On the least action principle.
Hamiltonian dynamics on fixed energy levels in the non convex case.

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Abstract
We review the classical Principle of the Least Action in a general context where the Hamilton function $H$ is possibly non convex. We show how van Groesen [6] principle follow as particular cases where $H$ is hyperregular and of homogeneous type. Homogeneous scalar field spacetimes in spherical symmetry are derived as an application.

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1 Introduction
It dates back to 1746 when Pierre Louis Moreau de Maupertuis expressed the principle that today we name after him. Let us state it in a modern fashion:

Maupertuis–Jacobi Principle. Let $(M,g)$ be a Riemannian manifold and $V : M \to \mathbb{R}$ be a $C^1$ function. Motions of particles between fixed points in $M$ with potential energy $V(x)$ and energy $E$ (in the set $\{V(x) < E\}$) are – up to a reparametrization – geodesics of the metric $g_E = 2(E - V(x))g$. 

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In other words, we would say today that a curve \( x : [0, T] \to M \) is a solution of Euler–Lagrange equation for the Lagrangian function \( \frac{1}{2}g(\dot{x}, \dot{x}) - V(x) \) if and only if it admits a reparameterization on the interval \([0, 1]\) which is a critical point of the functional \( \int_0^1 g_E(\dot{x}, \dot{x}) \, dt \) in the space of \( C^1 \) curves from \([0, 1]\) to \( M \) between \( x(0) \) and \( x(T) \).

Maupertuis discovered the above principle in quite an embryonic form, following largely incorrect arguments. Nevertheless, his underlying idea that Nature behaves minimizing a universal quantity – the action, as he called it – is “well in line with the cosmic spirit of the 18th century”, as remarked in [3], and probably influenced many of the contemporary mathematicians who worked on pioneering Calculus of Variations: among all Leonhard Euler, who correctly derived the principle, probably even before than Maupertuis himself, but nevertheless defended the work of the French scientist from other claims of principle’s fatherhood, especially Leibnitz’. Also Jacobi developed a correct form of the principle – the metric \( g_E \) is usually called Jacobi metric – and so did Lagrange.

As a matter of fact, the above principle can be seen as an applications to natural systems (i.e. ruled by Lagrangian function of the form “metric plus potential”) of a most general Principle of the Least Action. In this context, the Hamiltonian formulation of the principle seems more natural, and its classical version when \( M = \mathbb{R}^n \) is the following:

**Principle of the Least Action** (case \( \mathbb{R}^n \)). Let \( p = (p_i) \) and \( q = (q^i) \) be coordinates on \( \mathbb{R}^{2n}, i = 1, \ldots , n \). Let \( \Gamma(t) = (q(t), p(t)) \) be a curve in \( \mathbb{R}^{2n} \) and call \( q_0, q_1 \) the initial and the final point on \( \mathbb{R}^n \) of \( q(t) \).

Then, \( \Gamma \) is a solution of Hamilton equations for a Hamilton function \( H \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \), with \( E \) (the – constant – value of \( H(\Gamma(t)) \)) regular value for \( H \), if and only if \( \Gamma \) is a critical points of the functional

\[
G(\Gamma) = \int_{\Gamma} p \cdot dq
\]

in the space of \( C^1 \) curves \( (q(t), p(t)) \) on \( \mathbb{R}^{2n} \) with support in \( H^{-1}(E) \), such that \( q_0, q_1 \) are the initial and the final point on \( \mathbb{R}^n \) of \( q(t) \).

We observe that the interval of definition of the curves in the functional space is not fixed. The above functional arises in Hamiltonian dynamics from Poincaré–Cartan invariant form \( p \cdot dq - H dt \), when restricted to level sets of the Hamiltonian function \( H \), and has the property to be invariant by reparameterizations.

This classical result may be applied to recover Maupertuis–Jacobi principle in the context of natural systems (see Remark 4.4). But it turns out, that if one imposes the constraint in an integral form, i.e. one restricts to curves such that the integral of the Hamilton function along the curves is equal to a given constant, the same functional leads, for natural systems, to a new variational principle, where the functional is now given by \( \int_0^1 (E - V(x)) \, dt \cdot \int_0^1 g(\dot{x}, \dot{x}) \, dt \). This fact has been first established in the Riemannian (i.e., positive definite) case by van Groesen in [6, 7], where such functional framework is employed in the study of brake orbits for symmetric conservative system with smooth potentials.
In this review, we present (Sections 2 to 3) the Hamiltonian formulation of the Least Action Principle, recovering van Groesen results, in a general context of differentiable manifolds where we remove any convexity assumption on the Hamilton function $H$. The convexity assumption is replaced by a more general assumption of homogeneity type (see Definition 4.2), and the proof of the principle is obtained using techniques of infinite dimensional Banach manifold theory. Special attention has to be given to the fact that the class of Hamiltonians considered leads to a possibly non time-reversible dynamics (see Remark 3.1). The special case where $H$ is hyperregular will allow us to express (Section 4) van Groesen principle for natural system in the case where the “kinetic” part is given by a general nondegenerate metric. Motivations for this are explained in Section 5, where it is shown how van Groesen principle can be applied to study existence of spherically symmetric solutions of Einstein field equations in General Relativity, starting from Hilbert–Palatini functional.

