\times f)_\#)_p} \\ \mathfrak{S}_p(Y) & \xrightarrow[(D_Y)_p]{} & \mathfrak{S}_{p+1}(I \times Y) \end{array} \tag{3.3.16}$$

commutes for every $p \in \mathbb{Z}$, where $\mathrm{Id} \times f$ *is given by* $(t, x) \mapsto (t, f(x))$.

PROOF. For each topological space X and each $p \in \mathbb{Z}$ we must define a homomorphism

$$(D_X)_p\colon \mathfrak{S}_p(X) \longrightarrow \mathfrak{S}_{p+1}(I \times X);$$

for $p < 0$ we obviously set $(D_X)_p = 0$. For $p \geq 0$ we denote by Id_p the identity map of the space Δ_p; then Id_p is a singular p-simplex in Δ_p, and therefore $\mathrm{Id}_p \in \mathfrak{S}_p(\Delta_p)$. The construction of D_X must be such that the diagram (3.3.16) commutes, and this suggests the following definition:

$$(D_X)_p(T) = ((\mathrm{Id} \times T)_\#)_p \circ (D_{\Delta_p})_p(\mathrm{Id}_p), \tag{3.3.17}$$

for every singular p-simplex $T\colon \Delta_p \to X$ (observe that $T_\#(\mathrm{Id}_p) = T$); hence, we need to find the correct definition of

$$(D_{\Delta_p})_p(\mathrm{Id}_p) = a_p \in \mathfrak{S}_{p+1}(I \times \Delta_p), \tag{3.3.18}$$

for each $p \geq 0$. Keeping in mind the definition of chain homotopy (see (3.3.14)), our definition of a_p will have to be given in such a way that the identity

$$\partial a_p = (i_{\Delta_p})_\#(\mathrm{Id}_p) - (j_{\Delta_p})_\#(\mathrm{Id}_p) - (D_{\Delta_p})_{p-1} \circ \partial(\mathrm{Id}_p) \tag{3.3.19}$$

be satisfied for every $p \geq 0$ (we will omit some index to simplify the notation); observe that (3.3.19) is equivalent to:

$$\partial a_p = i_{\Delta_p} - j_{\Delta_p} - (D_{\Delta_p})_{p-1} \circ \partial(\mathrm{Id}_p). \tag{3.3.20}$$

Let us begin by finding $a_0 \in \mathfrak{S}_1(I \times \Delta_0)$ that satisfies (3.3.20), that is, a_0 must satisfy $\partial a_0 = i_{\Delta_0} - j_{\Delta_0}$; we compute as follows:

$$\varepsilon(i_{\Delta_0} - j_{\Delta_0}) = 0.$$

Since $\tilde{H}_0(I \times \Delta_0) = 0$ (see Example 3.3.9) we see that it is indeed possible to determine a_0 with the required property.

We now argue by induction; fix $r \geq 1$. Suppose that $a_p \in \mathfrak{S}_{p+1}(I \times \Delta_p)$ has been found for $p = 0, \ldots, r-1$ in such a way that condition (3.3.20) be satisfied, where $(D_X)_p$ is defined in (3.3.17) for every topological space X; it is then easy to see that the diagram (3.3.16) commutes. An easy computation that uses (3.3.16), (3.3.18) and (3.3.20) shows that:

$$\big((i_X)_\#\big)_p - \big((j_X)_\#\big)_p = \partial \circ (D_X)_p + (D_X)_{p-1} \circ \partial, \tag{3.3.21}$$

for $p = 0, \ldots, r-1$.

Now, we need to determine a_r that satisfies (3.3.20) (with $p = r$). It follows from (3.3.21), where we set $X = \Delta_r$ and $p = r-1$, that:

(3.3.22)

$$\begin{aligned}\partial \circ (D_{\Delta_r})_{r-1} \circ \partial(\mathrm{Id}_r)& \\ &= (i_{\Delta_r})_\# \circ \partial(\mathrm{Id}_r) - (j_{\Delta_r})_\# \circ \partial(\mathrm{Id}_r) - (D_{\Delta_r})_{r-2} \circ \partial \circ \partial(\mathrm{Id}_r) \\ &= \partial(i_{\Delta_r} - j_{\Delta_r});\end{aligned}$$

using (3.3.22) we see directly that

$$i_{\Delta_r} - j_{\Delta_r} - (D_{\Delta_r})_{r-1} \circ \partial(\mathrm{Id}_r) \in Z_r(I \times \Delta_r). \tag{3.3.23}$$

Since $H_r(I \times \Delta_r) = 0$ (see Example 3.3.9) it follows that (3.3.23) is an r-boundary; hence it is possible to choose a_r satisfying (3.3.20) (with $p = r$).

This concludes the proof. □

It is now easy to prove the homotopical invariance of the singular homology.

3.3.23. PROPOSITION. *If two continuous maps $f, g : X \to Y$ are homotopic, then the chain maps $f_\#$ and $g_\#$ are homotopic.*

PROOF. Let $H\colon f \cong g$ be a homotopy between f e g; by Lemma 3.3.22 there exists a chain homotopy $D_X\colon i_X \cong j_X$. Then, it is easy to see that we obtain a chain homotopy between $f_\#$ and $g_\#$ by considering, for each $p \in \mathbb{Z}$, the homomorphism

$$(H_\#)_{p+1} \circ (D_X)_p\colon \mathfrak{S}_p(X) \longrightarrow \mathfrak{S}_{p+1}(Y). \qquad \square$$

3.3.24. COROLLARY. *If $f, g\colon X \to Y$ are homotopic, then $f_* = g_*$.*

PROOF. It follows from Proposition 3.3.23 and from Lemma 3.3.21. $\square$

3.3.1. The Hurewicz's homomorphism. In this subsection we will show that the first singular homology group $H_1(X)$ of a topological space X can be computed from its fundamental group; more precisely, if X is arc-connected, we will show that $H_1(X)$ is the *abelianized group* of $\pi_1(X)$.

In the entire subsection we will assume familiarity with the notations and the concepts introduced in Section 3.1; we will consider a fixed topological space X.

Observing that the unit interval $I = [0, 1]$ coincides with the first standard simplex Δ_1, we see that every curve $\gamma \in \Omega(X)$ is a singular 1-simplex in X; then, $\gamma \in \mathfrak{S}_1(X)$. We will say that two singular 1-chains $c, d \in \mathfrak{S}_1(X)$ are *homologous* when $c - d \in B_1(X)$; this terminology will be used also in the case that c and d are not necessarily cycles.[4]

We begin with some Lemmas:

3.3.25. LEMMA. *Let $\gamma \in \Omega(X)$ and let $\sigma : I \to I$ be a continuous map. If $\sigma(0) = 0$ and $\sigma(1) = 1$, then $\gamma \circ \sigma$ is homologous to γ; if $\sigma(0) = 1$ and $\sigma(1) = 0$, then $\gamma \circ \sigma$ is homologous to $-\gamma$.*

PROOF. We suppose first that $\sigma(0) = 0$ and $\sigma(1) = 1$. Consider the singular 1-simplexes σ and $\ell(0, 1)$ in I (recall the definition of ℓ in (3.3.1)). Clearly, $\partial\big(\sigma - \ell(0, 1)\big) = 0$, i.e., $\sigma - \ell(0, 1) \in Z_1(I)$; since $H_1(I) = 0$ (see Example 3.3.9) it follows that $\sigma - \ell(0, 1) \in B_1(I)$. Consider the chain map

$$\gamma_\#\colon \mathfrak{S}(I) \longrightarrow \mathfrak{S}(X);$$

we have that $\gamma_\#(\sigma - \ell(0, 1)) \in B_1(X)$. But

$$\gamma_\#\big(\sigma - \ell(0, 1)\big) = \gamma \circ \sigma - \gamma \in B_1(X),$$

from which it follows that γ is homologous to $\gamma \circ \sigma$. The case $\sigma(0) = 1$, $\sigma(1) = 0$ is proven analogously, observing that $\sigma + \ell(0, 1) \in Z_1(I)$. $\square$

3.3.26. REMARK. In some situations we will consider singular 1-chains given by curves $\gamma\colon [a, b] \to X$ that are defined on an arbitrary closed interval $[a, b]$ (rather than on the unit interval I); in this case, with a slight abuse, we will denote by $\gamma \in \mathfrak{S}_1(X)$ the singular 1-simplex $\gamma \circ \ell(a, b)\colon I \to X$; it follows from Lemma 3.3.25 that $\gamma \circ \ell(a, b)$ is homologous to any reparameterization $\gamma \circ \sigma$ of γ, where $\sigma\colon I \to [a, b]$ is a continuous map such that $\sigma(0) = a$ and $\sigma(1) = b$ (see also Remark 3.1.4).

[4] Observe that a singular 1-chain c defines a homology class in $H_1(X)$ only if $c \in Z_1(X)$.

3.3.27. LEMMA. *If $\gamma, \mu \in \Omega(X)$ are such that $\gamma(1) = \mu(0)$, then $\gamma \cdot \mu$ is homologous to $\gamma + \mu$; moreover, for every $\gamma \in \Omega(X)$ we have that γ^{-1} is homologous to $-\gamma$ and for every $x_0 \in X$, $\mathfrak{o}_{x_0}$ is homologous to zero.*

PROOF. We will basically use the same idea that was used in the proof of Lemma 3.3.25. We have that $\ell\big(0, \frac{1}{2}\big) + \ell\big(\frac{1}{2}, 1\big) - \ell(0, 1) \in Z_1(I) = B_1(I)$; considering the chain map $(\gamma \cdot \mu)_\#$ we obtain:

$$(\gamma \cdot \mu)_\#\big(\ell(0, \tfrac{1}{2}) + \ell(\tfrac{1}{2}, 1) - \ell(0, 1)\big) = \gamma + \mu - \gamma \cdot \mu \in B_1(X),$$

from which it follows that $\gamma \cdot \mu$ is homologous to $\gamma + \mu$. The fact that γ^{-1} is homologous to $-\gamma$ follows from Lemma 3.3.25; finally, if $T \colon \Delta_2 \to X$ denotes the constant map with value x_0, we obtain $\partial T = \mathfrak{o}_{x_0} \in B_1(X)$. $\square$

3.3.28. LEMMA. *Let $K : I \times I \to X$ be a continuous map; considering the curves:*

$$\gamma_1 = K \circ \ell\big((0, 0), (1, 0)\big), \quad \gamma_2 = K \circ \ell\big((1, 0), (1, 1)\big),$$
$$\gamma_3 = K \circ \ell\big((1, 1), (0, 1)\big), \quad \gamma_4 = K \circ \ell\big((0, 1), (0, 0)\big),$$

we have that the singular 1*-chain $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is homologous to zero.*

PROOF. We have that $H_1(I \times I) = 0$ (see Example 3.3.9); moreover

$$\begin{aligned} \ell\big((0, 0), (1, 0)\big) + \ell\big((1, 0), (1, 1)\big) + \ell\big((1, 1), (0, 1)\big) \\ + \ell\big((0, 1), (0, 0)\big) \in Z_1(I \times I) = B_1(I \times I). \end{aligned} \tag{3.3.24}$$

The conclusion follows by applying $K_\#$ to (3.3.24). $\square$

We now relate the homotopy class and the homology class of a curve $\gamma \in \Omega(X)$.

3.3.29. COROLLARY. *If $\gamma, \mu \in \Omega(X)$ are homotopic with fixed endpoints, then γ is homologous to μ.*

PROOF. It suffices to apply Lemma 3.3.28 to a homotopy with fixed endpoints $K \colon \gamma \cong \mu$, keeping in mind Lemma 3.3.27. $\square$

3.3.30. REMARK. Let $A \subset X$ be a subset; if $\gamma : I \to X$ is a continuous curve with endpoints in A, i.e., $\gamma(0), \gamma(1) \in A$, then $\partial\gamma \in \mathfrak{S}_0(A)$, and therefore $\gamma \in Z_1(X, A)$ defines a homology class $\gamma + B_1(X, A)$ in $H_1(X, A)$. It follows from Lemma 3.3.28 (keeping in mind also Lemma 3.3.27) that if γ and μ are homotopic with free endpoints in A (recall Definition 3.1.25) then γ and μ define the same homology class in $H_1(X, A)$.

3.3.31. REMARK. If γ, μ are freely homotopic loops in X (see Remark 3.1.16) then it follows easily from Lemma 3.3.28 (keeping in mind also Lemma 3.3.27) that γ is homologous to μ.

We define a map:

$$\Theta \colon \overline{\Omega}(X) \longrightarrow \mathfrak{S}_1(X)/B_1(X) \tag{3.3.25}$$

by setting $\Theta([\gamma]) = \gamma + B_1(X)$ for every $\gamma \in \Omega(X)$; it follows from Corollary 3.3.29 that Θ is well defined, i.e., it does not depend on the choice of the representative of the homotopy class $[\gamma] \in \overline{\Omega}(X)$. Then, Lemma 3.3.27 tells us that:

(3.3.26)

$$\Theta([\gamma]\cdot[\mu]) = \Theta([\gamma]) + \Theta([\mu]), \quad \Theta\big([\gamma]^{-1}\big) = -\Theta([\gamma]), \quad \Theta([\mathfrak{o}_{x_0}]) = 0,$$

for every $\gamma, \mu \in \Omega(X)$ with $\gamma(1) = \mu(0)$ and for every $x_0 \in X$. If $\gamma \in \Omega(X)$ is a loop, then $\gamma \in Z_1(X)$; if we fix $x_0 \in X$, we see that Θ restricts to a map (also denoted by Θ):

(3.3.27) $$\Theta\colon \pi_1(X, x_0) \longrightarrow H_1(X).$$

It follows from (3.3.26) that (3.3.27) is a group homomorphism; this homomorphism is known as the *Hurewicz's homomorphism*. The Hurewicz's homomorphism is *natural* in the sense that, given a continuous map $f\colon X \to Y$ with $f(x_0) = y_0$, the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{\ \Theta\ } & H_1(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \pi_1(Y,y_0) & \xrightarrow[\ \Theta\ ]{} & H_1(Y) \end{array}$$

If $\lambda\colon I \to X$ is a continuous curve joining x_0 and x_1, then the Hurewicz's homomorphism fits well together with the isomorphism $\lambda_\#$ between the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ (see Proposition 3.1.11); more precisely, it follows from (3.3.26) that we have a commutative diagram:

(3.3.28)
$$\begin{array}{ccc} \pi_1(X,x_0) & & \\ & \searrow^{\Theta} & \\ {\scriptstyle\lambda_\#}\downarrow & & H_1(X) \\ & \nearrow_{\Theta} & \\ \pi_1(X,x_1). & & \end{array}$$

We are now ready to prove the main result of this subsection. We will first recall some definitions in group theory.

3.3.32. DEFINITION. If G is a group, the *commutator subgroup* of G, denoted by G', is the subgroup of G generated by all the elements of the form $ghg^{-1}h^{-1}$, with $g, h \in G$. The commutator subgroup G' is always a normal subgroup[5] of G, and therefore the quotient G/G' is always a group. We say that G/G' is the *abelianized group* of G.

The group G/G' is always abelian; as a matter of facts, if H is a normal subgroup of G, then the quotient group G/H is abelian if and only if $H \supset G'$.

[5]actually, the commutator subgroup G' of G is *invariant by every automorphism* of G.

3.3.33. THEOREM. *Let X be an arc-connected topological space. Then, for every $x_0 \in X$, the Hurewicz's homomorphism* (3.3.27) *is surjective, and its kernel is the commutator subgroup of $\pi_1(X, x_0)$; in particular, the first singular homology group $H_1(X)$ is isomorphic to the abelianized group of $\pi_1(X, x_0)$.*

PROOF. Since the quotient $\pi_1(X, x_0)/\mathrm{Ker}(\Theta) \cong \mathrm{Im}(\Theta)$ is abelian, it follows that $\mathrm{Ker}(\Theta)$ contains the commutator subgroup $\pi_1(X, x_0)'$, and therefore Θ defines a homomorphism by passage to the quotient:

$$\overline{\Theta}\colon \pi_1(X, x_0)/\pi_1(X, x_0)' \longrightarrow H_1(X);$$

our strategy will be to show that $\overline{\Theta}$ is an isomorphism.

For each $x \in X$, choose a curve $\eta_x \in \Omega(X)$ such that $\eta_x(0) = x_0$ and $\eta_x(1) = x$; we are now going to define a homomorphism

$$\Psi\colon \mathfrak{S}_1(X) \longrightarrow \pi_1(X, x_0)/\pi_1(X, x_0)';$$

since $\pi_1(X, x_0)/\pi_1(X, x_0)'$ is abelian and the singular 1-simplexes of X form a basis of $\mathfrak{S}_1(X)$ as a free abelian group, Ψ is well defined if we set

$$\Psi(\gamma) = q\big([\eta_{\gamma(0)}] \cdot [\gamma] \cdot [\eta_{\gamma(1)}]^{-1}\big), \quad \gamma \in \Omega(X),$$

where q denotes the quotient map

$$q\colon \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)/\pi_1(X, x_0)'.$$

We are now going to show that $B_1(X)$ is contained in the kernel of Ψ; to this aim, it suffices to show that $\psi(\partial T)$ is the neutral element of $\pi_1(X, x_0)/\pi_1(X, x_0)'$ for every singular 2-simplex T in X. We write:

$$\partial T = \gamma_0 - \gamma_1 + \gamma_2, \tag{3.3.29}$$

where $\gamma_0 = T \circ \ell(e_1, e_2)$, $\gamma_1 = T \circ \ell(e_0, e_2)$ and $\gamma_2 = T \circ \ell(e_0, e_1)$. Applying Ψ to both sides of (3.3.29) we obtain:

$$\begin{aligned} \Psi(\partial T) &= \Psi(\gamma_0)\Psi(\gamma_1)^{-1}\Psi(\gamma_2) \\ &= q\big([\eta_{T(e_1)}] \cdot [\gamma_0] \cdot [\gamma_1]^{-1} \cdot [\gamma_2] \cdot [\eta_{T(e_1)}]^{-1}\big). \end{aligned} \tag{3.3.30}$$

Writing $[\rho] = [\ell(e_1, e_2)] \cdot [\ell(e_2, e_0)] \cdot [\ell(e_0, e_1)] \in \overline{\Omega}(\Delta_2)$ then (3.3.30) implies that:

$$\Psi(\partial T) = q\big([\eta_{T(e_1)}] \cdot T_*([\rho]) \cdot [\eta_{T(e_1)}]^{-1}\big);$$

since $[\rho] \in \pi_1(\Delta_2, e_1)$, we have that $[\rho] = [\mathfrak{o}_{e_1}]$ (see Example 3.1.15), from which it follows $\Psi(\partial T) = q([\mathfrak{o}_{x_0}])$.

Then, we conclude that $B_1(X) \subset \mathrm{Ker}(\Psi)$, from which we deduce that Ψ passes to the quotient and defines a homomorphism

$$\overline{\Psi}\colon \mathfrak{S}_1(X)/B_1(X) \longrightarrow \pi_1(X, x_0)/\pi_1(X, x_0)'.$$

The strategy will now be to show that the restriction $\overline{\Psi}|_{H_1(X)}$ is an inverse for $\overline{\Theta}$. Let us compute $\overline{\Theta} \circ \Psi$; for $\gamma \in \Omega(X)$ we have:

$$\begin{aligned} (\overline{\Theta} \circ \Psi)(\gamma) &= \Theta([\eta_{\gamma(0)}]) + \Theta([\gamma]) - \Theta([\eta_{\gamma(1)}]) \\ &= \eta_{\gamma(0)} + \gamma - \eta_{\gamma(1)} + B_1(X). \end{aligned} \tag{3.3.31}$$

Define a homomorphism $\phi\colon \mathfrak{S}_0(X) \to \mathfrak{S}_1(X)$ by setting $\phi(x) = \eta_x$ for every singular 0-simplex $x \in X$; then (3.3.31) implies that:

$$\overline{\Theta} \circ \Psi = p \circ (\mathrm{Id} - \phi \circ \partial), \tag{3.3.32}$$

where $p\colon \mathfrak{S}_1(X) \to \mathfrak{S}_1(X)/B_1(X)$ denotes the quotient map and Id denotes the identity map of $\mathfrak{S}_1(X)$. If we restricts both sides of (3.3.32) to $Z_1(X)$ and passing to the quotient we obtain:

$$\overline{\Theta} \circ \overline{\Psi}|_{H_1(X)} = \mathrm{Id}.$$

Let us now compute $\overline{\Psi} \circ \overline{\Theta}$; for every loop $\gamma \in \Omega_{x_0}(X)$ we have:

$$(\overline{\Psi} \circ \overline{\Theta})\big(q([\gamma])\big) = \Psi(\gamma) = q([\eta_{x_0}])q([\gamma])q([\eta_{x_0}])^{-1} = q([\gamma]),$$

observing that $\pi_1(X, x_0)/\pi_1(X, x_0)'$ is abelian. It follows that:

$$\big(\overline{\Psi}|_{H_1(X)}\big) \circ \overline{\Theta} = \mathrm{Id},$$

which concludes the proof. □

3.3.34. REMARK. If X is arc-connected and $\pi_1(X)$ is abelian, it follows from Theorem 3.3.33 that the Hurewicz's homomorphism is an isomorphism of $\pi_1(X, x_0)$ onto $H_1(X)$; this fact "explains" why the fundamental groups with different base-points $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ can be canonically identified when the fundamental group of the space is abelian. The reader should compare this observation with Remark 3.1.13 and with the diagram (3.3.28).

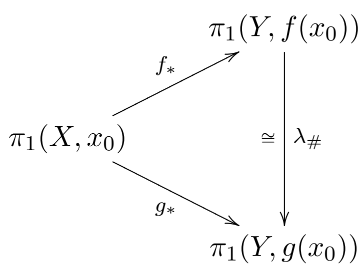
Exercises for Chapter 3

EXERCISE 3.1. Prove that every contractible space is arc-connected.

EXERCISE 3.2. Prove that a topological space X which is connected and locally arc-connected is arc-connected. Deduce that a connected (topological) manifold is arc-connected.

EXERCISE 3.3. Let $\gamma \in \Omega(X)$ be a loop and let $\lambda \in \Omega(X)$ be such that $\lambda(0) = \gamma(0)$; show that the loops γ and $\lambda^{-1} \cdot \gamma \cdot \lambda$ are freely homotopic.

EXERCISE 3.4. Let $f, g : X \to Y$ be homotopic maps and let $H : f \cong g$ be a homotopy from f to g; fix $x_0 \in X$ and set $\lambda(s) = H_s(x_0)$, $s \in I$. Show that the following diagram commutes:



EXERCISE 3.5. A continuous map $f\colon X \to Y$ is said to be a *homotopy equivalence* if there exists a continuous map $g\colon Y \to X$ such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y; in this case we say that g is a *homotopy inverse* for f. Show that if f is a homotopy equivalence then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism for every $x_0 \in X$.

EXERCISE 3.6. Show that X is contractible if and only if the map $f : X \to \{x_0\}$ is a homotopy equivalence.

EXERCISE 3.7. Prove that a homotopy equivalence induces an isomorphism in singular homology. Conclude that, if X is contractible, then $H_0(X) \cong \mathbb{Z}$ and $H_p(X) = 0$ for every $p \geq 1$.

EXERCISE 3.8. If $Y \subset X$, a continuous map $r : X \to Y$ is said to be a *retraction* if r restricts to the identity map of Y; in this case we say that Y is a *retract* of X. Show that if r is a retraction then $r_* : \pi_1(X, x_0) \to \pi_1(Y, x_0)$ is surjective for every $x_0 \in Y$. Show also that if Y is a retract of X then the inclusion map $\mathfrak{i} : Y \to X$ induces an injective homomorphism $\mathfrak{i}_* : \pi_1(Y, x_0) \to \pi_1(X, x_0)$ for every $x_0 \in Y$.

EXERCISE 3.9. Let G_1, G_2 be groups and $f : G_1 \to G_2$ a homomorphism. Prove that the sequence $0 \longrightarrow G_1 \xrightarrow{f} G_2 \longrightarrow 0$ is exact if and only if f is an isomorphism.

EXERCISE 3.10. Let $p : E \to B$ be a locally injective continuous map with E Hausdorff and let $f : X \to B$ be a continuous map defined in a connected topological space X. Given $x_0 \in X$, $e_0 \in E$, show that there exists at most one map $\hat{f} : X \to E$ with $p \circ \hat{f} = f$ and $\hat{f}(x_0) = e_0$. Show that if p is a covering map then the hypothesis that E is Hausdorff can be dropped.

EXERCISE 3.11. Let $X \subset I\!R^2$ be defined by:

$$X = \big\{(x, \sin(1/x)) : x > 0\big\} \cup \Big(\{0\} \times [-1, 1]\Big).$$

Show that X is connected but not arc-connected; compute the singular homology groups of X.

EXERCISE 3.12. Prove the Zig-Zag Lemma (Lemma 3.3.17).

EXERCISE 3.13. Let G be a group and let G act on a topological space X by homeomorphisms. We say that such action is *properly discontinuous* if for every $x \in X$ there exists a neighborhood U of x such that $gU \cap U = \emptyset$ for every $g \neq 1$, where $gU = \{g \cdot y : y \in U\}$. Let be given a properly discontinuous action of G in X and denote by X/G the set of orbits of G endowed with the quotient topology.

- Show that the quotient map $p : X \to X/G$ is a covering map with typical fiber G.
- Show that, if X is arc-connected, there exists an exact sequence of groups and group homomorphisms:

$$0 \longrightarrow \pi_1(X) \xrightarrow{p_*} \pi_1(X/G) \longrightarrow G \longrightarrow 0.$$

- If X is simply connected conclude that $\pi_1(X/G)$ is isomorphic to G.

EXERCISE 3.14. Let $X = \mathbb{R}^2$ be the Euclidean plane; for each $m, n \in \mathbb{Z}$ let $g_{m,n}$ be the homeomorphism of X given by:

$$g_{m,n}(x, y) = ((-1)^n x + m, y + n).$$

Set $G = \{g_{m,n} : m, n \in \mathbb{Z}\}$. Show that:

- G is a subgroup of the group of all homeomorphisms of X;
- show that X/G is homeomorphic to the *Klein bottle*;
- show that the natural action of G in X is properly discontinuous; conclude that the fundamental group of the Klein bottle is isomorphic to G;
- show that G is the *semi-direct product*[6] of two copies of $\mathbb{Z}$;
- compute the commutator subgroup of G and conclude that the first singular homology group of the Klein bottle is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$.

EXERCISE 3.15. Prove that if X and Y are arc-connected, then $H_1(X \times Y) \cong H_1(X) \oplus H_1(Y)$.

EXERCISE 3.16. Compute the relative homology group $H_2(D, \partial D)$, where D is the unit disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and ∂D is its boundary.

[6] Recall that a group G is the (inner) semi-direct product of two subgroups H and K if $G = HK$ with $H \cap K = \{1\}$ and K normal in G.

CHAPTER 4

The Maslov Index

4.1. Index of a Symmetric Bilinear Form

In this section we will define the index and the co-index of a symmetric bilinear form; in finite dimension, these numbers are respectively the number of negative and of positive entries of a symmetric matrix when it is diagonalized as in the Sylvester Inertia Theorem (Theorem 4.1.10). We will show some properties of these numbers.

In this Section, V will always denote a *real* vector space, not necessarily finite dimensional. Recall that $\mathrm{B_{sym}}(V)$ denotes the space of symmetric bilinear forms $B : V \times V \to I\!R$. We start with a definition:

4.1.1. DEFINITION. Let $B \in \mathrm{B_{sym}}(V)$; we say that B is:

- *positive definite* if $B(v, v) > 0$ for all $v \in V$, $v \neq 0$;
- *positive semi-definite* if $B(v, v) \geq 0$ for all $v \in V$;
- *negative definite* if $B(v, v) < 0$ for all $v \in V$, $v \neq 0$;
- *negative semi-definite* if $B(v, v) \leq 0$ for all $v \in V$.

