The theory of connections and G-structures. Applications to affine and isometric immersions

Paolo Piccione

Daniel V. Tausk

Dedicated to Prof. Francesco Mercuri on occasion of his 60th birthday

Contents

Preface

This book contains the notes of a short course given by the two authors at the 14th School of Differential Geometry, held at the *Universidade Federal da Bahia*, Salvador, Brazil, in July 2006. Our goal is to provide the reader/student with the necessary tools for the understanding of an immersion theorem that holds in the very general context of affine geometry. As most of our colleagues know, there is no better way for learning a topic than *teaching a course* about it and, even better, *writing a book* about it. This was precisely our original motivation for undertaking this task, that lead us *way* beyond our most optimistic previsions of writing a *short and concise* introduction to the machinery of fiber bundles and connections, and a self-contained *compact* proof of a general immersion theorem.

The original idea was to find a unifying language for several isometric immersion theorems that appear in the classical literature [**5**] (immersions into Riemannian manifolds with constant sectional curvature, immersions into Kähler manifolds of constant holomorphic curvature), and also some recent results (see for instance [**6, 7**]) concerning the existence of isometric immersions in more general Riemannian manifolds. The celebrated equations of Gauss, Codazzi and Ricci are well known necessary conditions for the existence of isometric immersions. Additional assumptions are needed in specific situations; the starting point of our theory was precisely the interpretation of such additional assumptions in terms of "structure preserving" maps, that eventually lead to the notion of *G-structure*. Giving a G-structure on an n-dimensional manifold M, where G is a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$, means that it is chosen a set of "preferred frames" of the tangent bundle of M on which G acts freely and transitively. For instance, giving an $\mathrm{O}(\mathbb{R}^n)$ structure is the same as giving a Riemannian metric on M by specifying which are the orthonormal frames of the metric.

The central result of the book is an immersion theorem into (infinitesimally) homogeneous affine manifolds endowed with a G-structure. The covariant derivative of the G-structure with respect to the given connection gives a tensor field on M, called the *inner torsion* of the G-structure, that plays a central role in our theory. *Infinitesimally homogeneous* means that the curvature and the torsion of the connection, as well as the inner torsion of the G-structure, are constant in frames of the G-structure. For instance, consider the case that M is a Riemannian manifold endowed with the Levi-Civita connection of its metric tensor, G is the orthogonal group and the G-structure is given by the set of orthonormal frames. Since parallel transport takes orthonormal frames to orthonormal frames, the inner torsion of this G-structure is zero. The condition that the curvature tensor should be constant in orthonormal frames is equivalent to the condition that M has constant sectional curvature, and we recover in this case the classical "fundamental theorem of isometric immersions in spaces of constant curvature". Similarly, if M is a Riemannian manifold endowed with an orthogonal almost complex structure, then one has a G-structure on M, where G is the unitary group, by considering the set of orthonormal complex frames of TM. In this case, the inner torsion of the G-structure relatively to the Levi-Civita connection of the Riemannian metric is the covariant derivative of the almost complex structure, which vanishes if and only if M is Kähler. Requiring that the curvature tensor be constant in orthonormal complex frames means that M has constant holomorphic curvature; in this context, our immersion theorem reproduces the classical result of isometric immersions into Kähler manifolds of constant holomorphic curvature. Another interesting example of G-structure that will be considered in detail in these notes is the case of Riemannian manifolds endowed with a distinguished unit vector field ξ; in this case, we obtain an immersion theorem into Riemannian manifolds with the property that both the curvature tensor and the covariant derivative of the vector field are constant in orthonormal frames whose first vector is ξ. This is the case in a number of important examples, like for instance all manifolds that are Riemannian products of a space form with a copy of the real line, as well as all homogeneous, simply-connected 3-dimensional manifolds whose isometry group has dimension 4. These examples were first considered in [**6**]. Two more examples will be studied in some detail. First, we will consider isometric immersions into Lie groups endowed with a left invariant semi-Riemannian metric tensor. These manifolds have an obvious 1-structure, given by the choice of a distinguished orthonormal left invariant frame; clearly, the curvature tensor is constant in this frame. Moreover, the inner torsion of the structure is simply the Christoffel tensor associated to this frame, which is also constant. The second example that will be treated in some detail is the case of isometric immersions into products of manifolds with constant sectional curvature; in this situation, the G-structure considered is the one consisting of orthonormal frames adapted to a smooth distribution.

The book was written under severe time restrictions. Needless saying that, in its present form, these notes carry a substantial number of lacks, imprecisions, omissions, repetitions, etc. One evident weak point of the book is the total absence of reference to the already existing literature on the topic. Most the material discussed in this book, as well as much of the notations employed, was simply created on the blackboard of our offices, and not much attention has been given to the possibility that different conventions might have been established by previous authors. Also, very little emphasis was given to the applications of the affine immersion theorem, that are presently confined to the very last section of Chapter 3, where a few isometric immersion theorems are discussed in the context of semi-Riemannian geometry. Applications to general affine geometry are not even mentioned in this book. Moreover, the reference list cited in the text is extremely reduced, and it does not reflect the intense activity of research produced in the last decades about affine geometry, submanifold theory, etc. In our apology, we must emphasize that the entire material exposed in these three long Chapters and two

Appendices started from zero and was produced in a period of seven months since the beginning of our project.

On the other hand, we are particularly proud of having been able to write a text which is basically self-contained, and in which very little prerequisite is assumed on the reader's side. Many preliminary topics discussed in these notes, that form the core of the book, have been treated in much detail, with the hope that the text might serve as a reference also for other purposes, beyond the problem of affine immersions. Particular care has been given to the theory of principal fiber bundles and principal connections, which are the basic tools for the study of many topics in differential geometry. The theory of vector bundles is deduced from the theory of principal fiber bundles via the principal bundle of frames. We feel we have done a good job in relating the notions of principal connections and of linear connections on vector bundles, via the notions of associated bundle and contraction map. A certain effort has been made to clarify some points that are sometimes treated without many details in other texts, like for instance the question of inducing connections on vector bundles constructed from a given one by functorial constructions. The question is treated formally in this text with the introduction of the notion of *smooth natural transformation* between functors, and with the proof of several results that allow one to give a formal justification for many types of computations using connections that are very useful in many applications. Also, we have tried to make the exposition of the material in such a way that generalizations to the infinite dimensional case should be easy to obtain. The *global immersion results* in this book have been proven using a general "globalization technique" that is explained in Appendix B in the language of pre-sheafs. An intensive effort has been made in order to maintain the (sometimes heavy) notations and terminology self-consistent throughout the text. The book has been written having in mind an hypothetical reader that would read it sequentially from the beginning to the end. In spite of this, lots of cross references have been added, and complete (and sometimes repetitive) statements have been chosen for each proposition proved.

Thanks are due to the Scientific Committee of the 14th School of Differential Geometry for giving the authors the opportunity to teach this course. We also want to thank the staff at IMPA for taking care of the publishing of the book, which was done in a very short time. The authors gratefully acknowledge the sponsorship by CNPq and Fapesp.

The two authors wish to dedicate this book to their colleague and friend Francesco Mercuri, in occasion of his 60th birthday. *Franco* has been to the two authors an example of careful dedication to research, teaching, and supervision of graduate students.

CHAPTER 1

Principal and associated fiber bundles

1.1. G-structures on sets

A field of mathematics is sometimes characterized by the category it works with. Of central importance among categories are the ones whose objects are sets endowed with some sort of structure and whose morphisms are maps that preserve the given structure. A structure on a set X is often described by a certain number of operations, relations or some distinguished collection of subsets of the set X. Following the ideas of the Klein program for geometry, a structure on a set X can also be described along the following lines: one fixes a *model space* X_0, which is supposed to be endowed with a canonical version of the structure that is being defined. Then, a collection P of bijective maps $p : X_0 \to X$ is given in such a way that if $p : X_0 \to X$, $q : X_0 \to X$ belong to P then the *transition map* $p^{-1} \circ q : X_0 \to X_0$ belongs to the group G of all automorphisms of the structure of the model space X_0. The set X thus inherits the structure from the model space X_0 via the given collection of bijective maps P. The maps $p \in P$ can be thought of as *parameterizations* of X.

To illustrate the ideas described above in a more concrete way, we consider the following example. We wish to endow a set V with the structure of an n-dimensional real vector space, where n is some fixed natural number. This is usually done by defining on V a pair of operations and by verifying that such operations satisfy a list of properties. Following the ideas explained in the paragraph above, we would instead proceed as follows: let P be a set of bijective maps $p : \mathbb{R}^n \to V$ such that:

(a) for $p, q \in P$, the map $p^{-1} \circ q : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism;
(b) for every $p \in P$ and every linear isomorphism $g : \mathbb{R}^n \to \mathbb{R}^n$, the bijective map $p \circ g : \mathbb{R}^n \to V$ is in P.

The set P can be thought of as being an n-dimensional real vector space structure on the set V. Namely, using P and the canonical vector space operations of $\mathbb{R}^n$, one can define vector space operations on the set V by setting:

$$v + w = p\big(p^{-1}(v) + p^{-1}(w)\big), \quad tv = p\big(tp^{-1}(v)\big), \tag{1.1.1}$$

for all $v, w \in V$ and all $t \in \mathbb{R}$, where $p \in P$ is fixed. Clearly condition (a) above implies that the operations on V defined by (1.1.1) do not depend on the choice of the bijection $p \in P$. Moreover, the fact that the vector space operations of $\mathbb{R}^n$ satisfy the standard vector space axioms implies that the operations defined on V also satisfy the standard vector space axioms. If V is endowed with the operations defined by (1.1.1) then the bijective maps $p : \mathbb{R}^n \to V$ belonging to P are linear isomorphisms; condition (b) above implies that P is actually the set of *all* linear isomorphisms from $\mathbb{R}^n$ to V. Thus every set of bijective maps P satisfying conditions (a) and (b) defines an n-dimensional real vector space structure on V. Conversely, every n-dimensional real vector space structure on V defines a set of bijections P satisfying conditions (a) and (b); just take P to be the set of all linear isomorphisms from $\mathbb{R}^n$ to V. Using the standard terminology from the theory of group actions, conditions (a) and (b) above say that P is an orbit of the right action of the general linear group $\mathrm{GL}(\mathbb{R}^n)$ on the set of all bijective maps $p : \mathbb{R}^n \to V$. The set P will be thus called a $\mathrm{GL}(\mathbb{R}^n)$-*structure* on the set V.