2 Basic notations

Let $M$ be a differentiable ($C^3$) manifold and denote by $\pi: TM \to M$, $\pi^*: TM^* \to M$ respectively the tangent and the cotangent bundle of $M$. For $q \in M$, the fibers $\pi^{-1}(q) = T_qM$ and $\pi^*\pi^{-1}(q) = T_qM^*$ are embedded submanifolds of $TM$ and $TM^*$ respectively, and given $v \in T_qM$ and $p \in T_qM^*$, the tangent spaces $T_v(T_qM) \subset T_v(TM)$ and $T_p(T_qM^*) \subset T_p(TM)$ will be called the vertical subspaces. These spaces will be denoted by $\text{Ver}_v$ and $\text{Ver}_v^*$ respectively; clearly, $\text{Ver}_v = \text{Ker}(d\pi_v)$ and $\text{Ver}_v^* = \text{Ker}(d\pi_v^*)$. One can identify canonically $\text{Ver}_v^* \cong T_qM^*$ and $\text{Ver}_v \cong T_qM$ for all $p \in T_qM^*$ and all $v \in T_qM$.

The canonical 1-form of $TM^*$, denoted by $\theta$, is defined by:

$$\theta_p(\alpha) = p(d\pi_p^*(\alpha)), \quad p \in TM^*, \quad \alpha \in T_p(TM^*);$$

the canonical 2-form $\omega$ of $TM^*$ is given by $-d\theta$. It is well known that $\omega$ is everywhere nondegenerate, i.e., $\omega$ is a symplectic form, and that $\text{Ver}_p^*$ is a Lagrangian subspace of the symplectic space $(T_p(TM), \omega_p)$, i.e., $\text{Ver}_p^*$ is a maximal subspace of $T_p(TM^*)$ on which $\omega_p$ vanishes identically. In particular, if $\omega_p(\alpha, \beta) = 0$ for all $\beta \in \text{Ver}_p^*$, then $\alpha \in \text{Ver}_p^*$.

Given an absolutely continuous curve $\Gamma: [a, b] \to TM^*$, the canonical action $G(\Gamma)$ is defined by the line integral:

$$G(\Gamma) = \int_a^b \theta(\Gamma') \, dt = \int_a^b \Gamma(d\pi^*(\Gamma')) \, dt; \quad (2.1)$$

clearly, $G$ in invariant by reparameterizations.

For $q_0, q_1 \in M$, we will denote by $\Lambda_{q_0, q_1}$ and $\Omega_{q_0, q_1}$ the infinite dimensional Banach manifolds:

$$\Lambda_{q_0, q_1} = \left\{ \gamma \in C^1([0, 1], M) : \gamma(0) = q_0, \gamma(1) = q_1 \right\},$$

$$\Omega_{q_0, q_1} = \left\{ \Gamma \in C^1([0, 1], TM^*) : \pi^*(\Gamma(0)) = q_0, \pi^*(\Gamma(1)) = q_1 \right\}.$$
Given $\Gamma \in \Omega_{q_0,q_1}^E$, the tangent space $T_\Gamma \Omega_{q_0,q_1}^E$ is identified with the Banach space of all $C^1$-vector fields $V : [0,1] \to T(TM^*)$ along $\Gamma$ and such that $V(0) \in \text{Ver}_{\Gamma(0)}^*$ and $V(1) \in \text{Ver}_{\Gamma(1)}^*$.

### 3 Hamiltonian dynamics

A smooth ($C^2$) map $H : TM^* \to \mathbb{R}$ will be called a Hamiltonian on $M$; the fiber derivative of $H$ is the map $\mathcal{F}H : TM^* \to TM$ defined by $\mathcal{F}H(\alpha) = d(H|_{T_aM^*})_\alpha$, where $\alpha \in T_aM^*$. The Hamiltonian $H$ is said to be regular if $\mathcal{F}H$ has no critical point, in which case $\mathcal{F}H$ is a local diffeomorphism between $TM^*$ and $TM$. When $\mathcal{F}H$ is a global diffeomorphism, we say that $H$ is hyperregular. The Hamiltonian vector field $\tilde{H}$ is the symplectic gradient of $H$, defined by $\omega(\tilde{H}, \cdot) = dH$; the fiber derivative $\mathcal{F}H$ can be expressed in terms of the Hamiltonian vector field $\tilde{H}$:

$$\mathcal{F}H(p) = d\pi^*_p(\tilde{H}(p)), \quad \forall p \in TM^*.$$  

An integral curve $\Gamma : [a,b] \to TM^*$ of $\tilde{H}$ is called a solution of the Hamiltonian $H$. If $\Gamma : [a,b] \to TM^*$ is a solution of $H$, then $H \circ \Gamma$ is constant:

$$\frac{d}{dt} H(\Gamma(t)) = dH((\Gamma(t)) \Gamma'(t) = \omega(\tilde{H}(\Gamma(t)), \tilde{H}(\Gamma(t))) = 0; \text{ we call such constant the energy of } \Gamma.$$

Assume that $H$ is a regular Hamiltonian and that $E$ is a regular value of $H$ with $H^{-1}(E) \neq \emptyset$; given $q_0, q_1 \in M$, we want to determine the solutions $\Gamma : [a,b] \to TM^*$ of $H$ having energy $E$ and such that $\pi^*(\Gamma(a)) = q_0, \pi^*(\Gamma(b)) = q_1$.

The Principle of Least Action – recalled in the introduction for the case $M = \mathbb{R}^n$ – states [2] that solutions of $H$ with energy $E$ correspond (after a suitable reparameterization) to the critical points of the restriction of the canonical action functional $\mathcal{G}$ in (2.1) to the submanifold:

$$\Omega_{q_0,q_1}^E = \{ \Gamma \in \Omega_{q_0,q_1} : H \circ \Gamma \equiv E \text{ on } [0,1] \}.$$  

An alternative variational principle for fixed energy Hamiltonian solutions has been introduced in [6] in the case of Hamiltonians $H$ on the manifold $M = \mathbb{R}^n$ that are strictly convex and even in the momenta. We will now discuss a similar variational principle in the general case of a regular Hamiltonian on an arbitrary manifold $M$.