We say that a subspace $W \subset V$ is *positive with respect to* B, or B*-positive*, if $B|_{W\times W}$ is positive definite; similarly, we say that W is *negative with respect to* B, or B*-negative*, if $B|_{W\times W}$ is negative definite.

The *index* of B, denoted by $n_-(B)$, is defined by:

(4.1.1) $$n_-(B) = \sup\big\{\dim(W) : W \text{ is a } B\text{-negative subspace of } V\big\}.$$

The index of B can be a non negative integer, or $+\infty$. The *co-index* of B, denoted by $n_+(B)$, is defined as the index of $-B$:

$$n_+(B) = n_-(-B).$$

Obviously, the co-index of B can be defined as the supremum of the dimensions of all B-positive subspaces of V. When at least one of the numbers $n_-(B)$ and $n_+(B)$ is finite, we define the *signature* of B by:

$$\mathrm{sgn}(B) = n_+(B) - n_-(B).$$

If $B \in \mathrm{B_{sym}}(V)$ and $W \subset V$ is a subspace, then clearly:

(4.1.2) $$n_-(B|_{W\times W}) \leq n_-(B), \quad n_+(B|_{W\times W}) \leq n_+(B).$$

The reader should now recall the definitions of *kernel* of a symmetric bilinear form B, denoted by $\mathrm{Ker}(B)$, and of *orthogonal complement* of a subspace $S \subset V$ with respect to B, denoted by $S^\perp$. Recall also that B is said to be *nondegenerate* if $\mathrm{Ker}(B) = \{0\}$.

Observe that in Section 1.1 we have considered only finite dimensional vector spaces, but obviously the definitions of kernel, orthogonal complement and nondegeneracy make sense for symmetric bilinear forms defined on an arbitrary vector space V. However, many results proven in Section 1.1 make an *essential* use of the finiteness of the vector space (see Example 1.1.12). For instance, observe that a bilinear form is nondegenerate if and only if its associated linear operator

$$V \ni v \longmapsto B(v,\cdot) \in V^* \tag{4.1.3}$$

is injective; if $\dim(V) = +\infty$, this does *not* imply that (4.1.3) is an isomorphism.

4.1.2. DEFINITION. Given $B \in \mathrm{B}_{\mathrm{sym}}(V)$, the *degeneracy* of B, denoted by $\mathrm{dgn}(B)$ is the possibly infinite dimension of $\mathrm{Ker}(B)$. We say that a subspace $W \subset V$ is *nondegenerate with respect to* B, or also that W is B*-nondegenerate*, if $B|_{W\times W}$ is nondegenerate.

4.1.3. EXAMPLE. Unlike the case of the index and the co-index (see (4.1.2)), the degeneracy of a symmetric bilinear form B is *not* monotonic with respect to the inclusion of subspaces. For instance, if $V = I\!R^2$ and B is the symmetric bilinear form:

$$B\big((x_1,y_1),(x_2,y_2)\big) = x_1x_2 - y_1y_2 \tag{4.1.4}$$

then $\mathrm{dgn}(B) = 0$; however, if W is the subspace generated by the vector $(1,1)$, we have:

$$\mathrm{dgn}(B|_{W\times W}) = 1 > 0 = \mathrm{dgn}(B).$$

On the other hand, if B is defined by

$$B\big((x_1,y_1),(x_2,y_2)\big) = x_1x_2$$

and if W is the subspaces generated by $(1,0)$, then

$$\mathrm{dgn}(B|_{W\times W}) = 0 < 1 = \mathrm{dgn}(B).$$

4.1.4. EXAMPLE. If $T : V_1 \to V_2$ is an isomorphism and if $B \in \mathrm{B}_{\mathrm{sym}}(V_1)$, then we can consider the push-forward of B, $T_\#(B) \in \mathrm{B}_{\mathrm{sym}}(V_2)$. Clearly, T maps B-positive subspaces of V_1 into $T_\#(B)$-positive subspaces of V_2, and B-negative subspaces of V_1 into $T_\#(B)$-negative subspaces of V_2; moreover, $\mathrm{Ker}(T_\#(B)) = T(\mathrm{Ker}(B))$. Hence we have:

$$n_+\big(T_\#(B)\big) = n_+(B), \quad n_-\big(T_\#(B)\big) = n_-(B), \quad \mathrm{dgn}\big(T_\#(B)\big) = \mathrm{dgn}(B).$$

4.1.5. REMARK. It follows from Proposition 1.1.10 and from remark 1.1.13 that if $W \subset V$ is a *finite dimensional* B-nondegenerate subspace, then $V = W \oplus W^\perp$, even in the case that $\dim(V) = +\infty$.

Recall that if $W \subset V$ is a subspace, then the *codimension* of W in V is defined by:

$$\mathrm{codim}_V(W) = \dim(V/W);$$

this number may be finite even when $\dim(W) = \dim(V) = +\infty$. The codimension of W in V coincides with the dimension of any complementary subspace of W in V.

The following Lemma and its Corollary are the basic tool for the computation of indices of bilinear forms:

4.1.6. LEMMA. *Let $B \in \mathrm{B}_{\mathrm{sym}}(V)$; if $Z \subset V$ is a subspace of V on which B is positive semi-definite, then:*

$$n_-(B) \le \textit{codim}_V(Z).$$

PROOF. If B is negative definite on a subspace W, then $W \cap Z = \{0\}$, and so the quotient map $q : V \to V/Z$ takes W isomorphically onto a subspace of V/Z. Hence, $\dim(W) \le \mathrm{codim}_V(Z)$. □

4.1.7. COROLLARY. *Suppose that $V = Z \oplus W$ with B positive semi-definite on Z and negative definite on W; then $n_-(B) = \dim(W)$.*

PROOF. Clearly, $n_-(B) \ge \dim(W)$.
From Lemma 4.1.6 it follows that $n_-(B) \le \mathrm{codim}_V(Z) = \dim(W)$. □

4.1.8. REMARK. Note that every result concerning the index of symmetric bilinear forms, like for instance Lemma 4.1.6 and Corollary 4.1.7, admits a corresponding version for the co-index of forms. For shortness, we will only state these results in the version for the index, and we will understand the version for the co-index. Similarly, results concerning negative (semi-)definite symmetric bilinear forms B can be translated into results for positive (semi-)definite symmetric forms by replacing B with $-B$.

4.1.9. PROPOSITION. *If $B \in \mathrm{B}_{\mathrm{sym}}(V)$ and $V = Z \oplus W$ with B positive definite in Z and negative definite in W, then B is nondegenerate.*

PROOF. Let $v \in \mathrm{Ker}(B)$; write $v = v_+ + v_-$ with $v_+ \in Z$ and $v_- \in W$. Then:

$$B(v, v_+) = B(v_+, v_+) + B(v_-, v_+) = 0, \tag{4.1.5}$$

$$B(v, v_-) = B(v_+, v_-) + B(v_-, v_-) = 0; \tag{4.1.6}$$

from (4.1.5) we get that $B(v_+, v_-) \le 0$, and from (4.1.6) we get $B(v_+, v_-) \ge 0$, from which it follows $B(v_+, v_-) = 0$. Then, (4.1.5) implies $v_+ = 0$ and (4.1.6) implies $v_- = 0$. □

4.1.10. THEOREM (Sylvester's Inertia Theorem). *Suppose $\dim(V) = n < +\infty$ and let $B \in \mathrm{B}_{\mathrm{sym}}(V)$; then, there exists a basis of V with respect to which the matrix form of B is given by:*

$$B \sim \begin{pmatrix} \mathrm{I}_p & 0_{p\times q} & 0_{p\times r} \\ 0_{q\times p} & -\mathrm{I}_q & 0_{q\times r} \\ 0_{r\times p} & 0_{r\times q} & 0_r \end{pmatrix}, \tag{4.1.7}$$

where $0_{\alpha\times\beta}$, 0_α and I_α denote respectively the zero $\alpha \times \beta$ matrix, the zero $\alpha \times \alpha$ matrix and the $\alpha \times \alpha$ identity matrix.

The numbers p, q and r are uniquely determined by the bilinear form B; we have:

$$n_+(B) = p, \quad n_-(B) = q, \quad \mathrm{dgn}(B) = r. \tag{4.1.8}$$

PROOF. The existence of a basis $(b_i)_{i=1}^n$ with respect to which B has the canonical form (4.1.7) follows from Theorem 1.1.14, after suitable rescaling of the vectors of the basis. To prove that p, q and r are uniquely determined by B, i.e., that they do not depend on the choice of the basis, it is actually enough to prove (4.1.8). To this aim, let Z be the subspace generated by the vectors $\{b_i\}_{i=1}^p \cup \{b_i\}_{i=p+q+1}^n$ and W the subspace generated by $\{b_i\}_{i=p+1}^{p+q}$; then $V = Z \oplus W$, B is positive semi-definite in Z and negative definite in W. It follows from Corollary 4.1.7 that $n_-(B) = \dim(W) = q$. Similarly, we get $n_+(B) = p$. It is easy to see that $\mathrm{Ker}(B)$ is generated by the vectors $\{b_i\}_{i=p+q+1}^n$ and we conclude that $\mathrm{dgn}(B) = r$. □

4.1.11. COROLLARY. *Let $B \in \mathrm{B}_{\mathrm{sym}}(V)$, with $\dim(V) < +\infty$. If g is an inner product in V and if $T \in \mathrm{Lin}(V)$ is such that $B = g(T\cdot, \cdot)$, then the index (resp., the co-index) of B is equal to the sum of the multiplicities of the negative (resp., the positive) eigenvalues of T; the degeneracy of B is equal to the multiplicity of the zero eigenvalue of T.*

PROOF. Since T is g-symmetric, there exists a g-orthonormal basis that diagonalizes T, and this diagonal matrix has in its diagonal entries the eigenvalues of T repeated with multiplicity. In such basis, the bilinear form B is represented by the same matrix. After suitable rescaling of the vectors of the basis, B will be given in the canonical form (4.1.7); this operation does not change the signs of the elements in the diagonal of the matrix that represents B. The conclusion now follows from Theorem 4.1.10 □

4.1.12. EXAMPLE. The conclusion of Corollary 4.1.11 holds in the more general case of a matrix T that represents B in any basis; indeed, observe that any basis is orthonormal with respect to some inner product of V. Recall that the *determinant* and the *trace* of a matrix are equal respectively to the product and the sum of its eigenvalues (repeated with multiplicity); in the case $\dim(V) = 2$ it follows that the determinant and the trace of a matrix that represents B in any basis determine uniquely the numbers $n_-(B)$, $n_+(B)$ and $\mathrm{dgn}(B)$.

4.1.13. LEMMA. *Suppose that $B \in \mathrm{B}_{\mathrm{sym}}(V)$ is positive semi-definite; then*

$$\mathrm{Ker}(B) = \{v \in V : B(v, v) = 0\}.$$

PROOF. Let $v \in V$ with $B(v, v) = 0$ and let $w \in V$ be arbitrary; we need to show that $B(v, w) = 0$. If v and w are linearly dependent, the conclusion is trivial; otherwise, v and w form the basis of a two-dimensional subspace of V in which the restriction of B is represented by the matrix:

$$\begin{pmatrix} B(v,v) & B(v,w) \\ B(v,w) & B(w,w) \end{pmatrix}. \tag{4.1.9}$$

It follows from Corollary 4.1.11 (see Example 4.1.12) that the determinant of (4.1.9) is non negative, that is:

$$B(v, w)^2 \leq B(v, v)B(w, w) = 0,$$

which concludes the proof. □

4.1.14. COROLLARY. *If $B \in \mathrm{B}_{\mathrm{sym}}(V)$ is positive semi-definite and nondegenerate, then B is positive definite.* □

We now prove a generalized version of the *Cauchy–Schwarz inequality* for symmetric bilinear forms:

4.1.15. PROPOSITION. *Let $B \in \mathrm{B}_{\mathrm{sym}}(V)$ and vectors $v, w \in V$ be given. We have:*

- *if v, w are linearly dependent or if v, w generate a B-degenerate two-dimensional subspace, then*
$$B(v,w)^2 = B(v,v)B(w,w);$$
- *if v, w generate a B-positive or B-negative two-dimensional subspace, then*
$$B(v,w)^2 < B(v,v)B(w,w);$$
- *if v, w generate a two-dimensional subspace where B has index equal to 1, then*
$$B(v,w)^2 > B(v,v)B(w,w);$$

the above possibilities are exhaustive and mutually exclusive.

PROOF. The case that v and w are linearly dependent is trivial; all the others follow directly from Corollary 4.1.11 (see also Example 4.1.12), keeping in mind that the matrix that represents the restriction of B to the subspace generated by v and w is given by (4.1.9). □

4.1.16. DEFINITION. Given $B \in \mathrm{B}_{\mathrm{sym}}(V)$, we say that two subspaces V_1 and V_2 of V are *orthogonal with respect to B*, or *B-orthogonal*, if $B(v_1, v_2) = 0$ for all $v_1 \in V_1$ and all $v_2 \in V_2$; a direct sum $V = V_1 \oplus V_2$ with V_1 and V_2 B-orthogonal will be called a *B-orthogonal decomposition* of V.

4.1.17. LEMMA. *Let $B \in \mathrm{B}_{\mathrm{sym}}(V)$; if $V = V_1 \oplus V_2$ is a B-orthogonal decomposition of V and if B is negative definite (resp., negative semi-definite) in V_1 and in V_2, then B is negative definite (resp., negative semi-definite) in V.*

PROOF. It is obtained from the following simple computation:
$$B(v_1 + v_2, v_1 + v_2) = B(v_1, v_1) + B(v_2, v_2), \quad v_1 \in V_1,\ v_2 \in V_2. \qquad \square$$

4.1.18. DEFINITION. Given $B \in \mathrm{B}_{\mathrm{sym}}(V)$, we say that a subspace $W \subset V$ is *maximal negative with respect to B* if W is B-negative and if it is not properly contained in any other B-negative subspace of V. Similarly, we say that $W \subset V$ is *maximal positive with respect to B* if W is B-positive and if it is not properly contained in any other B-positive subspace of V.

4.1.19. COROLLARY. *Let $B \in \mathrm{B}_{\mathrm{sym}}(V)$ and $W \subset V$ be a maximal negative subspace with respect to B. Then, if $Z \subset V$ is a subspace which is B-orthogonal to W, it follows that B is positive semi-definite in Z.*

PROOF. By Lemma 4.1.17, the sum of any non zero B-negative subspace of Z with W would be a B-negative subspace of V that contains properly W. The conclusion follows. □

Observe that Corollary 4.1.19 can be applied when $n_-(B) < +\infty$ and W is a B-negative subspace with $\dim(W) = n_-(B)$.

4.1.20. COROLLARY. *Given $B \in \mathrm{B}_{\mathrm{sym}}(V)$, then*

$$\dim(V) = n_+(B) + n_-(B) + \mathrm{dgn}(B).$$

PROOF. If either one of the numbers $n_+(B)$ or $n_-(B)$ is infinite, the result is trivial. Suppose then that both numbers are finite; let $W \subset V$ be a B-negative subspace with $\dim(W) = n_-(B)$ and let $Z \subset V$ be a B-positive subspace with $\dim(Z) = n_+(B)$. By Proposition 4.1.9 we have that B is nondegenerate in $Z \oplus W$, and it follows from Remark 4.1.5 that

$$V = Z \oplus W \oplus (Z \oplus W)^\perp.$$

By Corollary 4.1.19, we have that B is positive semi-definite and also negative semi-definite in $(Z \oplus W)^\perp$, hence B vanishes in $(Z \oplus W)^\perp$. It follows now that $\mathrm{Ker}(B) = (Z \oplus W)^\perp$, which concludes the proof. □

4.1.21. COROLLARY. *If $W \subset V$ is a maximal negative subspace with respect to $B \in \mathrm{B}_{\mathrm{sym}}(V)$, then $n_-(B) = \dim(W)$.*

PROOF. If $\dim(W) = +\infty$ the result is trivial; for the general case, it follows from Remark 4.1.5 that $V = W \oplus W^\perp$. By Corollary 4.1.19, B is positive semi-definite in $W^\perp$, and then the conclusion follows from Corollary 4.1.7. □

4.1.22. REMARK. We can now conclude that the "supremum" that appears in the definition of index in (4.1.1) is in facts a *maximum*, i.e., there always exists a B-negative subspace $W \subset V$ with $n_-(B) = \dim(W)$. If $n_-(B)$ is finite, this statement is trivial. If $n_-(B) = +\infty$, it follows from Corollary 4.1.21 that no finite-dimensional subspace of V is maximal B-negative. If there were no infinite-dimensional B-negative subspace of V, we could construct a strictly increasing sequence $W_1 \subset W_2 \subset \cdots$ of B-negative subspaces; then $W = \bigcup_{n \geq 1} W_n$ would be an infinite-dimensional B-negative subspace, in contradiction with the hypothesis.

As a matter of facts, it follows from Zorn's Lemma that every symmetric bilinear form admits a maximal negative subspace (see Exercise 4.1).

4.1.23. PROPOSITION. *Let $B \in \mathrm{B}_{\mathrm{sym}}(V)$; if $V = V_1 \oplus V_2$ is a B-orthogonal decomposition, then:*

$$n_+(B) = n_+\big(B|_{V_1 \times V_1}\big) + n_+\big(B|_{V_2 \times V_2}\big), \tag{4.1.10}$$

$$n_-(B) = n_-\big(B|_{V_1 \times V_1}\big) + n_-\big(B|_{V_2 \times V_2}\big), \tag{4.1.11}$$

$$\mathrm{dgn}(B) = \mathrm{dgn}\big(B|_{V_1 \times V_1}\big) + \mathrm{dgn}\big(B|_{V_2 \times V_2}\big). \tag{4.1.12}$$

PROOF. The identity (4.1.12) follows from

$$\mathrm{Ker}(B) = \mathrm{Ker}\big(B|_{V_1 \times V_1}\big) \oplus \mathrm{Ker}\big(B|_{V_2 \times V_2}\big).$$

Let us prove (4.1.11). If B has infinite index in V_1 or in V_2 the result is trivial; suppose then that these indices are finite. Let $W_i \subset V_i$ be a B-negative subspace with $n_-\big(B|_{V_i\times V_i}\big) = \dim(W_i)$, $i = 1, 2$. By Remark 4.1.5 we can find a B-orthogonal decomposition $V_i = Z_i \oplus W_i$; it follows from Corollary 4.1.19 that B must be positive semi-definite in Z_i. Then:

$$V = (W_1 \oplus W_2) \oplus (Z_1 \oplus Z_2),$$

where, by Lemma 4.1.17, B is negative definite in $W_1 \oplus W_2$ and positive semi-definite in $Z_1 \oplus Z_2$. The identity (4.1.11) now follows from Corollary 4.1.7; the identity (4.1.10) follows by replacing B with $-B$. □

4.1.24. COROLLARY. *Let $B \in \mathrm{B}_{\mathrm{sym}}(V)$ and let $N \subset \mathrm{Ker}(B)$; if $W \subset V$ is any complementary subspace to N then the following identities hold:*

$$\begin{gathered} n_+(B) = n_+\big(B|_{W\times W}\big), \quad n_-(B) = n_-\big(B|_{W\times W}\big), \\ \mathrm{dgn}(B) = \mathrm{dgn}\big(B|_{W\times W}\big) + \dim(N); \end{gathered} \tag{4.1.13}$$

if $N = \mathrm{Ker}(B)$ then B is nondegenerate in W.

PROOF. The identities (4.1.13) follow immediately from Proposition 4.1.23, because $V = W \oplus N$ is a B-orthogonal decomposition. If $N = \mathrm{Ker}(B)$, the nondegeneracy of B in W is obvious. □

4.1.25. REMARK. If N is a subspace of $\mathrm{Ker}(B)$ then we can define by passing to the quotient a symmetric bilinear form $\overline{B} \in \mathrm{B}_{\mathrm{sym}}(V/N)$:

$$\overline{B}(v_1 + N, v_2 + N) = B(v_1, v_2), \quad v_1, v_2 \in V.$$

If $W \subset V$ is any subspace complementary to N, we have an isomorphism $q : W \to V/N$ obtained by restriction of the quotient map; moreover, $\overline{B}$ is the push-forward of $B|_{W\times W}$ by q. It follows from Corollary 4.1.24 (see also Example 4.1.4) that

$$n_+(B) = n_+\big(\overline{B}\big), \quad n_-(B) = n_-\big(\overline{B}\big), \quad \mathrm{dgn}(B) = \mathrm{dgn}\big(\overline{B}\big) + \dim(N);$$

if $N = \mathrm{Ker}(B)$ then it follows also that $\overline{B}$ is nondegenerate.

4.1.26. EXAMPLE. Lemma 4.1.17 does *not* hold if the subspaces V_1 and V_2 are not B-orthogonal. For instance, if $V = \mathbb{R}^2$ and if we consider the symmetric bilinear form B given in (4.1.4), then $n_-(B) = n_+(B) = 1$, but we can write $\mathbb{R}^2$ as the direct sum of the subspaces generated respectively by $v_1 = (0, 1)$ and $v_2 = (1, 2)$, that are both B-negative.

In the next proposition we generalize the result of Lemma 4.1.17 by showing that if $V = V_1 \oplus V_2$, where V_1 and V_2 are B-negative subspaces such that the product of elements of V_1 with elements of V_2 is “relatively small with respect to their lengths”, then V is B-negative.

4.1.27. PROPOSITION. *Let $B \in \mathrm{B}_{\mathrm{sym}}(V)$ and assume that V is written as the direct sum of B-negative subspaces $V = V_1 \oplus V_2$; if for all $v_1 \in V_1$ and $v_2 \in V_2$, with $v_1, v_2 \neq 0$, it is*

$$B(v_1, v_2)^2 < B(v_1, v_1)B(v_2, v_2) \tag{4.1.14}$$

then B is negative definite in V.

PROOF. Let $v \in V$ be non zero and write $v = v_1 + v_2$, with $v_1 \in V_1$ and $v_2 \in V_2$. We need to show that $B(v,v) < 0$, and clearly it suffices to consider the case that both v_1 and v_2 are non zero. In this case, the hypothesis (4.1.14) together with Proposition 4.1.15 imply that the two-dimensional subspace generated by v_1 and v_2 is B-negative, which concludes the proof. □

4.1.28. REMARK. It can also be shown a version of Proposition 4.1.27 assuming only that B is negative semi-definite in V_1 and in V_2, and that

$$B(v_1,v_2)^2 \le B(v_1,v_1)B(v_2,v_2), \tag{4.1.15}$$

for all $v_1 \in V_1$, $v_2 \in V_2$. In this case, the conclusion is that B is negative semi-definite in V (see Exercise 4.2).

4.1.1. The evolution of the index of a one-parameter family of symmetric bilinear forms.

In this subsection we will study the evolution of the function $n_-(B(t))$, where $t \mapsto B(t)$ is a one parameter family of symmetric bilinear forms on a space V.

We make the convention that in this subsection V will always denote a *finite dimensional real vector space*:

$$\dim(V) < +\infty.$$

We choose an arbitrary norm in V denoted by $\|\cdot\|$; we then define the *norm of a bilinear form* $B \in \mathrm{B}(V)$ by setting:

$$\|B\| = \sup_{\substack{\|v\|\le 1\\ \|w\|\le 1}} |B(v,w)|.$$

Observe that, since V and $\mathrm{B}(V)$ are finite dimensional, then any norm in these spaces induces the same topology.

We will first show that the condition $n_-(B) \ge k$ (for some fixed k) is an *open* condition.

4.1.29. LEMMA. *Let $k \ge 0$ be fixed; the set of symmetric bilinear forms $B \in \mathrm{B}_{\mathrm{sym}}(V)$ such that $n_-(B) \ge k$ is open in $\mathrm{B}_{\mathrm{sym}}(V)$.*

PROOF. Let $B \in \mathrm{B}_{\mathrm{sym}}(V)$ with $n_-(B) \ge k$; then, there exists a k-dimensional B-negative subspace $W \subset V$. Since the unit sphere of W is compact, we have:

$$\sup_{\substack{v\in W\\ \|v\|=1}} B(v,v) = c < 0;$$

it now follows directly that if $A \in \mathrm{B}_{\mathrm{sym}}(V)$ and $\|A - B\| < |c|/2$ then A is negative definite in W, and therefore $n_-(A) \ge k$. □

4.1.30. COROLLARY. *Let $k \ge 0$ be fixed; the set of nondegenerate symmetric bilinear forms $B \in \mathrm{B}_{\mathrm{sym}}(V)$ such that $n_-(B) = k$ is open in $\mathrm{B}_{\mathrm{sym}}(V)$.*

PROOF. If $B \in \mathrm{B}_{\mathrm{sym}}(V)$ is nondegenerate and $n_-(B) = k$, then $n_+(B) = \dim(V) - k$ (see Corollary 4.1.20); by Lemma 4.1.29, for A in a neighborhood of B in $\mathrm{B}_{\mathrm{sym}}(V)$ we have $n_-(A) \ge k$ and $n_+(A) \ge \dim(V) - k$, from which we get $n_-(A) = k$ and $\mathrm{dgn}(A) = 0$. □

4.1.31. COROLLARY. *Let $t \mapsto B(t)$ be a continuous curve in $\mathrm{B}_{\mathrm{sym}}(V)$ defined in some interval $I \subset \mathbb{R}$; if $B(t)$ is nondegenerate for all $t \in I$, then $n_-(B(t))$ and $n_+(B(t))$ are constant in I.*

PROOF. By Corollary 4.1.30, the set of instants $t \in I$ such that $n_-(B(t)) = k$ is open in I for each $k = 0, \dots, \dim(V)$ fixed. The conclusion follows from the connectedness of I. □

Corollary 4.1.31 tells us that the index $n_-(B(t))$ and the co-index $n_+(B(t))$ can only change when $B(t)$ becomes degenerate; in the next Theorem we show how to compute this change when $t \mapsto B(t)$ is of class C^1:

4.1.32. THEOREM. *Let $B\colon\ [t_0, t_1[\ \to \mathrm{B}_{\mathrm{sym}}(V)$ be a curve of class C^1; write $N = \mathrm{Ker}\big(B(t_0)\big)$. Suppose that the bilinear form $B'(t_0)|_{N\times N}$ is nondegenerate; then there exists $\varepsilon > 0$ such that for $t \in\ ]t_0, t_0+\varepsilon[$ the bilinear form $B(t)$ is nondegenerate, and the following identities hold:*

$$n_+(B(t)) = n_+(B(t_0)) + n_+\big(B'(t_0)|_{N\times N}\big),$$
$$n_-(B(t)) = n_-(B(t_0)) + n_-\big(B'(t_0)|_{N\times N}\big).$$

The proof of Theorem 4.1.32 will follow easily from the following:

4.1.33. LEMMA. *Let $B\colon\ [t_0, t_1[\ \to \mathrm{B}_{\mathrm{sym}}(V)$ be a curve of class C^1; write $N = \mathrm{Ker}(B(t_0))$. If $B(t_0)$ is positive semi-definite and $B'(t_0)|_{N\times N}$ is positive definite, then there exists $\varepsilon > 0$ such that $B(t)$ is positive definite for $t \in\ ]t_0, t_0+\varepsilon[$.*

PROOF. Let $W \subset V$ be a subspace complementary to N; it follows from Corollary 4.1.24 that $B(t_0)$ is nondegenerate in W, and from Corollary 4.1.14 that $B(t_0)$ is positive definite in W. Choose any norm in V; since the unit sphere of W is compact, we have:

$$\inf_{\substack{w\in W\\ \|w\|=1}} B(t_0)(w,w) = c_0 > 0; \tag{4.1.16}$$

similarly, since $B'(t_0)$ is positive definite in N we have:

$$\inf_{\substack{n\in N\\ \|n\|=1}} B'(t_0)(n,n) = c_1 > 0. \tag{4.1.17}$$

Since B is continuous, there exists $\varepsilon > 0$ such that

$$\|B(t) - B(t_0)\| \le \frac{c_0}{2}, \quad t \in [t_0, t_0+\varepsilon[\,,$$

and it follows from (4.1.16) that:

$$\inf_{\substack{w\in W\\ \|w\|=1}} B(t)(w,w) \ge \frac{c_0}{2} > 0, \quad t \in [t_0, t_0+\varepsilon[\,. \tag{4.1.18}$$