Let us now present more explicitly the notions that were informally explained in the discussion above. To this aim, we quickly recall the basic terminology of the theory of group actions. Let G be a group with operation

$$G \times G \ni (g_1, g_2) \longmapsto g_1 g_2 \in G$$

and unit element $1 \in G$. Given an element $g \in G$, we denote by $L_g : G \to G$ and $R_g : G \to G$ respectively the *left translation map* and the *right translation map* defined by:

$$L_g(x) = gx, \quad R_g(x) = xg, \tag{1.1.2}$$

for all $x \in G$; we also denote by $\mathcal{I}_g : G \to G$ the *inner automorphism* of G defined by:

$$\mathcal{I}_g = L_g \circ R_g^{-1} = R_g^{-1} \circ L_g. \tag{1.1.3}$$

Given a set A then a *left action* of G on A is a map:

$$G \times A \ni (g, a) \longmapsto g \cdot a \in A$$

satisfying the conditions $1 \cdot a = a$ and $(g_1 g_2) \cdot a = g_1 \cdot (g_2 \cdot a)$, for all $g_1, g_2 \in G$, and all $a \in A$; similarly, a *right action* of G on A is a map:

$$A \times G \ni (a, g) \longmapsto a \cdot g \in A$$

satisfying the conditions $a \cdot 1 = a$ and $a \cdot (g_1 g_2) = (a \cdot g_1) \cdot g_2$, for all $g_1, g_2 \in G$, and all $a \in A$. Given a left action (resp., right action) of G on A then for every $a \in A$ we denote by $\beta_a : G \to A$ the map given by *action on the element* a, i.e., we set:

$$\beta_a(g) = g \cdot a, \tag{1.1.4}$$

(resp., $\beta_a(g) = a \cdot g$), for all $g \in G$. The set:

$$G_a = \beta_a^{-1}(a)$$

is a subgroup of G and is called the *isotropy group* of the element $a \in A$. The G-*orbit* (or, more simply, the *orbit*) of the element $a \in A$ is the set Ga (resp., aG) given by the image of G under the map β_a; a subset of A is called a G-*orbit* (or, more simply, an *orbit*) if it is equal to the G-orbit of some element of A. The set of all orbits constitute a partition of the set A. The map β_a induces a bijection from the set G/G_a of left (resp., right) cosets of the isotropy subgroup G_a onto the G-orbit of a. In particular, when the isotropy group G_a is trivial (i.e., when $G_a = \{1\}$) then the map β_a is a bijection from G onto the G-orbit of a. The action is said to be *transitive* if there is only one G-orbit, i.e., if the map β_a is surjective for some (and hence for any) $a \in A$. The action is said to be *free* if the isotropy group G_a is trivial for every $a \in A$. For each $g \in G$ we denote by $\gamma_g : A \to A$ the bijection of A corresponding to the *action of the element* g, i.e., we set:

$$\gamma_g(a) = g \cdot a, \tag{1.1.5}$$

(resp., $\gamma_g(a) = a \cdot g$), for all $a \in A$. If $\mathrm{Bij}(A)$ denotes the group of all bijective maps of A endowed with the operation of composition then the map:

$$G \ni g \longmapsto \gamma_g \in \mathrm{Bij}(A) \tag{1.1.6}$$

is a homomorphism (resp., a anti-homomorphism[1]). Conversely, every homomorphism (resp., every anti-homomorphism) (1.1.6) defines a left action (resp., a right action) of G on A by setting $g \cdot a = \gamma_g(a)$ (resp., $a \cdot g = \gamma_g(a)$), for all $g \in G$ and all $a \in A$. The action of G on A is said to be *effective* if the map (1.1.6) is injective, i.e., if $\bigcap_{a\in A} G_a = \{1\}$; more generally, given a subset A' of A, we say that the action of G is *effective on* A' if $\bigcap_{a\in A'} G_a = \{1\}$. The image of the map (1.1.6) is a subgroup of G and it will be denoted by G_{ef}. Notice that if the action is effective then G is isomorphic to G_{ef}; in the general case, G_{ef} is isomorphic to the quotient of G by the normal subgroup $\bigcap_{a\in A} G_a$.

We now proceed to the statement of the main definitions of the section. Given sets X_0 and X, we denote by $\mathrm{Bij}(X_0, X)$ the set of all bijections $p : X_0 \to X$. The group $\mathrm{Bij}(X_0)$ of all bijections of X_0 acts on the right on the set $\mathrm{Bij}(X_0, X)$ by composition of maps. The action of $\mathrm{Bij}(X_0)$ on $\mathrm{Bij}(X_0, X)$ is clearly free and transitive.

DEFINITION 1.1.1. Let X_0 be a set and G a subgroup of $\mathrm{Bij}(X_0)$. A *G-structure* on a set X is a subset P of $\mathrm{Bij}(X_0, X)$ which is a G-orbit. We say that the G-structure P is *modeled* upon the set X_0.

More explicitly, a G-structure on a set X is a nonempty subset P of $\mathrm{Bij}(X_0, X)$ satisfying the following conditions:

(a) $p^{-1} \circ q : X_0 \to X_0$ is in G, for all $p, q \in P$;
(b) $p \circ g : X_0 \to X$ is in P, for all $p \in P$ and all $g \in G$.

EXAMPLE 1.1.2. Given a natural number n, denote by I_n the set:

$$I_n = \{0, 1, \dots, n-1\}.$$

Let X be a set having n elements. By an *ordering* of the set X we mean a bijective map $p : I_n \to X$; notice that an ordering $p : I_n \to X$ of X can be identified with the n-tuple $\big(p(0), p(1), \dots, p(n-1)\big) \in X^n$. Denote by $S_n = \mathrm{Bij}(I_n)$ the symmetric group on n elements. The group S_n acts on the right on the set $\mathrm{Bij}(I_n, X)$ of all orderings of X. If G is a subgroup of S_n then a G-structure on X is a choice of a set of orderings $P \subset \mathrm{Bij}(I_n, X)$ which is an orbit of the

[1] Given groups G, H, then a *anti-homomorphism* $\phi : G \to H$ is a map satisfying the condition $\phi(g_1 g_2) = \phi(g_2)\phi(g_1)$, for all $g_1, g_2 \in G$.

action of G on $\mathrm{Bij}(I_n, X)$. For example, if $G = \{1\}$ is the trivial group then a G-structure on X is the same as the choice of one particular ordering $p : I_n \to X$ of X. If $G = S_n$ then there is only one G-structure on X, which is the entire set $\mathrm{Bij}(I_n, X)$. If $n \geq 2$ and $G = A_n \subset S_n$ is the group of even permutations then there are exactly two possible G-structures on X; if $n = 3$ and $X = \{a, b, c\}$, these G-structures are:

$$P = \big\{(a, b, c), (c, a, b), (b, c, a)\big\},$$

and:

$$P' = \big\{(a, c, b), (c, b, a), (b, a, c)\big\}.$$

If G is an arbitrary subgroup of S_n then the number of possible G-structures on X is equal to the index of G on S_n (see Exercise 1.3). If X is the set of vertices of an $(n-1)$-dimensional affine simplex and $G = A_n$ then the two possible G-structures of X are usually known as the two *orientations* of the given affine simplex.

If X_0 and X are arbitrary sets having the same cardinality, then bijective maps $p : X_0 \to X$ will also be called X_0-*orderings* of the set X. We remark that, when this terminology is used, it is not assumed that the set X_0 is endowed with some order relation.

EXAMPLE 1.1.3. Let V be an n-dimensional real vector space. A *frame* of V is a linear isomorphism $p : \mathbb{R}^n \to V$. Notice that p can be identified with the basis of V obtained as image under p of the canonical basis of $\mathbb{R}^n$; given a vector $v \in V$, the n-tuple $p^{-1}(v) \in \mathbb{R}^n$ contains the *coordinates* of v with respect to the frame p. Let $\mathrm{FR}(V)$ denote the set of all frames of V and let $\mathrm{GL}(\mathbb{R}^n)$ denote the *general linear group* of $\mathbb{R}^n$, i.e., the group of all linear isomorphisms of $\mathbb{R}^n$. Then $\mathrm{GL}(\mathbb{R}^n)$ is a subgroup of $\mathrm{Bij}(\mathbb{R}^n)$ and $\mathrm{FR}(V)$ is a $\mathrm{GL}(\mathbb{R}^n)$-structure on V modeled upon $\mathbb{R}^n$. Notice that given a set V and a $\mathrm{GL}(\mathbb{R}^n)$-structure P on V then there exists a unique n-dimensional real vector space structure on V such that $P = \mathrm{FR}(V)$. A $\mathrm{GL}(\mathbb{R}^n)$-structure on a set can thus be thought of as being the same as an n-dimensional real vector space structure on that set.

Let V_0 and V be arbitrary vector spaces having the same dimension and the same field of scalars; a linear isomorphism $p : V_0 \to V$ will be called a V_0-*frame* of V. Let $\mathrm{GL}(V_0)$ denote the *general linear group* of V_0, i.e., the group of all linear isomorphisms of V_0. Then $\mathrm{GL}(V_0)$ is a subgroup of $\mathrm{Bij}(V_0)$ and the set $\mathrm{FR}_{V_0}(V)$ of all V_0-frames of V is a $\mathrm{GL}(V_0)$-structure on the set V modeled upon V_0. Given a set V and a $\mathrm{GL}(V_0)$-structure on V then there exists a unique vector space structure on V such that $P = \mathrm{FR}_{V_0}(V)$.

EXAMPLE 1.1.4. Let M_0 and M be diffeomorphic differentiable manifolds and denote by $\mathrm{Diff}(M_0) \subset \mathrm{Bij}(M_0)$ the group of all diffeomorphisms of M_0. The set $\mathrm{Diff}(M_0, M)$ of all diffeomorphisms $p : M_0 \to M$ is a $\mathrm{Diff}(M_0)$-structure on M modeled upon M_0. Conversely, given a $\mathrm{Diff}(M_0)$-structure P on a set M then there exists a unique differentiable manifold structure on M such that P equals $\mathrm{Diff}(M_0, M)$.

In Exercise 1.4 the reader is asked to explore more examples like 1.1.3 and 1.1.4.