Let us consider the set:

$$\widetilde{\Omega}_{q_0,q_1}^E = \{ \Gamma : [0,1] \to TM^* : \pi^*(\Gamma(0)) = q_0, \pi^*(\Gamma(1)) = q_1, \int_0^1 H(\Gamma(s)) \, ds = E \}.$$  

Clearly, $\widetilde{\Omega}_{q_0,q_1}^E \supset \Omega_{q_0,q_1}^E$; we have the following:

**Lemma 3.1** If $E$ is a regular value of $H$, then $\widetilde{\Omega}_{q_0,q_1}^E$ is a smooth embedded submanifold of $\Omega_{q_0,q_1}$. For $\Gamma \in \widetilde{\Omega}_{q_0,q_1}^E$, the tangent space $T_\Gamma \widetilde{\Omega}_{q_0,q_1}^E$ is the space of all $C^1$-vector fields $V : [0,1] \to T(TM^*)$ along $\Gamma$, with $V(0) \in \text{Ver}_{\Gamma(0)}^*$, $V(1) \in \text{Ver}_{\Gamma(1)}^*$, and such that:

$$\int_0^1 dH(\Gamma(t))V(t) \, dt = \int_0^1 \omega(\tilde{H}(\Gamma(t)), V(t)) \, dt = 0.$$  

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Proof. The map $\tilde{H} : \Omega_{q_0, q_1} \ni \Gamma \mapsto \int_0^1 H(\Gamma(t)) \, dt \in \mathbb{R}$ is smooth. Let $\Gamma \in \tilde{\Omega}_{q_0, q_1}^E$ be fixed. Using the implicit function theorem, it will suffice to show that the differential $d\tilde{H}(\Gamma) : T_{\Gamma} \Omega_{q_0, q_1} \to \mathbb{R}$ is non zero, where

$$d\tilde{H}(\Gamma)V = \int_0^1 dH(\Gamma(t))V(t) \, dt,$$

and that the kernel $\text{Ker}(d\tilde{H}(\Gamma))$ splits in the Banach space $T_{\Gamma} \Omega_{q_0, q_1}$. This last fact follows immediately from the fact that $\text{Ker}(d\tilde{H}(\Gamma))$ has finite codimension in $T_{\Gamma} \Omega_{q_0, q_1}$.

Let us show that, for $\Gamma \in \tilde{\Omega}_{q_0, q_1}^E$, $d\tilde{H}(\Gamma)$ is non zero. We claim that there exists $t_0 \in \mathbb{R}$ with $H(\Gamma(t)) = E$; this follows easily from the fact that $\int_0^1 H(\Gamma(t)) \, dt = E$ and from the continuity of $H \circ \Gamma$. Since $E$ is a regular value of $H$, there exists $t_0 \in T_{\Gamma(t_0)}(TM^*)$ with $dH(\Gamma(t_0))v_0 \neq 0$, say, $dH(\Gamma(t_0))v_0 > 0$. Choose an arbitrary smooth vector field $W$ along $\Gamma$ with $W(t_0) = v_0$, and choose a smooth map $\psi : [0, 1] \to \mathbb{R}$ with $\psi(t) > 0$, and whose support is contained in a small neighborhood of $t_0$. Finally, for $t \in [0, 1]$, set $V(t) = \psi(t)W(t)$; by continuity, $dH(\Gamma(t))V(t) \geq 0$ on $[0, 1]$, and $dH(\Gamma(t))V(t) > 0$ for $t$ near $t_0$, hence $\int_0^1 dH(\Gamma(t))V(t) \, dt > 0$, which concludes the proof.

Remark 3.1 We observe that the general class of Hamiltonian dynamics considered in this paper is not necessarily time-reversible, in the sense that, given a solution $\Gamma : [0, T] \to TM^*$ of $H$, its backwards reparameterization $\Gamma^- : [0, T] \to TM^*$, defined by:

$$\Gamma^-(t) = \Gamma(T - t), \quad \forall t \in [0, T], \quad (3.1)$$

is not a solution of $H$, in general. In particular, the existence of a solution $\Gamma$ of $H$ having energy $E$ and going from some $p_0 \in TM^*$ to some other $p_1 \in TM^*$, does not imply the existence of some solution of $H$ with energy $E$ and going from $p_1$ to $p_0$. Time reversibility is guaranteed when $H$ is an even function in the momenta, i.e., when the restriction of $H$ to each fiber $T_qM^*$ is even for all $q \in M$.

Our next result will relate the solutions $\Gamma : [0, T] \to TM^*$ of $H$ with energy $E$ and with $\pi^*(\Gamma(0)) = q_0$, $\pi^*(\Gamma(1)) = q_1$, to the critical points of the canonical action functional $G$ in $\tilde{\Omega}_{q_0, q_1}^E$. However, we observe that if $\Gamma \in \tilde{\Omega}_{q_0, q_1}^E$ is a critical point of $G$, then its backwards reparameterization $\Gamma^-$ belongs to $\Omega_{q_1, q_0}^E$, and it is a critical point of $G$. As we have observed, solutions of $H$ do not share this property in general.