Since B is differentiable at t_0 we can write:

$$B(t) = B(t_0) + (t-t_0)B'(t_0) + r(t), \quad \text{with } \lim_{t\to t_0} \frac{r(t)}{t-t_0} = 0, \tag{4.1.19}$$

and then, by possibly choosing a smaller $\varepsilon > 0$, we get:

$$\|r(t)\| \le \frac{c_1}{2}(t - t_0), \quad t \in [t_0, t_0 + \varepsilon[\,; \tag{4.1.20}$$

from (4.1.17), (4.1.19) and (4.1.20) it follows:

$$\inf_{\substack{n \in N\\ \|n\|=1}} B(t)(n, n) \ge \frac{c_1}{2}(t - t_0), \quad t \in \,]t_0, t_0 + \varepsilon[\,. \tag{4.1.21}$$

From (4.1.18) and (4.1.21) it follows that $B(t)$ is positive definite in W and in N for $t \in \,]t_0, t_0 + \varepsilon[$; taking $c_3 = \|B'(t_0)\| + \frac{c_1}{2}$ we obtain from (4.1.19) and (4.1.20) that:

$$\big|B(t)(w, n)\big| \le (t - t_0)c_3, \quad t \in [t_0, t_0 + \varepsilon[\,, \tag{4.1.22}$$

provided that $w \in W$, $n \in N$ and $\|w\| = \|n\| = 1$. By possibly taking a smaller $\varepsilon > 0$, putting together (4.1.18), (4.1.21) and (4.1.22) we obtain:

$$\begin{aligned} B(t)(w, n)^2 \le (t - t_0)^2 c_3^2 < \frac{c_0 c_1}{4}(t - t_0)\\ \le B(t)(w, w)\, B(t)(v, v), \quad t \in \,]t_0, t_0 + \varepsilon[\,, \end{aligned} \tag{4.1.23}$$

for all $w \in W$, $n \in N$ with $\|w\| = \|n\| = 1$; but (4.1.23) implies:

$$B(t)(w, n)^2 < B(t)(w, w)\, B(t)(n, n), \quad t \in \,]t_0, t_0 + \varepsilon[\,,$$

for all $w \in W$, $n \in N$ non zero. The conclusion follows now from Proposition 4.1.27. □

PROOF OF THEOREM 4.1.32. By Theorem 4.1.10 there exists a decomposition $V = V_+ \oplus V_- \oplus N$ where V_+ and V_- are respectively a $B(t_0)$-positive and a $B(t_0)$-negative subspace; similarly, we can write $N = N_+ \oplus N_-$ where N_+ is a $B'(t_0)$-positive and N_- is a $B'(t_0)$-negative subspace. Obviously:

$$n_+(B(t_0)) = \dim(V_+), \quad n_-(B(t_0)) = \dim(V_-),$$
$$n_+\big(B'(t_0)|_{N\times N}\big) = \dim(N_+), \quad n_-\big(B'(t_0)|_{N\times N}\big) = \dim(N_-);$$

applying Lemma 4.1.33 to the restriction of B to $V_+ \oplus N_+$ and to the restriction of $-B$ to $V_- \oplus N_-$ we conclude that there exists $\varepsilon > 0$ such that $B(t)$ is positive definite in $V_+ \oplus N_+$ and negative definite in $V_- \oplus N_-$ for $t \in \,]t_0, t_0 + \varepsilon\,[$;the conclusion now follows from Corollary 4.1.7 and from Proposition 4.1.9. □

4.1.34. COROLLARY. *If $t \mapsto B(t) \in \mathrm{B}_{\mathrm{sym}}(V)$ is a curve of class C^1 defined in a neighborhood of the instant $t_0 \in \mathbb{R}$ and if $B'(t_0)|_{N\times N}$ is nondegenerate, where $N = \mathrm{Ker}\big(B(t_0)\big)$, then for $\varepsilon > 0$ sufficiently small we have:*

$$n_+(B(t_0 + \varepsilon)) - n_+(B(t_0 - \varepsilon)) = \mathrm{sgn}\big(B'(t_0)|_{N\times N}\big).$$

PROOF. It follows from Theorem 4.1.32 that for $\varepsilon > 0$ sufficiently small we have:

$$n_+(B(t_0 + \varepsilon)) = n_+(B(t_0)) + n_+\big(B'(t_0)|_{N\times N}\big); \tag{4.1.24}$$

applying Theorem 4.1.32 to the curve $t \mapsto B(-t)$ we obtain:

$$n_+(B(t_0 - \varepsilon)) = n_+(B(t_0)) + n_-\big(B'(t_0)|_{N\times N}\big). \tag{4.1.25}$$

The conclusion follows by taking the difference of (4.1.24) and (4.1.25). □

We will need a *uniform version* of Theorem 4.1.32 for technical reasons:

4.1.35. PROPOSITION. *Let $\mathcal{X}$ be a topological space and let be given a continuous map*

$$\mathcal{X} \times [t_0, t_1[\ni (\lambda, t) \longmapsto B_\lambda(t) = B(\lambda, t) \in \mathrm{B}_{\mathrm{sym}}(V)$$

differentiable in t, such that $\frac{\partial B}{\partial t}$ is also continuous in $\mathcal{X} \times [t_0, t_1[$.

Write $N_\lambda = \mathrm{Ker}\big(B_\lambda(t_0)\big)$; assume that $\dim(N_\lambda)$ does not depend on $\lambda \in \mathcal{X}$ and that $B'_{\lambda_0}(t_0) = \frac{\partial B}{\partial t}(\lambda_0, t_0)$ is nondegenerate in N_{λ_0} for some $\lambda_0 \in \mathcal{X}$. Then, there exists $\varepsilon > 0$ and a neighborhood $\mathfrak{U}$ of λ_0 in $\mathcal{X}$ such that $B'_\lambda(t_0)$ is nondegenerate on N_λ and such that $B_\lambda(t)$ is nondegenerate on V for every $\lambda \in \mathfrak{U}$ and for every $t \in\,]t_0, t_0 + \varepsilon[$.

PROOF. We will show first that the general case can be reduced to the case that N_λ does not depend on $\lambda \in \mathcal{X}$. To this aim, let $k = \dim(N_\lambda)$, that by hypothesis does not depend on λ. Since the kernel of a bilinear form coincides with the kernel of its associated linear operator, it follows from Proposition 2.4.10 that the map $\lambda \mapsto N_\lambda \in G_k(V)$ is continuous in $\mathcal{X}$; now, using Proposition 2.4.6 we find a continuous map $A\colon \mathfrak{U} \to \mathrm{GL}(V)$ defined in a neighborhood $\mathfrak{U}$ of λ_0 in $\mathcal{X}$ such that for all $\lambda \in \mathfrak{U}$, the isomorphism $A(\lambda)$ takes N_{λ_0} onto N_λ. Define:

$$\overline{B}_\lambda(t) = A(\lambda)^{\#}\big(B_\lambda(t)\big) = B_\lambda(t)\big(A(\lambda)\cdot, A(\lambda)\cdot\big),$$

for all $\lambda \in \mathfrak{U}$ and all $t \in [t_0, t_1[$. Then, $\mathrm{Ker}\big(\overline{B}_\lambda(t_0)\big) = N_{\lambda_0}$ for all $\lambda \in \mathfrak{U}$; moreover, the map $\overline{B}$ defined in $\mathfrak{U} \times [t_0, t_1[$ satisfies the hypotheses of the Proposition, and the validity of the thesis for $\overline{B}$ will imply the validity of the thesis also for B.

The above argument shows that there is no loss of generality in assuming that:

$$\mathrm{Ker}(B_\lambda(t_0)) = N,$$

for all $\lambda \in \mathcal{X}$. We split the remaining of the proof into two steps.

(1) *Suppose that $B_{\lambda_0}(t_0)$ is positive semi-definite and that $B'_{\lambda_0}(t_0)$ is positive definite in N.*

Let W be a subspace complementary to N in V; then $B_{\lambda_0}(t_0)$ is positive definite in W. It follows that $B_\lambda(t_0)$ is positive definite in W and that $B'_\lambda(t_0)$ is positive definite in N for all λ in a neighborhood $\mathfrak{U}$ of λ_0 in $\mathcal{X}$. Observe that, by hypothesis, $\mathrm{Ker}(B_\lambda(t_0)) = N$ for all $\lambda \in \mathfrak{U}$. Then, for all $\lambda \in \mathfrak{U}$, Lemma 4.1.33 gives us the existence of a positive number $\varepsilon(\lambda)$ such that $B_\lambda(t)$ is positive definite for all $t \in\,]t_0, t_0 + \varepsilon(\lambda)[$; we only need to look more closely at the estimates done in the proof of Lemma 4.1.33 to see that it is possible to choose $\varepsilon > 0$ independently of λ, when λ runs in a sufficiently small neighborhood of λ_0 in $\mathcal{X}$.

The only estimate that is delicate appears in (4.1.20). Formula (4.1.19) defines now a function $r_\lambda(t)$; for each $\lambda \in \mathfrak{U}$, we apply the mean value inequality to the function $t \mapsto \sigma(t) = B_\lambda(t) - tB'_\lambda(t_0)$ and we obtain:

$$\begin{aligned}\|\sigma(t) - \sigma(t_0)\| = \|r_\lambda(t)\| &\le (t - t_0) \sup_{s\in[t_0,t]} \|\sigma'(s)\| \\ &= (t - t_0) \sup_{s\in[t_0,t]} \|B'_\lambda(s) - B'_\lambda(t_0)\|.\end{aligned}$$

With the above estimate it is now easy to get the desired conclusion.

(2) *Let us prove the general case.*

Keeping in mind that $\mathrm{Ker}(B_\lambda(t_0)) = N$ does not depend on $\lambda \in \mathcal{X}$, we repeat the proof of Theorem 4.1.32 replacing $B(t_0)$ by $B_\lambda(t_0)$, $B'(t_0)$ by $B'_{\lambda_0}(t_0)$ and $B(t)$ by $B_\lambda(t)$; we use step (1) above instead of Lemma 4.1.33 and the proof is completed.

□

4.1.36. EXAMPLE. Theorem 4.1.32 and its Corollary 4.1.34 *do not hold* without the hypothesis that $B'(t_0)$ be nondegenerate in $N = \mathrm{Ker}(B(t_0))$; counterexamples are easy to produce by considering diagonal matrices $B(t) \in \mathrm{B_{sym}}(\mathbb{R}^n)$. A naive analysis of the case in which the bilinear forms $B(t)$ are simultaneously diagonalizable would suggest the conjecture that when $B'(t_0)$ is degenerate in $\mathrm{Ker}(B(t_0))$ then it would be possible to determine the variation of the co-index of $B(t)$ when t passes through t_0 by using higher order terms on the Taylor expansion of $B(t)|_{N\times N}$ around $t = t_0$. The following example show that this is *not* possible.

Consider the curves $B_1, B_2 \colon \mathbb{R} \to \mathrm{B_{sym}}(\mathbb{R}^2)$ given by:

$$B_1(t) = \begin{pmatrix} 1 & t \\ t & t^3 \end{pmatrix}, \quad B_2(t) = \begin{pmatrix} 1 & t^2 \\ t^2 & t^3 \end{pmatrix};$$

we have $B_1(0) = B_2(0)$ and $N = \mathrm{Ker}\big(B_1(0)\big) = \mathrm{Ker}\big(B_2(0)\big) = \{0\} \oplus \mathbb{R}$. Observe that $B_1(t)|_{N\times N} = B_2(t)|_{N\times N}$ for all $t \in \mathbb{R}$, so that the Taylor expansion of B_1 coincides with that of B_2 in N; on the other hand, for $\varepsilon > 0$ sufficiently small, we have:

$$\begin{aligned} n_+(B_1(\varepsilon)) - n_+(B_1(-\varepsilon)) &= 1 - 1 = 0, \\ n_+(B_2(\varepsilon)) - n_+(B_2(-\varepsilon)) &= 2 - 1 = 1. \end{aligned}$$

Our next goal is to prove that the basis provided by Sylvester's Inertia Theorem can be written as a differentiable function of the parameter t when $B(t)$ depends differentiably on t. Towards this goal, we consider the action of the general linear group $\mathrm{GL}(V)$ in the space $\mathrm{B_{sym}}(V)$ given by:

(4.1.26)
$$\mathrm{GL}(V) \times \mathrm{B_{sym}}(V) \ni (T, B) \longmapsto T_\#(B) = B(T^{-1}\cdot, T^{-1}\cdot) \in \mathrm{B_{sym}}(V);$$

it follows from Sylvester's Inertia Theorem (Theorem 4.1.10) that the orbits of this action are the sets:

$$\mathrm{B}^{p,q}_{\mathrm{sym}}(V) = \{B \in \mathrm{B_{sym}}(V) : n_+(B) = p,\ n_-(B) = q\},$$

with $p+q=0,1,\ldots,\dim(V)$. Moreover, for p and q fixed, the sets

$$\{B\in \mathrm{B}_{\mathrm{sym}}(V) : n_+(B)\ge p,\ n_-(B)\ge q\}\quad \text{and}$$
$$\{B\in \mathrm{B}_{\mathrm{sym}}(V) : n_+(B)\le p,\ n_-(B)\le q\}$$

are respective an open and a closed subset of $\mathrm{B}_{\mathrm{sym}}(V)$, by Lemma 4.1.29. It follows that the set $\mathrm{B}^{p,q}_{\mathrm{sym}}(V)$ is locally closed in $\mathrm{B}_{\mathrm{sym}}(V)$. From these observation we deduce the following

4.1.37. LEMMA. *The set* $\mathrm{B}^{p,q}_{\mathrm{sym}}(V)$ *is a connected embedded submanifold of* $\mathrm{B}_{\mathrm{sym}}(V)$ *for any integers* $p,q\ge 0$ *with* $p+q=0,1,\ldots,\dim(V)$.

PROOF. The fact that $\mathrm{B}^{p,q}_{\mathrm{sym}}(V)$ is an embedded submanifold of $\mathrm{B}_{\mathrm{sym}}(V)$ follows from Theorem 2.1.12. The connectedness of $\mathrm{B}^{p,q}_{\mathrm{sym}}(V)$ follows from the fact that the restriction of the action (4.1.26) to $\mathrm{GL}_+(V)$ is still transitive in $\mathrm{B}^{p,q}_{\mathrm{sym}}(V)$; this last statement follows from the fact that, once an orientation has been fixed in V, the basis $(b_i)_{i=1}^n$ given by the Sylvester's Inertia Theorem can be chosen positively oriented (possibly replacing b_1 with $-b_1$). □

4.1.38. COROLLARY. *The set of nondegenerate symmetric bilinear forms in* V *is an open subset of* $\mathrm{B}_{\mathrm{sym}}(V)$ *whose (arc-)connected components are the sets* $\mathrm{B}^{k,n-k}_{\mathrm{sym}}(V)$, $k=0,1,\ldots,n$, *where* $n=\dim(V)$.

PROOF. It follows from Corollary 4.1.30 and Lemma 4.1.37. □

We finally obtain the desired extension of Sylvester's Inertia Theorem:

4.1.39. PROPOSITION. *Given a curve* $B\colon [a,b]\to \mathrm{B}_{\mathrm{sym}}(V)$ *of class* C^k ($0\le k\le +\infty$) *such that the integers* $n_-(B(t))$ *and* $n_+(B(t))$ *do not depend on* $t\in[a,b]$, *then there exist maps* $b_i\colon [a,b]\to V$ *of class* C^k, $i=1,\ldots,n$, *such that for each* $t\in[a,b]$ *the vectors* $(b_i(t))_{i=1}^n$ *form a basis of* V *in which* $B(t)$ *assumes the canonical form* (4.1.7).

PROOF. Let p and q be such that $n_+(B(t))=p$, $n_-(B(t))=q$ for all $t\in[a,b]$; keeping in mind the transitive action (4.1.26) of $\mathrm{GL}(V)$ on $\mathrm{B}^{p,q}_{\mathrm{sym}}(V)$, it follows from Corollary 2.1.15 that, for $B_0\in \mathrm{B}^{p,q}_{\mathrm{sym}}(V)$ fixed, the map

$$\mathrm{GL}(V)\ni T\longmapsto T_\#(B_0)=B_0(T^{-1}\cdot,T^{-1}\cdot)\in \mathrm{B}^{p,q}_{\mathrm{sym}}(V)$$

is a differentiable fibration. It follows from Remark 2.1.18 that there exists a map $T\colon [a,b]\to \mathrm{GL}(V)$ of class C^k such that $T(t)_\#(B_0)=B(t)$ for all $t\in[a,b]$. Choosing a basis $(b_i)_{i=1}^n$ of V with respect to which B_0 has the canonical form (4.1.7), we define $b_i(t)=T(t)\cdot b_i$ for $i=1,\ldots,n$ and $t\in[a,b]$. This concludes the proof. □

4.2. Definition and Computation of the Maslov Index

In this section we will introduce the Maslov index (relative to a fixed Lagrangian subspace L_0) of a curve in the Lagrangian Grassmannian of a symplectic space (V,ω); this index is an integer number that corresponds to a sort of algebraic count of the intersections of this curve with the subset $\Lambda^{\ge 1}(L_0)$.

The definition of Maslov index will be given in terms of relative homology, and we will therefore assume familiarity with the machinery introduced in Section 3.3. We will use several properties of the Lagrangian Grassmannian Λ that were discussed in Section 2.5 (especially from Subsection 2.5.1). It will be needed to compute the fundamental group of Λ, and to this aim we will use the homotopy long exact sequence of a fibration, studied in Section 3.2. This computation follows the same line of the examples that appear in Subsection 3.2.1; following the notations of that subsection, we will omit for simplicity the specification of the basepoint of the fundamental groups studied. As a matter of facts, all the fundamental groups that will appear are abelian, so that the fundamental groups corresponding to different choices of basepoint can be canonically identified (see Corollary 3.1.12 and Remarks 3.1.13 and 3.3.34). Finally, in order to relate the fundamental group of Λ with its first singular homology group we will use the Hurewicz's homomorphism, presented in Subsection 3.3.1.

Throughout this section we will consider a fixed symplectic space (V,ω), with $\dim(V) = 2n$; we will denote by Λ the Lagrangian Grassmannian of this symplectic space. All the curves considered will be tacitly meant to be "continuous curves"; moreover, we will often use the fact that any two Lagrangian subspaces admit a common complementary Lagrangian subspace (see Remark 2.5.18).

We know that the Lagrangian Grassmannian Λ is diffeomorphic to the quotient $\mathrm{U}(n)/\mathrm{O}(n)$ (see Corollary 2.5.12). Consider the homomorphism:

$$d = \det{}^2\colon \mathrm{U}(n) \longrightarrow S^1,$$

where $S^1 \subset \mathbb{C}$ denotes the unit circle; if $A \in \mathrm{O}(n)$ then clearly $\det(A) = \pm 1$, hence $\mathrm{O}(n) \subset \mathrm{Ker}(d)$. It follows that d induces, by passing to the quotient, a map:

$$\bar{d}\colon \mathrm{U}(n)/\mathrm{O}(n) \longrightarrow S^1, \tag{4.2.1}$$

given by $\bar{d}(A \cdot \mathrm{O}(n)) = \det^2(A)$. We have the following:

4.2.1. PROPOSITION. *The fundamental group of the Lagrangian Grassmannian $\Lambda \cong \mathrm{U}(n)/\mathrm{O}(n)$ is infinite cyclic; more explicitly, the map* (4.2.1) *induces an isomorphism:*

$$\bar{d}_*\colon \pi_1\big(\mathrm{U}(n)/\mathrm{O}(n)\big) \xrightarrow{\ \cong\ } \pi_1(S^1) \cong \mathbb{Z}.$$

PROOF. It follows from Corollary 2.1.16 that $\bar{d}$ is a fibration with typical fiber $\mathrm{Ker}(d)/\mathrm{O}(n)$. It is easy to see that the action of $\mathrm{SU}(n)$ on $\mathrm{Ker}(d)/\mathrm{O}(n)$ by left translation is transitive, and that the isotropy group of the class $1 \cdot \mathrm{O}(n)$ of the neutral element is $\mathrm{SU}(n) \cap \mathrm{O}(n) = \mathrm{SO}(n)$; it follows from Corollary 2.1.9 that we have a diffeomorphism

$$\mathrm{SU}(n)/\mathrm{SO}(n) \cong \mathrm{Ker}(d)/\mathrm{O}(n)$$

induced by the inclusion of $\mathrm{SU}(n)$ in $\mathrm{Ker}(d)$. Since $\mathrm{SU}(n)$ is simply connected and $\mathrm{SO}(n)$ is connected, it follows easily from the homotopy long exact sequence of the fibration $\mathrm{SU}(n) \to \mathrm{SU}(n)/\mathrm{SO}(n)$ that $\mathrm{SU}(n)/\mathrm{SO}(n)$ is simply connected. Then, $\mathrm{Ker}(d)/\mathrm{O}(n)$ is also simply connected, and the homotopy exact sequence of the fibration $\bar{d}$ becomes:

$$0 \longrightarrow \pi_1\big(\mathrm{U}(n)/\mathrm{O}(n)\big) \xrightarrow[\cong]{\bar{d}_*} \pi_1(S^1) \longrightarrow 0$$

This concludes the proof. □

4.2.2. COROLLARY. *The first singular homology group $H_1(\Lambda)$ of Λ is infinite cyclic.*

PROOF. Since Λ is arc-connected and $\pi_1(\Lambda)$ is abelian, it follows from Theorem 3.3.33 that the Hurewicz's homomorphism is an isomorphism:

$$\Theta\colon \pi_1(\Lambda) \xrightarrow{\cong} H_1(\Lambda) \tag{4.2.2}$$ □

4.2.3. COROLLARY. *For a fixed Lagrangian $L_0 \in \Lambda$, the inclusion*

$$\mathfrak{q}\colon (\Lambda, \emptyset) \longrightarrow (\Lambda, \Lambda^0(L_0))$$

induces an isomorphism:

$$\mathfrak{q}_*\colon H_1(\Lambda) \xrightarrow{\cong} H_1(\Lambda, \Lambda^0(L_0)); \tag{4.2.3}$$

in particular, $H_1(\Lambda, \Lambda^0(L_0))$ is infinite cyclic.

PROOF. It follows from Remark 2.5.3 and from Example 3.3.19. □

Let $\ell : [a, b] \to \Lambda$ be a curve with endpoints in $\Lambda^0(L_0)$, i.e., $\ell(a), \ell(b) \in \Lambda^0(L_0)$; then, ℓ defines a relative homology class in $H_1(\Lambda, \Lambda^0(L_0))$ (see Remarks 3.3.30 and 3.3.26). Our goal is now to show that the transverse orientation of $\Lambda^1(L_0)$ given in Definition 2.5.19 induces a canonical choice of a generator of the infinite cyclic group $H_1(\Lambda, \Lambda^0(L_0))$. Once this choice is made, we will be able to associate an integer number to each curve in Λ with endpoints in $\Lambda^0(L_0)$.

4.2.4. EXAMPLE. If we analyze the steps that lead us to the conclusion that $H_1(\Lambda, \Lambda^0(L_0))$ is isomorphic to $\mathbb{Z}$ we can compute explicitly a generator for this group. In first place, the curve

$$[\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \ni t \longmapsto A(t) = \begin{pmatrix} e^{it} & & & \\ & i & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & i \end{pmatrix} \in \mathrm{U}(n)$$

projects onto a closed curve $\overline{A}(t) = A(t) \cdot \mathrm{O}(n)$ in $\mathrm{U}(n)/\mathrm{O}(n)$; moreover,

$$[\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \ni t \longmapsto \det{}^2\big(A(t)\big) = (-1)^{n-1} e^{2it} \tag{4.2.4}$$

is a generator of the fundamental group of the unit circle S^1. It follows from Proposition 4.2.1 that $\overline{A}$ defines a generator of the fundamental group of $\mathrm{U}(n)/\mathrm{O}(n)$.

Denoting by $\Lambda(\mathbb{R}^{2n})$ the Lagrangian Grassmannian of the symplectic space $\mathbb{R}^{2n}$ endowed with the canonical symplectic form, it follows from Proposition 2.5.11 that a diffeomorphism $\mathrm{U}(n)/\mathrm{O}(n) \cong \Lambda(\mathbb{R}^{2n})$ is given explicitly by:

$$\mathrm{U}(n)/\mathrm{O}(n) \ni A \cdot \mathrm{O}(n) \longmapsto A\big(\mathbb{R}^n \oplus \{0\}^n\big) \in \Lambda(\mathbb{R}^{2n}).$$

The Lagrangian $A(t)\big(\mathbb{R}^n \oplus \{0\}^n\big)$ is generated by the vectors[1]

$$\{e_1 \cos(t) + e_{n+1}\,\sin(t), e_{n+2}, \ldots, e_{2n}\},$$

where $(e_j)_{j=1}^{2n}$ denotes the canonical basis of $\mathbb{R}^{2n}$.

The choice of a symplectic basis $(b_j)_{j=1}^{2n}$ of V induces a diffeomorphism of Λ onto $\Lambda(\mathbb{R}^{2n})$ in an obvious way. Consider the Lagrangian $\ell(t)$ given by:

$$\ell(t) = \mathbb{R}\big(b_1 \cos(t) + b_{n+1}\,\sin(t)\big) + \sum_{j=n+2}^{2n} \mathbb{R} b_j; \tag{4.2.5}$$

then, the curve

$$[\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \ni t \longmapsto \ell(t) \in \Lambda \tag{4.2.6}$$

is a generator of $\pi_1(\Lambda)$. By the definition of the Hurewicz's homomorphism (see (3.3.25)) we have that the same curve (4.2.6) defines a generator of $H_1(\Lambda)$; since the isomorphism (4.2.3) is induced by inclusion, we have that the curve (4.2.6) is also a generator of $H_1(\Lambda, \Lambda^0(L_0))$.

4.2.5. LEMMA. *Let $A \in \mathrm{Sp}(V, \omega)$ be a symplectomorphism of (V, ω) and consider the diffeomorphism (also denoted by A) of Λ induced by the action of A; then the induced homomorphism in homology:*

$$A_*\colon H_p(\Lambda) \longrightarrow H_p(\Lambda)$$

is the identity map for all $p \in \mathbb{Z}$.

PROOF. Since $\mathrm{Sp}(V, \omega)$ is arc-connected, there exists a curve

$$[0, 1] \ni s \longmapsto A(s) \in \mathrm{Sp}(V, \omega)$$

such that $A(0) = A$ and $A(1) = \mathrm{Id}$. Define

$$[0, 1] \times \Lambda \ni (s, L) \longmapsto H_s(L) = A(s) \cdot L \in \Lambda;$$

then $H\colon A \cong \mathrm{Id}$ is a homotopy. The conclusion follows from Corollary 3.3.24. □

4.2.6. COROLLARY. *Let $L_0 \in \Lambda$ be a Lagrangian subspace of (V, ω) and let $A \in \mathrm{Sp}(V, \omega, L_0)$ (recall* (2.5.15)*); then the homomorphism*

$$A_*\colon H_1(\Lambda, \Lambda^0(L_0)) \longrightarrow H_1(\Lambda, \Lambda^0(L_0))$$

is the identity map.

PROOF. It follows from Lemma 4.2.5 and from the following commutative diagram:

$$\begin{array}{ccc} H_1(\Lambda) & \xrightarrow{A_* = \mathrm{Id}} & H_1(\Lambda) \\ {\scriptstyle \mathfrak{q}_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathfrak{q}_*} \\ H_1(\Lambda, \Lambda^0(L_0)) & \xrightarrow[A_*]{} & H_1(\Lambda, \Lambda^0(L_0)) \end{array}$$

where $\mathfrak{q}_*$ is given in (4.2.3). □

[1]The complex matrix $A(t)$ must be seen as a linear endomorphism of $\mathbb{R}^{2n}$; therefore, we need the identification of $n \times n$ complex matrices with $2n \times 2n$ real matrices (see Remark 1.2.9).