EXAMPLE 1.1.5. If X_0 is a set and G is a subgroup of $\mathrm{Bij}(X_0)$ then the set G itself is a G-structure on X_0; namely, G is the G-orbit of the identity map of X_0. The set G is called the *canonical* G-structure of the model space X_0. Notice that the canonical $\mathrm{GL}(\mathbb{R}^n)$-structure of $\mathbb{R}^n$ is identified with the canonical real vector space structure of $\mathbb{R}^n$.

Let X_0 be a set, G be a subgroup of $\mathrm{Bij}(X_0)$ and H be a subgroup of G. If P is a G-structure on a set X then P is a union of H-orbits; any one of this H-orbits is an H-structure on X. On the other hand, if $\mathcal{G}$ is a subgroup of $\mathrm{Bij}(X_0)$ containing G then there exists exactly one $\mathcal{G}$-orbit containing P (see Exercise 1.2); it's the only $\mathcal{G}$-structure on X containing P. We state the following:

DEFINITION 1.1.6. If P is a G-structure on a set X and H is a subgroup of G then an H-structure Q on X contained in P is said to be a *strengthening*of the G-structure P. We also say that P is a *weakening*of the H-structure Q.

Thus if H is a proper subgroup of G there are several ways of strengthening a G-structure P into an H-structure (it follows from the result of Exercise 1.3 that the number of such strengthenings is precisely the index of H in G); on the other hand, there is only one way of weakening an H-structure into a G-structure. In order to strengthen a structure one has to introduce new information; in order to weaken a structure, one has just to "forget" about something. In this sense, G-structures are stronger when the group G is smaller. The largest possible group G, which is $\mathrm{Bij}(X_0)$, gives *no structure at all*; namely, if X has the same cardinality as X_0 then there exists exactly one $\mathrm{Bij}(X_0)$-structure on X, which is the entire set $\mathrm{Bij}(X_0, X)$. On the other extreme, if $G = \{1\}$ is the trivial group containing only the identity map of X_0 then a G-structure on X is the same as a bijection $p : X_0 \to X$; it allows one to identify the set X with the model set X_0.

The following particularization of Definition 1.1.1 is the one that we will be more interested in.

DEFINITION 1.1.7. Let V_0, V be vector spaces having the same dimension and the same field of scalars. Given a subgroup G of $\mathrm{GL}(V_0)$ then by a *G-structure on the vector space* V we mean a G-structure P on the set V that strengthens the $\mathrm{GL}(V_0)$-structure $\mathrm{FR}_{V_0}(V)$ of V.

In other words, if G is a subgroup of $\mathrm{GL}(V_0)$, a G-structure on a vector space V is a G-structure P on the set V such that every $p \in P$ is a linear isomorphism from V_0 to V.

EXAMPLE 1.1.8. Let V be an n-dimensional real vector space endowed with an inner product $\langle \cdot, \cdot \rangle_V$, i.e., a positive definite symmetric bilinear form. A frame $p : \mathbb{R}^n \to V$ is called *orthonormal* if it is a linear isometry, i.e., if:

$$\langle p(x), p(y) \rangle_V = \langle x, y \rangle,$$

for all $x, y \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the canonical (positive definite) inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^n$. Equivalently, p is orthonormal if it carries the canonical basis of $\mathbb{R}^n$ to an orthonormal basis of V. Let $\mathrm{O}(\mathbb{R}^n)$ denote the *orthogonal group* of $\mathbb{R}^n$, i.e., the subgroup of $\mathrm{GL}(\mathbb{R}^n)$ consisting of all linear isometries of $\mathbb{R}^n$. The set $\mathrm{FR}^{\mathrm{o}}(V)$ of all orthonormal frames of V is an $\mathrm{O}(\mathbb{R}^n)$-structure on the vector space V modeled upon $\mathbb{R}^n$. Conversely, given an n-dimensional real vector space V and an $\mathrm{O}(\mathbb{R}^n)$-structure P on V then there exists a unique inner product $\langle\cdot,\cdot\rangle_V$ on V such that $P = \mathrm{FR}^{\mathrm{o}}(V)$.

Let V_0 and V be finite-dimensional real vector spaces having the same dimension, endowed with inner products $\langle\cdot,\cdot\rangle_{V_0}$ and $\langle\cdot,\cdot\rangle_V$, respectively; a V_0-frame $p : V_0 \to V$ of V is called *orthonormal* if p is a linear isometry. Let $\mathrm{O}(V_0, \langle\cdot,\cdot\rangle_{V_0})$ denote the *orthogonal group* of V_0, i.e., the subgroup of $\mathrm{GL}(V_0)$ consisting of all linear isometries. The set $\mathrm{FR}^{\mathrm{o}}_{V_0}(V)$ of all orthonormal V_0-frames of V is an $\mathrm{O}(V_0, \langle\cdot,\cdot\rangle_{V_0})$-structure on V modeled upon V_0. Conversely, given a real vector space V and an $\mathrm{O}(V_0, \langle\cdot,\cdot\rangle_{V_0})$-structure P on V then there exists a unique inner product $\langle\cdot,\cdot\rangle_V$ on V such that $P = \mathrm{FR}^{\mathrm{o}}_{V_0}(V)$.

EXAMPLE 1.1.9. Let V be a real vector space. A bilinear form:

$$V \times V \ni (v, w) \longmapsto \langle v, w\rangle_V \in \mathbb{R}$$

on V is said to be *nondegenerate* if $\langle v, w\rangle_V = 0$ for all $w \in V$ implies $v = 0$. The *index* of a symmetric bilinear form $\langle\cdot,\cdot\rangle_V$ on V is defined by:

$$\mathrm{n}_-\big(\langle\cdot,\cdot\rangle_V\big) = \sup\big\{\dim(W) : W \text{ is a subspace of } V \text{ and } \langle\cdot,\cdot\rangle_V \text{ is negative definite on } W\big\}.$$

An *indefinite inner product* $\langle\cdot,\cdot\rangle_V$ on V is a nondegenerate symmetric bilinear form on V. For instance, the *Minkowski bilinear form* of index r in $\mathbb{R}^n$, defined by:

$$\langle x, y\rangle = \sum_{i=1}^{n-r} x_i y_i - \sum_{i=n-r+1}^{n} x_i y_i,$$

for all $x, y \in \mathbb{R}^n$, is an indefinite inner product of index r in $\mathbb{R}^n$. If $\langle\cdot,\cdot\rangle_V$ is an indefinite inner product on V then we denote by $\mathrm{O}(V, \langle\cdot,\cdot\rangle_V)$ the subgroup of $\mathrm{GL}(V)$ consisting of all linear isometries $T : V \to V$, i.e.:

$$\mathrm{O}(V, \langle\cdot,\cdot\rangle_V) = \big\{T \in \mathrm{GL}(V) : \langle T(v), T(w)\rangle_V = \langle v, w\rangle_V, \text{ for all } v, w \in V\big\}.$$

We call $\mathrm{O}(V, \langle\cdot,\cdot\rangle_V)$ the *orthogonal group* of V; when the indefinite inner product $\langle\cdot,\cdot\rangle_V$ is given by the context, we will write simply $\mathrm{O}(V)$. If $\langle\cdot,\cdot\rangle$ is the Minkowski bilinear form of index r in $\mathbb{R}^n$ then the orthogonal group $\mathrm{O}(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ will also be denoted by $\mathrm{O}_r(\mathbb{R}^n)$.

Let V_0 and V be finite-dimensional real vector spaces having the same dimension, endowed with indefinite inner products $\langle\cdot,\cdot\rangle_{V_0}$ and $\langle\cdot,\cdot\rangle_V$, respectively; assume that $\langle\cdot,\cdot\rangle_{V_0}$ and $\langle\cdot,\cdot\rangle_V$ have the same index. A V_0-frame $p : V_0 \to V$ of V is called *orthonormal* if p is a linear isometry. The set $\mathrm{FR}^{\mathrm{o}}_{V_0}(V)$ of all orthonormal V_0-frames of V is an $\mathrm{O}(V_0, \langle\cdot,\cdot\rangle_{V_0})$-structure on V modeled upon V_0. Conversely, given a real vector space V and an $\mathrm{O}(V_0, \langle\cdot,\cdot\rangle_{V_0})$-structure P on V

then there exists a unique indefinite inner product $\langle\cdot,\cdot\rangle_V$ on V, having the same index as $\langle\cdot,\cdot\rangle_{V_0}$, such that $P = \mathrm{FR}^{\mathrm{o}}_{V_0}(V)$. If $\langle\cdot,\cdot\rangle_V$ has index r, $V_0 = \mathbb{R}^n$ and $\langle\cdot,\cdot\rangle_{V_0}$ is the Minkowski bilinear form of index r then we write $\mathrm{FR}^{\mathrm{o}}(V)$ instead of $\mathrm{FR}^{\mathrm{o}}_{V_0}(V)$.

EXAMPLE 1.1.10. Let V_0, V be finite dimensional vector spaces having the same dimension and the same field of scalars; let W_0 be a subspace of V_0 and W be a subspace of V such that W_0 has the same dimension as W. A V_0-frame $p \in \mathrm{FR}_{V_0}(V)$ of V is said to be *adapted* to (W_0, W) if $p(W_0) = W$. The set $\mathrm{FR}_{V_0}(V; W_0, W)$ of all V_0-frames of V that are adapted to (W_0, W) is a $\mathrm{GL}(V_0; W_0)$-structure on the vector space V modeled upon V_0, where $\mathrm{GL}(V_0; W_0)$ denotes the subgroup of the general linear group $\mathrm{GL}(V_0)$ consisting of all linear isomorphisms $T : V_0 \to V_0$ such that $T(W_0) = W_0$. If V_0 and V are endowed with positive definite or indefinite inner products, we set:

$$\mathrm{FR}^{\mathrm{o}}_{V_0}(V; W_0, W) = \mathrm{FR}_{V_0}(V; W_0, W) \cap \mathrm{FR}^{\mathrm{o}}_{V_0}(V),$$
$$\mathrm{O}(V_0; W_0) = \mathrm{GL}(V_0; W_0) \cap \mathrm{O}(V_0).$$

If the set $\mathrm{FR}^{\mathrm{o}}_{V_0}(V; W_0, W)$ is nonempty then it is an $\mathrm{O}(V_0; W_0)$-structure on the vector space V modeled upon V_0.