Proposition 3.1 Let $H$ be a regular Hamiltonian, $E$ a regular value of $H$, and let $q_0, q_1 \in M$ be fixed. Given a solution $\Gamma : [0, T] \to TM^*$ of $H$ with energy $E$ and satisfying $\pi^*(\Gamma(0)) = q_0$, $\pi^*(\Gamma(T)) = q_1$, then its affine reparameterization $\Gamma : [0, 1] \to TM^*$, defined by $\Gamma(s) = \Gamma(st)$, is a critical point of the restriction to $\tilde{\Omega}_{q_0, q_1}^E$ of the canonical action functional $G$. Conversely, given a critical point $\Gamma$ of the restriction to $\tilde{\Omega}_{q_0, q_1}^E$ of the canonical action functional $G$, then there exists
Let \( T > 0 \) such that, defining \( \Gamma : [0, T] \to TM^* \) to be the affine reparameterization of \( \tilde{\Gamma} \) on \([0, T]\), either \( \Gamma \) or its backwards reparameterization \( \Gamma^- \) is a solution of \( H \) having energy \( E \).

**Proof.** For \( \tilde{\Gamma} \in \tilde{\Omega}^E_{q_0, q_1} \), if \( \alpha \in T_{\tilde{\Gamma}} \tilde{\Omega}^E_{q_0, q_1} \) is a variational vector field along \( \tilde{\Gamma} \), one computes:

\[
dG(\tilde{\Gamma}) \alpha = \int_0^1 \theta(\alpha, \tilde{\Gamma}') \, ds + \theta(\alpha)\mid_0^1 = \int_0^1 \omega(\tilde{\Gamma}', \alpha) \, ds;
\]

observe that \( \theta(\alpha(0)) = \theta(\alpha(1)) = 0 \) because \( \alpha(0) \) and \( \alpha(1) \) are vertical. Using the Lagrange multiplier method to the constraint \( \int_0^1 H(\tilde{\Gamma}) \, ds = E \), we obtain that \( \tilde{\Gamma} \) is a critical point of the restriction of \( G \) to \( \tilde{\Omega}^E_{q_0, q_1} \) if and only if there exists \( \lambda \in \mathbb{R} \) such that:

\[
\int_0^1 \left[ \omega(\tilde{\Gamma}', \alpha) - \lambda \cdot dH(\tilde{\Gamma}) \alpha \right] \, ds = \int_0^1 \omega(\tilde{\Gamma}' - \lambda \cdot H(\tilde{\Gamma}), \alpha) \, ds = 0
\]

for all \( \alpha \in T_{\tilde{\Gamma}} \tilde{\Omega}^E_{q_0, q_1} \). The fundamental lemma of Calculus of Variations implies easily that this is equivalent to \( \tilde{\Gamma}' - \lambda \tilde{H}(\tilde{\Gamma}) = 0 \) on \([0, 1]\). The case \( \lambda = 0 \) is trivial and uninteresting. If \( \lambda > 0 \), then set \( T = \lambda \) to obtain the solution \( \Gamma(t) = \tilde{\Gamma}(t/T) \) of \( H \). If \( \lambda < 0 \), then set \( T = -\lambda \) and \( \Gamma^-(t) = \tilde{\Gamma}(1 - t/T), \, t \in [0, T] \). Clearly, \( H \circ \Gamma^- \equiv E \), and:

\[
(\Gamma^-)'(t) = -\frac{1}{T} \tilde{\Gamma}'(1 - t/T) = -\frac{\lambda}{T} H(\tilde{\Gamma}(1 - t/T)) = H(\Gamma^-(t)), \quad \forall \, t \in [0, T].
\]

This concludes the proof.

**Remark 3.2** From the proof of Proposition 3.1 one sees that, given a critical point \( \tilde{\Gamma} \) of the restriction to \( \tilde{\Omega}^E_{q_0, q_1} \) of the canonical action functional \( G \), then the corresponding solution \( \Gamma \) of \( H \) is defined on the interval \([0, T]\), where \( T \) is the absolute value of the Lagrange multiplier \( \lambda \) of \( \tilde{\Gamma} \).

### 4 The hyperregular case: variational problem in configuration space

Let us discuss now a procedure that, in the case of a hyperregular Hamiltonian, allows to reduce the variational problem in phase space given in Proposition 3.1 to an alternative variational problem in the configuration space, in the spirit of [6, Section 4]. Let us assume that \( H : TM^* \to \mathbb{R} \) is a hyperregular Hamiltonian function, and that \( E \) is a regular value of \( H \).

**Definition 4.1** Let \( \Gamma \) be a curve in \( \tilde{\Omega}^E_{q_0, q_1} \); a vector field \( V \in T\tilde{\Omega}^E_{q_0, q_1} \) is said to be a vertical variation of \( \Gamma \) if \( V(t) \in \text{Ver}_{\tilde{\Gamma}(t)} \) for all \( t \in [0, 1] \). The curve \( \Gamma \) is said to be vertically critical for the functional \( G \) in \( \tilde{\Omega}^E_{q_0, q_1} \), if \( dG(\Gamma)V = 0 \) for all vertical variation \( V \) of \( \Gamma \); the set of vertically critical curves \( \Gamma \) in \( \tilde{\Omega}^E_{q_0, q_1} \) is denoted with \( \mathcal{V}^E_{q_0, q_1} \).
Let us introduce the following notion for hyperregular Hamiltonians:

**Definition 4.2** Let $H : TM^* \to \mathbb{R}$ be a hyperregular Hamiltonian function and let $E$ be a regular value of $H$. We will say that $H$ is of homogeneous type at $E$ if for all curve $\gamma : [0, 1] \to M$ of class $C^1$ there exists a unique $\lambda_{\gamma,E} \in \mathbb{R}^+$ (or in $\mathbb{R}^-$) such that:

$$
\int_0^1 H\left((\bar{F}H^{-1})(\lambda_{\gamma,E}^{-1}\gamma'(t))\right) dt = E,
$$

and if the map $C^1([0, 1], M) \ni \gamma \mapsto \lambda_{\gamma,E} \in \mathbb{R}^+$ is of class $C^1$.