4.2.7. EXAMPLE. Consider a Lagrangian decomposition (L_0, L_1) of V and let L be an element in the domain of the chart φ_{L_0,L_1}, i.e., $L \in \Lambda^0(L_1)$. It follows directly from the definition of φ_{L_0,L_1} (see (2.5.3)) that the kernel of the symmetric bilinear form $\varphi_{L_0,L_1}(L) \in \mathrm{B_{sym}}(L_0)$ is $L_0 \cap L$, that is:

$$\mathrm{Ker}\big(\varphi_{L_0,L_1}(L)\big) = L_0 \cap L. \tag{4.2.7}$$

Then, we obtain that for each $k = 0, \dots, n$ the Lagrangian L belongs to $\Lambda^k(L_0)$ if and only if the kernel of $\varphi_{L_0,L_1}(L)$ has dimension k, that is:

$$\varphi_{L_0,L_1}\big(\Lambda^0(L_1) \cap \Lambda^k(L_0)\big) = \{B \in \mathrm{B_{sym}}(L_0) : \mathrm{dgn}(B) = k\}.$$

In particular, we have $L \in \Lambda^0(L_0)$ if and only if $\varphi_{L_0,L_1}(L)$ is nondegenerate.

4.2.8. EXAMPLE. Let $t \mapsto \ell(t)$ be a curve in Λ differentiable at $t = t_0$ and let (L_0, L_1) be a Lagrangian decomposition of V with $\ell(t_0) \in \Lambda^0(L_1)$. Then, for t in a neighborhood of t_0 we also have $\ell(t) \in \Lambda^0(L_1)$ and we can therefore define $\beta(t) = \varphi_{L_0,L_1}(\ell(t)) \in \mathrm{B_{sym}}(L_0)$. Let us determine the relation between $\beta'(t_0)$ and $\ell'(t_0)$; by Lemma 2.5.7 we have:

$$\beta'(t_0) = \mathrm{d}\varphi_{L_0,L_1}(\ell(t_0)) \cdot \ell'(t_0) = \big(\eta^{L_1}_{\ell(t_0),L_0}\big)_* \cdot \ell'(t_0).$$

Since $\eta^{L_1}_{\ell(t_0),L_0}$ fixes the points of $L_0 \cap \ell(t_0)$, we obtain in particular that the symmetric bilinear forms $\beta'(t_0) \in \mathrm{B_{sym}}(L_0)$ and $\ell'(t_0) \in \mathrm{B_{sym}}(\ell(t_0))$ coincide on $L_0 \cap \ell(t_0)$.

4.2.9. LEMMA. *Let $L_0 \in \Lambda$ be a fixed Lagrangian; assume given two curves*

$$\ell_1, \ell_2 \colon [a, b] \longrightarrow \Lambda$$

with endpoints in $\Lambda^0(L_0)$. Suppose that there exists a Lagrangian subspace $L_1 \in \Lambda$ complementary to L_0 such that $\Lambda^0(L_1)$ contains the images of both curves ℓ_1, ℓ_2; if we have

$$n_+\big(\varphi_{L_0,L_1}(\ell_1(t))\big) = n_+\big(\varphi_{L_0,L_1}(\ell_2(t))\big), \tag{4.2.8}$$

for $t = a$ and $t = b$, then the curves ℓ_1, ℓ_2 are homologous in $H_1(\Lambda, \Lambda^0(L_0))$.

PROOF. It follows from (4.2.8) and from Corollary 4.1.38 that there exist curves:

$$\sigma_1, \sigma_2 \colon [0, 1] \longrightarrow \mathrm{B_{sym}}(L_0)$$

such that $\sigma_1(t)$ and $\sigma_2(t)$ are nondegenerate for all $t \in [0, 1]$ and also:

$$\sigma_1(0) = \varphi_{L_0,L_1}(\ell_1(a)), \qquad \sigma_1(1) = \varphi_{L_0,L_1}(\ell_2(a)),$$
$$\sigma_2(0) = \varphi_{L_0,L_1}(\ell_1(b)), \qquad \sigma_2(1) = \varphi_{L_0,L_1}(\ell_2(b)).$$

Define $m_i = \varphi^{-1}_{L_0,L_1} \circ \sigma_i$, $i = 1, 2$; it follows from Example 4.2.7 that m_1 and m_2 have image in the set $\Lambda^0(L_0)$ and therefore they are homologous to zero in $H_1(\Lambda, \Lambda^0(L_0))$. Consider the concatenation $\ell = m_1^{-1} \cdot \ell_1 \cdot m_2$; it follows from Lemma 3.3.27 that ℓ_1 and ℓ are homologous in $H_1(\Lambda, \Lambda^0(L_0))$. We have that ℓ and ℓ_2 are curves in $\Lambda^0(L_1)$ with the same endpoints, and since $\Lambda^0(L_1)$ is homeomorphic to the vector space $\mathrm{B_{sym}}(L_0)$ it follows that ℓ and ℓ_2 are homotopic with fixed endpoints. By Corollary 3.3.29 we have that ℓ and ℓ_2 are homologous, which concludes the proof. □

4.2.10. DEFINITION. Let $\ell : [a, b] \to \Lambda$ be a curve of class C^1. We say that ℓ *intercepts transversally* the set $\Lambda^{\geq 1}(L_0)$ at the instant $t = t_0$ if $\ell(t_0) \in \Lambda^1(L_0)$ and $\ell'(t_0) \notin T_{\ell(t_0)}\Lambda^1(L_0)$; we say that such transverse intersection is *positive* (resp., *negative*) if the class of $\ell'(t_0)$ in the quotient $T_{\ell(t_0)}\Lambda/T_{\ell(t_0)}\Lambda^1(L_0)$ defines a positively oriented (resp., a negatively oriented) basis (recall Definition 2.5.19).

From Theorem 2.5.16 it follows that ℓ intercepts $\Lambda^{\geq 1}(L_0)$ transversally at the instant $t = t_0$ if and only if $\ell(t_0) \in \Lambda^1(L_0)$ and the symmetric bilinear form $\ell'(t_0)$ is non zero in the space $L_0 \cap \ell(t_0)$; such intersection will be positive (resp., negative) if $\ell'(t_0)$ is positive definite (resp., negative definite) in $L_0 \cap \ell(t_0)$.

4.2.11. LEMMA. *Let $L_0 \in \Lambda$ be a Lagrangian subspace and let*

$$\ell_1, \ell_2 \colon [a, b] \longrightarrow \Lambda$$

be curves of class C^1 with endpoints in $\Lambda^0(L_0)$ that intercept $\Lambda^{\geq 1}(L_0)$ only once; suppose that such intersection is transverse and positive. Then, we have that ℓ_1 and ℓ_2 are homologous in $H_1(\Lambda, \Lambda^0(L_0))$, and either one of these curves defines a generator of $H_1(\Lambda, \Lambda^0(L_0)) \cong \mathbb{Z}$.

PROOF. Thanks to Lemma 3.3.25, we can assume that ℓ_1, ℓ_2 intercept $\Lambda^1(L_0)$ at the same instant $t_0 \in\,]a, b[$. By Proposition 1.4.38 there exists a symplectomorphism $A \in \mathrm{Sp}(V, \omega, L_0)$ such that $A(\ell_1(t_0)) = \ell_2(t_0)$. It follows from Corollary 4.2.6 that $A \circ \ell_1$ and ℓ_1 are homologous in $H_1(\Lambda, \Lambda^0(L_0))$; note that also $A \circ \ell_1$ intercepts $\Lambda^{\geq 1}(L_0)$ only at the instant t_0 and that such intersection is transverse and positive (see Proposition 2.5.20).

The above argument shows that there is no loss of generality in assuming $\ell_1(t_0) = \ell_2(t_0)$. By Lemma 3.3.27, it is enough to show that the restriction $\ell_1|_{[t_0-\varepsilon, t_0+\varepsilon]}$ is homologous to $\ell_2|_{[t_0-\varepsilon, t_0+\varepsilon]}$ for some $\varepsilon > 0$. Let $L_1 \in \Lambda$ be a common complementary Lagrangian to $\ell_1(t_0)$ and L_0; for t in a neighborhood of t_0 we can write $\beta_i(t) = \varphi_{L_0, L_1} \circ \ell_i(t)$, $i = 1, 2$. By Example 4.2.8 we have that $\beta_i'(t_0)$ and $\ell_i'(t_0)$ coincide in $L_0 \cap \ell_i(t_0) = \mathrm{Ker}\big(\beta_i(t_0)\big)$ (see (4.2.7)); since by hypothesis $\ell_i'(t_0)$ is positive definite in the unidimensional space $L_0 \cap \ell_i(t_0)$, it follows from Theorem 4.1.32 (see also (4.1.25)) that for $\varepsilon > 0$ sufficiently small we have

$$n_+\big(\beta_i(t_0 + \varepsilon)\big) = n_+\big(\beta_i(t_0)\big) + 1, \quad n_+\big(\beta_i(t_0 - \varepsilon)\big) = n_+\big(\beta_i(t_0)\big). \tag{4.2.9}$$

Since $\beta_1(t_0) = \beta_2(t_0)$, it follows from (4.2.9) that

$$n_+\big(\beta_1(t_0 + \varepsilon)\big) = n_+\big(\beta_2(t_0 + \varepsilon)\big), \quad n_+\big(\beta_1(t_0 - \varepsilon)\big) = n_+\big(\beta_2(t_0 - \varepsilon)\big),$$

for $\varepsilon > 0$ sufficiently small. Now, it follows from Lemma 4.2.9 that the curve $\ell_1|_{[t_0-\varepsilon, t_0+\varepsilon]}$ is homologous to the curve $\ell_2|_{[t_0-\varepsilon, t_0+\varepsilon]}$ in $H_1(\Lambda, \Lambda^0(L_0))$. This concludes the proof of the first statement of the thesis.

To prove the second statement it suffices to exhibit a curve ℓ that has a unique intersection with $\Lambda^{\geq 1}(L_0)$, being such intersection transverse and positive, so that ℓ defines a generator of $H_1(\Lambda, \Lambda^0(L_0))$. Let $(b_j)_{j=1}^{2n}$ be a symplectic basis of V

such that $(b_j)_{j=1}^n$ is a basis of L_0 (see Lemma 1.4.35); consider the generator ℓ of $H_1(\Lambda, \Lambda^0(L_0))$ described in (4.2.5) and (4.2.6). It is easy to see that ℓ intercepts $\Lambda^{\geq 1}(L_0)$ only at the instant $t = \pi$ and $L_0 \cap \ell(\pi)$ is the unidimensional space generated by b_1; moreover, an easy calculation shows that:

$$\ell'(\pi)(b_1, b_1) = \omega(b_{n+1}, b_1) = -1; \tag{4.2.10}$$

it follows that ℓ^{-1} has a unique intersection with $\Lambda^{\geq 1}(L_0)$ and that this intersection is transverse and positive. By Lemma 3.3.27, the curve ℓ^{-1} is also a generator of $H_1(\Lambda, \Lambda^0(L_0))$, which concludes the proof. □

4.2.12. DEFINITION. Let $L_0 \in \Lambda$ be a fixed Lagrangian; we define an isomorphism

$$\mu_{L_0} \colon H_1(\Lambda, \Lambda^0(L_0)) \xrightarrow{\cong} \mathbb{Z} \tag{4.2.11}$$

as follows: choose a curve ℓ of class C^1 in Λ with endpoints in $\Lambda^0(L_0)$ such that ℓ has a unique intersection with $\Lambda^{\geq 1}(L_0)$ and such that this intersection is transverse and positive. Define μ_{L_0} by requiring that the homology class of ℓ be taken into the element $1 \in \mathbb{Z}$; by Lemma 4.2.11 the isomorphism (4.2.11) is well defined, i.e., independent of the choice of the curve ℓ.

Suppose now that $\ell : [a, b] \to \Lambda$ is an *arbitrary* curve with endpoints in $\Lambda^0(L_0)$, then we denote by $\mu_{L_0}(\ell) \in \mathbb{Z}$ the integer number that corresponds to the homology class of ℓ by the isomorphism (4.2.11); the number $\mu_{L_0}(\ell)$ is called the *Maslov index* of the curve ℓ relative to the Lagrangian L_0.

In the following Lemma we list some of the properties of the Maslov index:

4.2.13. LEMMA. *Let $\ell \colon [a, b] \to \Lambda$ be a curve with endpoints in $\Lambda^0(L_0)$; then we have:*

(1) *if $\sigma \colon [a', b'] \to [a, b]$ is a continuous map with $\sigma(a') = a$, $\sigma(b') = b$ then $\mu_{L_0}(\ell \circ \sigma) = \mu_{L_0}(\ell)$;*
(2) *if $m \colon [a', b'] \to \Lambda$ is a curve with endpoints in $\Lambda^0(L_0)$ such that $\ell(b) = m(a')$, then $\mu_{L_0}(\ell \cdot m) = \mu_{L_0}(\ell) + \mu_{L_0}(m)$;*
(3) *$\mu_{L_0}(\ell^{-1}) = -\mu_{L_0}(\ell)$;*
(4) *if $\mathrm{Im}(\ell) \subset \Lambda^0(L_0)$ then $\mu_{L_0}(\ell) = 0$;*
(5) *if $m \colon [a, b] \to \Lambda$ is homotopic to ℓ with free endpoints in $\Lambda^0(L_0)$ (see Definition 3.1.25) then $\mu_{L_0}(\ell) = \mu_{L_0}(m)$;*
(6) *there exists a neighborhood $\mathcal{U}$ of ℓ in $C^0([a, b], \Lambda)$ endowed with the compact-open topology such that, if $m \in \mathcal{U}$ has endpoints in $\Lambda^0(L_0)$, then $\mu_{L_0}(\ell) = \mu_{L_0}(m)$.*

PROOF. Property (1) follows from Lemma 3.3.25; Properties (2) and (3) follow from Lemma 3.3.27. Property (4) follows immediately from the definition of the group $H_1(\Lambda, \Lambda^0(L_0))$ (see (3.3.7)). Property (5) follows from Remark 3.3.30 and Property (6) follows from Theorem 3.1.27 and from Property (5). □

4.2.14. EXAMPLE. The Maslov index $\mu_{L_0}(\ell)$ can be seen as the *intersection number* of the curve ℓ with the subset $\Lambda^{\geq 1}(L_0) \subset \Lambda$; indeed, it follows from Lemma 4.2.13 (more specifically, from Properties (2), (3) and (4)) that if

$\ell\colon [a,b] \to \Lambda$ is a curve of class C^1 with endpoints in $\Lambda^0(L_0)$ that has only transverse intersections with $\Lambda^{\geq 1}(L_0)$ then the Maslov index $\mu_{L_0}(\ell)$ is the number of positive intersections of ℓ with $\Lambda^{\geq 1}(L_0)$ minus the number of negative intersections of ℓ with $\Lambda^{\geq 1}(L_0)$. As a matter of facts, these numbers are finite (see Example 4.2.17 below). In Corollary 4.2.18 we will give a generalization of this result.

We will now establish an explicit formula for the Maslov index μ_{L_0} in terms of a chart φ_{L_0,L_1} of Λ:

4.2.15. THEOREM. *Let $L_0 \in \Lambda$ be a Lagrangian subspace and let $\ell : [a,b] \to \Lambda$ be a given curve with endpoints in $\Lambda^0(L_0)$. If there exists a Lagrangian $L_1 \in \Lambda$ complementary to L_0 such that the image of ℓ is contained in $\Lambda^0(L_1)$, then the Maslov index $\mu_{L_0}(\ell)$ of ℓ is given by:*

$$\mu_{L_0}(\ell) = n_+\big(\varphi_{L_0,L_1}(\ell(b))\big) - n_+\big(\varphi_{L_0,L_1}(\ell(a))\big).$$

PROOF. By Lemma 4.2.9, it suffices to determine for each $i,j = 0,1,\ldots,n$ a curve $\beta_{i,j}\colon [0,1] \to \mathrm{B}_{\mathrm{sym}}(L_0)$ such that:

$$n_+\big(\beta_{i,j}(0)\big) = i, \qquad \mathrm{dgn}\big(\beta_{i,j}(0)\big) = 0, \tag{4.2.12}$$

$$n_+\big(\beta_{i,j}(1)\big) = j, \qquad \mathrm{dgn}\big(\beta_{i,j}(1)\big) = 0 \tag{4.2.13}$$

and such that the curve $\ell_{i,j} = \varphi_{L_0,L_1}^{-1} \circ \beta_{i,j}$ satisfies $\mu_{L_0}(\ell_{i,j}) = j - i$. If $i = j$, we simply take $\beta_{i,i}$ to be any constant curve such that $\beta_{i,i}(0)$ is nondegenerate and such that $n_+\big(\beta_{i,i}(0)\big) = i$.

Property (3) in the statement of Lemma 4.2.13 implies that there is no loss of generality in assuming $i < j$. Let us start with the case $j = i+1$; choose any basis of L_0 and define $\beta_{i,i+1}(t)$ as the bilinear form whose matrix representation in this basis is given by:

$$\beta_{i,i+1}(t) \sim \mathrm{diag}(\underbrace{1,1,\ldots,1}_{i \text{ times}}, t-\tfrac{1}{2}, \underbrace{-1,-1,\ldots,-1}_{n-i-1 \text{ times}}), \quad t \in [0,1],$$

where $\mathrm{diag}(\alpha_1,\ldots,\alpha_n)$ denotes the diagonal matrix with entries $\alpha_1,\ldots,\alpha_n$.

Then, we have:

$$n_+\big(\beta_{i,i+1}(0)\big) = i, \qquad \mathrm{dgn}\big(\beta_{i,i+1}(0)\big) = 0,$$
$$n_+\big(\beta_{i,i+1}(1)\big) = i+1, \qquad \mathrm{dgn}\big(\beta_{i,i+1}(1)\big) = 0;$$

moreover $\beta_{i,i+1}(t)$ is degenerate only at $t = \frac{1}{2}$ and the derivative $\beta'_{i,i+1}(\frac{1}{2})$ is positive definite in the unidimensional space $\mathrm{Ker}\big(\beta_{i,i+1}(\frac{1}{2})\big)$. It follows from Examples 4.2.7 and 4.2.8 that $\ell_{i,i+1}$ intercepts $\Lambda^{\geq 1}(L_0)$ only at $t = \frac{1}{2}$, and that such intersection is transverse and positive. By definition of Maslov index, we have:

$$\mu_{L_0}(\ell_{i,i+1}) = 1;$$

and this completes the construction of the curve $\beta_{i,j}$ in the case $j = i+1$.

Let us look now at the case $j > i+1$. For each $i = 0,\ldots,n$, let $B_i \in \mathrm{B}_{\mathrm{sym}}(L_0)$ be a nondegenerate symmetric bilinear form with $n_+(B_i) = i$; choose any curve $\tilde\beta_{i,i+1}\colon [0,1] \to \mathrm{B}_{\mathrm{sym}}(L_0)$ with $\tilde\beta_{i,i+1}(0) = B_i$ and $\tilde\beta_{i,i+1}(1) = B_{i+1}$

for $i = 0, \ldots, n-1$. It follows from Lemma 4.2.9 and from the first part of the proof that $\tilde{\ell}_{i,i+1} = \varphi_{L_0,L_1}^{-1} \circ \tilde{\beta}_{i,i+1}$ satisfies $\mu_{L_0}\big(\tilde{\ell}_{i,i+1}\big) = 1$; for $j > i+1$ define:

$$\beta_{i,j} = \tilde{\beta}_{i,i+1} \cdot \tilde{\beta}_{i+1,i+2} \cdot \cdots \cdot \tilde{\beta}_{j-1,j}.$$

Then, the curve $\beta_{i,j}$ satisfies (4.2.12), (4.2.13) and from Property (2) in the statement of Lemma 4.2.13 it follows that $\mu_{L_0}(\ell_{i,j}) = j - i$.

This concludes the proof. $\square$

4.2.16. DEFINITION. Given a curve $t \mapsto \ell(t) \in \Lambda$ of class C^1 we say that ℓ has a *nondegenerate intersection* with $\Lambda^{\geq 1}(L_0)$ at the instant $t = t_0$ if $\ell(t_0) \in \Lambda^{\geq 1}(L_0)$ and $\ell'(t_0)$ is nondegenerate in $L_0 \cap \ell(t_0)$.

4.2.17. EXAMPLE. If a curve ℓ in Λ has a nondegenerate intersection with $\Lambda^{\geq 1}(L_0)$ at the instant $t = t_0$, then this intersection is *isolated*, i.e., $\ell(t) \in \Lambda^0(L_0)$ for $t \neq t_0$ sufficiently close to t_0. To see this, choose a common complementary Lagrangian $L_1 \in \Lambda$ to L_0 and $\ell(t_0)$ and apply Theorem 4.1.32 to the curve $\beta = \varphi_{L_0,L_1} \circ \ell$, keeping in mind Examples 4.2.7 and 4.2.8.

Since $\Lambda^{\geq 1}(L_0)$ is closed in Λ, it follows that if a curve $\ell : [a,b] \to \Lambda$ has only nondegenerate intersections with $\Lambda^{\geq 1}(L_0)$, then $\ell(t) \in \Lambda^{\geq 1}(L_0)$ only at a finite number of instants $t \in [a,b]$.

We have the following corollary to Theorem 4.2.15:

4.2.18. COROLLARY. *Let $L_0 \in \Lambda$ be a Lagrangian subspace and let be given a curve $\ell\colon [a,b] \to \Lambda$ of class C^1 with endpoints in $\Lambda^0(L_0)$ that has only nondegenerate intersections with $\Lambda^{\geq 1}(L_0)$. Then, $\ell(t) \in \Lambda^{\geq 1}(L_0)$ only at a finite number of instants $t \in [a,b]$ and the following identity holds:*

$$\mu_{L_0}(\ell) = \sum_{t \in [a,b]} \operatorname{sgn}\big(\ell'(t)|_{(L_0 \cap \ell(t)) \times (L_0 \cap \ell(t))}\big).$$

PROOF. It follows from Example 4.2.17 that $\ell(t) \in \Lambda^{\geq 1}(L_0)$ only at a finite number of instants $t \in [a,b]$. Let $t_0 \in \,]a,b[$ be such that $\ell(t_0) \in \Lambda^{\geq 1}(L_0)$; keeping in mind Property (2) and (4) in the statement of Lemma 4.2.13, it suffices to prove that:

$$\mu_{L_0}\big(\ell|_{[t_0-\varepsilon, t_0+\varepsilon]}\big) = \operatorname{sgn}\big(\ell'(t_0)|_{(L_0 \cap \ell(t_0)) \times (L_0 \cap \ell(t_0))}\big),$$

for $\varepsilon > 0$ sufficiently small. Choose a common complementary $L_1 \in \Lambda$ of L_0 and $\ell(t_0)$; for t in a neighborhood of t_0 we can write $\beta(t) = \varphi_{L_0,L_1}(\ell(t))$. The conclusion now follows from Theorem 4.2.15 and from Corollary 4.1.34, keeping in mind Examples 4.2.7 and 4.2.8. $\square$

In Example 4.2.17 we have seen that a nondegenerate intersection of a curve ℓ of class C^1 with $\Lambda^{\geq 1}(L_0)$ at an instant t_0 is isolated, i.e., there exists $\varepsilon > 0$ such that $\ell(t) \notin \Lambda^{\geq 1}(L_0)$ for $t \in [t_0 - \varepsilon, t_0[\, \cup \,]t_0, t_0 + \varepsilon]$. For technical reasons, we will need (in Proposition 5.2.8) a slightly stronger result and we will prove next that the choice of such $\varepsilon > 0$ can be made *uniformly* with respect to a parameter.

4.2.19. LEMMA. *Let $\mathcal{X}$ be a topological space and suppose that it is given a continuous map:*

$$\mathcal{X} \times [t_0, t_1[\ \ni (\lambda, t) \longmapsto \ell_\lambda(t) = \ell(\lambda, t) \in \Lambda$$

which is differentiable in the variable t and such that $\frac{\partial \ell}{\partial t} : \mathcal{X} \times [t_0, t_1[\to T\Lambda$ is also continuous. Fix a Lagrangian $L_0 \in \Lambda$; suppose that $\dim(\ell(\lambda, t_0) \cap L_0)$ is independent of $\lambda \in \mathcal{X}$ and that the curve $\ell_{\lambda_0} = \ell(\lambda_0, \cdot)$ has a nondegenerate intersection with $\Lambda^{\geq 1}(L_0)$ at $t = t_0$ for some $\lambda_0 \in \mathcal{X}$. Then, there exists $\varepsilon > 0$ and a neighborhood $\mathfrak{U}$ of λ_0 in $\mathcal{X}$ such that, for all $\lambda \in \mathfrak{U}$, ℓ_λ has a nondegenerate intersection with $\Lambda^{\geq 1}(L_0)$ at t_0 and such that $\ell(\lambda, t) \in \Lambda^0(L_0)$ for all $\lambda \in \mathfrak{U}$ and all $t \in]t_0, t_0 + \varepsilon]$.

PROOF. Choose a common complementary Lagrangian L_1 of L_0 and $\ell(\lambda_0, t_0)$ and define $\beta(\lambda, t) = \varphi_{L_0, L_1}(\ell(\lambda, t))$ for t in a neighborhood of t_0 and λ in a neighborhood of λ_0 in $\mathcal{X}$. Then, β is continuous, it is differentiable in t, and the derivative $\frac{\partial \beta}{\partial t}$ is continuous. The conclusion follows now applying Proposition 4.1.35 to the map β, keeping in mind Examples 4.2.7 and 4.2.8. □

4.2.20. REMARK. A more careful analysis of the definition of the transverse orientation of $\Lambda^1(L_0)$ in Λ (Definition 2.5.19) shows that the choice of the sign made for the isomorphism μ_{L_0} is actually determined by the choice of a sign in the symplectic form ω. More explicitly, if we replace ω by $-\omega$, which does not affect the definition of the set Λ, then we obtain a change of sign for the isomorphisms ρ_{L_0, L_1} and ρ_L (defined in formulas (1.4.11) and (1.4.13)). Consequently, this change of sign induces a change of sign in the charts φ_{L_0, L_1} (defined in formula (2.5.3)) and in the isomorphism (2.5.12) that identifies $T_L\Lambda$ with $\mathrm{B}_{\mathrm{sym}}(L)$.

The conclusion is that changing the sign of ω causes an inversion of the transverse orientation of $\Lambda^1(L_0)$ in Λ, which inverts the sign of the isomorphism μ_{L_0}.

4.2.21. REMARK. The choice of a Lagrangian subspace $L_0 \in \Lambda$ defines an isomorphism:

$$\mu_{L_0} \circ \mathfrak{q}_* \colon H_1(\Lambda) \xrightarrow{\cong} \mathbb{Z}, \tag{4.2.14}$$

where $\mathfrak{q}_*$ is given in (4.2.3). We claim that this isomorphism does not indeed depend on the choice of L_0; for, let $L_0' \in \Lambda$ be another Lagrangian subspace. By Corollary 1.4.28, there exists a symplectomorphism $A \in \mathrm{Sp}(V, \omega)$ such that $A(L_0) = L_0'$; we have the following commutative diagram (see Lemma 4.2.5):

$$\begin{array}{ccc}
H_1(\Lambda) & \xrightarrow{A_* = \mathrm{Id}} & H_1(\Lambda) \\
\downarrow{\scriptstyle \mathfrak{q}_*} & & \downarrow{\scriptstyle \mathfrak{q}_*} \\
H_1(\Lambda, \Lambda^0(L_0)) & \xrightarrow{A_*} & H_1(\Lambda, \Lambda^0(L_0')) \\
& {\scriptstyle \mu_{L_0}} \searrow \quad \swarrow {\scriptstyle \mu_{L_0'}} & \\
& \mathbb{Z} &
\end{array}$$

where the commutativity of the lower triangle follows from Remark 2.5.21. This proves the claim. Observe that if $\gamma\colon [a,b] \to \Lambda$ is a loop, i.e., $\gamma(a) = \gamma(b)$, then, since γ defines a homology class in $H_1(\Lambda)$,we obtain the equality:

$$\mu_{L_0}(\gamma) = \mu_{L_0'}(\gamma),$$

for any pair of Lagrangian subspaces $L_0, L_0' \in \Lambda$.