EXAMPLE 1.1.11. Let V_0, V be vector spaces having the same dimension and the same field of scalars. Let $v_0 \in V_0$, $v \in V$ be fixed nonzero vectors. A V_0-frame $p \in \mathrm{FR}_{V_0}(V)$ of V is said to be *adapted* to (v_0, v) if $p(v_0) = v$. The set $\mathrm{FR}_{V_0}(V; v_0, v)$ of all V_0-frames of V that are adapted to (v_0, v) is a $\mathrm{GL}(V_0; v_0)$-structure on the vector space V modeled upon V_0, where $\mathrm{GL}(V_0; v_0)$ denotes the subgroup of $\mathrm{GL}(V_0)$ consisting of all linear isomorphisms $T : V_0 \to V_0$ such that $T(v_0) = v_0$. If V_0 and V are real vector spaces endowed with positive definite or indefinite inner products, we set:

$$\mathrm{FR}^{\mathrm{o}}_{V_0}(V; v_0, v) = \mathrm{FR}_{V_0}(V; v_0, v) \cap \mathrm{FR}^{\mathrm{o}}_{V_0}(V),$$
$$\mathrm{O}(V_0; v_0) = \mathrm{GL}(V_0; v_0) \cap \mathrm{O}(V_0).$$

If the set $\mathrm{FR}^{\mathrm{o}}_{V_0}(V; v_0, v)$ is nonempty then it is an $\mathrm{O}(V_0; v_0)$-structure on the vector space V modeled upon V_0.

EXAMPLE 1.1.12. Let V be a real vector space endowed with a *complex structure* J, i.e., J is a linear endomorphism of V such that J^2 equals minus the identity map of V. The *canonical complex structure* J_0 of $\mathbb{R}^{2n}$ is defined by:

$$J_0(x, y) = (-y, x),$$

for all $x, y \in \mathbb{R}^n$. If V_0, V are real vector spaces with the same dimension endowed with complex structures J_0 and J, respectively then the set:

$$\mathrm{FR}^{\mathrm{c}}_{V_0}(V) = \big\{p \in \mathrm{FR}_{V_0}(V) : p \circ J_0 = J \circ p\big\}$$

is a $\mathrm{GL}(V_0, J_0)$-structure on the vector space V modeled upon V_0, where $\mathrm{GL}(V_0, J_0)$ denotes the subgroup of $\mathrm{GL}(V_0)$ consisting of all linear isomorphisms of V_0 that commute with J_0. Conversely, if P is a $\mathrm{GL}(V_0, J_0)$-structure on the vector space V then there exists a unique complex structure J on V such that $P = \mathrm{FR}^{\mathrm{c}}_{V_0}(V)$.

An element p of $\mathrm{FR}^{\mathrm{c}}_{V_0}(V)$ is called a *complex frame* of V. When V_0 is equal to $\mathbb{R}^{2n}$ endowed with its canonical complex structure, we write $\mathrm{FR}^{\mathrm{c}}(V)$ instead of $\mathrm{FR}^{\mathrm{c}}_{V_0}(V)$.

Let $\langle\cdot,\cdot\rangle_V$ be a positive definite or indefinite inner product on V. Assume that J is anti-symmetric with respect to $\langle\cdot,\cdot\rangle_V$, i.e.:

$$\langle J(v), w\rangle_V + \langle v, J(w)\rangle_V = 0,$$

for all $v, w \in V$. The *unitary group* of V with respect to J and $\langle\cdot,\cdot\rangle_V$ is defined by:

$$\mathrm{U}(V, J, \langle\cdot,\cdot\rangle_V) = \mathrm{O}(V, \langle\cdot,\cdot\rangle_V) \cap \mathrm{GL}(V, J).$$

We write also $\mathrm{U}(V)$ when J and $\langle\cdot,\cdot\rangle_V$ are fixed by the context. If $\mathbb{R}^{2n}$ is endowed with the canonical complex structure J_0 and the indefinite inner product:

$$(1.1.7)\quad \langle (x,y),(x',y')\rangle = \sum_{i=1}^{n-r}(x_i x_i' + y_i y_i') - \sum_{i=n-r+1}^{n}(x_i x_i' + y_i y_i'), \quad x, y, x', y' \in \mathbb{R}^n,$$

of index $2r$ then the unitary group $\mathrm{U}(\mathbb{R}^{2n}, J_0, \langle\cdot,\cdot\rangle)$ will be denoted by $\mathrm{U}_r(\mathbb{R}^{2n})$. Given finite dimensional real vector spaces V_0, V having the same dimension, and endowed respectively with indefinite inner products $\langle\cdot,\cdot\rangle_{V_0}$, $\langle\cdot,\cdot\rangle_V$ having the same index and complex structures $J_0 : V_0 \to V_0$, $J : V \to V$ anti-symmetric with respect to $\langle\cdot,\cdot\rangle_{V_0}$, $\langle\cdot,\cdot\rangle_V$ respectively then we set:

$$\mathrm{FR}^{\mathrm{u}}_{V_0}(V) = \big\{p \in \mathrm{FR}^{\mathrm{o}}_{V_0}(V) : p \circ J_0 = J \circ p\big\}.$$

The set $\mathrm{FR}^{\mathrm{u}}_{V_0}(V)$ is a $\mathrm{U}(V_0, J_0, \langle\cdot,\cdot\rangle_{V_0})$-structure on the vector space V. Conversely, if P is a $\mathrm{U}(V_0, J_0, \langle\cdot,\cdot\rangle_{V_0})$-structure on the vector space V then there exists a unique indefinite inner product $\langle\cdot,\cdot\rangle_V$ on V and a unique complex structure $J : V \to V$ anti-symmetric with respect to $\langle\cdot,\cdot\rangle_V$ such that $P = \mathrm{FR}^{\mathrm{u}}_{V_0}(V)$. When V_0 is $\mathbb{R}^{2n}$ endowed with its canonical complex structure and the indefinite inner product (1.1.7) we write $\mathrm{FR}^{\mathrm{u}}(V)$ instead of $\mathrm{FR}^{\mathrm{u}}_{V_0}(V)$.

Let us now define the natural morphisms of the category of sets endowed with G-structure.

DEFINITION 1.1.13. Let X_0 be a set, G be a subgroup of $\mathrm{Bij}(X_0)$ and let X, Y be sets endowed with G-structures P and Q respectively. A map $f : X \to Y$ is said to be *G-structure preserving* if $f \circ p$ is in Q, for all $p \in P$.

REMARK 1.1.14. Notice that if $f \circ p$ is in Q for *some* $p \in P$ then the map $f : X \to Y$ is G-structure preserving; namely, every other element of P is of the form $p \circ g$ with $g \in G$ and $f \circ (p \circ g) = (f \circ p) \circ g$ is also in Q.

The composite of G-structure preserving maps is again a G-structure preserving map; moreover, every G-structure preserving map is bijective and its inverse is also a G-structure preserving map (see Exercise 1.5). We denote by

$\mathrm{Iso}_G(X, Y')$ the set of all G-structure preserving maps from X to Y and we set $\mathrm{Iso}_G(X) = \mathrm{Iso}_G(X, X)$.

EXAMPLE 1.1.15. Let V_0, V, W be vector spaces having the same dimension and the same field of scalars. If V and W are regarded respectively as sets endowed with the $\mathrm{GL}(V_0)$-structures $\mathrm{FR}_{V_0}(V)$ and $\mathrm{FR}_{V_0}(W)$ then a map $f : V \to W$ is $\mathrm{GL}(V_0)$-structure preserving if and only if f is a linear isomorphism. Assume that V_0, V and W are finite-dimensional real vector spaces endowed with inner products. If V and W are regarded as sets endowed with the $\mathrm{O}(V_0)$-structures $\mathrm{FR}^{\mathrm{o}}_{V_0}(V)$ and $\mathrm{FR}^{\mathrm{o}}_{V_0}(W)$ respectively then a map $f : V \to W$ is $\mathrm{O}(V_0)$-structure preserving if and only if f is a linear isometry.

Notice that if V_0, V, W are vector spaces, G is a subgroup of $\mathrm{GL}(V_0)$ and if $P \subset \mathrm{FR}_{V_0}(V)$ and $Q \subset \mathrm{FR}_{V_0}(W)$ are G-structures on V and W respectively then every G-structure preserving map $f : V \to W$ is automatically a linear isomorphism.

EXAMPLE 1.1.16. Let M_0, M, N be differentiable manifolds with M and N both smoothly diffeomorphic to M_0. If the sets M and N are endowed respectively with the $\mathrm{Diff}(M_0)$-structures $\mathrm{Diff}(M_0, M)$ and $\mathrm{Diff}(M_0, N)$ then a map $f : M \to N$ is $\mathrm{Diff}(M_0)$-structure preserving if and only if it is a smooth diffeomorphism.

See Exercise 1.6 for more examples like 1.1.15 and 1.1.16.

EXAMPLE 1.1.17. Let X_0, X be sets, G be a subgroup of $\mathrm{GL}(X_0)$ and P be a G-structure on the set X. If the model space X_0 is endowed with its canonical G-structure (recall Example 1.1.5) then the G-structure preserving maps $f : X_0 \to X$ are precisely the elements of the G-structure P, i.e.:

$$\mathrm{Iso}_G(X_0, X) = P. \tag{1.1.8}$$

EXAMPLE 1.1.18. Let X_0 be a set, G, G' be subgroups of $\mathrm{Bij}(X_0)$ such that $G \subset G'$, P, Q be G-structures on sets X, Y respectively and P', Q' be G'-structures that weaken respectively P and Q. If a map $f : X \to Y$ is G-structure preserving then it is also G'-structure preserving, i.e., $\mathrm{Iso}_G(X, Y) \subset \mathrm{Iso}_{G'}(X, Y)$.

1.2. Principal spaces and fiber products

Principal spaces are the algebraic structures that will play the role of the fibers of the principal bundles, to be introduced later on Section 1.3. Principal spaces bare the same relation to groups as affine spaces bare to vector spaces. Recall that an *affine space* is a nonempty set A endowed with a free and transitive action of the additive abelian group of a vector space V. The affine space A can be identified with the vector space V once a point of A (a *origin*) is chosen. In a similar way, a principal space is, roughly speaking, an object that becomes a group once the position of the unit element is chosen.