Examples of hyperregular Hamiltonians of homogeneous type will be discussed in next Section. We will assume for the remainder of this section that $E$ is a regular value of $H$.

**Lemma 4.1** Assume that $H$ is of homogeneous type at $E$ and, for all curve $\gamma : [0, 1] \to M$ of class $C^1$, denote by $\Gamma^E_\gamma : [0, 1] \to TM^*$ the curve given by:

$$
\Gamma^E_\gamma(t) = (\bar{F}H^{-1})(\lambda_{\gamma,E}^{-1}\gamma'(t)), \quad \forall t \in [0, 1].
$$

If $q_0 \neq q_1$, a curve $\Gamma \in \bar{\Omega}^E_{q_0,q_1}$ is vertically critical if and only if $\Gamma = \Gamma^E_\gamma$, where $\gamma = \pi^* \circ \Gamma$. When $q_0 = q_1$, the same conclusion holds for curves $\Gamma$ whose image is not entirely contained in the fiber $\pi^*(q_0)$.

**Proof.** An immediate calculation shows that vertical criticality for a curve $\Gamma$ is equivalent to the existence of a real number $\lambda$ such that:

$$
\int_0^1 \omega\left(\Gamma'(t) - \lambda \cdot \bar{H}(\Gamma(t)), V(t)\right) dt = 0
$$

for all vertical variations $V$ of $\Gamma$. Since vertical subspaces are Lagrangian, this condition is easily seen to be equivalent to the fact that $\Gamma'(t) - \lambda \cdot \bar{H}(\Gamma(t))$ be vertical for all $t$, i.e.,

$$
d\pi^*_{\Gamma(t)}(\Gamma'(t) - \lambda \cdot \bar{H}(\Gamma(t))) = 0.
$$

This gives:

$$
\gamma'(t) = d\pi^*_{\Gamma(t)}(\Gamma'(t)) = \lambda \cdot d\pi^*_{\Gamma(t)}(\bar{H}(\Gamma(t))) = \lambda \cdot \bar{F}H(\Gamma(t)), \quad \forall t \in [0, 1].
$$

The assumption that the image of $\Gamma$ is not entirely contained in a fiber of $TM^*$ implies that $\lambda \neq 0$, hence (4.3) is equivalent to:

$$
\Gamma(t) = (\bar{F}H)^{-1}(\lambda^{-1}\gamma'(t)), \quad \forall t \in [0, 1].
$$

The conclusion follows easily. We will use the notation $\text{Crit}(f)$ to denote the set of critical points of a smooth (real-valued) map $f$.

**Proposition 4.1** Let $E$ be a regular value of $H$, and let $H$ be of homogeneous type at $E$; then, the set $\mathcal{V}^E_{q_0,q_1}$ of vertically critical curves in $\bar{\Omega}^E_{q_0,q_1}$ is a smooth embedded submanifold of $\bar{\Omega}^E_{q_0,q_1}$. Moreover, $\text{Crit}\left(G_{|\mathcal{V}^E_{q_0,q_1}}\right) = \text{Crit}\left(G_{|\bar{\Omega}^E_{q_0,q_1}}\right)$. 

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Proof. The map \( \tilde{\Omega}^E_{g_0,q_1} \ni \gamma \mapsto \pi^* \circ \Gamma \in \Lambda_{g_0,q_1} \) is smooth, and the homogeneity assumption tells us that it admits as right-inverse the smooth map \( \gamma \mapsto \Gamma^E \) given in (4.2). This proves\(^1\) that \( \Lambda_{g_0,q_1} \ni \gamma \mapsto \Gamma^E \in \tilde{\Omega}^E_{g_0,q_1} \) is an embedding, that will be denoted by \( i_E \), and thus that \( \mathcal{V}^E_{g_0,q_1} \) is a smooth embedded submanifold of \( \tilde{\Omega}^E_{g_0,q_1} \).

As to the second part of the statement, observe first that, by definition, it must be \( \text{Crit}(\mathcal{G}|_{\tilde{\Omega}^E_{g_0,q_1}}) \subset \mathcal{V}^E_{g_0,q_1} \), hence \( \text{Crit}(\mathcal{G}|_{\tilde{\Omega}^E_{g_0,q_1}}) \subset \text{Crit}(\mathcal{G}|_{\mathcal{V}^E_{g_0,q_1}}) \). Assume that \( \gamma \in \Lambda_{g_0,q_1} \) is such that \( \Gamma^E \gamma = i_E(\gamma) \) belongs to \( \text{Crit}(\mathcal{G}|_{\mathcal{V}^E_{g_0,q_1}}) \). Then, \( d\mathcal{G}(\Gamma^E \gamma) \) vanishes on the image of the differential \( d\mathcal{G}(\gamma) \), and, by definition of \( \mathcal{V}^E_{g_0,q_1} \), it also vanishes on the space of vertical variations along \( \Gamma^E \gamma \). In order to conclude that \( \Gamma^E \gamma \) belongs to \( \text{Crit}(\mathcal{G}|_{\tilde{\Omega}^E_{g_0,q_1}}) \) it suffices to observe that the image of \( d\mathcal{G}(\gamma) \) and the space of vertical variations of \( \Gamma^E \gamma \) span the entire tangent space \( T_{\Gamma^E \gamma} \tilde{\Omega}^E_{g_0,q_1} \). This concludes the proof.