4.2.22. REMARK. Let J be a complex structure in V compatible with ω; consider the inner product $g = \omega(\cdot, J\cdot)$ and the Hermitian product $g_{\rm s}$ in (V, J) defined in (1.4.10). Let $\ell_0 \in \Lambda$ be a Lagrangian subspace; Proposition 2.5.11 tells us that the map

$$\mathrm{U}(V,J,g_{\rm s})/\mathrm{O}\big(\ell_0, g|_{\ell_0\times\ell_0}\big) \ni A\cdot \mathrm{O}\big(\ell_0, g|_{\ell_0\times\ell_0}\big) \longmapsto A(\ell_0) \in \Lambda \tag{4.2.15}$$

is a diffeomorphism. As in (4.2.1), we can define a map

$$\bar{d}\colon \mathrm{U}(V,J,g_{\rm s})/\mathrm{O}\big(\ell_0, g|_{\ell_0\times\ell_0}\big) \longrightarrow S^1$$

obtained from

$$d = \det{}^2\colon \mathrm{U}(V,J,g_{\rm s}) \longrightarrow S^1$$

by passage to the quotient; then the map $\bar{d}$ induces an isomorphism $\bar{d}_*$ of the fundamental groups. Indeed, by Remark 1.4.30 we can find a basis of V that puts all the objects $(V,\omega,J,g,g_{\rm s},\ell_0)$ simultaneously in their canonical forms, and then everything works as in Proposition 4.2.1. The isomorphism $\bar{d}_*$ together with the diffeomorphism (4.2.15) and the choice of (3.2.24) (or, equivalently, of (4.2.4)) as a generator of $\pi_1(S^1) \cong H_1(S^1)$ produce an isomorphism (see also (4.2.2)):

$$\mathfrak{u} = \mathfrak{u}_{J,\ell_0}\colon H_1(\Lambda) \xrightarrow{\ \cong\ } \mathbb{Z};$$

this isomorphism does not indeed depend on the choice of J and of ℓ_0. To see this, choose another complex structure J' in V compatible with ω and another Lagrangian subspace $\ell_0' \in \Lambda$; we then obtain an isomorphism $\mathfrak{u}' = \mathfrak{u}_{J',\ell_0'}$. From Remark 1.4.30 it follows that there exists a symplectomorphism $A \in \mathrm{Sp}(V,\omega)$ that takes ℓ_0 onto ℓ_0' and that is $\mathbb{C}$-linear from (V,J) into (V,J'); then, it is easy to see that the following diagram commutes:

$$\begin{array}{ccc} H_1(\Lambda) & & \\ {\scriptstyle A_*}\downarrow & \searrow^{\mathfrak{u}} & \\ & & \mathbb{Z} \\ & \nearrow_{\mathfrak{u}'} & \\ H_1(\Lambda) & & \end{array}$$

By Lemma 4.2.5 we have that $A_* = \mathrm{Id}$ and the conclusion follows.

As a matter of facts, formula (4.2.10) shows that the isomorphism $\mathfrak{u}$ has the opposite sign of the isomorphism (4.2.14) obtained by using the transverse orientation of $\Lambda^1(L_0)$ in Λ.

Exercises for Chapter 4

EXERCISE 4.1. Prove that every symmetric bilinear form $B \in \mathrm{B}_{\mathrm{sym}}(V)$ admits a maximal negative subspace.

EXERCISE 4.2. Suppose that $B \in \mathrm{B}_{\mathrm{sym}}(V)$, with $V = V_1 \oplus V_2$ and B is negative semi-definite in V_1 and in V_2. If the inequality (4.1.15) holds for all $v_1 \in V_1$ and $v_2 \in V_2$, then B is negative semi-definite in V.

EXERCISE 4.3. Let V be an n-dimensional real vector space, $B \in \mathrm{B}_{\mathrm{sym}}(V)$ a symmetric bilinear form and assume that the matrix representation of B in some basis $\{v_1, \ldots, v_n\}$ of V is given by:

$$\begin{pmatrix} X & Z \\ Z^* & Y \end{pmatrix},$$

where X is a $k \times k$ symmetric matrix and Y is a $(n-k) \times (n-k)$ symmetric matrix. Prove that, if X is invertible, then:

$$n_-(B) = n_-(X) + n_-(Y - Z^* X^{-1} Z), \quad \mathrm{dgn}(B) = \mathrm{dgn}(Y - Z^* X^{-1} Z),$$
$$\text{and} \quad n_+(B) = n_+(X) + n_+(Y - Z^* X^{-1} Z).$$

EXERCISE 4.4. Let V be a finite dimensional real vector space and let $U, Z \in \mathrm{B}_{\mathrm{sym}}(V)$ be nondegenerate symmetric bilinear forms on V such that $U - Z$ is also nondegenerate. Prove that $U^{-1} - Z^{-1}$ is nondegenerate and that:

$$n_-(Z) - n_-(U) = n_-(Z^{-1} - U^{-1}) - n_-(U - Z).$$

EXERCISE 4.5. Consider the space $\mathbb{R}^{2n}$ endowed with its canonical symplectic form ω; define an isomorphism $\mathcal{O} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by $\mathcal{O}(x, y) = (x, -y)$, for all $x, y \in \mathbb{R}^n$. Show that $\mathcal{O}^{\#}(\omega) = -\omega$ and conclude that $\mathcal{O}$ induces a diffeomorphism of the Lagrangian Grassmannian Λ to itself. Show that the homomorphism:

$$\mathcal{O}_* : H_1(\Lambda) \longrightarrow H_1(\Lambda)$$

is equal to minus the identity map (compare with Remark 4.2.20).

EXERCISE 4.6. Let $L_0 \in \Lambda$ be a Lagrangian in the symplectic space (V, ω) and let $A : [a, b] \to \mathrm{Sp}(V, \omega, L_0)$, $\ell : [a, b] \to \Lambda$ be continuous curves such that $\ell(a), \ell(b) \in \Lambda^0(L_0)$. Prove that the curve $\tilde{\ell} = A \circ \ell : [a, b] \to \Lambda$ is homologous to ℓ in $H_1(\Lambda, \Lambda^0(L_0))$.

EXERCISE 4.7. Let L_0 be a Lagrangian subspace of (V, ω) and let $L_1, \ell : [a, b] \to \Lambda$ be curves such that:

- $L_1(t)$ is transverse to L_0 and to $\ell(t)$ for all $t \in [a, b]$;
- $\ell(a)$ and $\ell(b)$ are transverse to L_0.

Show that the Maslov index $\mu_{L_0}(\ell)$ of the curve ℓ is equal to:

$$\mu_{L_0}(\ell) = n_+\big(\varphi_{L_0, L_1(b)}(\ell(b))\big) - n_+\big(\varphi_{L_0, L_1(a)}(\ell(a))\big).$$

EXERCISE 4.8. Let $L_0, L_1, L_2, L_3 \in \Lambda$ be four Lagrangian subspaces of the symplectic space (V, ω), with $L_0 \cap L_1 = L_0 \cap L_2 = L_0 \cap L_3 = L_2 \cap L_3 = \{0\}$. Recall the definition of the map $\rho_{L_0,L_1} : L_1 \to L_0^*$ given in (1.4.11), the definition of pull-back of a bilinear form given in Definition 1.1.2 and the definition of the chart φ_{L_0,L_1} of Λ given in (2.5.3). Prove that the following identity holds:

$$\varphi_{L_0,L_1}(L_3) - \varphi_{L_1,L_0}(L_2) = (\rho_{L_0,L_1})^{\#}\big(\varphi_{L_0,L_3}(L_2)^{-1}\big).$$

EXERCISE 4.9. As in Exercise 4.8, prove that the following identity holds:

$$n_+\big(\varphi_{L_0,L_3}(L_2)\big) = n_+\big(\varphi_{L_1,L_0}(L_3) - \varphi_{L_1,L_0}(L_2)\big).$$

EXERCISE 4.10. Let (L_0, L_1) be a Lagrangian decomposition of the symplectic space (V, ω) and let $\ell : [a, b] \to \Lambda$ be a continuous curve with endpoints in $\Lambda^0(L_0)$. Suppose that there exists a Lagrangian $L_* \in \Lambda$ such that $\mathrm{Im}(\ell) \subset \Lambda^0(L_*)$. Prove that the Maslov index $\mu_{L_0}(\ell)$ is given by the following formula:

$$\mu_{L_0}(\ell) = n_-\big(\varphi_{L_1,L_0}(\ell(b)) - \varphi_{L_1,L_0}(L_*)\big) - n_-\big(\varphi_{L_1,L_0}(\ell(a)) - \varphi_{L_1,L_0}(L_*)\big).$$

EXERCISE 4.11. Define the following symplectic form $\overline{\omega}$ in $\mathbb{R}^{4n}$:

$$\overline{\omega}\big((v_1, w_1), (v_2, w_2)\big) = \omega(v_1, v_2) - \omega(w_1, w_2), \quad v_1, w_1, v_2, w_2 \in \mathbb{R}^{2n}.$$

where ω is the canonical symplectic form of $\mathbb{R}^{2n}$. Prove that $A \in \mathrm{Lin}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is a symplectomorphism of $(\mathbb{R}^{2n}, \omega)$ if and only if its graph $\mathrm{Gr}(A)$ is a Lagrangian subspace of $(\mathbb{R}^{4n}, \overline{\omega})$. Show that the map $\mathrm{Sp}(2n, \mathbb{R}) \ni A \mapsto \mathrm{Gr}(A) \in \Lambda(\mathbb{R}^{4n}, \overline{\omega})$ is a diffeomorphism onto an open subset.

EXERCISE 4.12. Prove that the set $\big\{T \in \mathrm{Sp}(2n, \mathbb{R}) : T(L_0) \cap L_0 = \{0\}\big\}$ is an open dense subset of $\mathrm{Sp}(2n, \mathbb{R})$ with two connected components.

EXERCISE 4.13. Define:

$$\Gamma_+ = \Big\{T \in \mathrm{Sp}(2n, \mathbb{R}) : \det(T - \mathrm{Id}) > 0\Big\};$$
$$\Gamma_- = \Big\{T \in \mathrm{Sp}(2n, \mathbb{R}) : \det(T - \mathrm{Id}) < 0\Big\}.$$

Prove that Γ_+ and Γ_- are open and connected subsets of $\mathrm{Sp}(2n, \mathbb{R})$ (see Exercise 4.15 for more properties of the sets Γ_+ and Γ_-).

EXERCISE 4.14. Consider the set:

$$\mathrm{E} = \Big\{T \in \mathrm{Sp}(2n, \mathbb{R}) : \det(T - \mathrm{Id}) \neq 0,\ T(L_0) \cap L_0 = \{0\}\Big\}.$$

Prove that E is a dense open subset of $\mathrm{Sp}(2n, \mathbb{R})$ having $2(n+1)$ connected components. Prove that each connected component contains an element $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A = 0$ and B in diagonal form.

EXERCISE 4.15. Recall from Exercise 4.13 the definition of the sets $\Gamma_+, \Gamma_- \subset \mathrm{Sp}(2n, \mathbb{R})$. Prove that $A \in \Gamma_+ \cup \Gamma_-$ if and only if $\mathrm{Gr}(A)$ is a Lagrangian in $\Lambda^0(\Delta) \subset \Lambda(\mathbb{R}^{4n}, \overline{\omega})$, where Δ is the *diagonal* of $\mathbb{R}^{4n} = \mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. Conclude that any loop in $\Gamma_+ \cup \Gamma_-$ is homotopic to a constant in $\mathrm{Sp}(2n, \mathbb{R})$.

CHAPTER 5

Some Applications to Differential Systems

5.1. Symplectic Differential Systems

In this section we will always consider the symplectic space $I\!R^n \oplus I\!R^{n*}$ endowed with its canonical symplectic form ω. Recall from Subsection 2.1.1 that the Lie algebra $\mathrm{sp}(I\!R^n \oplus I\!R^{n*}, \omega)$ can be identified with the set of $2n \times 2n$ real matrices X of the form:

$$X = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}, \quad B, C \text{ symmetric,} \tag{5.1.1}$$

where A, B, C are $n \times n$ matrices and A^* is the transpose of A. The matrices A, B, C and A^* can be identified with linear operators $A \in \mathrm{Lin}(I\!R^n)$, $B \in \mathrm{Lin}(I\!R^{n*}, I\!R^n)$, $C \in \mathrm{Lin}(I\!R^n, I\!R^{n*})$ and $A^* \in \mathrm{Lin}(I\!R^{n*})$; we can also identify B with a symmetric bilinear form in $I\!R^{n*}$ and C with a symmetric bilinear form in $I\!R^n$.

We will be interested in homogeneous systems of linear differential equations of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} v(t) \\ \alpha(t) \end{pmatrix} = X(t) \begin{pmatrix} v(t) \\ \alpha(t) \end{pmatrix}, \quad t \in [a, b], \tag{5.1.2}$$

where $X\colon [a, b] \to \mathrm{sp}(I\!R^n \oplus I\!R^{n*}, \omega)$, $v\colon [a, b] \to I\!R^n$ e $\alpha\colon [a, b] \to I\!R^{n*}$. For all $t \in [a, b]$, the linear operator $X(t)$ determines operators $A(t)$, $B(t)$ and $C(t)$ as in (5.1.1); we can then rewrite (5.1.2) more concisely in the form:

$$\begin{cases} v' = Av + B\alpha, \\ \alpha' = Cv - A^*\alpha, \end{cases} \tag{5.1.3}$$

where the variable t is omitted for simplicity.

5.1.1. DEFINITION. A homogeneous linear system of differential equations of the form (5.1.3), where $A : [a, b] \to \mathrm{Lin}(I\!R^n)$, $B : [a, b] \to \mathrm{B_{sym}}(I\!R^{n*})$ and $C : [a, b] \to \mathrm{B_{sym}}(I\!R^n)$ are smooth functions and $B(t)$ is nondegenerate for all $t \in [a, b]$ is called a *symplectic differential system*. If $X(t)$ denotes the matrix defined by $A(t)$, $B(t)$ and $C(t)$ as in (5.1.1), we say that X is the *coefficient matrix* of the symplectic system (5.1.3), and the maps A, B and C will be called the *components* of X.

In general, we will identify the symplectic differential system (5.1.3) with its coefficient matrix X; for instance, we will say that (v, α) *is a solution of* X meaning that (v, α) is a solution of (5.1.3), or, equivalently, of (5.1.2).

In the rest of this section we will consider a fixed symplectic differential system $X : [a,b] \to \mathrm{sp}(I\!R^n \oplus I\!R^{n*}, \omega)$ with components A, B and C.

5.1.2. REMARK. Since $B(t)$ is nondegenerate for all t, then its index does not depend on t, and so we can write:

$$n_-(B(t)) = n_-(B(t)^{-1}) = k, \quad t \in [a,b].$$

Recall that the linear operator $B(t)^{-1}$ can be identified with a symmetric bilinear form in $I\!R^n$, which is the push-forward of the bilinear form $B(t)$ by the linear operator $B(t)$.

Given a smooth map $v : [a,b] \to I\!R^n$, there exists at most one map $\alpha : [a,b] \to I\!R^{n*}$ such that (v,α) is a solution of X; for, the invertibility of $B(t)$ allows to solve the first equation of (5.1.3) for α in terms of v. We can therefore define for *any smooth map* $v : [a,b] \to I\!R^n$ a smooth map $\alpha_v : [a,b] \to I\!R^{n*}$ by the following formula:

$$\alpha_v(t) = B(t)^{-1}\big(v'(t) - A(t)v(t)\big). \tag{5.1.4}$$

5.1.3. DEFINITION. Given a smooth map $v : [a,b] \to I\!R^n$, we say that v is a *solution* of the symplectic differential system X if (v, α_v) is a solution of X.

From the elementary theory of ordinary linear differential equations we know that, given $v_0 \in I\!R^n$ and $\alpha_0 \in I\!R^{n*}$, there exists a unique solution (v,α) of X in $[a,b]$ satisfying $v(a) = v_0$ and $\alpha(a) = \alpha_0$ (see for instance [**4**, Theorem 5.1, Chapter 1]); therefore we have a well defined linear isomorphism:

$$\Phi(t) : I\!R^n \oplus I\!R^{n*} \to I\!R^n \oplus I\!R^{n*}$$

such that

$$\Phi(t)\big((v(a),\alpha(a)\big) = \big(v(t),\alpha(t)\big),$$

for any solution (v,α) of X. We then obtain a smooth curve $[a,b] \ni t \mapsto \Phi(t) \in \mathrm{GL}(I\!R^n \oplus I\!R^{n*})$ that satisfies:

$$\Phi'(t) = X(t) \circ \Phi(t), \text{ for all } t \in [a,b] \quad \text{and} \quad \Phi(a) = \mathrm{Id}. \tag{5.1.5}$$

5.1.4. DEFINITION. The map Φ determined by (5.1.5) is called the *fundamental matrix* of the symplectic differential system X.

Since X takes values in the Lie algebra of the symplectic group, it follows from (5.1.5) that Φ takes values in the symplectic group (see Remark 2.1.4), that is, the fundamental matrix of the symplectic differential system X is a differentiable map:

$$\Phi\colon [a,b] \longrightarrow \mathrm{Sp}(I\!R^n \oplus I\!R^{n*}, \omega).$$

The fact that $\Phi(t)$ is a symplectomorphism is expressed by the following identity

$$\omega\big((v(t),\alpha_v(t)),(w(t),\alpha_w(t))\big) = \alpha_w(t)\cdot v(t) - \alpha_v(t)\cdot w(t) = \text{constant}, \tag{5.1.6}$$

for any solutions v and w of X.

Let $\ell_0 \subset I\!R^n \oplus I\!R^{n*}$ be Lagrangian subspace; let us consider the following initial condition for the system (5.1.3):

$$(v(a),\alpha(a)) \in \ell_0. \tag{5.1.7}$$

Recalling Exercise 1.11, there exists a bijection between the set of Lagrangian subspaces $\ell_0 \subset I\!R^n \oplus I\!R^{n*}$ and the set of pairs (P, S), where $P \subset I\!R^n$ is a subspace and $S \in \mathrm{B}_{\mathrm{sym}}(P)$ is a symmetric bilinear form; such bijection is determined by the identity:

$$\ell_0 = \big\{(v, \alpha) \in I\!R^n \oplus I\!R^{n*} : v \in P,\ \alpha|_P + S(v) = 0\big\}, \tag{5.1.8}$$

where S is identified with a linear operator $S\colon P \to P^*$. In terms of the pair (P, S), the initial condition (5.1.7) can be rewritten in the form:

$$v(a) \in P, \quad \alpha(a)|_P + S\big(v(a)\big) = 0. \tag{5.1.9}$$

5.1.5. DEFINITION. We call (5.1.7) (respectively, (5.1.9)) the *Lagrangian initial condition* determined by the Lagrangian ℓ_0 (respectively, by the pair (P, S)); if (v, α) is a solution of X that satisfies (5.1.7), or, equivalently, (5.1.9), we say that (v, α) is a *solution of the pair* (X, ℓ_0), or also that (v, α) is a (X, ℓ_0)-*solution*. We will denote by $\mathbb{V} = \mathbb{V}(X, \ell_0)$ the set of solutions of (X, ℓ_0), that is:

$$\mathbb{V}(X, \ell_0) = \mathbb{V} = \big\{v : v \text{ is a solution of } (X, \ell_0)\big\}. \tag{5.1.10}$$

Clearly, $\mathbb{V}$ is a subspace of the space of all maps $v\colon [a, b] \to I\!R^n$; moreover:

$$\dim(\mathbb{V}) = \dim(\ell_0) = n.$$

We will fix for the rest of the section a Lagrangian $\ell_0 \subset I\!R^n \oplus I\!R^{n*}$ and we will denote by (P, S) the pair corresponding to ℓ_0 as in (5.1.8).

For each $t \in [a, b]$ we define the following subspace of $I\!R^n$:

$$\mathbb{V}[t] = \big\{v(t) : v \in \mathbb{V}\big\} \subset I\!R^n; \tag{5.1.11}$$

It is easy to see that:

$$\mathbb{V}[a] = P. \tag{5.1.12}$$

It follows directly from (5.1.6) that, given solutions v and w of (X, ℓ_0), then:

$$\alpha_v(t) \cdot w(t) = \alpha_w(t) \cdot v(t),$$

for each $t \in [a, b]$; then, if v is a solution of (X, ℓ_0) with $v(t) = 0$, the functional $\alpha_v(t)$ annihilates the space $\mathbb{V}[t]$. Conversely, if $\alpha_0 \in I\!R^{n*}$ is a functional that annihilates $\mathbb{V}[t]$, it follows from (5.1.6) that if v is the unique solution of X such that $v(t) = 0$ and $\alpha_v(t) = \alpha_0$, then

$$0 = \omega\big((v(t), \alpha_v(t)), (w(t), \alpha_w(t))\big) = \omega\big((v(a), \alpha_v(a)), (w(a), \alpha_w(a))\big),$$

for all (X, ℓ_0)-solution w; hence, $(v(a), \alpha_v(a))$ is ω-orthogonal to ℓ_0 and therefore v is a solution of (X, ℓ_0). These observations show that the annihilator of $\mathbb{V}[t]$ is given by:

$$\mathbb{V}[t]^o = \big\{\alpha_v(t) : v \in \mathbb{V} \text{ and } v(t) = 0\big\}, \tag{5.1.13}$$

for all $t \in [a, b]$; keeping in mind (5.1.4), it follows directly from (5.1.13) that the orthogonal complement of $\mathbb{V}[t]$ with respect to $B(t)^{-1}$ is given by:

$$\mathbb{V}[t]^\perp = B(t)\big(\mathbb{V}[t]^o\big) = \big\{v'(t) : v \in \mathbb{V} \text{ and } v(t) = 0\big\}. \tag{5.1.14}$$

5.1.6. DEFINITION. We say that $t \in \,]a,b]$ is a *focal instant* for the pair (X,ℓ_0) (or that t is a (X,ℓ_0)-*focal instant*) if there exists a non zero solution $v \in \mathbb{V}$ of (X,ℓ_0) such that $v(t)=0$; the dimension of the space of solutions $v \in \mathbb{V}$ such that $v(t)=0$ is called the *multiplicity* of the focal instant t, and it is denoted by $\mathrm{mul}(t)$. The *signature* of the focal instant t, denoted by $\mathrm{sgn}(t)$, is defined as the signature of the restriction of the symmetric bilinear form $B(t)^{-1}$ to the space $\mathbb{V}[t]^{\perp}$, that is:

$$\mathrm{sgn}(t) = \mathrm{sgn}\big(B(t)^{-1}|_{\mathbb{V}[t]^{\perp}\times\mathbb{V}[t]^{\perp}}\big),$$

where the orthogonal complement $\mathbb{V}[t]^{\perp}$ is taken relatively to the bilinear form $B(t)^{-1}$. We say that the focal instant t is *nondegenerate* if $B(t)^{-1}$ is nondegenerate on $\mathbb{V}[t]^{\perp}$. If $t \in \,]a,b]$ is not a (X,ℓ_0)-focal instant, we define $\mathrm{mul}(t)=0$ and $\mathrm{sgn}(t)=0$; if the pair (X,ℓ_0) has only a finite number of focal instants, we define the *focal index* of (X,ℓ_0) as the integer number:

$$\mathrm{i}_{\mathrm{foc}}(X,\ell_0) = \mathrm{i}_{\mathrm{foc}} = \sum_{t\in]a,b]} \mathrm{sgn}(t).$$

5.1.7. REMARK. For all $t \in \,]a,b]$ we have:

$$\mathrm{mul}(t) = \dim\big(\mathbb{V}[t]^o\big) = \mathrm{codim}_{\mathbb{R}^n}\mathbb{V}[t], \tag{5.1.15}$$

and in particular t is (X,ℓ_0)-focal if and only if $\mathbb{V}[t] \ne \mathbb{R}^n$; indeed, it follows from (5.1.13) that the map

$$\big\{v \in \mathbb{V} : v(t)=0\big\} \ni v \longmapsto \alpha_v(t) \in \mathbb{V}[t]^o \subset \mathbb{R}^{n*}$$

is an isomorphism. Keeping in mind Remark 5.1.2, we see that the symmetric bilinear form $B(t)^{-1}|_{\mathbb{V}[t]^{\perp}\times\mathbb{V}[t]^{\perp}}$ is the push-forward of $B(t)|_{\mathbb{V}[t]^o\times\mathbb{V}[t]^o}$ by the isomorphism

$$B(t)|_{\mathbb{V}[t]^o}\colon \mathbb{V}[t]^o \longrightarrow \mathbb{V}[t]^{\perp};$$

Then, we conclude that the signature of a (X,ℓ_0)-focal instant t coincides with:

$$\mathrm{sgn}(t) = \mathrm{sgn}\big(B(t)|_{\mathbb{V}[t]^o\times\mathbb{V}[t]^o}\big);$$

moreover, a focal instant t is nondegenerate if and only if $\mathbb{V}[t]^o$ is a nondegenerate subspace for $B(t)$. Observe also that Corollary 1.1.11 implies that a focal instant t is nondegenerate if and only if $B(t)^{-1}$ is nondegenerate in $\mathbb{V}[t]$.

5.1.8. DEFINITION. We say that the Lagrangian initial condition determined by the Lagrangian subspace ℓ_0 (or, equivalently, by the pair (P,S)) is *nondegenerate* if the bilinear form $B(a)^{-1}$ is nondegenerate on P. In this case, we also say that the pair (X,ℓ_0) has a *nondegenerate initial condition*.

5.1.9. REMARK. In Definition 5.1.6 we have explicitly excluded the possibility that $t=a$ be a (X,ℓ_0)-focal instant. Nevertheless, if we admit for the moment the terminology of Definition 5.1.6 also for $t=a$, we see that the nondegeneracy for the initial condition determined by ℓ_0 is indeed equivalent to the nondegeneracy of $t=a$ as a focal instant. Arguing as in Remark 5.1.7 we see that the nondegeneracy of the initial condition determined by ℓ_0 is equivalent to the nondegeneracy of $B(a)$ in the annihilator of P in $\mathbb{R}^n$, and also to the nondegeneracy of $B(a)^{-1}$ in

$P^\perp$, where the orthogonal complement is taken with respect to $B(a)^{-1}$. Observe that if $P = \mathbb{R}^n$, which is equivalent to ℓ_0 being transversal to $\{0\}^n \oplus \mathbb{R}^{n*}$, or if $P = \{0\}$, which is equivalent to $\ell_0 = \{0\}^n \oplus \mathbb{R}^{n*}$, then automatically (X, ℓ_0) has nondegenerate initial condition.

5.1.10. EXAMPLE. If $g : [a, b] \to \mathrm{B}_{\mathrm{sym}}(\mathbb{R}^n)$ and $R : [a, b] \to \mathrm{Lin}(\mathbb{R}^n)$ are differentiable maps with $g(t)$ nondegenerate and $R(t)$ a $g(t)$-symmetric operator for all t, then the homogeneous linear differential equation:

$$g(t)^{-1}\big(g(t) \cdot v'(t)\big)' = R(t) \cdot v(t), \quad t \in [a, b], \tag{5.1.16}$$

is called a *Morse-Sturm equation*. If g is constant, then (5.1.16) can be written in a simplified form:

$$v''(t) = R(t) \cdot v(t), \quad t \in [a, b]. \tag{5.1.17}$$

Defining $\alpha(t) = g(t) \cdot v'(t)$, we can rewrite (5.1.16) as a system of differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v(t) \\ \alpha(t) \end{pmatrix} = \begin{pmatrix} 0 & g(t)^{-1} \\ g(t) \circ R(t) & 0 \end{pmatrix}\begin{pmatrix} v(t) \\ \alpha(t) \end{pmatrix}, \quad t \in [a, b]. \tag{5.1.18}$$

The system (5.1.18) is a symplectic differential system, with $A(t) = 0$, $B(t) = g(t)^{-1}$ and $C(t) = g(t) \circ R(t)$. In general, any symplectic differential system X with $A = 0$ will be identified with a Morse–Sturm equation with $g(t) = B(t)^{-1}$ and $R(t) = B(t) \circ C(t)$.