The name "principal space" comes from the idea that any set with G-structure can be obtained from a principal space through a natural construction that we call the *fiber product*. Fiber products will be studied in Subsection 1.2.1.

DEFINITION 1.2.1. A *principal space* consists of a nonempty set P and a group G acting freely and transitively on P on the right. We call G the *structural group* of the principal space P.

Observe that the condition that the action of G on P be free and transitive means that for every $p, p' \in P$ there exists a unique element $g \in G$ with $p' = p \cdot g$; we say that g *carries* p to p'. The unique element g of G that carries p to p' is denoted by $p^{-1}p'$. The operation:

$$P \times P \ni (p, p') \longmapsto p^{-1}p' \in G$$

is analogous to the operation of *difference of points* in the theory of affine spaces. Notice that it's the *whole expression* $p^{-1}p'$ that has a meaning; for a general principal space, we cannot write just p^{-1}, although in most concrete examples the object p^{-1} is indeed defined (but it's *not* an element of the principal space P).

EXAMPLE 1.2.2. Any group G is a principal space with structural group G, if we let G act on itself by right translations.

EXAMPLE 1.2.3. Let G be a group and H be a subgroup of G. For any $g \in G$, the left coset gH is a principal space with structural group H.

EXAMPLE 1.2.4. Given a natural number n and a set X with n elements then the set $\mathrm{Bij}(I_n, X)$ of all orderings of X (recall Example 1.1.2) is a principal space with structural group S_n. More generally, if X_0 and X are sets having the same cardinality then the set $\mathrm{Bij}(X_0, X)$ of all X_0-orderings of X is a principal space with structural group $\mathrm{Bij}(X_0)$.

EXAMPLE 1.2.5. Let V be an n-dimensional real vector space. The set $\mathrm{FR}(V)$ of all frames of V (recall Example 1.1.3) is a principal space with structural group $\mathrm{GL}(\mathbb{R}^n)$. More generally, if V_0 and V are arbitrary vector spaces having the same dimension and the same field of scalars then the set $\mathrm{FR}_{V_0}(V)$ of all V_0-frames of V is a principal space with structural group $\mathrm{GL}(V_0)$.

In Exercise 1.8 the reader is asked to generalize Examples 1.2.4 and 1.2.5.

EXAMPLE 1.2.6. Let X_0 be a set, G be a subgroup of $\mathrm{Bij}(X_0)$ and P be a G-structure on a set X. Then P is a principal space with structural group G. Notice that, since $P = \mathrm{Iso}_G(X_0, X)$ (see Example 1.1.17), we are again dealing with a particular case of the situation described in Exercise 1.8.

EXAMPLE 1.2.7. Let P and Q be principal spaces with structural groups G and H respectively. The cartesian product $P \times Q$ can be naturally regarded as a principal space with structural group $G \times H$; the right action of $G \times H$ on $P \times Q$ is defined by:

$$(p, q) \cdot (g, h) = (p \cdot g, q \cdot h),$$

for all $(p, q) \in P \times Q$ and all $(g, h) \in G \times H$.

DEFINITION 1.2.8. Let P be a principal space with structural group G and let H be a subgroup of G. If $Q \subset P$ is an H-orbit then Q is itself a principal space with structural group H; we call Q a *principal subspace* of P.

The result of Exercise 1.3 implies that the number of principal subspaces of P with structural group H is equal to the index of H in G.

EXAMPLE 1.2.9. Let X_0 and X be sets having the same cardinality and let G be a subgroup of $\mathrm{Bij}(X_0)$. The set $\mathrm{Bij}(X_0, X)$ is a principal space with structural group $\mathrm{Bij}(X_0)$; the principal subspaces of $\mathrm{Bij}(X_0, X)$ with structural group G are precisely the G-structures of X. If P is a G-structure on X and H is a subgroup of G then the H-structures on X that strengthen P are precisely the principal subspaces of P with structural group H.

DEFINITION 1.2.10. Let P, Q be principal spaces with the same structural group G. A map $t : P \to Q$ is called a *left translation* if

$$t(p \cdot g) = t(p) \cdot g,$$

for all $p \in P$ and all $g \in G$. The set of all left translations $t : P \to Q$ will be denoted by $\mathrm{Left}(P, Q)$.

If we think of the structural group G as being the group of *right translations* of a principal space, then left translations are precisely the maps that commute with right translations.

Notice that the composite of left translations is again a left translation; moreover, a left translations is always bijective and its inverse is also a left translation (see Exercise 1.9). If $t : P \to P$ is a left translation from a principal space P to itself, we say simply that t is a *left translation of* P. The set $\mathrm{Left}(P, P)$ of all left translations of P is a group under composition and it will be denoted simply by $\mathrm{Left}(P)$.

EXAMPLE 1.2.11. If P is a principal space with structural group G then for all $p \in P$ the map $\beta_p : G \to P$ of action on the element p (recall (1.1.4)) is a left translation that carries the unit element $1 \in G$ to $p \in P$.

We think informally of β_p as being the identification between the principal space P and the structural group G that arises by declaring $p \in P$ to be the unit element; this is analogous to the identification between an affine space and the corresponding vector space that arises by declaring a point of the affine space to be the origin.

Let us compare the identifications β_p and $\beta_{p'}$ of G with P that arise from different choices of points $p, p' \in P$. If $g = p^{-1}p'$ is the element of G that carries p to p' then we have the following commutative diagram:

(1.2.1)
$$\begin{tikzcd} G \arrow[dr, "\beta_p"] \arrow[dd, "L_{g^{-1}}"'] & \\ & P \\ G \arrow[ur, "\beta_{p'}"'] & \end{tikzcd}$$

Diagram 1.2.1 says that two different identifications of a principal space P with its structural group G differ by a left translation of G. This is the same that happens in the theory of affine spaces: two different choices of origin for an affine space A give identifications with the corresponding vector space V that differ by a translation of V. Obviously, since the additive group of a vector space is abelian, there is no distinction between right and left translations in the theory of affine spaces and vector spaces.

A left translation is uniquely determined by its value at a point of its domain. More explicitly, we have the following:

LEMMA 1.2.12. *Let P, Q be principal spaces with the same structural group G. Given $p \in P$, $q \in Q$ then there exists a unique left translation $t \in \mathrm{Left}(P, Q)$ with $t(p) = q$.*

PROOF. Clearly $t = \beta_q \circ \beta_p^{-1}$ is a left translation from P to Q such that $t(p) = q$ (see Example 1.2.11). To prove uniqueness, let $t_1, t_2 \in \mathrm{Left}(P, Q)$ be given with $t_1(p) = t_2(p)$; then:

$$t_1(p \cdot g) = t_1(p) \cdot g = t_2(p) \cdot g = t_2(p \cdot g),$$

for all $g \in G$, so that $t_1 = t_2$. □

Lemma 1.2.12 implies that the canonical left action of the group of left translations $\mathrm{Left}(P)$ on P is free and transitive. Given $p, p' \in P$ then the unique left translation $t \in \mathrm{Left}(P)$ with $t(p) = p'$ is denoted by $p'p^{-1}$.

EXAMPLE 1.2.13. If G is a group then the left translations of the principal space G (recall Example 1.2.2) are just the left translations of the group G, i.e., the maps $L_g : G \to G$ with $g \in G$. Namely, the associativity of the multiplication of G implies that the maps L_g are left translations of the principal space G; conversely, if $t : G \to G$ is a left translation of the principal space G then Lemma 1.2.12 implies that $t = L_g$, with $g = t(1)$. Thus:

$$\mathrm{Left}(G) = \big\{L_g : g \in G\big\}.$$

Obviously the map $g \mapsto L_g$ gives an isomorphism from the group G onto the group $\mathrm{Left}(G)$ of left translations of G.

EXAMPLE 1.2.14. We have seen in Example 1.2.11 that if P is a principal space with structural group G then the maps $\beta_p : G \to P$, $p \in P$ are left translations. It follows from Lemma 1.2.12 that these are in fact the *only* left translations from G to P, i.e.:

$$\mathrm{Left}(G, P) = \big\{\beta_p : p \in P\big\}.$$

EXAMPLE 1.2.15. Let P, Q be principal spaces with the same structural group G and let $p \in P$, $q \in Q$ be fixed. If $t : P \to Q$ is a left translation then the composition $\beta_q^{-1} \circ t \circ \beta_p : G \to G$ is also a left translation and therefore, by Example 1.2.13, there exists a unique $g \in G$ with $\beta_q^{-1} \circ t \circ \beta_p = L_g$. This situation is illustrated by the following commutative diagram:

$$\begin{tikzcd} P \arrow[r, "t"] & Q \\ G \arrow[u, "\beta_p", "\cong"'] \arrow[r, "L_g"'] & G \arrow[u, "\cong", "\beta_q"'] \end{tikzcd} \tag{1.2.2}$$

We see that a choice of elements $p \in P$, $q \in Q$ induces a bijection between the set $\mathrm{Left}(P, Q)$ and the group G; such bijection associates to each $t \in \mathrm{Left}(P, Q)$ the element $g \in G$ that makes diagram (1.2.2) commutative. When $P = Q$, the bijection just described between $\mathrm{Left}(P)$ and G is an isomorphism of groups and we will denote it by $\mathcal{I}_p$. More explicitly, for each $p \in P$ we define the map $\mathcal{I}_p$ by:

$$\mathcal{I}_p : G \ni g \longmapsto \beta_p \circ L_g \circ \beta_p^{-1} \in \mathrm{Left}(P). \tag{1.2.3}$$

We see that the group of left translations $\mathrm{Left}(P)$ is isomorphic to the structural group G (the group of right translations of P), but the isomorphism is in general *not canonical*: it depends on the choice of an element $p \in P$. For $p, p' \in P$, the group isomorphisms $\mathcal{I}_p$ and $\mathcal{I}_{p'}$ differ by an inner automorphism of G; namely, the following diagram commutes:

$$\begin{tikzcd} G \arrow[dr, "\mathcal{I}_p"] \arrow[dd, "\mathcal{I}_{g^{-1}}"'] & \\ & \mathrm{Left}(P) \\ G \arrow[ur, "\mathcal{I}_{p'}"'] & \end{tikzcd} \tag{1.2.4}$$

where $g = p^{-1}p'$ is the element of G that carries p to p'.