**Proposition 4.2** Let \( \Gamma \in \tilde{\Omega}^E_{g_0,q_1} \) be fixed, and let \( \gamma : [0,1] \to M \) be given by \( \gamma = \pi^* \circ \Gamma ; \) thus \( g_0 = \gamma(0) \) and \( q_1 = \gamma(1) \). Then, \( \Gamma \) is a critical point of the restriction of \( \mathcal{G} \) to \( \tilde{\Omega}^E_{g_0,q_1} \), if and only if:

(a) \( \Gamma = \Gamma^E \gamma \);

(b) \( \gamma \) is a critical point of the functional:

\[
\gamma \mapsto \mathcal{L}_E(\gamma) = \mathcal{G}(\Gamma^E \gamma)
\]  

in the space \( \Lambda_{g_0,q_1} \).

**Proof.** From Lemma 4.1, the map \( \Lambda_{g_0,q_1} \ni \gamma \mapsto \Gamma^E \gamma \in \mathcal{V}^E_{g_0,q_1} \) is a diffeomorphism. The conclusion follows readily from Proposition 4.1.

**Remark 4.1** Clearly, one can extend the notion of Hamiltonian of homogeneous type by requiring the existence of the constant \( \lambda_{\gamma,E} \) satisfying (4.1) only for curves belonging to some fixed open subset of \( \Lambda_{g_0,q_1} \) for some \( g_0,q_1 \in M \). Under these circumstances, the statements of Lemma 4.1, Proposition 4.1 and Proposition 4.2 remain valid, with the obvious appropriate modifications.

### 4.1 A example: natural Hamiltonian systems

Let \( M \) be a differentiable manifold endowed with a semi-Riemannian (i.e., non necessarily positive definite) metric tensor \( g \). For all \( q \in M \), the metric tensor gives a linear (self-adjoint) isomorphism \( g_q : T_q M \to T_q^* M^* \), whose inverse \( g_q^{-1} : T_q^* M^* \to T_q M \) can be thought as a nondegenerate symmetric bilinear form on \( T_q M \). A tangent vector \( v \in TM \) will be called *timelike* (resp., *spacelike*) if \( g(v,v) < 0 \) (resp., \( g(v,v) > 0 \)); similarly, a (piecewise) \( C^1 \) curve \( \gamma : [a,b] \to M \) will be called

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\(^1\)If \( h : N_1 \to N_2 \) is a smooth map between manifolds \( N_1 \) and \( N_2 \), that admits a smooth left-inverse, then \( h \) is an embedding.
are open subsets of $\Lambda^q$ and one computes immediately:

$$V = \frac{1}{2} g^{-1}(p,p) + V(q).$$

This is a hyperregular Hamiltonian; for each timelike (resp., spacelike) if $\dot{\gamma} \in E$, we also observe that the natural Hamiltonian $H : TM^* \to \mathbb{R}$ associated to this setup is given by:

$$H(q,p) = \frac{1}{2} g^{-1}(p,p) + V(q).$$

It is a simple computation based on (4.7), from which one obtains:

$$\frac{1}{2 \lambda^2} = \left( \int_0^1 [E - V(\gamma)] \, dt \right) \left( \int_0^1 g(\gamma', \gamma') \, dt \right)^{-1}.$$

The result of Proposition 4.3 motivates the following:

**Proposition 4.3** Let $H : TM^* \to \mathbb{R}$ be the natural Hamiltonian (4.5) and let $E \in \mathbb{R}$ be a regular value of $H$ in $M$. Then, $H$ is of homogeneous type at $E$ on the open subset of $C^1([0,1], M)$ given by:

$$A_E = \left\{ \gamma \in C^1([0,1], M) : \left( \int_0^1 [E - V(\gamma)] \, dt \right) \cdot \left( \int_0^1 g(\gamma', \gamma') \, dt \right) > 0 \right\}.$$ 

In particular, given $q_0, q_1 \in M$, if $E \geq \sup_{\Lambda^+_{q_0, q_1}} V$ (resp., $E \leq \inf_{\Lambda_{q_0, q_1}} V$) then $H$ is of homogeneous type at $E$ on $\Lambda^+_{q_0, q_1}$ (resp., on $\Lambda_{q_0, q_1}$).

**Proof.** It is a simple computation based on (4.7), from which one obtains:

$$\frac{1}{2 \lambda^2} = \left( \int_0^1 [E - V(\gamma)] \, dt \right) \left( \int_0^1 g(\gamma', \gamma') \, dt \right)^{-1}.$$
Definition 4.3 Let $H : TM^* \to \mathbb{R}$ be the natural Hamiltonian (4.5) and let $E \in \mathbb{R}$ be a regular value of $H$. A solution $\Gamma : [0, T] \to TM^*$ of $H$ having energy $E$ will be called standard if its affine reparameterization $\tilde{H} : [0, 1] \to TM^*$ belongs to the set $\mathcal{A}_E$ in (4.8).

A simple computation shows the well known fact that solutions $\Gamma : [0, T] \to TM^*$ of this Hamiltonian project onto curves $\gamma = \pi^* \circ \Gamma : [0, T] \to M$ that satisfy the second order equation:

$$\frac{D}{dt} \dot{\gamma} = \nabla V(\gamma),$$  \hspace{1cm} (4.10)

where $\frac{D}{dt}$ denotes the covariant differentiation along $\gamma$ in the Levi–Civita connection of $g$, while $\nabla V$ is the $g$-gradient of $V$. We will call standard a solution $\gamma : [0, T] \to M$ of (4.10) whose corresponding solution $\Gamma : [0, T] \to TM^*$ of $H$ is standard in the sense of Definition 4.3.