5.2. The Maslov Index of a Symplectic Differential System

In this section we show that if X is a symplectic differential system and ℓ_0 is a Lagrangian subspace of $\mathbb{R}^n \oplus \mathbb{R}^{n*}$, then we can associate in a natural way a curve in the Lagrangian Grassmannian Λ to the pair (X, ℓ_0), and under suitable hypotheses we can associate a *Maslov index* to the pair (X, ℓ_0). We will *always* denote by L_0 the Lagrangian subspace:

$$L_0 = \{0\}^n \oplus \mathbb{R}^{n*} \subset \mathbb{R}^n \oplus \mathbb{R}^{n*}$$

and by Λ the Lagrangian Grassmannian of the symplectic space $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ endowed with its canonical symplectic structure. As usual, we will denote by A, B and C the components of X and by Φ its fundamental matrix.

We define a differentiable curve $\ell\colon [a, b] \to \Lambda$ by setting:

$$\ell(t) = \Phi(t)(\ell_0), \tag{5.2.1}$$

for all $t \in [a, b]$; more explicitly, we have:

$$\ell(t) = \{(v(t), \alpha_v(t)) : v \in \mathbb{V}\}. \tag{5.2.2}$$

Observe that $\ell(a) = \ell_0$; keeping in mind (5.2.2) and (5.1.13) we see that:

$$L_0 \cap \ell(t) = \{0\}^n \oplus \mathbb{V}[t]^o, \tag{5.2.3}$$

for all $t \in [a, b]$. We have the following:

5.2.1. LEMMA. *An instant $t \in \,]a, b]$ is (X, ℓ_0)-focal if and only if $\ell(t) \in \Lambda^{\geq 1}(L_0)$; moreover, $\ell(t) \in \Lambda^k(L_0)$ if and only if $\mathrm{mul}(t) = k$.*

PROOF. It follows easily from (5.2.3) and (5.1.15). □

Lemma 5.2.1 is a first indication that the properties of the focal instants of the pair (X, ℓ_0) may be investigated by looking at the intersections of the curve ℓ with $\Lambda^{\geq 1}(L_0)$. In order to make more explicit the relations between the focal instants of (X, ℓ_0) and such intersections we now compute the derivative of ℓ.

The linearization of the natural action of the symplectic group $\mathrm{Sp}(\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega)$ in the Lagrangian Grassmannian Λ gives us an anti-homomorphism of the Lie algebra $\mathrm{sp}(\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega)$ into the Lie algebra of differentiable vector fields in Λ. These concepts were defined in Subsection 2.1.3, and we will use here the notations of that subsection.

The identity (5.1.5) tells us that Φ is an integral curve of the time-dependent vector field $(t, g) \mapsto X(t)^R(g)$ in the Lie group $\mathrm{Sp}(\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega)$; from Remark 2.1.22 and from (5.2.1) it then follows that ℓ is an integral curve of the time-dependent vector field $(t, m) \mapsto X(t)^*(m)$ in Λ, i.e.,

$$\ell'(t) = X(t)^*(\ell(t)), \tag{5.2.4}$$

for all $t \in [a, b]$. Proposition 2.5.9 gives us:

$$X(t)^*(L) = \omega(X(t)\cdot, \cdot)|_{L\times L}, \tag{5.2.5}$$

for all $L \in \Lambda$. Then, putting together (5.2.4) and (5.2.5) we obtain:

$$\begin{aligned} \ell'(t)\big((0,\alpha),(0,\beta)\big) &= \omega\big(X(t)(0,\alpha),(0,\beta)\big) \\ &= \omega\big((B(t)\alpha, -A(t)^*\alpha),(0,\beta)\big) = B(t)(\alpha,\beta), \end{aligned} \tag{5.2.6}$$

for any $(0, \alpha), (0, \beta) \in L_0 \cap \ell(t)$. We have therefore shown the following:

5.2.2. LEMMA. *For all $t \in [a, b]$, the restriction of the symmetric bilinear form $B(t) \in \mathrm{B}_{\mathrm{sym}}(\mathbb{R}^{n*})$ to $\mathbb{V}[t]^o$ coincides with the push-forward of the restriction of $\ell'(t) \in \mathrm{B}_{\mathrm{sym}}(\ell(t))$ to $L_0 \cap \ell(t)$ by the isomorphism:*

$$L_0 \cap \ell(t) \ni (0, \alpha) \longmapsto \alpha \in \mathbb{V}[t]^o$$

PROOF. It follows from (5.2.3) and (5.2.6). □

5.2.3. COROLLARY. *An (X, ℓ_0)-focal instant $t \in\,]a, b]$ is nondegenerate if and only if ℓ has a nondegenerate intersection with $\Lambda^{\geq 1}(L_0)$ at the instant t; moreover,*

$$\mathrm{sgn}(t) = \mathrm{sgn}\big(\ell'(t)|_{L_0\cap\ell(t)}\big).$$

Also, the pair (X, ℓ_0) has nondegenerate initial condition if and only if ℓ either has a nondegenerate intersection with $\Lambda^{\geq 1}(L_0)$ at the instant $t = a$ or $\ell(a) \notin \Lambda^{\geq 1}(L_0)$.

PROOF. It follows from Remark 5.1.7 and Remark 5.1.9. □

5.2.4. COROLLARY. *If $t_0 \in\,]a, b]$ is a nondegenerate (X, ℓ_0)-focal instant, then it is* isolated, *i.e., no instant $t \neq t_0$ sufficiently close to t_0 is (X, ℓ_0)-focal. Moreover, if the initial condition of (X, ℓ_0) is nondegenerate, then there are no (X, ℓ_0)-focal instants in a neighborhood of $t = a$. If (X, ℓ_0) has nondegenerate initial condition and if it has only nondegenerate focal instants, then (X, ℓ_0) has only a finite number of focal instants.*

PROOF. It follows from Corollary 5.2.3, Lemma 5.2.2 and Example 4.2.17. □

We now want to define the Maslov index of a pair (X, ℓ_0); essentially, it will be defined as the Maslov index $\mu_{L_0}(\ell)$ of the curve ℓ. The problem is that $\mu_{L_0}(\ell)$ only makes sense if ℓ has endpoints in $\Lambda^0(L_0)$; assuming that $t = b$ is not a (X, ℓ_0)-focal instant, we have that $\ell(b) \in \Lambda^0(L_0)$ (see Lemma 5.2.1). However, in general $\ell(a) = \ell_0$ may be in $\Lambda^{\geq 1}(L_0)$; to overcome this problem the idea is to "erase" a short initial portion of the curve ℓ. More precisely, let us give the following:

5.2.5. DEFINITION. If the pair (X, ℓ_0) has nondegenerate initial condition and if the final instant $t = b$ is not (X, ℓ_0)-focal, then we define the *Maslov index* $\mathrm{i}_{\mathrm{maslov}}(X, \ell_0)$ of the pair (X, ℓ_0) by setting:

$$\mathrm{i}_{\mathrm{maslov}}(X, \ell_0) = \mathrm{i}_{\mathrm{maslov}} = \mu_{L_0}\big(\ell|_{[a+\varepsilon, b]}\big),$$

where $\varepsilon > 0$ is chosen in such a way that there are no (X, ℓ_0)-focal instants in the interval $[a, a + \varepsilon]$.

From Corollary 5.2.4 it follows that, indeed, there exists $\varepsilon > 0$ such that (X, ℓ_0) does not have focal instants in the interval $[a, a + \varepsilon]$. Moreover, the definition of $\mathrm{i}_{\mathrm{maslov}}$ does not depend on the choice of ε (see Exercise 5.4).

Generically, the Maslov index $\mathrm{i}_{\mathrm{maslov}}(X, \ell_0)$ can be thought as a sort of *algebraic count* of the focal instants of (X, ℓ_0):

5.2.6. PROPOSITION. *Suppose that (X, ℓ_0) has nondegenerate initial condition and that $t = b$ is not (X, ℓ_0)-focal. If (X, ℓ_0) has only nondegenerate focal instants, then the focal index coincides with the Maslov index:*

$$\mathrm{i}_{\mathrm{foc}}(X, \ell_0) = \mathrm{i}_{\mathrm{maslov}}(X, \ell_0).$$

PROOF. It follows directly from Corollary 4.2.18 and Corollary 5.2.3. □

5.2.7. EXAMPLE. If $B(t)$ is positive definite for some (hence for all) $t \in [a, b]$ then (X, ℓ_0) automatically has nondegenerate initial condition; moreover, every (X, ℓ_0)-focal instant $t \in \,]a, b]$ is nondegenerate and $\mathrm{sgn}(t) = \mathrm{mul}(t)$. Hence, if $t = b$ is not (X, ℓ_0)-focal, it follows from Proposition 5.2.6 that

$$\mathrm{i}_{\mathrm{maslov}}(X, \ell_0) = \sum_{t \in]a, b[} \mathrm{mul}(t) < +\infty.$$

One of the fundamental properties of the Maslov index of a pair is its *stability*; we have the following:

5.2.8. PROPOSITION. *Let $\mathcal{X}$ be a topological space and assume that for all $\lambda \in \mathcal{X}$ it is given a symplectic differential system X_λ such that the map:*

$$\mathcal{X} \times [a, b] \ni (\lambda, t) \longmapsto X_\lambda(t) \in \mathrm{sp}(\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega)$$

is continuous; let $\ell_0 \colon \mathcal{X} \to \Lambda$ be a continuous curve in the Lagrangian Grassmannian such that $\dim(L_0 \cap \ell_0(\lambda))$ does not depend on $\lambda \in \mathcal{X}$. If for some $\lambda_0 \in \mathcal{X}$ the pair $(X_{\lambda_0}, \ell_0(\lambda_0))$ has nondegenerate initial condition and $t = b$ is not $(X_{\lambda_0}, \ell_0(\lambda_0))$-focal, then there exists a neighborhood $\mathfrak{U}$ of λ_0 in $\mathcal{X}$ such that for all $\lambda \in \mathfrak{U}$ we have:

- $(X_\lambda, \ell_0(\lambda))$ *has nondegenerate initial condition;*
- *the instant* $t = b$ *is not* $(X_\lambda, \ell_0(\lambda))$*-focal;*
- $\mathrm{i}_{\mathrm{maslov}}(X_\lambda, \ell_0(\lambda)) = \mathrm{i}_{\mathrm{maslov}}(X_{\lambda_0}, \ell_0(\lambda_0))$.

PROOF. Denote by Φ_λ the fundamental matrix of the symplectic differential system X_λ. It follows from standard theory on continuous dependence with respect to a parameter of solutions of differential equations that the map $(\lambda, t) \mapsto \Phi_\lambda(t)$ is continuous in $\mathcal{X} \times [a, b]$. Define $\ell_\lambda(t) = \Phi_\lambda(t)(\ell_0(\lambda))$; then clearly $(\lambda, t) \mapsto \ell_\lambda(t)$ is a continuous map in $\mathcal{X} \times [a, b]$ and it follows from (5.2.4) that also $(\lambda, t) \mapsto \ell'_\lambda(t)$ is continuous in $\mathcal{X} \times [a, b]$. In particular,since $\Lambda^0(L_0)$ is open, we have that $\ell_\lambda(b) \in \Lambda^0(L_0)$ for λ in a neighborhood of λ_0 in $\mathcal{X}$, and therefore for such values of λ the instant $t = b$ is not $(X_\lambda, \ell_0(\lambda))$-focal (see Lemma 5.2.1). Keeping in mind Corollary 5.2.3, it follows directly from Lemma 4.2.19 that there exists $\varepsilon > 0$ and a neighborhood $\mathfrak{U}$ of λ in $\mathcal{X}$ such that $(X_\lambda, \ell_0(\lambda))$ has nondegenerate initial condition, and such that there are no $(X_\lambda, \ell_0(\lambda))$-focal instants in the interval $[a, a + \varepsilon]$ for all $\lambda \in \mathfrak{U}$. Hence,

$$\mathrm{i}_{\mathrm{maslov}}(X_\lambda, \ell_0(\lambda)) = \mu_{L_0}\big(\ell_\lambda|_{[a+\varepsilon,b]}\big)$$

for all λ in a neighborhood of λ_0 in $\mathcal{X}$. It follows from Remark 3.1.20 that the map $\lambda \mapsto \ell_\lambda \in C^0([a + \varepsilon, b], \Lambda)$ is continuous when we consider $C^0([a + \varepsilon, b], \Lambda)$ endowed with the compact-open topology. By Property (6) in the statement of Lemma 4.2.13 we conclude that $\mathrm{i}_{\mathrm{maslov}}(X_\lambda, \ell_0(\lambda))$ is constant when λ runs in a neighborhood of λ_0 in $\mathcal{X}$; this concludes the proof. □

5.2.9. COROLLARY. *Suppose that it is given a sequence* $(X_n)_{n\ge1}$ *of differentiable maps* $X_n : [a, b] \to \mathrm{sp}(I\!R^n \oplus I\!R^{n*}, \omega)$ *that converges uniformly to a differentiable map* X*; assume that* X *and* X_n *are symplectic differential systems for all* n*. Let also be given a sequence* (ℓ_0^n) *of Lagrangian subspaces that converges to some* $\ell_0 \in \Lambda$*, where* $\dim(L_0 \cap \ell_0^n) = \dim(L_0 \cap \ell_0)$ *for all* n*. Then, if* (X, ℓ_0) *has nondegenerate initial condition and if* $t = b$ *is not a* (X, ℓ_0)*-focal instant, then for all* n *sufficiently large also* (X_n, ℓ_0^n) *has nondegenerate initial condition and* $t = b$ *is not* (X_n, ℓ_0^n)*-focal, and:*

$$\mathrm{i}_{\mathrm{maslov}}(X_n, \ell_0^n) = \mathrm{i}_{\mathrm{maslov}}(X_0, \ell_0).$$

PROOF. Consider the topological space $\mathcal{X} = I\!N \cup \{+\infty\}$, where $\mathfrak{U} \subset \mathcal{X}$ is open if and only if $\mathfrak{U} \subset I\!N$ or $\mathcal{X} \backslash \mathfrak{U}$ is a finite subset of $I\!N$. Setting $X_{+\infty} = X$, then it is easy to see that the hypotheses of Proposition 5.2.8 are satisfied for $\lambda_0 = +\infty$. The conclusion follows. □

5.3. The Maslov Index of semi-Riemannian Geodesics and Hamiltonian Systems

In this section we show how the theory of symplectic differential systems appears in several fields of geometry. In Subsection 5.3.1 we show how every geodesic in a semi-Riemannian manifold determines in a natural way a Morse-Sturm equation; such system is essentially obtained from the Jacobi equation along the geodesic through a parallel trivialization of the tangent bundle of the semi-Riemannian manifold along the geodesic. In Subsection 5.3.2 we show that symplectic differential systems appear also as linearizations of Hamiltonian systems; we develop the theory in a very abstract and general formalism, using arbitrary symplectic manifolds. Finally, in Subsection 5.3.3 we give references for some further developments of the theory.

5.3.1. Geodesics in a semi-Riemannian manifold. Let M be a differentiable manifold; a *semi-Riemannian metric* in M is a differentiable $(2,0)$-tensor field $\mathfrak{g}$ such that for every $m \in M$, $\mathfrak{g}_m$ is a nondegenerate symmetric bilinear form on T_mM. The pair $(M,\mathfrak{g})$ is called a *semi-Riemannian manifold*; when $\mathfrak{g}_m$ is positive definite for every $m \in M$ we say that $\mathfrak{g}$ is a *Riemannian metric* and that $(M,\mathfrak{g})$ is a *Riemannian manifold*. It is well known that there exists a unique *connection* ∇ on the tangent bundle TM of M which is torsion-free and such that $\mathfrak{g}$ is parallel; such connection is called the *Levi-Civita connection*. The *curvature tensor* of ∇ is defined by:

$$\mathcal{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for every differentiable vector fields X,Y,Z in M. Given a differentiable curve $\gamma : I \to M$ defined in some interval $I \subset I\!R$ and given a differentiable vector field $\mathfrak{v} : I \to TM$ *along* γ, i.e., $\mathfrak{v}(t) \in T_{\gamma(t)}M$ for every $t \in I$, then we denote by $\frac{\mathrm{D}\mathfrak{v}}{\mathrm{d}t}$ (or just by $\mathfrak{v}'$) the covariant derivative of $\mathfrak{v}$ along γ; a *geodesic* is defined as a differentiable curve $\gamma : I \to M$ whose derivative γ' is parallel, i.e., $\gamma'' = 0$. A differentiable vector field $\mathfrak{v}$ along a geodesic $\gamma\colon [a,b] \to M$ is called a *Jacobi vector field* if it satisfies the differential equation:

$$\frac{\mathrm{D}^2}{\mathrm{d}t^2}\mathfrak{v}(t) = \mathcal{R}\big(\gamma'(t),\mathfrak{v}(t)\big)\gamma'(t), \quad t \in [a,b]; \tag{5.3.1}$$

equation (5.3.1) is known as the *Jacobi equation*. Set $\dim(M) = n$ and chose parallel vector fields $Z_i\colon [a,b] \to TM$ along γ, $i = 1,\ldots,n$ such that $(Z_i(t))_{i=1}^n$ form a basis of $T_{\gamma(t)}M$ for some (and hence for all) $t \in [a,b]$; we say that $(Z_i)_{i=1}^n$ is a *parallel trivialization* of the tangent bundle TM along γ. The parallel trivialization $(Z_i)_{i=1}^n$ induces a bijection between the set of differentiable vector fields $\mathfrak{v}\colon [a,b] \to TM$ along γ and the set of differentiable maps $v\colon [a,b] \to I\!R^n$ given by:

$$\mathfrak{v}(t) = \sum_{i=1}^n v_i(t)Z_i(t), \quad t \in [a,b], \tag{5.3.2}$$

where $v(t) = (v_1(t),\ldots,v_n(t)) \in I\!R^n$; taking the covariant derivative along γ on both sides of (5.3.2) we get that if $\mathfrak{v}$ corresponds to v then the covariant derivative $\mathfrak{v}'$ of $\mathfrak{v}$ corresponds to the (standard) derivative v' of v by means of the bijection induced by the parallel trivialization. For each $t \in [a,b]$ we define a nondegenerate symmetric bilinear form $g(t) \in \mathrm{B}_{\mathrm{sym}}(I\!R^n)$ and a linear operator $R(t) \in \mathrm{Lin}(I\!R^n)$

whose matrices with respect to the canonical basis of $\mathbb{R}^n$ satisfy the identities:

$$g_{ij}(t) = \mathfrak{g}\big(Z_i(t), Z_j(t)\big), \quad \mathcal{R}\big(\gamma'(t), Z_j(t)\big)\gamma'(t) = \sum_{i=1}^{n} R_{ij}(t) Z_i(t),$$

for every $i, j = 1, \dots, n$. Observe that, since both $\mathfrak{g}$ and the vector fields Z_i are parallel, the *symmetric bilinear form* $g(t)$ *does not depend on* t; moreover, standard symmetry properties of the curvature tensor imply that $R(t)$ is g-symmetric for every $t \in [a, b]$. We can therefore consider the Morse-Sturm equation:

$$v''(t) = R(t)v(t), \quad t \in [a, b]; \tag{5.3.3}$$

moreover, it is easy to see that *a vector field* $\mathfrak{v}$ *along* γ *is a Jacobi vector field iff the map* $v\colon [a, b] \to \mathbb{R}^n$ *defined by* (5.3.2) *is a solution of* (5.3.3). In Example 5.1.10 we have mentioned that every Morse-Sturm equation can be identified with a symplectic differential system X with components $A(t) = 0$, $B(t) = g(t)^{-1}$ and $C(t) = g \circ R(t)$; observe that $\alpha_v(t) = g(v(t)) \in \mathbb{R}^{n*}$ for every t.

Consider now a submanifold $\mathcal{P} \subset M$; the *second fundamental form* of P at a point $p \in P$ in a normal direction $n \in T_pP^\perp$ (where the orthogonal complement is taken with respect to $\mathfrak{g}$) is the symmetric bilinear form $\mathcal{S}_n$ on T_pP defined by:

$$\mathcal{S}_n(v, w) = \mathfrak{g}(\nabla_v W, n), \quad v, w \in T_pP,$$

where W is any differential vector field which is tangent to P and such that $W(p) = w$. Suppose now that we have a geodesic $\gamma\colon [a, b] \to M$ with $\gamma(a) \in \mathcal{P}$ and $\gamma'(a) \in T_{\gamma(a)}\mathcal{P}^\perp$; a vector field $v\colon [a, b] \to TM$ along γ is called a *$\mathcal{P}$-Jacobi field* if v is Jacobi and satisfy the condition:

$$v(a) \in P \quad \text{and} \quad \mathfrak{g}\big(v'(a), \cdot\big)|_{T_{\gamma(a)}\mathcal{P}} + \mathcal{S}_{\gamma'(a)}\big(v(a), \cdot\big) = 0 \in T_{\gamma(a)}\mathcal{P}^*.$$

The basis $(Z_i(a))_{i=1}^n$ of $T_{\gamma(a)}M$ induces an isomorphism from $T_{\gamma(a)}M$ to $\mathbb{R}^n$ which takes $T_{\gamma(a)}\mathcal{P}$ onto some subspace $P \subset \mathbb{R}^n$; moreover, there exists a unique symmetric bilinear form $S \in \mathrm{B}_{\mathrm{sym}}(P)$ which is the push-forward of $\mathcal{S}_{\gamma'(a)}$ by (the restriction of) such isomorphism. The pair (P, S) therefore defines a Lagrangian subspace $\ell_0 \subset \mathbb{R}^n \oplus \mathbb{R}^{n*}$ as in (5.1.8); it is easily seen that a vector field $\mathfrak{v}$ along γ is $\mathcal{P}$-Jacobi if and only if the corresponding map $v\colon [a, b] \to \mathbb{R}^n$ defined by (5.3.2) is a solution of (X, ℓ_0). In semi-Riemannian geometry one usually defines that a point $\gamma(t)$, $t \in \left]a, b\right]$ is *$\mathcal{P}$-focal* along γ when there exists a non zero $\mathcal{P}$-Jacobi vector field $\mathfrak{v}$ along γ such that $\mathfrak{v}(t) = 0$; the dimension of the space of $\mathcal{P}$-Jacobi fields $\mathfrak{v}$ along γ such that $\mathfrak{v}(t) = 0$ is called the *multiplicity* of the $\mathcal{P}$-focal point $\gamma(t)$. Moreover, for each $t \in \left]a, b\right]$ one considers the space:

$$\mathbb{J}[t] = \big\{\mathfrak{v}(t) : \mathfrak{v} \text{ is } \mathcal{P}\text{-Jacobi along } \gamma\big\} \subset T_{\gamma(t)}M.$$

The *signature* of the $\mathcal{P}$-focal point $\gamma(t)$ is defined as the signature of the restriction of the metric $\mathfrak{g}$ to the orthogonal complement $\mathbb{J}[t]^\perp$ of $\mathbb{J}[t]$; the $\mathcal{P}$-focal point $\gamma(t)$ is called *nondegenerate* if the space $\mathbb{J}[t]^\perp$ (or equivalently, $\mathbb{J}[t]$) is nondegenerate for $\mathfrak{g}$. When there are only a finite number of $\mathcal{P}$-focal points along γ we define the *focal index* of the geodesic γ with respect to $\mathcal{P}$ as the sum of the signatures of the $\mathcal{P}$-focal points along γ. The following facts are obvious:

- an instant $t \in \,]a, b]$ is (X, ℓ_0)-focal iff $\gamma(t)$ is a $\mathcal{P}$-focal point; the multiplicity and signature of t as a (X, ℓ_0)-focal instant or of $\gamma(t)$ as a $\mathcal{P}$-focal point coincide;
- the focal index of the pair (X, ℓ_0) coincide with the focal index of the geodesic γ with respect to $\mathcal{P}$;
- a (X, ℓ_0)-focal instant $t \in \,]a, b]$ is nondegenerate if and only if the $\mathcal{P}$-focal point $\gamma(t)$ is non-degenerate;
- the initial condition of (X, ℓ_0) is nondegenerate if and only if $T_{\gamma(a)}\mathcal{P}$ is a nondegenerate subspace for $\mathfrak{g}$.

If $\mathfrak{g}$ is nondegenerate on $T_{\gamma(a)}\mathcal{P}$ and if $\gamma(b)$ is not a $\mathcal{P}$-focal point we define the *Maslov index* of the geodesic γ with respect to $\mathcal{P}$ as the Maslov index of the pair (X, ℓ_0). From Proposition 5.2.6 it follows immediately the following:

5.3.1. PROPOSITION. *Let $\gamma\colon [a, b] \to M$ be a geodesic starting orthogonally to a submanifold $\mathcal{P} \subset M$; suppose that the metric $\mathfrak{g}$ is nondegenerate on $T_{\gamma(a)}\mathcal{P}$ and that there are only nondegenerate $\mathcal{P}$-focal points along γ. Then, if $\gamma(b)$ is not $\mathcal{P}$-focal, the Maslov index of γ coincides with the focal index of γ with respect to $\mathcal{P}$.* □

Observe that if $(M, \mathfrak{g})$ is Riemannian then the focal index of a geodesic γ is simply the sum of the multiplicities of the $\mathcal{P}$-focal points along γ; this (non-negative) integer is sometimes called the *geometric index* of the geodesic γ. The geometric index of a geodesic is one of the numbers which enters into the statement of the celebrated *Morse Index Theorem*.