REMARK 1.2.16. Let P be a principal space with structural group G and let $g \in G$ be fixed. If P is identified with G by means of the map $\beta_p : G \to P$ for some choice of $p \in P$, then the map $\gamma_g : P \to P$ given by the action of g (recall (1.1.5)) is identified with the map $R_g : G \to G$ of right translation by g; more explicitly, we have a commutative diagram:

$$\begin{tikzcd} P \arrow[r, "\gamma_g"] & P \\ G \arrow[u, "\beta_p", "\cong"'] \arrow[r, "R_g"'] & G \arrow[u, "\cong", "\beta_p"'] \end{tikzcd}$$

We could also identify the domain of γ_g with G via β_p and the counter-domain of γ_g with G via $\beta_{p \cdot g}$; this yields an identification of γ_g with the inner automorphism $\mathcal{I}_{g^{-1}}$ of G, which is illustrated by the commutative diagram:

$$\begin{tikzcd} P \arrow[r, "\gamma_g"] & P \\ G \arrow[u, "\beta_p", "\cong"'] \arrow[r, "\mathcal{I}_{g^{-1}}"'] & G \arrow[u, "\cong", "\beta_{p\cdot g}"'] \end{tikzcd} \tag{1.2.5}$$

EXAMPLE 1.2.17. Let V_0, V, W be vector spaces having the same dimension and the same field of scalars. Given a linear isomorphism $T : V \to W$ then the map

$$T_* : \mathrm{FR}_{V_0}(V) \longrightarrow \mathrm{FR}_{V_0}(W)$$

given by composition with T on the left is a left translation. Moreover, every left translation $t : \mathrm{FR}_{V_0}(V) \to \mathrm{FR}_{V_0}(W)$ is equal to T_* for a unique linear isomorphism $T : V \to W$. To prove this, choose any $p \in \mathrm{FR}_{V_0}(V)$ and let $T : V \to W$ be the unique linear isomorphism such that $T \circ p = t(p)$; it follows from Lemma 1.2.12 that $t = T_*$. We conclude that the rule $T \mapsto T_*$ defines a bijection from the set of linear isomorphisms $T : V \to W$ onto the set of left translations $t : \mathrm{FR}_{V_0}(V) \to \mathrm{FR}_{V_0}(W)$. If $V = W$, we obtain a bijection:

$$\mathrm{GL}(V) \ni T \longmapsto T_* \in \mathrm{Left}\big(\mathrm{FR}_{V_0}(V)\big); \tag{1.2.6}$$

such bijection is in fact a group isomorphism. We will therefore, from now on, *always identify the groups* $\mathrm{GL}(V)$ *and* $\mathrm{Left}\big(\mathrm{FR}_{V_0}(V)\big)$ *via the isomorphism* (1.2.6). Notice that, under such identification, for any given $p \in \mathrm{FR}_{V_0}(V)$, the isomorphism $\mathcal{I}_p : \mathrm{GL}(V_0) \to \mathrm{Left}\big(\mathrm{FR}_{V_0}(V)\big) \cong \mathrm{GL}(V)$ is given by:

$$\mathcal{I}_p(g) = p \circ g \circ p^{-1} \in \mathrm{GL}(V), \quad g \in \mathrm{GL}(V_0). \tag{1.2.7}$$

It may be instructive to solve Exercise 1.15 now.

EXAMPLE 1.2.18. Let X_0 be a set, G be a subgroup of $\mathrm{Bij}(X_0)$ and P, Q be respectively a G-structure on a set X and a G-structure on a set Y. Then P and Q are principal spaces with structural group G. If $f : X \to Y$ is a G-structure preserving map then the map $f_* : P \to Q$ given by composition with f on the left is a left translation. Arguing as in Example 1.2.17, we see that every left translation from P to Q is of the form f_* for a unique G-structure preserving map $f : X \to Y$; in other words, the map:

$$\mathrm{Iso}_G(X,Y) \ni f \longmapsto f_* \in \mathrm{Left}(P,Q)$$

is a bijection. Moreover, for $X = Y$, $P = Q$, the map:

$$\mathrm{Iso}_G(X) \ni f \longmapsto f_* \in \mathrm{Left}(P) \tag{1.2.8}$$

is a group isomorphism. We will from now on *always identify the groups* $\mathrm{Iso}_G(X)$ *and* $\mathrm{Left}(P)$ *via the isomorphism* (1.2.8).

In Exercises 1.11 and 1.12 the reader is asked to generalize the idea of Examples 1.2.17 and 1.2.18 to a more abstract context.

If P and Q are G-structures on sets X and Y respectively, H is a subgroup of G and P', Q' are H-structures that strengthen respectively P and Q then the set $\mathrm{Left}(P,Q)$ is identified with the set $\mathrm{Iso}_G(X,Y)$ and the set $\mathrm{Left}(P',Q')$ is identified with the set $\mathrm{Iso}_H(X,Y)$. Since $\mathrm{Iso}_H(X,Y)$ is a subset of $\mathrm{Iso}_G(X,Y)$, we should have an identification of $\mathrm{Left}(P',Q')$ with a subset of $\mathrm{Left}(P,Q)$. This is the objective of our next lemma.

LEMMA 1.2.19. *Let P, Q be principal spaces with structural group G and let $P' \subset P$, $Q' \subset Q$ be principal subspaces with structural group $H \subset G$. Then every left translation $t : P' \to Q'$ extends uniquely to a left translation $\bar{t} : P \to Q$. The map:*

$$\mathrm{Left}(P',Q') \ni t \longmapsto \bar{t} \in \mathrm{Left}(P,Q) \tag{1.2.9}$$

is injective and its image is the set:

$$\big\{s \in \mathrm{Left}(P,Q) : s(P') \subset Q'\big\}. \tag{1.2.10}$$

Moreover, if $P = Q$ and $P' = Q'$ then the map (1.2.9) *is an injective group homomorphism and therefore its image* (1.2.10) *is a subgroup of* $\mathrm{Left}(P)$.

PROOF. Let $t \in \mathrm{Left}(P', Q')$ be given and choose any $p \in P'$; then, by Lemma 1.2.12, there exists a unique left translation $\bar{t} : P \to Q$ with $\bar{t}(p) = t(p)$. For any $g \in H$ we have $\bar{t}(p \cdot g) = \bar{t}(p) \cdot g = t(p) \cdot g = t(p \cdot g)$, which proves that $\bar{t}$ is an extension of t; clearly, $\bar{t}$ is the unique left translation that extends t. We have thus established that the map (1.2.9) is well-defined; obviously, such map is injective and its image is contained in (1.2.10). Given any $s \in \mathrm{Left}(P, Q)$ with $s(P') \subset Q'$ then the map $t : P' \to Q'$ obtained by restricting s is a left translation and thus $s = \bar{t}$. This proves that the image of (1.2.9) is equal to (1.2.10). Finally, if $P = Q$, $P' = Q'$ and $t_1, t_2 \in \mathrm{Left}(P')$ then $\bar{t}_1 \circ \bar{t}_2$ is a left translation that extends $t_1 \circ t_2$; hence $\overline{t_1 \circ t_2} = \bar{t}_1 \circ \bar{t}_2$ and (1.2.9) is a group homomorphism. $\square$

Under the conditions of the statement of Lemma 1.2.19, *we will from now on always identify the set* $\mathrm{Left}(P', Q')$ *with the subset* (1.2.10) *of* $\mathrm{Left}(P, Q)$ *via the map* (1.2.9). In particular, the group $\mathrm{Left}(P')$ is identified with a subgroup of $\mathrm{Left}(P)$. Under such identification, the canonical left action of $\mathrm{Left}(P')$ on P' is identified with the restriction of the canonical left action of $\mathrm{Left}(P)$ on P. Observe also that the identification we have made here is consistent with the identifications made in Example 1.2.18. More explicitly, if P and Q are G-structures on sets X and Y respectively, H is a subgroup of G and P', Q' are H-structures that strengthen respectively P and Q then the following diagram commutes:

$$\begin{tikzcd} \mathrm{Iso}_G(X,Y) \arrow[r, "f\mapsto f_*", "\cong"'] & \mathrm{Left}(P,Q) \\ \mathrm{Iso}_H(X,Y) \arrow[u, "\text{inclusion}"] \arrow[r, "\cong", "f\mapsto f_*"'] & \mathrm{Left}(P',Q') \arrow[u, "(1.2.9)"'] \end{tikzcd} \tag{1.2.11}$$

In Exercise 1.24 the reader is asked to generalize Lemma 1.2.19.

REMARK 1.2.20. Let P be a principal space with structural group G; for each $p \in P$, we have an isomorphism $\mathcal{I}_p : G \to \mathrm{Left}(P)$ (recall (1.2.3)). For the sake of this discussion, let us write $\mathcal{I}_p^P$ instead of just $\mathcal{I}_p$. If Q is a principal subspace of P with structural group $H \subset G$ then for each $p \in Q$ we also have an isomorphism $\mathcal{I}_p^Q : H \to \mathrm{Left}(Q)$. For a fixed $p \in Q$, we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Left}(Q) \arrow[r, "t\mapsto\bar{t}"] & \mathrm{Left}(P) \\ H \arrow[u, "\mathcal{I}_p^Q", "\cong"'] \arrow[r, "\text{inclusion}"'] & G \arrow[u, "\cong", "\mathcal{I}_p^P"'] \end{tikzcd} \tag{1.2.12}$$

This means that, identifying $\mathrm{Left}(Q)$ with a subgroup of $\mathrm{Left}(P)$ then the isomorphism $\mathcal{I}_p^Q$ is just a restriction of the isomorphism $\mathcal{I}_p^P$.

In Exercise 1.25 the reader is asked to generalize this.

DEFINITION 1.2.21. Let G, H be groups, P be a G-principal space and Q be an H-principal space. A map $\phi : P \to Q$ is said to be a *morphism of principal spaces* if there exists a group homomorphism $\phi_0 : G \to H$ such that:

$$\phi(p \cdot g) = \phi(p) \cdot \phi_0(g), \tag{1.2.13}$$

for all $p \in P$ and all $g \in G$. We call ϕ_0 the *group homomorphism subjacent to the morphism* ϕ.

The fact that the action of H on Q is free implies that map $\phi_0 : G \to H$ such that equality (1.2.13) holds for all $p \in P$, $g \in G$ is unique.