An application of Proposition 4.2 gives us the following:

Corollary 4.1 Let $H : TM^* \to \mathbb{R}$ be the natural Hamiltonian (4.5) and let $E \in \mathbb{R}$ be a regular value of $H$ in $M$. Standard solutions $\gamma : [0, T] \to M$ of the second order equation (4.10) having energy $E$, satisfying $\gamma(0) = q_0$ and $\gamma(T) = q_1$ correspond, up to an affine reparameterization, to critical points of the functional:

$$\tilde{L}_E(x) = \left( \int_0^1 \left[ E - V(x) \right] \, ds \right) \cdot \left( \int_0^1 g(\dot{x}, \dot{x}) \, ds \right)$$  \hspace{1cm} (4.11)

in the space $\Lambda_{q0,q1} \cap \mathcal{A}_E$. Given a critical point $x$ of $\tilde{L}_E$ in $\Lambda_{q0,q1} \cap \mathcal{A}_E$, the corresponding solution $\gamma$ of (4.10) is obtained by setting $\gamma(t) = x(t/T)$, where:

$$T = \frac{1}{\sqrt{2}} \left( \int_0^1 \left[ E - V(x) \right] \, ds \right)^{\frac{1}{2}}.$$

(4.12)

Proof. By Proposition 3.1 and Proposition 4.2, the required solutions $\gamma$ are obtained by an affine reparameterization of critical points of the functional $\mathcal{L}_E : \Lambda_{q0,q1} \to \mathbb{R}$ given in (4.4). Using (2.1), (4.2), (4.4), (4.6) and (4.9), one computes:

$$\mathcal{L}_E(x) = \int_0^1 \Gamma_x(\dot{x}) \, ds = \int_0^1 (FH^{-1})(\lambda_\gamma^{-1} \dot{x}) \dot{x} \, ds$$

$$= \sqrt{2} \left| \int_0^1 \left[ E - V(x) \right] \, ds \right|^{\frac{1}{2}} \int_0^1 g(\dot{x}, \dot{x}) \, ds$$

$$= \sqrt{2} \left| \int_0^1 \left[ E - V(x) \right] \, ds \right|^{\frac{1}{2}} \cdot \left| \int_0^1 g(\dot{x}, \dot{x}) \, ds \right|^{\frac{1}{2}}.$$  \hspace{1cm} (4.13)

The conclusion follows easily, observing that $\tilde{L}_E(x) = \frac{1}{2} \mathcal{L}_E(x)^2$, and that $\mathcal{L}_E(x) = G(\Gamma_x) \neq 0$ on $\mathcal{A}_E$. Keeping in mind the observation in Remark 3.2, (4.12) follows easily from (4.9).
Remark 4.2 We observe that there may exist solutions that are not obtained from the variational principle in Corollary 4.1. If the Hamiltonian (4.5) admits a solutions $\Gamma$ which projects onto a lightlike curve $\gamma$ in $M$, then such $\Gamma$ is not standard; for instance, this is the case when the potential $V$ vanishes identically and $\gamma$ is a lightlike geodesic in $(M,g)$.

Remark 4.3 An immediate inspection of (4.13) shows that, in the case of the natural Hamiltonian (4.5), if $x$ is a critical point of $L_E$ in $\Lambda_{q_0,q_1}$, then the value of the “final instant” $T$ given in (4.12) is equal to the derivative $\frac{d}{dT} L_E(x)$, recovering a similar result in the convex case (see [6, formula (4.2)]).

Remark 4.4 Also, one can show that Maupertuis–Jacobi principle stated at the beginning of the introduction is a consequence of the Principle of Least Action. Indeed, the (natural) Hamiltonian function $H_E$ associated with the Jacobi metric $g_E = 2(E - V(x))g$ is

$$ H_E(q,p) = g_E^{-1}(p,p) = \frac{1}{2(E - V(q))} g^{-1}(p,p). $$

The key observation is that, given $E$ regular value for $H$, the sets $\Sigma_E = H^{-1}(E)$ and $H_E^{-1}(1)$ coincide, and then the two Hamiltonian vector fields $H(p), H_E(p)$ have the same integral curves, up to reparameterization [1], on the set $\Sigma_E$, since these two situations corresponds to the same variational problem (of course, one has to consider the set of curves in $TM^*$ such that the projection on $M$ lives in the set $\{V(x) < E\}$). Observe that the action functional $G(\Gamma)$ is canonical, and does not depend on any Hamilton function.

Moreover, we can observe that Maupertuis–Jacobi principle can be extended to cases where $g$ is semi–Riemannian, with the appropriate modifications due to the signature of $g$.

5 Homogeneous scalar fields in General Relativity

In the previous section, we have considered the case of natural systems, where the metric $g$ is possibly non definite, for instance Lorentzian. One may argue that, in General Relativity, the study of dynamics of moving particles under the effect of some potential does not need a “natural” approach as in Galilean dynamics, since the effect of the force fields appears as a contribution inside the metric itself, through the Einstein equations

$$ R - \frac{1}{2} S g = 8\pi T, \quad (5.1) $$

where $R$ and $S$ are respectively Ricci tensor field and scalar curvature function of $g$ (see [9]), and $T$ is the energy–momentum tensor field.

