G-structures and affine immersions

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1^o Congres(s)o Latino-Americano de Grupos de Lie en(m) Geometria

Outline.

1. G-structures

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2. Principal spaces and fiber products

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2. Principal spaces and fiber products
3. Principal fiber bundles
4. Connections
5. Inner torsion of a *G*-structure

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5. Inner torsion of a G-structure
6. Immersion theorems

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Given a G-structure P on X and a G-structure Q on Y, a map $f : X \to Y$ is *G-structure preserving* if $f \circ p \in Q$ for all $p \in P$.

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Given a G-structure $P \subset \mathrm{Bij}(X_0, X)$ and a subgroup $H \subset G$, there are $[G : H]$ strengthening H-structures of P.

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Given a representation $\rho : G \to \mathrm{GL}(V_0)$ (i.e., a left action of G on V_0 by linear isomorphisms) and a G-principal space P, the set:

$$\widehat{P} \subset \mathrm{FR}_{V_0}(P \times_G V_0)$$

consisting of all bijections $\widehat{p} : V_0 \to P \times_G V_0$ is a $\mathrm{GL}(V_0)$-structure. Hence, $P \times_G V_0$ has the structure of a vector space.

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$\mathrm{Ver}_p = \mathrm{Ker}\big(\mathrm{d}\Pi_p\big) \subset T_pP$ *vertical space*;

canonical isomorphism $\mathrm{d}\beta_p(1) : \mathfrak{g} \xrightarrow{\cong} \mathrm{Ver}_p P$.

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- *pull-backs*: $\Pi : P \to M$ principal fiber bundle, $f : M' \to M$ smooth map, $f^*P = \bigcup_{y \in M'} \big(\{y\} \times P_{f(y)}\big)$.

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Associated bundle: $P \times_G N = \bigcup_{x \in M} P_x \times_G N$

Definition

A *vector bundle* consists of:

- a set E (*total space*)
- a differentiable manifold M (*base manifold*)
- a map $\pi : E \to M$ (*projection*)
- a finite dimensional vector space E_0 (*typical fiber*)
- a vector space structure on each fiber $E_x = \pi^{-1}$ isomorphic to E_0

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Def.: A *G-structure on E* is a G-principal subbundle of $\mathrm{FR}(E)$.

Outline

Principal connections

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$$\boxed{\mathrm{Ker}(\omega_p) = \mathrm{Hor}_p, \quad \omega_p|_{\mathrm{Ver}_p} = \mathrm{d}\beta_p(1)^{-1} : \mathrm{Ver}_p \xrightarrow{\cong} \mathfrak{g}}$$

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Properties of principal connections

- can be *pushed forward*
- induce connections on all associated bundles

Connections on vector bundles

Definition

A *connection* on the vector bundle E is an $\mathbb{R}$-bilinear map
$\nabla : \Gamma(TM) \times \Gamma(E) \ni (X, \epsilon) \longmapsto \nabla_X \epsilon \in \Gamma(E)$

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- $\boxed{\text{Connections on } E} \quad \Longleftrightarrow \quad \boxed{\text{Principal connections on } \mathrm{FR}(E)}$

Curvature and torsion

Curvature tensor of ∇: $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$

$$\boxed{R(X,Y)\epsilon = \nabla_X \nabla_Y \epsilon - \nabla_Y \nabla_X \epsilon - \nabla_{[X,Y]}\epsilon}$$

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Given $\iota : TM \to E$ vector bundle morphism, ι-*torsion tensor*: $T^\iota : \Gamma(TM) \times \Gamma(TM) \to \Gamma(E)$

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When $E = TM$, *torsion*: $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.

$\boxed{\nabla \text{ is } \textit{symmetric} \text{ if } T = 0}$

Gauss, Codazzi and Ricci equations

$\pi_1 : E^1 \to M$ and $\pi_2 : E^2 \to M$ vector bundle.
Whitney sum: $\pi : E = E_1 \oplus E_2 \to M$, $\mathrm{pr}_i : E \to E^i$ projection.

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Ricci equation

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Since $\mathrm{d}\Pi_p : T_pP \xrightarrow{\cong} T_xM$ and $(\mathrm{d}\Pi_p, \omega_p)(T_pP) \cap \mathfrak{gl}(\mathbb{R}^k) = \mathfrak{g}$, there exists $L : T_xM \to \mathfrak{gl}(\mathbb{R}^k)/\mathfrak{g}$ linear s.t.:

$$\boxed{T_pP \cong \big\{(v, X) \in T_xM \oplus \mathfrak{gl}(\mathbb{R}^k) : L(v) = X + \mathfrak{g}\big\}}$$

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$$\boxed{T_pP \subset T_p\big(\mathrm{FR}(E)\big) = \mathrm{Hor}_p \oplus \mathrm{Ver}_p \xrightarrow[\cong]{(\mathrm{d}\Pi_p,\omega_p)} T_xM \oplus \mathfrak{gl}(\mathbb{R}^k)}$$

