REVIEW MR 2 383 373

A. Portaluri, *Maslov index for Hamiltonian systems*, Electron. J. Differential Equations **2008**, No. 09, 1–10.

The paper in question is the final version of several manuscripts by the same author that circulated on the internet over the last years. It is asserted that a *new* and *simpler* proof of one of the preliminary results in ref. [Nostre-Marques, Piccione, Tausk, MR1978789] is given. Such formula relates the Conley–Zehnder index of a linear symplectic system with its Maslov index relative to the choice of a fixed Lagrangian. The reader will hardly find any original statement in the present paper, whose relevant material can be entirely recovered from the existing literature. A technical analysis of the result might help the reader putting the contribution of this paper in the right perspective.

The formula in question expresses the difference between the *Maslov index* and the *Conley–Zehnder index* of a symplectic system as the signature of a certain symmetric bilinear form, which is obtained using the block coefficients of the time-one flow of the system. Both the formula and the correct expression for this bilinear form were studied in the original reference [Nostre-Marques, Piccione, Tausk, MR1978789], where the result was used to prove an index theorem. Let us point out that the Maslov index is a symplectic invariant associated to symplectic systems and to the choice of a fixed Lagrangian space L_0 of reference. Such Lagrangian space is usually called *vertical*, and the motivation for this is that the most natural occurrence of such objects arises from the linearization of the Hamilton equations along a solution of a Hamiltonian in a cotangent space TM^*. In this case, the reference Lagrangian L_0 correspond to the vertical subbundle of the tangent bundle $T(TM^*)$.

Recently, it has been observed that the difference between the Maslov index and the Conley–Zehnder index can also be described as a *Kashiwara index*, which is another symplectic invariant known in the literature. Also the Kashiwara index is defined as the signature of a certain symmetric bilinear form, and the study of its relations with the notions of Maslov and Conley–Zehnder index involves essentially the same arguments in [Nostre-Marques, Piccione, Tausk, MR1978789]. Several formulas relating the three indices appear in the literature, being hard (and irrelevant) to assess the authorship of these formulas, which certainly does not belong to the author of the present paper. Among *many others*, the reader will find a clear survey in the classical reference [S. Cappell, R. Lee, E. Miller MR1263126], or in more recent works, such as [de Gosson, MR2154859], and [M. de Gosson, S. de Gosson, MR2285155]. In the unpublished note `arXiv:math/0306187`, which contains some of the material reproduced by the author in the present paper, Proposition 3.27 gives a very general statement about an equality satisfied by the three indices that applies as a special case to the present situation. Very likely, similar results appear elsewhere, as they are well established among specialists.

The author's *idea* for this *new* and *simpler* proof of the formula is to make an explicit calculation in terms of elementary linear algebra to show that the Kashiwara index equals the index of the symmetric bilinear form found in [Nostre-Marques, Piccione, Tausk, MR1978789]. Anybody with a little familiarity with the objects of the game would easily understand that the direct and self-contained proof of the equality given in [Nostre-Marques, Piccione, Tausk, MR1978789] and a proof of the equality using the theory of Kashiwara index are perfectly equivalent tasks. The proof in [Nostre-Marques, Piccione, Tausk, MR1978789] contains exactly the same elements needed to develop the tools of Kashiwara index that are implicitly assumed by the author, and thus Portaluri's approach provides no simplification whatsoever. At any case, the task undertaken by the author is a matter of simple definitions of the objects in question and elementary linear algebra; one could raise legitimate doubts on whether such calculation is a sufficient motivation for publishing a paper on a research journal. To be *very precise*, once the entire background assumed by the author is set up, all one needs to do is to compute the restriction of a given quadratic form of $\mathbb{R}^{4n}$ to the direct sum of three given n-planes of $\mathbb{R}^{4n}$. Even admitting that a professional mathematician were not able to work this out by himself, the reader will not be surprised to find out that also this elementary calculation already appears in the literature, like for instance in Reference [M. de Gosson, S. de Gosson, MR2285155], Section 3.4, Proposition 8. In this article the authors also compare their result with some other classical notions, like the *index of concavity* of a peridioc Hamiltonian solution, the *index of inertia*, etc.; one can reasonably expect that very similar formulas are also well established in the mathematical physics realm. Again, such formula is a only a preliminary result in [M. de Gosson, S. de Gosson, MR2285155], where the authors use it to establish a product formula for paths of symplectomorphisms and discuss other applications. The core of Portaluri's computation, Lemma 3.1, which consists of a two page long algebraic proof ("multiply on the right by BB^{-1}"... "multiply on the left by $[B^T]^{-1}B^T$" sort of arguments) of the symmetry of a certain bilinear form, already appears with a one line proof in the original Nostre-Marques/Piccione/Tausk reference. The same result is mentioned and used in several of de Gosson papers, and possibly in many others. It appears also in the exercise list of a students textbook Lagrangian Grassmannians and Maslov index, written in 2000 and authored by this reviewer and D. Tausk, available at the address: `http://www.ime.usp.br/~piccione/Downloads/NotasXIEscola.pdf` (see Exercise 1.12, page 30, with solution on page 152).