Nevertheless, a nice example of a natural system provided in a GR context is given by solutions of homogeneous scalar fields [5]. The general scalar field space-time is a Lorentzian manifold $(N,g)$ such that the metric $g$ satisfies the Einstein
field equation (5.1), with \( T \) completely determined by a scalar function \( \phi \) on \( M \), and a potential function \( V(\phi) \). The expression for the energy–momentum tensor, evaluated on a couple \((u, v)\) of vectors belonging to the tangent space \( T_x N \) at a point \( x \in N \), reads

\[
4\pi T(x)[u, v] = d\phi(x)[u] \cdot d\phi(x)[v] - \left[ \frac{1}{2} g(\nabla\phi(x), \nabla\phi(x)) + V(\phi(x)) \right] g(u, v), \tag{5.2}
\]

where \( \nabla\phi \) denotes the Lorentzian gradient vector field of \( \phi \).

We make the assumptions that the spacetime is spatially homogeneous with Bianchi I type symmetry [8]. This allows us to choose a convenient coordinate system \( x = (x^0 = t, x^1, x^2, x^3) \), such that the metric can be written in the form

\[
g = -dt \otimes dt + a^2(t) \left[ dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \right], \tag{5.3}
\]

and the scalar field \( \phi \) that determines \( T \) (5.2) is a function \( \phi = \phi(t) \) of the variable \( t \) only. Equation (5.1), which is in principle a fully nonlinear second order system of PDE, in this case results in a system of two ODE’s in the unknown function \((a(t), \phi(t))\) (see [4]):

\[
-\frac{3\dot{a}^2}{a^2} = -\dot{\phi}^2 + 2V(\phi), \tag{5.4a}
\]

\[
-\frac{\ddot{a}^2 + 2a\ddot{a}}{a^2} = \dot{\phi}^2 - 2V(\phi). \tag{5.4b}
\]

Solutions of (5.1) are in general, as well known, related to critical points of the Hilbert–Palatini action functional

\[
I = \int_N \sqrt{-\det g} \left( L_g + L_f \right) dV, \tag{5.5}
\]

where the Lagrangian \( L_g \) is given by

\[
L_g = \frac{1}{16\pi} S = \frac{3}{8\pi} \frac{\dot{a}^2(t) + a(t) \ddot{a}(t)}{a^2(t)}. \tag{5.6}
\]

and accounts for the contribution of gravitation, whereas the function \( L_f \) depends on the source of matter, and takes the form

\[
L_f = -\frac{1}{4\pi} \left( \frac{1}{2} g(\nabla\phi, \nabla\phi) + V(\phi) \right) = \frac{1}{8\pi} \dot{\phi}^2(t) - 2V(\phi(t)). \tag{5.7}
\]

We are interested in determining solutions between \((a(0), \phi(0))\) and \((a(T), \phi(T))\) prescribed configurations, letting the comoving arrival time \( T \) free in principle. Since all the unknown functions depends on \( t \) only, we can reduce the volume integral in (5.5) to an integral made on the interval \([0, T]\) of definition of \( a(t) \) and \( \phi(t) \):

\[
L(a, \phi) = \int_0^T \left[ 3(\dot{a}^2(t) + a(t) \ddot{a}(t)) a(t) + a^3(t) (\dot{\phi}^2(t) - 2V(\phi(t))) \right] dt. \tag{5.8}
\]
We integrate by parts the term in (5.8) containing $\ddot{a}(t)$, and ignore the contribution of the boundary term $3a^2\dot{a}|_{t=0}^T$ coming from the integration, that amounts to modify the functional (5.8), adding the contribution $\frac{1}{16\pi} \int_{\partial N} K$ of the trace of the extrinsic curvature $K$ integrated along the boundary $\partial N$ of the spacetime [10]. All in all, we obtain – changing the sign overall –

$$\mathcal{L}(a, \phi) = \int_0^T 3a(t)\dot{a}^2(t) - a^3(t)\dot{\phi}^2(t) + 2a^3(t)V(\phi(t)) \, dt,$$

(5.9)

Considering the manifold $M = \mathbb{R}^+ \times \mathbb{R}$, with coordinates $(a, \phi)$, we observe that the integrand function in (5.9) above represents a natural Lagrangian, where the “kinetic” part is given by a Lorentzian metric. Moreover, it is easily shown that solutions of Euler–Lagrange equation for this Lagrangian, with null “energy”, satisfy (5.4a)–(5.4b).

Corollary 4.1 applied to this case, and the functional (4.11), that in this case reads

$$\tilde{\mathcal{L}}_0(a, \phi) = \left( \int_0^1 3a(t)\dot{a}^2(t) - a^3(t)\dot{\phi}^2(t) \, dt \right) \cdot \left( \int_0^1 2a^3(t)V(\phi(t)) \, dt \right),$$

(5.10)

is used in [5] to find solutions of the problem. Actually, we can observe that the functional is degenerate near the boundary $\partial M$ of $M$, where $a$ vanishes. This situation corresponds to singular behavior of the spacetime, which has physical interest in its own (see [4]). However, in [5] the functional $\tilde{\mathcal{L}}_0$ is studied to find regular solutions of scalar field:

**Theorem 5.1** Let $(a_0, \phi_0)$ and $(a_1, \phi_1) \in M = \mathbb{R}^+ \times \mathbb{R}$, be such that

$$3 \min\{a_0, a_1\}(a_1 - a_0)^2 > \max\{a_0, a_1\}(\phi_1 - \phi_0)^2,$$

(5.11)

and let $V \in C^1(\mathbb{R}, \mathbb{R}^+)$. Then, there exists $T > 0$ and $(a(t), \phi(t)) \in C^2([0, T], M)$ solutions of (5.4a)–(5.4b) with the boundary conditions $a(0) = a_0, a(T) = a_1, \phi(0) = \phi_0, \phi(T) = \phi_1$.

**References**