A semi-Riemannian manifold $(M, \mathfrak{g})$ is called *Lorentzian* if the metric $\mathfrak{g}$ has index 1 at every point; four-dimensional Lorentzian manifolds are mathematical models for general relativistic spacetimes. A vector $v \in TM$ is said to be *time-like*, *lightlike* or *spacelike* respectively when $\mathfrak{g}(v, v)$ is negative, zero or positive. Similarly, we say that a geodesic γ is timelike, lightlike or spacelike when $\gamma'(t)$ is respectively timelike, lightlike or spacelike for all t. We have the following:

5.3.2. LEMMA. *Let $\gamma\colon [a, b] \to M$ be a timelike or a lightlike (non constant) geodesic in a Lorentzian manifold $(M, \mathfrak{g})$ starting orthogonally to a submanifold $\mathcal{P} \subset M$; assume that $\mathfrak{g}$ is nondegenerate on $T_{\gamma(a)}\mathcal{P}$ (which is always the case if γ is timelike). Then, for every $t \in \,]a, b]$ the space $\mathbb{J}[t]$ is $\mathfrak{g}$-positive.*

PROOF. Set $\mathfrak{v}(t) = (t-a)\gamma'(t)$; then $\mathfrak{v}$ is a $\mathcal{P}$-Jacobi field and therefore $\gamma'(t) \in \mathbb{J}[t]$ for $t \in \,]a, b]]$. It follows that $\mathbb{J}[t]$ is contained in the orthogonal complement of $\mathbb{R}\gamma'(t)$; if γ is timelike, this implies that $\mathfrak{g}$ is positive definite on $\mathbb{J}[t]$. If γ is lightlike we still have to show that $\gamma'(t)$ is not in $\mathbb{J}[t]$; observe first that, since $T_{\gamma(a)}\mathcal{P}$ is $\mathfrak{g}$-nondegenerate it cannot be $\gamma'(a) \in T_{\gamma(a)}\mathcal{P}$ and therefore it cannot be $T_{\gamma(a)}\mathcal{P}^{\perp} \subset \big(\mathbb{R}\gamma'(a)\big)^{\perp}$; hence we can find a Jacobi field $\mathfrak{v}$ along γ with $\mathfrak{v}(a) = 0$ and $\mathfrak{v}'(a)$ orthogonal to $T_{\gamma(a)}\mathcal{P}$ but not orthogonal to $\gamma'(a)$. Then $\mathfrak{v}$ is a $\mathcal{P}$-Jacobi field and it cannot be that $\mathfrak{v}(t)$ is orthogonal to $\gamma'(t)$; for, $s \mapsto \mathfrak{g}(\mathfrak{v}(s), \gamma'(s))$ is an affine map which vanishes at $s = a$ and hence it cannot have another zero at $s = t$, since this would imply $\mathfrak{g}(\mathfrak{v}, \gamma') \equiv 0$ and $\mathfrak{g}(\mathfrak{v}'(a), \gamma'(a)) = 0$. This proves that $\gamma'(t) \not\in \mathbb{J}[t]^{\perp}$ and completes the proof. □

5.3.3. COROLLARY. *Under the hypotheses of Lemma 5.3.2, the geometric index and the focal index of γ with respect to $\mathcal{P}$ coincide.* □

5.3.2. Hamiltonian systems. In this subsection we will consider the following setup. Let $(\mathcal{M},\omega)$ be a *symplectic manifold*, i.e., $\mathcal{M}$ is a smooth manifold and ω is a smooth closed skew-symmetric nondegenerate two-form on $\mathcal{M}$, so that ω_m is a = symplectic form on T_mM for each $m \in M$. We set $\dim(\mathcal{M}) = 2n$. Let $H : U \mapsto I\!R$ be a smooth function defined in an open set $U \subseteq I\!R \times \mathcal{M}$; we will call such function a *Hamiltonian* in $(\mathcal{M},\omega)$. For each $t \in I\!R$, we denote by H_t the map $m \mapsto H(t,m)$ defined in the open set $U_t \subseteq \mathcal{M}$ consisting of those $m \in \mathcal{M}$ such that $(t,m) \in U$. We denote by $\vec{H}$ the smooth time-dependent vector field in $\mathcal{M}$ defined by $\mathrm{d}H_t(m) = \omega(\vec{H}(t,m),\cdot)$ for all $(t,m) \in U$; let F denote the maximal flow of the vector field $\vec{H}$ defined on an open set of $I\!R \times I\!R \times \mathcal{M}$ taking values in $\mathcal{M}$, i.e., for each $m \in \mathcal{M}$ and $t_0 \in I\!R$, the curve $t \mapsto F(t,t_0,m)$ is a maximal integral curve of $\vec{H}$ and $F(t_0,t_0,m) = m$. This means that $F(\cdot,t_0,m)$ is a maximal solution of the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t,t_0,m) = \vec{H}(t,F(t,t_0,m)), \quad F(t_0,t_0,m) = m.$$

Recall that F is a smooth map; we also write F_{t,t_0} for the map $m \mapsto F(t,t_0,m)$; observe that F_{t,t_0} is a diffeomorphism between open subsets of $\mathcal{M}$.

We recall that a *symplectic chart* in $\mathcal{M}$ is a local chart (q,p) taking values in $I\!R^n \oplus I\!R^{n*}$ whose differential at each point is a symplectomorphism from the tangent space of $\mathcal{M}$ to $I\!R^n \oplus I\!R^{n*}$ endowed with the canonical symplectic structure. We write $q = (q_1,\ldots,q_n)$ and $p = (p_1,\ldots,p_n)$; we denote by $\{\frac{\partial}{\partial q_i},\frac{\partial}{\partial p_j}\}$, $i,j = 1,\ldots,n$ the corresponding local referential of $T\mathcal{M}$, and by $\{\mathrm{d}q_i,\mathrm{d}p_j\}$ the local referential of $T\mathcal{M}^*$. By *Darboux's Theorem*, there always exists an atlas of symplectic charts.

In a given symplectic chart (q,p), we have:

$$\omega = \sum_{i=1}^n \mathrm{d}q_i \wedge \mathrm{d}p_i, \quad \vec{H} = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\right).$$

Let $\mathcal{P}$ be a *Lagrangian submanifold* of $\mathcal{M}$, i.e., $T_m\mathcal{P}$ is a Lagrangian subspace of $T_m\mathcal{M}$ for every $m \in \mathcal{P}$. We fix an integral curve $\Gamma : [a,b] \mapsto \mathcal{M}$ of $\vec{H}$, so that $\Gamma(t) = F(t,a,\Gamma(a))$ for all $t \in [a,b]$. We also say that Γ is a *solution of the Hamilton equation*, i.e., in a symplectic chart $\Gamma(t) = (q(t),p(t))$:

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{\partial H}{\partial p}, \\[2ex] \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\dfrac{\partial H}{\partial q}. \end{cases} \tag{5.3.4}$$

We assume that Γ starts at $\mathcal{P}$, that is $\Gamma(a) \in \mathcal{P}$. Finally, we will consider a fixed smooth distribution $\mathfrak{L}$ in $\mathcal{M}$ such that $\mathfrak{L}_m$ is a Lagrangian subspace of $T_m\mathcal{M}$ for all $m \in \mathcal{M}$.

The basic example to keep in mind for the above setup is the case where $\mathcal{M}$ is the cotangent bundle TM^* of some smooth manifold M endowed with the canonical symplectic structure, $\mathcal{P}$ is the annihilator TP^o of some smooth submanifold P of M, and $\mathfrak{L}$ is the distribution consisting of the *vertical* subspaces, i.e., the subspaces tangent to the fibers of TM^*.

The Hamiltonian flow F_{t,t_0} is a symplectomorphism:

5.3.4. PROPOSITION. *The symplectic form ω is invariant by the Hamiltonian flow F, i.e., $F_{t,t_0}^*\omega = \omega$ for all (t, t_0).*

In the case of a time-independent Hamiltonian, the result of Proposition 5.3.4 follows easily from the formula for the *Lie derivative* of forms:

$$\mathbb{L}_{\vec{H}} = \mathrm{d}\, i_{\vec{H}} + i_{\vec{H}}\, \mathrm{d}.$$

For the general case, the proof is based on the following elementary Lemma which says how to compute the derivative of the pull-back of forms by a one-parameter family of functions:

5.3.5. LEMMA. *Let G be a smooth map on an open subset of $\mathbb{R} \times \mathcal{M}$ taking values in $\mathcal{M}$ and let η be a smooth r-form on $\mathcal{M}$. For each $t \in \mathbb{R}$, denote by G_t the map $m \mapsto G(t, m)$ and by $\tilde{G}^t\eta$ the $(r-1)$-form on an open subset of $\mathcal{M}$ given by:*

$$(\tilde{G}^t\eta)_m = \mathrm{d}G_t(m)^*\, i_v\, \eta_{G(t,m)},$$

where $v = \frac{\mathrm{d}}{\mathrm{d}t} G(t, m)$ and i_v is the interior product *(or* contraction *in the first variable) of a form with the vector v. Then, for all $m \in \mathcal{M}$ we have:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, (G_t^*\eta)_m = \mathrm{d}(\tilde{G}^t\eta)_m + \tilde{G}^t(\mathrm{d}\eta)_m. \tag{5.3.5}$$

PROOF. The two sides of equality (5.3.5) are $\mathbb{R}$-linear maps of η which have the same behavior with respect to exterior derivative and exterior products. Moreover, they agree on 0-forms. The conclusion follows from the fact that, locally, every r-form is a linear combination of products of derivatives of 0-forms. □

PROOF OF PROPOSITION 5.3.4. We fix an instant $t_0 \in \mathbb{R}$; we consider the map $G(t, m) = F(t, t_0, m)$ and we apply Lemma 5.3.5 to $\eta = \omega$. Observe that, for each t, the 1-form $\tilde{G}^t\omega$ is equal to $G_t^*(\mathrm{d}H_t)$. From (5.3.5), we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, (F_{t,t_0}^*\omega)_m = 0,$$

and so $F_{t,t_0}^*\omega$ is independent of t. The conclusion follows from the fact that F_{t_0,t_0} is the identity map. □

A sextuplet $(\mathcal{M}, \omega, H, \mathfrak{L}, \Gamma, \mathcal{P})$ where $(\mathcal{M}, \omega)$ is a symplectic manifold, H is a (time-dependent) Hamiltonian function defined on an open subset of $\mathbb{R} \times \mathcal{M}$, $\mathfrak{L}$ is a smooth distribution of Lagrangians in $\mathcal{M}$, $\Gamma : [a, b] \mapsto \mathcal{M}$ is an integral curve of $\vec{H}$ and $\mathcal{P}$ is a Lagrangian submanifold of $\mathcal{M}$ with $\Gamma(a) \in \mathcal{P}$, will be called *a set of data for the Hamiltonian problem*.

We give some more basic definitions.

5.3.6. DEFINITION. A vector field ρ along Γ in $\mathcal{M}$ is said to be a *solution for the linearized Hamilton (LinH) equations* if it satisfies:

$$\rho(t) = \mathrm{d}F_{t,a}(\Gamma(a))\,\rho(a). \tag{5.3.6}$$

We also say that ρ is a $\mathcal{P}$*-solution* for the (LinH) equations if in addition it satisfies $\rho(a) \in T_{\Gamma(a)}\mathcal{P}$.

5.3.7. DEFINITION. A point $\Gamma(t)$, $t \in]a,b]$ is said to be a $\mathcal{P}$*-focal point* along Γ if there exists a non zero $\mathcal{P}$-solution ρ for the (LinH) equations such that $\rho(t) \in \mathfrak{L}_{\Gamma(t)}$. The *multiplicity* of a $\mathcal{P}$-focal point $\Gamma(t)$ is the dimension of the vector space of such ρ's.

5.3.8. DEFINITION. A *symplectic $\mathfrak{L}$-trivialization* of $T\mathcal{M}$ along Γ is a smooth family of symplectomorphisms $\phi(t) : I\!R^n \oplus I\!R^{n*} \mapsto T_{\Gamma(t)}\mathcal{M}$ such that, for all $t \in [a,b]$, $\phi(t)(L_0) = \mathfrak{L}_{\Gamma(t)}$, where $L_0 = \{0\} \oplus I\!R^{n*}$.

The existence of symplectic $\mathfrak{L}$-trivializations along Γ is easily established with elementary arguments, using the fact that $T\mathcal{M}$ restricts to a trivial vector bundle along Γ.

We will be interested also in the quotient bundle $T\mathcal{M}/\mathfrak{L}$ and its dual bundle. We have an obvious canonical identification of the dual $(T\mathcal{M}/\mathfrak{L})^*$ with the annihilator $\mathfrak{L}^o \subset T\mathcal{M}^*$; moreover, using the symplectic form, we will identify $\mathfrak{L}^o$ with $\mathfrak{L}$ by the isomorphism:

$$T_m\mathcal{M} \ni \rho \mapsto \omega(\cdot,\rho) \in T_m\mathcal{M}^*, \quad m \in \mathcal{M}. \tag{5.3.7}$$

A symplectic $\mathfrak{L}$-trivialization ϕ induces a trivialization of the quotient bundle $T\mathcal{M}/\mathfrak{L}$ along Γ, namely, for each $t \in [a,b]$ we define an isomorphism $\mathcal{Z}_t : I\!R^n \mapsto T_{\Gamma(t)}\mathcal{M}/\mathfrak{L}_{\Gamma(t)}$:

$$\mathcal{Z}_t(x) = \phi(t)(x,0) + \mathfrak{L}_{\Gamma(t)}, \quad x \in I\!R^n. \tag{5.3.8}$$

Given a symplectic $\mathfrak{L}$-trivialization ϕ of $T\mathcal{M}$ along Γ, we define a smooth curve $\Phi : [a,b] \mapsto \mathrm{Sp}(2n, I\!R)$ by:

$$\Phi(t) = \phi(t)^{-1} \circ \mathrm{d}F_{t,a}(\Gamma(a)) \circ \phi(a). \tag{5.3.9}$$

The fact that $\Phi(t)$ is a symplectomorphism follows from Proposition 5.3.4.

We now define a smooth curve $X : [a,b] \mapsto \mathrm{sp}(2n, I\!R)$ by setting:

$$X(t) = \Phi'(t)\Phi(t)^{-1}; \tag{5.3.10}$$

As customary, the components of the matrix X will be denoted by A, B and C. Finally, we define a Lagrangian subspace ℓ_0 of $I\!R^n \oplus I\!R^{n*}$ by:

$$\ell_0 = \phi(a)^{-1}(T_{\Gamma(a)}\mathcal{P}). \tag{5.3.11}$$

5.3.9. DEFINITION. The *canonical bilinear form* of $(\mathcal{M}, \omega, H, \mathfrak{L}, \Gamma, \mathcal{P})$ is a family of symmetric bilinear forms $H_{\mathfrak{L}}(t)$ on $(T_{\Gamma(t)}\mathcal{M}/\mathfrak{L}_{\Gamma(t)})^* \simeq \mathfrak{L}^o_{\Gamma(t)} \simeq \mathfrak{L}_{\Gamma(t)}$ given by:

$$H_{\mathfrak{L}}(t) = \mathcal{Z}_t \circ B(t) \circ \mathcal{Z}_t^*, \tag{5.3.12}$$

where $\mathcal{Z}$ is the trivialization of $T\mathcal{M}/\mathfrak{L}$ relative to some symplectic $\mathfrak{L}$-trivialization ϕ of $T\mathcal{M}$ and B is the upper-right $n \times n$ block of the map X in (5.3.10). In Exercise 5.5 the reader is asked to prove that the right hand side of (5.3.12) does *not* depend on the choice of the symplectic $\mathfrak{L}$-trivialization of $T\mathcal{M}$.

We say that the set of data $(\mathcal{M}, \omega, H, \mathfrak{L}, \Gamma, \mathcal{P})$ is *nondegenerate* if $H_{\mathfrak{L}}(t)$ is nondegenerate for all $t \in [a, b]$. In this case, we can also define the symmetric bilinear form $H_{\mathfrak{L}}(t)^{-1}$ on $T_{\Gamma(t)}\mathcal{M}/\mathfrak{L}_{\Gamma(t)}$.

Given a nondegenerate set of data $(\mathcal{M}, \omega, H, \mathfrak{L}, \Gamma, \mathcal{P})$, let us consider the pair (X, ℓ_0) defined by (5.3.10) and (5.3.11). It is easily seen that the submanifold $\mathcal{P}$ and the space P defined by ℓ_0 as in (5.1.8) are related by the following:

$$\mathcal{Z}_a(P) = \pi(T_{\Gamma(a)}\mathcal{P}),$$

where $\pi : T_{\Gamma(a)}\mathcal{M} \mapsto T_{\Gamma(a)}\mathcal{M}/\mathfrak{L}_{\Gamma(a)}$ is the quotient map. We set:

$$\mathcal{P}_0 = \pi(T_{\Gamma(a)}\mathcal{P}). \tag{5.3.13}$$

To define the signature of a $\mathcal{P}$-focal point along Γ, we need to introduce the following space:

(5.3.14)

$$\mathfrak{V}[t] = \big\{\rho(t) : \rho \text{ is a } \mathcal{P}\text{-solution of the (LinH) equation}\big\} \cap \mathfrak{L}_{\Gamma(t)}, \quad t \in [a, b].$$

Using the isomorphism (5.3.7), it is easy to see that $\mathfrak{V}[a]$ is identified with the annihilator $(T_{\Gamma(a)}\mathcal{P} + \mathfrak{L}_a)^o$. It is easily seen that a point $\Gamma(t)$ is $\mathcal{P}$-focal if and only if $\mathfrak{V}[t]$ is not zero and that the dimension of $\mathfrak{V}[t]$ is precisely the multiplicity of $\Gamma(t)$.

5.3.10. DEFINITION. Let $\Gamma(t)$ be a $\mathcal{P}$-focal point along the solution Γ. The *signature* $\operatorname{sgn}(\Gamma(t))$ is the signature of the restriction of $H_{\mathfrak{L}}(t)$ to $\mathfrak{V}[t] \subset \mathfrak{L}_t \simeq \mathfrak{L}_t^o$. $\Gamma(t)$ is said to be a nondegenerate $\mathcal{P}$-focal point if such restriction is nondegenerate. If Γ has only a finite number of $\mathcal{P}$-focal points, we define the *focal index* $\mathrm{i}_{\mathrm{foc}}(\Gamma)$ as:

$$\mathrm{i}_{\mathrm{foc}}(\Gamma) = \sum_{t \in \,]a,b]} \operatorname{sgn}(\Gamma(t)). \tag{5.3.15}$$

5.3.11. DEFINITION. Given a set of data $(\mathcal{M}, \omega, H, \mathfrak{L}, \Gamma, \mathcal{P})$ such that:

- $H_{\mathfrak{L}}(a)^{-1}$ is nondegenerate on $\mathcal{P}_0 = \pi\big(T_{\Gamma(a)}\mathcal{P}\big)$, where $\pi : T_{\Gamma(a)}\mathcal{M} \to T_{\Gamma(a)}\mathcal{M}/\mathfrak{L}_{\Gamma(a)}$ is the quotient map;
- $\Gamma(b)$ is not a $\mathcal{P}$-focal point.

We define the *Maslov index* $\mathrm{i}_{\mathrm{maslov}}(\Gamma)$ as the Maslov index of any pair (X, ℓ_0) associated to it by a symplectic $\mathfrak{L}$-trivialization of $T\mathcal{M}$ along Γ.

5.3.3. Further developments. In this short subsection we indicate some recent results by the authors of this book that contain further development of the theory of the Maslov index and its applications to semi-Riemannian geometry and Hamiltonian system.

In [**29**] prove the stability of the geometric index for timelike and lightlike Lorentzian geodesics (that follows from Corollary 5.3.3), it is given a counterexample to the equality of the Maslov index and of the focal index of a geodesic in the case that there are degenerate focal points. It is also studied the problem of characterizing those curves of Lagrangians that arise from the Jacobi equation along a semi-Riemannian geodesic.

In [**12**] it is proven a Lorentzian extension of the Morse Index Theorem for geodesics of all causal character in a stationary Lorentzian manifold $\mathcal{M}$, or, more generally, for geodesics that admit a timelike Jacobi field. It is considered the case of a geodesic with initial endpoint variable in a submanifold $\mathcal{P}$ of $\mathcal{M}$. Moreover, under suitable compactness assumptions it is developed an infinite dimensional Morse theory for geodesics with fixed endpoints in a stationary Lorentzian manifold. A version of the index theorem for periodic geodesics in stationary Lorentzian manifolds is proven in [**25**]. The Morse Index Theorem for timelike or lightlike geodesics in any Lorentzian manifold is proven in [**2, 8**]; for the case of both endpoints variable see [**34**].

In [**36**] the authors develop the Morse Index Theorem for the general case of a non periodic solution of a possibly time-dependent Hamiltonian system; it is used a suitable assumption that generalize the assumption of stationarity for the metric used in [**12**].

A general version of the semi-Riemannian Morse Index Theorem is proven in [**45**]; the Maslov index is proven to be equal to the difference of the index and of the coindex of suitable restrictions of the index form.

Exercises for Chapter 5

EXERCISE 5.1. Consider the symplectic differential system given in formula (5.1.3) and initial condition (5.1.7), with $X\colon [a,b] \to \mathrm{sp}(I\!R^n \oplus I\!R^{n*}, \omega)$ *real-analytic*. [1] Prove that either every instant $t \in \,]a,b]$ is (X,ℓ_0)-focal, or else there are only a finite number of (X,ℓ_0)-focal instants. Prove that if the initial condition is nondegenerate, then there are only a finite number of (X,ℓ_0)-focal instants.

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[1] A map $f\colon U \to I\!R^n$ defined in an open subset $U \subset I\!R^m$ is said to be *real-analytic* if for all $x^0 \in U$, in a neighborhood of x^0 we can write f as the sum of a *power series* centered at x^0, i.e.,

$$f(x) = \sum_{\lambda} a_\lambda (x_1 - x_1^0)^{\lambda_1} \cdots (x_m - x_m^0)^{\lambda_m},$$

for x near x^0, where $\lambda = (\lambda_1, \ldots, \lambda_m)$ runs over the set of all m-tuples of non negative integer numbers. A power series centered at x^0 and convergent in a neighborhood of x^0, converges absolutely and uniformly in a (possibly smaller) neighborhood of x^0. It follows that the series above can be differentiated termwise; in particular, every real-analytic function is C^∞, and the coefficient a_λ is given by the *Taylor's formula*:

$$a_\lambda = \frac{1}{\lambda_1! \cdots \lambda_m!} \frac{\partial^{|\lambda|} f}{\partial x_1^{\lambda_1} \ldots \partial x_m^{\lambda_m}} (x_1^0, \ldots, x_m^0),$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_m$. It follows easily from the two formulas above that the set of points of U where f and all its partial derivatives are zero is open and closed in U. In particular, *a real-analytic function on a connected domain which is zero in a non empty open set is identically zero.*

EXERCISE 5.2. Consider the isomorphism $\mathcal{O}\colon \mathbb{R}^n \oplus \mathbb{R}^{n*} \to \mathbb{R}^n \oplus \mathbb{R}^{n*}$ defined by $\mathcal{O}(v,\alpha) = (v,-\alpha)$; in terms of matrices:

$$\mathcal{O} = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix},$$

where I denotes the $n \times n$ identity matrix. Define:

$$X^{\mathrm{op}} = \mathcal{O} \circ X \circ \mathcal{O};$$

prove that X^{op} is a symplectic differential system and compute its components A^{op}, B^{op} and C^{op}, and its fundamental matrix Φ^{op}. The system X^{op} is called the *opposite symplectic differential system of* X. Characterize the solutions of X^{op} in terms of those of X.

EXERCISE 5.3. In the notations of Exercise 5.2, prove that $\mathcal{O} : \mathbb{R}^n \oplus \mathbb{R}^{n*} \to \mathbb{R}^n \oplus \mathbb{R}^{n*}$ is *not* a symplectomorphism (with respect to the canonical symplectic structure), but it takes Lagrangian subspaces into Lagrangian subspaces. Given $\ell_0 \in \Lambda(\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega)$, denote by ℓ_0^{op} the Lagrangian $\mathcal{O}(\ell_0)$, which is called the *opposite Lagrangian subspace of* ℓ_0. Denote by (P,S) and $(P^{\mathrm{op}}, S^{\mathrm{op}})$ respectively the pair associated to ℓ_0 and to the opposite Lagrangian subspace ℓ_0^{op}. Determine the relation between P and P^{op} and between S and S^{op}; prove that (X,ℓ_0) has a nondegenerate initial condition if and only if $(X^{\mathrm{op}}, \ell_0^{\mathrm{op}})$ does, and, in this case, find the relation between the focal index (whenever defined) and the Maslov index of (X,ℓ_0) and of $(X^{\mathrm{op}}, \ell_0^{\mathrm{op}})$.

EXERCISE 5.4. Prove that, in Definition 5.2.5, the Maslov index of the curve $\ell|_{[a+\varepsilon,b]}$ does not depend on the choice of $\varepsilon > 0$ provided that there are no (X,ℓ_0)-focal instants in the interval $]a, a+\varepsilon]$.

EXERCISE 5.5. Show that the symmetric bilinear form $H_{\mathfrak{L}}$ introduced in 5.3.9 does not indeed depend on the choice of the symplectic trivialization.

APPENDIX A

Answers and Hints to the exercises

A.1. From Chapter 1

Exercise 1.1. The naturality of the isomorfism (1.1.1) means that:

$$\begin{array}{ccc} \mathrm{Lin}(V, W^*) & \xrightarrow{\cong} & \mathrm{B}(V, W) \\ {\scriptstyle \mathrm{Lin}(L,M^*)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{B}(L,M)} \\ \mathrm{Lin}(V_1, W_1^*) & \xrightarrow[\cong]{} & \mathrm{B}(V_1, W_1) \end{array}$$

where $L \in \mathrm{Lin}(V_1, V)$, $M \in \mathrm{Lin}(W_1, W)$ and the horizontal arrows in the diagram are suitable versions of the isomorphism (1.1.1).

Exercise 1.2. Write $B = B_\mathrm{s} + B_\mathrm{a}$, with $B_\mathrm{s}(v, w) = \frac{1}{2}\big(B(v, w) + B(w, v)\big)$ and $B_\mathrm{a}(v, w) = \frac{1}{2}\big(B(v, w) - B(w, v)\big)$.

Exercise 1.3. Use formula (1.2.1).

Exercise 1.4. Every $v \in V$ can be written uniquely as $v = \sum_{j \in \mathcal{J}} z_j b_j$, where $z_j = x_j + i\, y_j$, $x_j, y_j \in I\!R$; therefore v can be written uniquely as a linear combination of the b_j's and of the $J(b_j)$'s as $v = \sum_{j \in \mathcal{J}} x_j b_j + y_j J(b_j)$.

Exercise 1.5. The uniqueness follows from the fact that $\iota(V)$ generates $V^{\mathbb{C}}$ as a complex vector space. For the existence define $\tilde{f}(v) = f \circ \iota^{-1} \circ \Re(v) + i\, f \circ \iota^{-1} \circ \Im(v)$, where $\Re$ and $\Im$ are the real part and the imaginary part operator relative to the real form $\iota(V)$ of $V^{\mathbb{C}}$.

Exercise 1.6. Use Proposition 1.3.3 to get maps $\phi : V_1^{\mathbb{C}} \to V_2^{\mathbb{C}}$ and $\psi : V_2^{\mathbb{C}} \to V_1^{\mathbb{C}}$ such that $\phi \circ \iota_1 = \iota_2$ and $\psi \circ \iota_2 = \iota_1$; the uniqueness of Proposition 1.3.3 gives the uniqueness of the ϕ. Using twice again the uniqueness in Proposition 1.3.3, one concludes that $\psi \circ \phi = \mathrm{Id}$ and $\phi \circ \psi = \mathrm{Id}$.

Exercise 1.7. If $\mathcal{Z} = U^{\mathbb{C}}$, then obviously $\mathfrak{c}(\mathcal{Z}) \subset \mathcal{Z}$. Conversely, if $\mathfrak{c}(\mathcal{Z}) \subset \mathcal{Z}$ then $\Re(\mathcal{Z})$ and $\Im(\mathcal{Z})$ are contained in $U = \mathcal{Z} \cap V$. It follows easily that $\mathcal{Z} = U^{\mathbb{C}}$.

Exercise 1.8. In the case of multi-linear operators, diagram (1.3.2) becomes:

$$\begin{array}{ccc} V_1^{\mathbb{C}} \times \cdots \times V_p^{\mathbb{C}} & & \\ {\scriptstyle \iota_1 \times \cdots \times \iota_r}\Big\uparrow & \searrow^{\tilde{f}} & \\ V_1 \times \cdots \times V_p & \xrightarrow[f]{} & \mathcal{W} \end{array}$$

The identities (1.3.5) still hold when $T^{\mathbb{C}}$ is replaced by $T^{\underline{\mathbb{C}}}$; observe that the same conclusion does *not* hold for the identities (1.3.3) and (1.3.4).

Lemma 1.3.10 generalizes to the case of multi-linear operators; observe that such generalization gives us as corollary natural isomorphisms between the complexification of the tensor, exterior and symmetric powers of V and the corresponding powers of $V^{\mathbb{C}}$.

Lemma 1.3.11 can be directly generalized to the case that $\mathcal{S}$ is an anti-linear, multi-linear or sesquilinear operator; in the anti-linear (respectively, sesquilinear) $T^{\mathbb{C}}$ must be replaced by $T^{\underline{\mathbb{C}}}$ (respectively, by $T^{\mathbb{C}_s}$). For a $\mathbb{C}$-multilinear operator $\mathcal{S} : V_1^{\mathbb{C}} \times \cdots \times V_p^{\mathbb{C}} \to V^{\mathbb{C}}$ (or if $p = 2$ and $\mathcal{S}$ is sesquilinear) the condition that $\mathcal{S}$ *preserves real forms* becomes:

$$\mathcal{S}(V_1 \times \cdots \times V_p) \subset V,$$

while the condition of *commuting with conjugation* becomes:

$$\mathcal{S}(\mathfrak{c}\cdot, \ldots, \mathfrak{c}\cdot) = \mathfrak{c} \circ \mathcal{S}.$$

Exercise 1.9. If $\mathcal{B} \in \mathrm{B}(\mathcal{V})$, then $\mathcal{B}(v, v) = -\mathcal{B}(iv, iv)$.