The composition $\psi \circ \phi$ of morphisms of principal spaces ϕ and ψ with subjacent group homomorphisms ϕ_0 and ψ_0 is a morphism of principal spaces with subjacent group homomorphism $\psi_0 \circ \phi_0$ (see Exercise 1.16). A morphism of principal spaces ϕ is bijective if and only if its subjacent group homomorphism ϕ_0 is bijective (see Exercise 1.17). A bijective morphism of principal spaces is called an *isomorphism of principal spaces*. If ϕ is an isomorphism of principal spaces with subjacent group homomorphism ϕ_0 then ϕ^{-1} is also an isomorphism of principal spaces with subjacent group homomorphism ϕ_0^{-1} (see Exercise 1.18).

EXAMPLE 1.2.22. If P is a principal space with structural group G and $Q \subset P$ is a principal subspace with structural group $H \subset G$ then the inclusion map from Q to P is a morphism of principal spaces whose subjacent group homomorphism is the inclusion map from H to G.

There is a natural notion of quotient of a principal space and the quotient map is another example of a morphism of principal spaces. See Exercise 1.21 for the details.

EXAMPLE 1.2.23. If P, Q are principal spaces with the same structural group G then the left translations $t : P \to Q$ are precisely the morphisms of principal spaces whose subjacent group homomorphism is the identity map of G.

1.2.1. Fiber products. If X is a set endowed with a G-structure then the set of all G-structure preserving maps from the model space X_0 to X is a principal space with structural group G (recall Example 1.2.6). Thus, to each set X endowed with a G-structure there corresponds a principal space with structural group G. The notion of fiber product that we study in this subsection provides us with a construction that goes in the opposite direction.

Before we give the definition of fiber product, we need the following:

DEFINITION 1.2.24. Let G be a group. By a *G-space* we mean a set N endowed with a left action of G. The subgroup G_{ef} of $\mathrm{Bij}(N)$ given by the image of the homomorphism $G \ni g \mapsto \gamma_g \in \mathrm{Bij}(N)$ corresponding to the action of G on N is called the *effective group* of the G-space N.

Let G be a group, P be a principal space with structural group G and N be a G-space. We have a left-action of G on the cartesian product $P \times N$ defined by:

$$g \cdot (p, n) = (p \cdot g^{-1}, g \cdot n), \tag{1.2.14}$$

for all $g \in G$, $p \in P$ and all $n \in N$. Denote by $[p, n]$ the G-orbit of an element (p, n) of $P \times N$ and by $P \times_G N$ the set of all G-orbits. We call $P \times_G N$ the *fiber product* of the principal space P with the G-space N. Notice that for all $p \in P$, $g \in G$, $n \in N$ we have the equality:

$$[p \cdot g, n] = [p, g \cdot n]. \tag{1.2.15}$$

We will use also the following alternative notation for the fiber product $P \times_G N$:

$$P \underset{\sim}{\times} N \stackrel{\text{def}}{=} P \times_G N,$$

where there is no interest in emphasizing the group G. Notice that the abbreviated notation $P \underset{\sim}{\times} N$ should cause no confusion, since the structural group G is encoded in the principal space P.

Let us now show that the fiber product $P \times_G N$ is naturally endowed with a G_{ef}-structure modeled upon N. We need the following:

LEMMA 1.2.25. *If P is a principal space with structural group G and N is a G-space then for each $p \in P$ the map:*

$$\hat{p} : N \ni n \longmapsto [p, n] \in P \times_G N \tag{1.2.16}$$

is bijective.

PROOF. Given $n, n' \in N$ with $[p, n] = [p, n']$ then there exists $g \in G$ with $g \cdot (p, n) = (p, n')$. This means that $p = p \cdot g^{-1}$ and $n' = g \cdot n$. Since the action of G on P is free, the equality $p = p \cdot g^{-1}$ implies $g = 1$ and therefore $n = n'$. Let us now show that $\hat{p}$ is surjective. An arbitrary element of $P \times_G N$ is of the form $[q, n]$, with $q \in P$, $n \in N$. Since the action of G on P is transitive, there exists $g \in G$ with $q = p \cdot g$. Hence $\hat{p}(g \cdot n) = [p, g \cdot n] = [p \cdot g, n] = [q, n]$. □

Given $p \in P$, $g \in G$ and setting $q = p \cdot g$ then equality (1.2.15) means that the following diagram commutes:

$$\begin{tikzcd} N \arrow[dr, "\hat{p}"] & \\ & P \times_G N \\ N \arrow[uu, "\gamma_g"] \arrow[ur, "\hat{q}"'] & \end{tikzcd} \tag{1.2.17}$$

It follows that the map:

$$\mathfrak{H} : P \ni p \longmapsto \hat{p} \in \mathrm{Bij}(N, P \times_G N) \tag{1.2.18}$$

is a morphism of principal spaces whose subjacent group homomorphism is the map $G \ni g \mapsto \gamma_g \in \mathrm{Bij}(N)$. The image of (1.2.18) is the set:

$$\widehat{P} = \big\{\hat{p} : p \in P\big\} \subset \mathrm{Bij}(N, P \times_G N). \tag{1.2.19}$$

By the result of Exercise 1.19, $\widehat{P}$ is a principal subspace of $\mathrm{Bij}(N, P \times_G N)$ with structural group G_{ef}. Thus, $\widehat{P}$ is a G_{ef}-structure on the fiber product $P \times_G N$ modeled upon N (recall Example 1.2.9). From now on, *we will always consider the fiber product $P \times_G N$ to be endowed with the G_{ef}-structure $\widehat{P}$.*

Observe that the map (1.2.18) is injective if and only if the action of G on N is effective (see Exercise 1.17); in this case, the map $P \ni p \mapsto \hat{p} \in \widehat{P}$ is an isomorphism of principal spaces. In the general case, $\widehat{P}$ is isomorphic to a quotient of the principal space P (see Exercises 1.21 and 1.23).

EXAMPLE 1.2.26. Let G be a group and V_0 be a vector space. A *representation* of G in V_0 is a group homomorphism $\rho : G \to \mathrm{GL}(V_0)$. Notice that a representation of G in V_0 is the same as a left action of G on the set V_0 such that for every $g \in G$ the action of g on V_0 is a linear map. In particular, a representation ρ of G in V_0 makes V_0 into a G-space with effective group $G_{\mathrm{ef}} = \rho(G)$. Let P be a principal space with structural group G. The fiber product $P \times_G V_0$ is endowed with the G_{ef}-structure $\widehat{P}$; since G_{ef} is a subgroup of $\mathrm{GL}(V_0)$, the G_{ef}-structure $\widehat{P}$ can be weakened to a $\mathrm{GL}(V_0)$-structure on $P \times_G V_0$. Such $\mathrm{GL}(V_0)$-structure makes the fiber product $P \times_G V_0$ into a vector space isomorphic to V_0 (recall Example 1.1.3). The $\mathrm{GL}(V_0)$-structure of $P \times_G V_0$ then becomes the set of V_0-frames $\mathrm{FR}_{V_0}(P \times_G V_0)$ of $P \times_G V_0$ and therefore $\widehat{P}$ is contained in $\mathrm{FR}_{V_0}(P \times_G V_0)$; in other words, for every $p \in P$ the map $\hat{p} : V_0 \to P \times_G V_0$ is a linear isomorphism.

EXAMPLE 1.2.27. Let G be a group, P be a principal space with structural group G and N be a differentiable manifold. Assume that we are given a left action of G on N by diffeomorphisms, i.e., the subgroup G_{ef} of $\mathrm{Bij}(N)$ is contained in the group $\mathrm{Diff}(N)$ of all diffeomorphisms of N (this is the case, for instance, if G is a Lie group and the action $G \times N \to N$ is smooth). Thus N is a G-space and the fiber product $P \times_G N$ is endowed with the G_{ef}-structure $\widehat{P}$, which can be weakened to a $\mathrm{Diff}(N)$-structure. Such $\mathrm{Diff}(N)$-structure makes $P \times_G N$ into a differentiable manifold (recall Example 1.1.4) and $\widehat{P}$ is contained in $\mathrm{Diff}(N, P \times_G N)$; in other words, for every $p \in P$ the map $\hat{p} : N \to P \times_G N$ is a diffeomorphism.

We will show now that *any* set with G-structure is naturally isomorphic to a suitable fiber product. Let us start with a concrete example.

EXAMPLE 1.2.28. Let V_0, V be vector spaces having the same dimension and the same field of scalars; consider the principal space $\mathrm{FR}_{V_0}(V)$ with structural group $\mathrm{GL}(V_0)$. The vector space V_0 is a $\mathrm{GL}(V_0)$-space in a obvious way and the fiber product $\mathrm{FR}_{V_0}(V) \underset{\sim}{\times} V_0$ is endowed with a $\mathrm{GL}(V_0)$-structure that makes it into a vector space isomorphic to V_0. Such fiber product is in fact naturally isomorphic to V; more explicitly, the *contraction map* $\mathcal{C}^V$ defined by:

$$\mathcal{C}^V : \mathrm{FR}_{V_0}(V) \underset{\sim}{\times} V_0 \ni [p, v] \longmapsto p(v) \in V$$

is a (well-defined) linear isomorphism.

The idea behind Example 1.2.28 is generalized by the following:

LEMMA 1.2.29. *Let X_0 be a set, G be a subgroup of* $\mathrm{Bij}(X_0)$ *and P be a G-structure on a set X. The inclusion map of G in* $\mathrm{Bij}(X_0)$ *determines a left action of G on X_0, so that X_0 is a G-space with effective group $G_{\mathrm{ef}} = G$. Then, the* contraction map $\mathcal{C}^X$ *defined by:*

$$\mathcal{C}^X : \mathrm{Iso}_G(X_0, X) \times_G X_0 = P \times_G X_0 \ni [p, x] \longmapsto p(x) \in X$$

is a (well-defined) G-structure preserving map (recall (1.1.8)*).*

PROOF. If $[p, x] = [q, y]$ then $q = p \circ g^{-1}$ and $y = g(x)$, for some $g \in G$; thus $p(x) = q(y)$ and the contraction map $\mathcal{C}^X$ is well-defined. To prove that it is G-structure preserving, choose any $p \in P$ and observe that the diagram:

$$\begin{tikzcd} P \times_G X_0 \arrow[rr, "\mathcal{C}^X"] & & X \\ & X_0 \arrow[ul, "\hat{p}"] \arrow[ur, "p"'] & \end{tikzcd}$$

commutes. Hence $\mathcal{C}^X$ is a composition of G-structure preserving maps and it is therefore itself G-structure preserving. $\square$

In Exercise 1.39 the reader is asked to generalize Lemma 1.2.29 to a more abstract context.