A special situation occurs in Section 2 of the paper, where the author attempts a calculation of the Maslov index of a symplectic system with constant coefficients using a change of coordinates that puts the system in *normal form*. This is a very natural technique for computing the Conley–Zehnder index of a system, as described in several articles and books, and the idea of using this technique for the case of Maslov index is obviously the only original contribution of the author to the entire theory. No other author has ever done this, because it is absolutely evident that the desired change of coordinates does not preserve the vertical Lagrangian employed in the definition of Maslov index, so that passing to a normal form produces a Maslov index relative to some other unknown Lagrangian. In the two previous versions of the manuscript this problem had been completely overlooked, leading the author to a whole series of incorrect results, and the mistake was pointed out by several other authors. In the present paper the author, rather than removing the incorrect part, decides to modify the result in an attempt to make a meaningful statement out of senseless computation. The result (Proposition 2.10), based on a detailed analysis of the 2×2 matrix $\begin{pmatrix} \cos \alpha x & -\sin \alpha x \\ \sin \alpha x & \cos \alpha x \end{pmatrix}$, gives an awkward formula for the Maslov index of the system relative to "the vertical Lagrangian $L_0' = \oplus_{j=1}^p L_0^j$ of the symplectic space $(V, \oplus_{j=1}^p \omega_j)$ where ω_j is the standard symplectic form in $\mathbb{R}^{2m}$ for $m = 1, 2$ corresponding to the decomposition of V into 2 and 4 dimensional $\psi(1)$-invariant symplectic subspaces." The use of the adjective "vertical" here for the Lagrangian L_0' is highly questionable, and misleading on the real meaning of such result. Indeed, this is a camouflaged and unprecise way of saying that the given formula produces the vertical Maslov index of a system whose coefficient matrix is in normal form, which has the same mathematical depth as giving a formula for the eigenvalues of a diagonal matrix. Certainly, whoever might be interested in determining the Maslov index of a system in normal form will not need to read this paper to compute it. In addition, whatever mathematical information Proposition 2.10 may contain, reasonably none, it is not the author of the present paper who should be given any credit for.

In conclusion, the paper contains no new idea, no new result, no new proof and no new application. Nonetheless, in spite of the fact that the total mathematical information delivered by this paper to the scientific community constitutes the empty set, reading this paper and all related material (see Reviewer's note below) may be very instructive for those who want to reflect about the fragility and the vulnerability of our refereeing/publishing system.

Reviewer's note. It must be said that the paper in question is the final outcome of a series of manuscripts released by the author over the last years, that the interested reader will find on the arXiv at `xxx.arxiv.org`, see arXiv:math/0405153v1, arXiv:math/0405153v2, arXiv:math/0405153v3 and arXiv:math/0405153v4.

The latest public version of Portaluri's manuscript before publication in the Electronic Journal of Differential Equations is arXiv:math/0405153v3, where a first attempt of proving the desired formula lead the author to an incorrect result, obtained by an incorrect linear algebraic proof (see arXiv:math/0405153v3, Theorem 6.3) (in fact, the result was also stated under incorrect assumptions). Analogous (incorrect) statements appeared in the older versions of the manuscript. The published version of the manuscript correspond to the preprint arXiv:math/0405153v6 (posted on the arXiv by the author only after publication on the Electronic Journal of Differential Equations), a revised version of arXiv:math/0405153v3 modified according to a report written by this reviewer. This reviewer's observations on the several incorrect formulas produced by the author over the years, as well as their alleged applications to semi-Riemannian Geometry, can be found in arXiv:math/0505160v1, Section 5.4, and especially in arXiv:0705.3171v1. Among other things, in this note the reviewer draws the extensive work of de Gosson to the attention of the author, who seems to have simply ignored the large amount of material already written on the topic.

This reviewer considers the present paper very incorrect, both from a mathematical and from an ethical point of view.