Exercise 1.10. Use (1.4.4).

Exercise 1.11. Set $2n = \dim(V)$ and let $P \subset L_1$ be a subspace and $S \in \mathrm{B}_{\mathrm{sym}}(P)$ be given. To see that the second term in (1.4.14) defines an n-dimensional subspace of V choose any complementary subspace W of P in L_1 and observe that the map:

$$L \ni v + w \longmapsto \big(v, \rho_{L_1,L_0}(w)|_Q\big) \in P \oplus Q^*, \quad v \in L_1,\ w \in L_0,$$

is an isomorphism. To show that L is isotropic, hence Lagrangian, one uses the symmetry of S:

(A.1.1)

$$\omega(v_1{+}w_1, v_2{+}w_2) = \rho_{L_1,L_0}(w_1){\cdot}v_2 - \rho_{L_1,L_0}(w_2){\cdot}v_1 = S(v_1, v_2) - S(v_2, v_1) = 0,$$

for all $v_1, v_2 \in L_1$, $w_1, w_2 \in L_0$ with $v_1 + w_1, v_2 + w_2 \in L$.

Conversely, let L be any Lagrangian; set $P = \pi_1(L)$, where $\pi_1 : V \to L_1$ is the projection relative to the direct sum decomposition $V = L_0 \oplus L_1$. If $v \in P$ and $w_1, w_2 \in L_0$ are such that $v + w_1, v + w_2 \in L$, then $w_1 - w_2 \in L \cap L_0$; since $P \subset L + L_0$, it follows that the functionals $\rho_{L_1,L_0}(w_1)$ and $\rho_{L_1,L_0}(w_2)$ coincide in P. Conclude that if one chooses $w \in L_0$ such that $v + w \in L$, then the functional $S(v) = \rho_{L_1,L_0}(w)|_P \in P^*$ does not depend on the choice of w. One obtains a linear map $S : P \mapsto P^*$; using the fact that L is isotropic the computation (A.1.1) shows that S is symmetric. The uniqueness of the pair (P, S) is trivial.

Exercise 1.12. The equality $T(0, \alpha) = (0, \beta)$ holds iff $B\alpha = 0$ and $-A^*\alpha = \beta$. If B is invertible, then clearly the only solution is $\alpha = 0$; conversely, if B is not invertible, then there exists a non zero solution α of the equations.

Since B^*D is symmetric, then so is $B^{*-1}(B^*D)B^{-1} = DB^{-1}$. Moreover, since DB^{-1} is symmetric, then so is $A^*DB^{-1}A$; substituting $A^*D = (\mathrm{Id} + B^*C)^*$, we get that the matrix $(\mathrm{Id}{+}B^*C)^*B^{-1}A = (\mathrm{Id}{+}C^*B)B^{-1}A = B^{-1}A+$

C^*A is symmetric. Since C^*A is symmetric, then $B^{-1}A$ is symmetric. Finally, substituting $C = B^{*-1}(D^*A - \mathrm{Id})$ and using the fact that $DB^{-1} = B^{*-1}D^*$, we get $C - DB^{-1}A - B^{-1} = B^{*-1}D^*A - B^{*-1} - DB^{-1}A - B^{-1} = -B^{*-1} - B^{-1}$, which is clearly symmetric.

Exercise 1.13. T^* is symplectic iff, in the matrix representations with respect to a symplectic basis, it is $T\omega T^* = \omega$; this is easily established using the equalities $T^*\omega T = \omega$ and $\omega^2 = -\mathrm{Id}$.

Exercise 1.14. Clearly, if $P, O \in \mathrm{Sp}(2n, I\!R)$ then $M = PO \in \mathrm{Sp}(2n, I\!R)$. Conversely, recall from (1.4.6) that M is symplectic if and only if $M = \omega^{-1}M^*J$; applying this formula to $M = PO$ we get:

$$PO = \omega^{-1}P^*O^*\omega = \omega^{-1}P^*\omega \cdot \omega^{-1}O^*\omega.$$

Since ω is an orthogonal matrix, then $\omega^{-1}P^*\omega$ is again symmetric and positive definite, while $\omega^{-1}O^*\omega$ is orthogonal. By the uniqueness of the polar decomposition, we get $P = \omega^{-1}P^*\omega$ and $O = \omega^{-1}O^*\omega$ which, by (1.4.6), implies that both P and O are symplectic.

Exercise 1.15. Use Remark 1.4.7: a symplectic map $T : V_1 \oplus V_2 \to V$ must be injective. Use a dimension argument to find a counterexample to the construction of a symplectic map on a direct sum whose values on each summand is prescribed.

Exercise 1.16 $\omega(Jv, Jw) = \omega(J^2w, v) = -\omega(w, v) = \omega(v, w)$.

Exercise 1.17. If J is g-anti-symmetric $g(Jv, Jw) = -g(v, J^2w) = g(v, w)$.

Exercise 1.18. Use induction on $\dim(\mathcal{V})$ and observe that the g_{s}-orthogonal complement of an eigenspace of $\mathcal{V}$ is invariant by $\mathcal{T}$. Note that if $\mathcal{T}$ is Hermitian, the its eigenvalues are real; if $\mathcal{T}$ is anti-Hermitian, then its eigenvalues are pure imaginary.

Exercise 1.19. The linearity of ρ_{L_0,L_1} is obvious. Since $\dim(L_1) = \dim(L_0^*)$, it suffices to show that ρ_{L_0,L_1} is surjective. To this aim, choose $\alpha \in L_0^*$ and extend α to the unique $\tilde{\alpha} \in V^*$ such that $\tilde{\alpha}(w) = 0$ for all $w \in L_1$. Since ω is nondegenerate on V, there exists $v \in V$ such that $\tilde{\alpha} = \omega(v, \cdot)$. Since L_1 is maximal isotropic it must be $v \in L_1$, and ρ_{L_0,L_1} is surjective.

Exercise 1.20. Clearly, $\pi(L)$ is isotropic in $(S^\perp/S, \overline{\omega})$. Now, to compute the dimension of $\pi(L)$ observe that:

$$\begin{aligned}
&\dim\big(\pi(L \cap S^\perp)\big) = \dim(L \cap S^\perp) - \dim(L \cap S),\\
&(L \cap S)^\perp = L^\perp + S^\perp = L + S^\perp,\\
&\dim(L \cap S) + \dim\big((L \cap S)^\perp\big) = \dim(V),\\
&\frac{1}{2}\dim(S^1/S) = \frac{1}{2}\dim(V) - \dim(S) = \dim(L) - \dim(S).
\end{aligned}$$

The conclusion follows easily.

A.2. From Chapter 2

Exercise 2.1. Suppose that X is locally compact, Hausdorff and second countable. Then, one can write X as a countable union of compact sets K_n, $n \in \mathbb{N}$, such that K_n is contained in the interior $\operatorname{int}(K_{n+1})$ of K_{n+1} for all n. Set $C_1 = K_1$ and $C_n = K_n \setminus \operatorname{int}(K_{n-1})$ for $n \geq 2$. Let $X = \bigcup_\lambda U_\lambda$ be an open cover of X; for each n, cover C_n with a finite number of open sets V_μ such that

- each V_μ is contained in some U_λ;
- each V_μ is contained in $C_{n-1} \cup C_n \cup C_{n+1}$.

It is easily seen that $X = \bigcup_\mu V_\mu$ is a locally finite open refinement of $\{U_\lambda\}_\lambda$.

Now, assume that X is locally compact, Hausdorff, paracompact, connected and locally second countable. We can find a locally finite open cover $X = \bigcup_\lambda U_\lambda$ such that each U_λ has compact closure. Construct inductively a sequence of compact sets K_n, $n \geq 1$, in the following way: K_1 is any non empty compact set, K_{n+1} is the union (automatically finite) of all $\overline{U}_\lambda$ such that $U_\lambda \cap K_n$ is non empty. Since $K_n \subset \operatorname{int}(K_{n+1})$, it follows that $\bigcup_n K_n$ is open; since $\bigcup_n K_n$ is the union of a locally finite family of closed sets, then $\bigcup_n K_n$ is closed. Since X is connected, $X = \bigcup_n K_n$. Each K_n can be covered by a finite number of second countable open sets, hence X is second countable.

Exercise 2.2. Let $p \in P$ be fixed; by the local form of immersions there exist open sets $U \subset M$ and $V \in N$, with $f_0(p) \in V \subset U$, and a differentiable map $r : U \to V$ such that $r|_V = \mathrm{Id}$. Since f_0 is continuous, there exists a neighborhood W of p in P with $f_0(W) \subset V$. Then, $f_0|_W = r \circ f|_W$.

Exercise 2.3. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be differentiable atlases for N which induce the topology τ and such that the inclusions $\mathfrak{i}_1 : (N, \mathcal{A}_1) \to M$ and $\mathfrak{i}_2 : (N, \mathcal{A}_2) \to M$ are differentiable immersions. Apply the result of Exercise 2.2 with $f = \mathfrak{i}_1$ and with $f = \mathfrak{i}_2$; conclude that $\mathrm{Id} : (N, \mathcal{A}_1) \to (N, \mathcal{A}_2)$ is a diffeomorphism.

Exercise 2.4. The proof follows from the following characterization of local closedness: S is locally closed in the topological space X if and only if every point $p \in S$ has a neighborhood V in X such that $V \cap S$ is closed in V.

Exercise 2.5. From (2.1.14) it follows easily that the curve:

$$t \longmapsto \exp(tX) \cdot m$$

is an integral line of X^*.

Exercise 2.6. Repeat the argument in Remark 2.2.5, by observing that the union of a countable family of *proper* subspaces of $\mathbb{R}^n$ is a *proper* subset of $\mathbb{R}^n$. To see this use the Baire's Lemma.

Exercise 2.7. It is the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of matrices whose lower left $(n-k) \times k$ block is zero.

Exercise 2.8. Set $k = \dim(L \cap L_0)$. Consider a (not necessarily symplectic) basis $(b_i)_{i=1}^{2n}$ of V such that $(b_i)_{i=1}^{n}$ is a basis of L_0 and $(b_i)_{i=n-k+1}^{2n-k}$ is a basis of L. Define an extension of B by setting $B(b_i, b_j) = 0$ if either i or j does not belong to $\{n-k+1, \ldots, 2n-k\}$.

Exercise 2.9. The proof can be done in three steps:

- choose a partition $a = t_0 < t_1 < \cdots < t_k = b$ of the interval $[a, b]$ such that for all $i = 1, \ldots, k-1$ the portion $\gamma|_{[t_{i-1}, t_{i+1}]}$ of γ has image contained in an open set $U_i \subset B$ on which the fibration is trivial (an argument used in the proof of Theorem 3.1.23);
- observe that a trivialization α_i of the fibration over the open set U_i induces a bijection between the lifts of $\gamma|_{[t_{i-1}, t_{i+1}]}$ and the maps $f : [t_{i-1}, t_{i+1}] \to F$;
- construct $\overline{\gamma} : [a, b] \to E$ inductively: assuming that a lift $\overline{\gamma}_i$ of $\gamma|_{[a, t_i]}$ is given, define a lift $\overline{\gamma}_{i+1}$ of $\gamma|_{[a, t_{i+1}]}$ in such a way that $\overline{\gamma}_{i+1}$ coincides with $\overline{\gamma}_i$ on the interval $[a, t_{i-1} + \varepsilon]$ for some $\varepsilon > 0$ (use the local trivialization α_i and a local chart in F).

Exercise 2.10. The map is differentiable because it is the inverse of a chart. Using the technique in Remark 2.3.4, one computes the differential of the map $T \mapsto \mathrm{Gr}(T)$ as:

$$\mathrm{Lin}(I\!R^n, I\!R^m) \ni Z \longmapsto q \circ Z \circ \pi_1|_{\mathrm{Gr}(T)} \in T_{\mathrm{Gr}(T)} G_n(n+m),$$

where π_1 is the first projection of the decomposition $I\!R^n \oplus I\!R^m$ and $q : I\!R^m \to I\!R^{n+m}/Gr(T)$ is given by:

$$q(x) = (0, x) + Gr(T).$$

Exercise 2.11. Use the result of Exercise 2.9 and the fact that $\mathrm{GL}(n, I\!R)$ is the total space of a fibration over $G_k(n)$.

Exercise 2.12. An isomorphism $A \in \mathrm{GL}(n, I\!R)$ acts on the element $(W, \mathcal{O}) \in G_k^+(n)$ and produces the element $(A(W), \mathcal{O}')$ where $\mathcal{O}'$ is the unique orientation on $A(W)$ which makes

$$A|_W : (W, \mathcal{O}) \longrightarrow (A(W), \mathcal{O}')$$

a positively oriented isomorphism. The transitivity is proven using an argument similar to the one used in the proof of Proposition 2.4.2.

Exercise 2.13. Fix_{L_0} is a closed subgroup of $\mathrm{Sp}(V, \omega)$, hence it is a Lie subgroup. Let $L_1, L_1' \in \Lambda^0(L_0)$ be given. Fix a basis $\mathcal{B}$ of L_0; this basis extends in a unique way to a symplectic basis $\mathcal{B}_1$ in such a way that the last n vectors of such basis are in L_1 (see the proof of Lemma 1.4.35). Similarly, $\mathcal{B}$ extends in a unique way to a symplectic basis $\mathcal{B}_1'$ whose last n vectors are in L_1'. The unique symplectomorphism T of (V, ω) which fixes L_0 and maps L_1 onto L_1' is determined by the condition that T maps $\mathcal{B}_1$ to $\mathcal{B}_1'$.

Exercise 2.14. Use (1.4.7) and (1.4.8) on page 21.

Exercise 2.15. Use formulas (2.5.6) and (2.5.7) on page 56: choose $\tilde{L}_1 \in \Lambda$ with $\tilde{L}_1 \cap L_0 = \{0\}$ and set $\tilde{B} = \varphi_{L_0, \tilde{L}_1}(L)$. Now, solve for L_1 the equation:

$$B = \varphi_{L_0, L_1}\big(\varphi_{L_0, \tilde{L}_1}^{-1}(\tilde{B})\big) = \Big(\tilde{B}^{-1} - (\rho_{L_0, \tilde{L}_1})_{\#}\big((\varphi_{\tilde{L}_1, L_0}(L_1)\big)\Big)^{-1}.$$

A.3. From Chapter 3

Exercise 3.1. A homotopy H between the identity of X and a constant map $f \equiv x_0$ drags any given point of X to x_0.

Exercise 3.2. For each $x_0 \in X$, the set $\{y \in X : \exists$ a continuous curve $\gamma : [0,1] \to X$ with $\gamma(0) = x_0,\ \gamma(1) = y\}$ is open and closed, since X is locally arc-connected.

Exercise 3.3. Define $\lambda_s(t) = \lambda((1-s)t)$ and $H_s = (\lambda_s^{-1} \cdot \gamma) \cdot \lambda_s$. Observe that H_1 is a reparameterization of γ.

Exercise 3.4. If $[\gamma] \in \pi_1(X, x_0)$ then H induces a free homotopy between the loops $f \circ \gamma$ and $g \circ \gamma$ in such a way that the base point travels through the curve λ; use Exercise 3.3.

Exercise 3.5. Using the result of Exercise 3.4, it is easily seen that $g_* \circ f_*$ and $f_* \circ g_*$ are isomorphisms.

Exercise 3.6. The inclusion of $\{x_0\}$ in X is a homotopy inverse for f iff X is contractible.

Exercise 3.7. If g is a homotopy inverse for f, then it follows from Corollary 3.3.24 that $g_* \circ f_* = \mathrm{Id}$ and $f_* \circ g_* = \mathrm{Id}$.

Exercise 3.8. It follows from $r_* \circ \mathfrak{i}_* = \mathrm{Id}$.

Exercise 3.9. Do you really need a hint for this Exercise?

Exercise 3.10. If $\hat{f}_1$ and $\hat{f}_2$ are such that $p \circ \hat{f}_1 = p \circ \hat{f}_2 = f$ then the set $\{x : \hat{f}_1(x) = \hat{f}_2(x)\}$ is open (because p is locally injective) and closed (because E is Hausdorff).

Exercise 3.11. X is connected because it is the closure of the graph of $f(x) = \sin(1/x)$, $x > 0$, which is connected. The two arc-connected components of X are the graph of f and the segment $\{0\} \times [-1, 1]$. Both connected components are contractible, hence $H_0(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $H_p(X) = 0$ for all $p \geq 1$.

Exercise 3.12. See [**31**, §24, Chapter 3].

Exercise 3.13. First, if $U \subset X$ is open then $p(U)$ is open in X/G since $p^{-1}(p(U)) = \bigcup_{g \in G} gU$; moreover, if U is such that $gU \cap U = \emptyset$ for every $g \neq 1$ then p is a trivial fibration over the open set $p(U)$. This proves that p is a covering map. The other statements follow from the long exact homotopy sequence of p and more specifically from Example 3.2.21.

Exercise 3.14. The restriction of the quotient map $p\colon X \to X/G$ to the unit square I^2 is still a quotient map since I^2 is compact and X/G is Hausdorff; this gives the more familiar construction of the Klein bottle. To see that the action of G on X is properly discontinuous take for every $x \in X = \mathbb{R}^2$ the open set U (see Exercise 3.13) as an open ball of radius $\frac{1}{2}$.

Exercise 3.15. Use Example 3.2.10 and Theorem 3.3.33.

Exercise 3.16. Use the exact sequence $0 = H_2(D) \longrightarrow H_2(D, \partial D) \longrightarrow H_1(\partial D) \longrightarrow H_1(D) = 0$.

A.4. From Chapter 4

Exercise 4.1. It follows from Zorn's Lemma, observing that the union of any increasing net of B-negative subspaces is B-negative.

Exercise 4.2. The proof is analogous to that of Proposition 4.1.27, observing that if $v_1, v_2 \in V$ are linearly independent vectors such that $B(v_i, v_i) \le 0$, $i = 1, 2$, and such that (4.1.5) holds, then B is negative semi-definite in the two-dimensional subspace generated by v_1 and v_2 (see Example 4.1.12).

Exercise 4.3. Let V_1 be the k-dimensional subspace of V generated by the vectors $\{v_1, \ldots, v_k\}$ and V_2 be the $(n-k)$-dimensional subspace generated by $\{v_{k+1}, \ldots, v_n\}$; since X is invertible, then $B|_{V_1 \times V_1}$ is nondegenerate, hence, by Propositions 1.1.10 and 4.1.23, $n_\pm(B) = n_\pm(B|_{V_1\times V_1}) + n_\pm(B|_{V_1^\perp \times V_1^\perp})$. One computes:

$$V_1^\perp = \Big\{(-X^{-1}Zw_2, w_2) : w_2 \in V_2\Big\},$$

and $B|_{V_1^\perp\times V_1^\perp}$ is represented by the matrix $Y - Z^*X^{-1}Z$.

Exercise 4.4. Set $W = V \oplus V$ and define the nondegenerate symmetric bilinear form $B \in \mathrm{B}_{\mathrm{sym}}(W)$ by $B((a_1, b_1), (a_2, b_2)) = Z(a_1, a_2) - U(b_1, b_2)$. Let $\Delta \subset W$ denote the diagonal $\Delta = \{(v, v) : v \in V\}$; identifying V with Δ by $v \to (v, v)$, one computes easily $B|_\Delta = Z - U$, which is nondegenerate. Moreover, identifing V with $\Delta^\perp$ by $V \ni V \to (v, U^{-1}Zv) \in \Delta^\perp$, it is easily seen that $B|_{\Delta^\perp} = Z(Z^{-1} - U^{-1})Z$. The conclusion follows.

Exercise 4.5. Compute $\mathcal{O}_*$ on a generator of $H_1(\Lambda)$.

Exercise 4.6. The map $[0,1] \times [a,b] \ni (s,t) \mapsto A((1-s)t + sa) \cdot \ell(t) \in \Lambda$ is a homotopy with free endpoints between $\tilde{\ell}$ and the curve $A(a) \circ \ell$. Using Remark 3.3.30 one gets that $\tilde{\ell}$ and $A(a)\circ\ell$ are homologous in $H_1(\Lambda, \Lambda^0(L_0))$; the conclusion follows from Corollary 4.2.6.

Exercise 4.7. Using the result of Exercise 2.13 we find a curve $A\colon [a,b] \to \mathrm{Sp}(V, \omega)$ such that $A(t)(L_2) = L_1(t)$ for all t and for some fixed $L_2 \in \Lambda^0(L_0)$; it is easily seen that $\varphi_{L_0, L_1(t)}(\ell(t)) = \varphi_{L_0,L_2}\big(A(t)^{-1}(\ell(t))\big)$. The conclusion follows from Theorem 4.2.15 and Exercise 4.6.

Exercise 4.8. Using formula (2.5.11) one obtains:

$$\varphi_{L_3,L_0}(L_2) = -(\rho_{L_0,L_3})^{\#}\big(\varphi_{L_0,L_3}(L_2)^{-1}\big); \tag{A.4.1}$$

from (2.5.5) it follows that:

$$\varphi_{L_1,L_0} \circ (\varphi_{L_3,L_0})^{-1}(B) = \varphi_{L_1,L_0}(L_3) + \big(\eta^{L_0}_{L_1,L_3}\big)^{\#}(B) \in \mathrm{B}(L_1), \tag{A.4.2}$$

for any symmetric bilinear form $B \in \mathrm{B}(L_3)$. It is easy to see that:

$$\rho_{L_0,L_3} \circ \eta^{L_0}_{L_1,L_3} = \rho_{L_0,L_1}; \tag{A.4.3}$$

and the conclusion follows by setting $B = \varphi_{L_3,L_0}(L_2)$ in (A.4.2) and then using (A.4.1) and (A.4.3).

Exercise 4.9. See Examples 4.1.4 and 1.1.4.

Exercise 4.10. By Theorem 4.2.15, it is

$$\mu_{L_0}(\ell) = n_+\big(\varphi_{L_0,L_*}(\ell(b))\big) - n_+\big(\varphi_{L_0,L_*}(\ell(a))\big);$$

Conclude using the result of Exercise 4.9 where $L_3 = L_*$, setting $L_2 = \ell(a)$ and then $L_2 = \ell(b)$.

Exercise 4.11. Use Exercise 2.10.

Exercise 4.12. Observe that the map $p\colon \mathrm{Sp}(I\!R^n \oplus I\!R^{n*}, \omega) \to \Lambda$ given by $p(T) = T(\{0\} \oplus I\!R^{n*})$ is a fibration; the set in question is the inverse image by p of the dense subset $\Lambda^0(L_0)$ of Λ (see Remark 2.5.18). The reader can prove a general result that the inverse image by the projection of a dense subset of the basis of a fibration is dense in the total space. For the connectedness matter see the suggested solution of Exercise 4.13 below.

Exercise 4.13 and **Exercise 4.14.** These are the hardest problems on the book. The basic idea is the following; write every symplectic matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with B invertible as a product of the form:

$$T = \begin{pmatrix} 0 & B \\ -B^{*-1} & D \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ U & \mathrm{I} \end{pmatrix},$$

with $D = S \circ B$ and S, U symmetric $n \times n$ matrices. Observe that the set of symplectic matrices T with B invertible is diffeomorphic to the set of triples (S, U, B) in $\mathrm{B}_{\mathrm{sym}}(I\!R^n) \times \mathrm{B}_{\mathrm{sym}}(I\!R^n) \times \mathrm{GL}(n, I\!R)$. Using also some density arguments (like the result of Exercise 4.12) the reader should be able to complete the details.

Exercise 4.15. The map $\Phi : \mathrm{Sp}(2n, I\!R) \to \Lambda(I\!R^{4n})$ given by $\Phi(t) = \mathrm{Gr}(T)$ induces a map

$$\Phi_* : \pi_1\big(\mathrm{Sp}(2n, I\!R)\big) \cong \mathbb{Z} \longrightarrow \pi_1(\Lambda(I\!R^{4n}) \cong \mathbb{Z}\big)$$

which is injective (it is the multiplication by 2, up to a sign). This is easily checked by computing Φ_* on a generator of $\pi_1\big(\mathrm{Sp}(2n, I\!R)\big)$ (see Remarks 4.2.21 and 4.2.22). It follows that if a loop in $\mathrm{Sp}(2n, I\!R)$ has image by Φ which is contractible in $\Lambda(I\!R^{4n})$ then the original loop is contractible in $\mathrm{Sp}(2n, I\!R)$. Now, use that $\Lambda^0(\Delta)$ is diffeomorphic to a Euclidean space.

A.5. From Chapter 5

Exercise 5.1. If X is real-analytic, then also the fundamental matrix $t \mapsto \Phi(t)$ is real-analytic in $[a, b]$. If $(b_i)_{i=1}^n$ is a basis of the Lagrangian ℓ_0, then the (X, ℓ_0)-focal instants are the zeroes in $]a, b]$ of the real-analytic function:

$$[a, b] \ni t \longmapsto \det\big((\pi_1 \circ \Phi(t)) \cdot b_1, \ldots, (\pi_1 \circ \Phi(t)) \cdot b_n\big),$$

where $\pi_1\colon I\!R^n \oplus I\!R^{n*} \to I\!R^n$ denotes the projection onto the first coordinate. If the initial condition is nondegenerate, then by Corollary 5.2.4 there are no (X, ℓ_0)-focal instants in a neighborhood of $t = a$.

Exercise 5.2. $A^{\rm op} = A$, $B^{\rm op} = -B$, $C^{\rm op} = -C$ and

$$\Phi^{\rm op} = \mathcal{O} \circ \Phi \circ \mathcal{O}. \tag{A.5.1}$$

Moreover, (v, α) is a solution of $X^{\rm op}$ iff $\mathcal{O} \circ (v, \alpha) = (v, -\alpha)$ is a solution of $X^{\rm op}$.

Exercise 5.3. It is $\mathcal{O}^{\#}(\omega) = -\omega$, which proves that $\mathcal{O}$ is not a symplectomorphism, but it takes Lagrangian subspaces into Lagrangian subspaces. It is $P^{\rm op} = P$, $S^{\rm op} = -S$, $\mathrm{i}_{\rm foc}(X^{\rm op}, \ell_0^{\rm op}) = -\mathrm{i}_{\rm foc}(X, \ell_0)$ and $\mathrm{i}_{\rm maslov}(X^{\rm op}, \ell_0^{\rm op}) = -\mathrm{i}_{\rm maslov}(X, \ell_0)$. For the focal indexes, it suffices to observe that (X, ℓ_0) and $(X^{\rm op}, \ell_0^{\rm op})$ have the same focal instants, with the same multiplicity but opposite signature. As to the Maslov indexes, one first observe that, using (A.5.1), the curve $\ell^{\rm op}$ is given by $\mathcal{O} \circ \ell$; $\mathcal{O}$ gives a diffeomorphism of the Lagrangian Grassmannian Λ, $\mathcal{O}(L_0) = L_0$ and so $\mathcal{O}$ leaves $\Lambda^0(L_0)$ invariant. One gets an isomorphism $\mathcal{O}_* : H_1(\Lambda, \Lambda^0(L_0)) \to H_1(\Lambda, \Lambda^0(L_0))$; it follows from the result of Exercise 4.5 that $\mathcal{O}_* = -\mathrm{Id}$.

Exercise 5.4. If $0 < \varepsilon < \varepsilon'$ and if there are no (X, ℓ_0)-focal instants in $]a, a + \varepsilon']$, then $\ell|_{[a+\varepsilon, a+\varepsilon']}$ is a curve in $\Lambda^0(L_0)$, and therefore $\mu_{L_0}\big(\ell|_{[a+\varepsilon, a+\varepsilon']}\big) = 0$.

Exercise 5.5. See [**36**], in the remarks after Definition 3.1.6.

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