Since $\mathrm{d}\Pi_p : T_pP \xrightarrow{\cong} T_xM$ and $(\mathrm{d}\Pi_p, \omega_p)(T_pP) \cap \mathfrak{gl}(\mathbb{R}^k) = \mathfrak{g}$, there exists $L : T_xM \to \mathfrak{gl}(\mathbb{R}^k)/\mathfrak{g}$ linear s.t.:

$$\boxed{T_pP \cong \big\{(v, X) \in T_xM \oplus \mathfrak{gl}(\mathbb{R}^k) : L(v) = X + \mathfrak{g}\big\}}$$

OBS 1: $\boxed{L = 0} \iff \boxed{T_pP \cong T_x \oplus \mathfrak{g}} \iff \boxed{\mathrm{Hor}_p \subset T_pP}$

Inner torsion via principal fiber bundles

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$\boxed{\mathfrak{I}^P_x = \overline{Ad}_p \circ L : T_xM \longrightarrow \mathfrak{gl}(E_x)/\mathfrak{g}_x \text{ does not depend on } p!}$ **inner torsion**

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$\mathfrak{I}^P_x : T_xM \to \mathfrak{gl}(E_x)/\mathfrak{g}_x$ is given by:

$$\begin{tikzcd}
T_xM \arrow[r, "\bar\omega_x"'] \arrow[rrr, bend left, "\mathfrak{I}^P_x"] \arrow[dr, "\Gamma_x"'] & \mathfrak{gl}(\mathbb{R}^k) \arrow[r, "\text{quotient}"'] \arrow[d, "\mathrm{Ad}_p"] & \mathfrak{gl}(\mathbb{R}^k)/\mathfrak{g} \arrow[r, "\overline{\mathrm{Ad}}_p"'] & \mathfrak{gl}(E_x)/\mathfrak{g}_x \\
& \mathfrak{gl}(E_x) \arrow[urr, "\text{quotient}"] & &
\end{tikzcd}$$

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An explicit computation using local sections of E that are constant in some orthonormal frame $s : U \to P$ gives:

$$\boxed{\mathfrak{I}_x^P(v) = \tfrac{1}{2}\big(\Gamma(v) + \Gamma(v)^*\big) = -\tfrac{1}{2}\nabla_v g \in \mathrm{sym}(E_x)}$$

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Lemma

$\mathfrak{I}_x^P = 0$ iff g is ∇-parallel.

Outline

1. G-structures
2. Principal spaces and fiber products
3. Principal fiber bundles
4. Connections
5. Inner torsion of a G-structure
6. Immersion theorems
7. Examples

Affine immersions

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Definition

An *affine immersion* of $(M, E, \widehat{\nabla})$ into $(\overline{M}, \overline{\nabla})$ is a pair (f, L), where $f : M \to \overline{M}$ is a smooth map, $L : \widehat{E} \to f^*T\overline{M}$ is a connection preserving vector bundle isomorphism with: $\boxed{L_x|_{T_xM} = \mathrm{d}f_x, \quad \forall\, x \in M.}$

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Uniqueness: If M is connected, given (f^1, L^1) and (f^2, L^2) with $f^1(x_0) = f^2(x_0)$ and $L^1(x_0) = L^2(x_0)$, then $(f^1, L^1) = (f^2, L^2)$.

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$\sigma : T_xM \to T_yM$ G-structure preserving,
$\mathcal{I}_\sigma : \mathrm{GL}(T_xM) \ni T \mapsto \sigma \circ T \circ \sigma^{-1} \in \mathrm{GL}(T_yM)$.
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Infinitesimally homogeneous manifolds – 2

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(M, ∇, P) is *infinitesimally homogeneous* if for all $\sigma : T_xM \to T_yM$ G-structure preserving:

- $\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}_x^P = \mathfrak{I}_y^P \circ \sigma$
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Theorem

(M, ∇, P) is infinitesimally homogeneous iff $\mathfrak{I}^P$, T and R are constant in frames of the G-structure P.

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Then, for all $x_0 \in M$, $y_0 \in M$, $\sigma_0 : \widehat{E}_x \to T_{y_0}\overline{M}$ G-structure preserving, there exist a locally defined affine immersion (f, L) of (M, E, ∇) into $(\overline{M}, \overline{\nabla})$ with $f(x_0) = y_0$, $L(x_0) = \sigma_0$, and such that L is G-structure preserving.

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If M is simply connected and $(\overline{M}, \overline{\nabla})$ is geodesically complete, then the affine immersion is global.

Outline

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Theorem

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Theorem

$\mathfrak{I}^P = 0$ iff (M, g, J) is Kähler. (M, ∇, P) is infinitesimally homogeneous iff g has constant holomorphic curvature.

An example with non vanishing inner torsion

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Lemma

$\mathfrak{I}^{\overline{P}} = 0$ iff g and ξ are parallel.

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Lemma

$\mathfrak{I}^{\overline{P}} = 0$ *iff* g *and* ξ *are parallel.* $(\overline{M}, \overline{\nabla}, \overline{P})$ *is infinitesimally homogeneous iff* R *and* $\nabla\xi$ *can be written in terms of* g *and* ξ *only.*

3-dimensional homogeneous manifolds (B. Daniel)

$(\overline{M}, \overline{g})$ 3-dimensional homogeneous manifold with 4-dim. isometry group (includes: Berger spheres, Heisenberg space Nil_3, $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$, products $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$)

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