We finish the section by defining some natural notions of induced maps on fiber products.

Let P, Q be principal spaces with structural groups G and H respectively; let $\phi : P \to Q$ be a morphism of principal spaces with subjacent group homomorphism $\phi_0 : G \to H$. If N is an H-space then we can also regard N as a G-space by considering the action of G on N defined by:

$$g \cdot n = \phi_0(g) \cdot n, \tag{1.2.20}$$

for all $g \in G$ and all $n \in N$. We define a map:

$$\hat{\phi} : P \times_G N \longrightarrow Q \times_H N$$

induced by ϕ by setting:

$$\hat{\phi}\big([p, n]\big) = [\phi(p), n],$$

for all $p \in P$ and all $n \in N$. The map $\hat{\phi}$ is well-defined; namely, given $g \in G$ then:

$$\begin{aligned} [\phi(p \cdot g^{-1}), g \cdot n] &= [\phi(p) \cdot \phi_0(g)^{-1}, g \cdot n] \\ &\overset{(1.2.20)}{=} [\phi(p) \cdot \phi_0(g)^{-1}, \phi_0(g) \cdot n] = [\phi(p), n], \end{aligned}$$

for all $p \in P$ and all $n \in N$. Notice that the following diagram:

$$\begin{tikzcd} P \times N \arrow[r, "\phi \times \mathrm{Id}"] \arrow[d, "\text{quotient map}"'] & Q \times N \arrow[d, "\text{quotient map}"] \\ P \times_G N \arrow[r, "\hat{\phi}"'] & Q \times_H N \end{tikzcd} \tag{1.2.21}$$

commutes.

We can also define an induced map on fiber products in a more general setting. We need the following:

DEFINITION 1.2.30. Let G, H be groups, N be a G-space and N' be an H-space. Given a group homomorphism $\phi_0 : G \to H$ then a map $\kappa : N \to N'$ is said to be ϕ_0-*equivariant*[2] if:

$$\kappa(g \cdot n) = \phi_0(g) \cdot \kappa(n),$$

for all $n \in N$ and all $g \in G$.

Let P, Q be principal spaces with structural groups G and H respectively and let $\phi : P \to Q$ be a morphism of principal spaces with subjacent group homomorphism $\phi_0 : G \to H$. Let N be a G-space and N' be an H-space and assume that we are given a ϕ_0-equivariant map $\kappa : N \to N'$. We define a map:

$$\phi \mathbin{\underset{\sim}{\times}} \kappa : P \times_G N \longrightarrow Q \times_H N'$$

induced by ϕ and κ by setting:

$$(\phi \mathbin{\underset{\sim}{\times}} \kappa)\big([p, n]\big) = [\phi(p), \kappa(n)],$$

for all $p \in P$ and all $n \in N$. The map $\phi \mathbin{\underset{\sim}{\times}} \kappa$ is well-defined; namely, given $g \in G$ then:

$$[\phi(p \cdot g^{-1}), \kappa(g \cdot n)] = [\phi(p) \cdot \phi_0(g)^{-1}, \phi_0(g) \cdot \kappa(n)] = [\phi(p), \kappa(n)],$$

for all $p \in P$ and all $n \in N$. Notice that the following diagram commutes:

$$\begin{tikzcd}
P \times N \arrow[r, "\phi \times \kappa"] \arrow[d, "\text{quotient map}"'] & Q \times N' \arrow[d, "\text{quotient map}"] \\
P \times_G N \arrow[r, "\phi \mathbin{\underset{\sim}{\times}} \kappa"'] & Q \times_H N'
\end{tikzcd}$$

Observe that if $N = N'$ and if the action of G on N is defined by (1.2.20) then the identity map of N is ϕ_0-equivariant and the induced map $\phi \mathbin{\underset{\sim}{\times}} \mathrm{Id}$ is just $\hat{\phi}$.

The induced map $\phi \mathbin{\underset{\sim}{\times}} \kappa$ retains many properties of the map κ, as is shown by the following:

LEMMA 1.2.31. *Let P, Q be principal spaces with structural groups G and H respectively and let $\phi : P \to Q$ be a morphism of principal spaces with subjacent group homomorphism $\phi_0 : G \to H$. Let N be a G-space and N' be an H-space and assume that we are given a ϕ_0-equivariant map $\kappa : N \to N'$. Then, for all $p \in P$, the following diagram commutes:*

(1.2.22)
$$\begin{tikzcd}
P \times_G N \arrow[r, "\phi \mathbin{\underset{\sim}{\times}} \kappa"] & Q \times_H N' \\
N \arrow[u, "\hat{p}"] \arrow[r, "\kappa"'] & N' \arrow[u, "\hat{q}"']
\end{tikzcd}$$

[2]If N' is regarded as a G-space with action defined as in (1.2.20) then the condition of κ being ϕ_0-equivariant is equivalent to the condition of κ being G-equivariant in the sense defined in Exercise 1.35.

where $q = \phi(p)$. In particular, the map κ is injective (resp., surjective) if and only if the induced map $\phi \underset{\sim}{\times} \kappa$ is injective (resp., surjective).

PROOF. Given $n \in N$ then:

$$(\phi \underset{\sim}{\times} \kappa)\big(\hat{p}(n)\big) = (\phi \underset{\sim}{\times} \kappa)\big([p, n]\big) = [q, \kappa(n)] = \hat{q}\big(\kappa(n)\big),$$

so that diagram (1.2.22) commutes. The claim relating the injectivity and the surjectivity of the maps κ and $\phi \underset{\sim}{\times} \kappa$ follows by observing that the maps $\hat{p}$ and $\hat{q}$ are bijective. □

COROLLARY 1.2.32. *Let P, Q be principal spaces with structural groups G and H respectively; let $\phi : P \to Q$ be a morphism of principal spaces with subjacent group homomorphism $\phi_0 : G \to H$. Let N be an H-space and let us regard N also as a G-space by considering the action of G on N defined by* (1.2.20). *Then the induced map $\hat{\phi} : P \times_G N \to Q \times_H N$ is bijective and for all $p \in P$, the following diagram commutes:*

(1.2.23)
$$\begin{tikzcd} P \times_G N \arrow[rr, "\hat{\phi}"] & & Q \times_H N \\ & N \arrow[ul, "\hat{p}"] \arrow[ur, "\hat{q}"'] & \end{tikzcd}$$

where $q = \phi(p)$.

PROOF. Apply Lemma 1.2.31 with κ the identity map of N. □

In Exercise 1.30 we ask the reader to prove that the induced map $\hat{\phi}$ is structure preserving, in a suitable sense.

EXAMPLE 1.2.33. Let V_0, V, W_0, W be vector spaces having the same field of scalars; assume that V_0 (resp., that W_0) has the same dimension as V (resp., as W). Let P, Q be principal spaces with structural groups G and H, respectively and let $\rho : G \to \mathrm{GL}(V_0)$, $\rho' : H \to \mathrm{GL}(W_0)$ be representations. Assume that we are given a morphism of principal spaces $\phi : P \to Q$ with subjacent group homomorphism $\phi_0 : G \to H$ and a linear map $T_0 : V_0 \to W_0$. Clearly, T_0 is ϕ_0-equivariant if and only if:

$$T_0 \circ \rho(g) = \rho'\big(\phi_0(g)\big) \circ T_0,$$

for all $g \in G$. If T_0 is ϕ_0-equivariant, we obtain an induced map:

$$\phi \underset{\sim}{\times} T_0 : P \times_G V_0 \longrightarrow Q \times_H W_0.$$

We have seen in Example 1.2.26 that the fiber products $P \times_G V_0$ and $Q \times_H W_0$ are vector spaces. We claim that the induced map $\phi \underset{\sim}{\times} T_0$ is linear. Namely, choose any $p \in P$ and set $q = \phi(p)$; the analogue of commutative diagram (1.2.22) in this context is:

$$\begin{tikzcd} P \times_G V_0 \arrow[r, "\phi \underset{\sim}{\times} T_0"] & Q \times_G W_0 \\ V_0 \arrow[u, "\hat{p}"] \arrow[r, "T_0"'] & W_0 \arrow[u, "\hat{q}"'] \end{tikzcd}$$

The linearity of $\phi \underset{\sim}{\times} T_0$ follows from the fact that the maps $\hat{p}$ and $\hat{q}$ are linear isomorphisms. Observe that, if $V_0 = W_0$, T_0 is the identity map of V_0 and $\rho = \rho' \circ \phi_0$ then the induced map $\phi \underset{\sim}{\times} T_0$ is equal to $\hat{\phi}$; thus, the map $\hat{\phi} : P \times_G V_0 \to Q \times_H V_0$ is a linear isomorphism.

1.3. Principal fiber bundles

Let M be a differentiable manifold, G be a Lie group, P be a set and let $\Pi : P \to M$ be a map; for each $x \in M$ we denote by P_x the subset $\Pi^{-1}(x)$ of P and we call it the *fiber* of P over x. Assume that for each $x \in M$ we are given a right action of G on the fiber P_x that makes it into a principal space with structural group G; equivalently, assume that the map Π is surjective and that we are given a right action

$$P \times G \ni (p, g) \longmapsto p \cdot g \in P \tag{1.3.1}$$

of G on P such that $\Pi(p \cdot g) = \Pi(p)$ for all $p \in P$, $g \in G$ and such that for all $p, q \in P$ with $\Pi(p) = \Pi(p)$ there exists a unique $g \in G$ with $p \cdot g = q$.

By a *local section* of Π we mean a map $s : U \to P$ defined on an open subset U of M such that $\Pi \circ s$ is the inclusion map of U in M; this means that $s(x)$ is a point of the fiber P_x, for all $x \in U$. A local section s of Π whose domain is the entire manifold M will be called a *section* (or *global section*) of Π. Given local sections $s_1 : U_1 \to P$, $s_2 : U_2 \to P$ of Π then there exists a unique map $g : U_1 \cap U_2 \to G$ such that $s_2(x) = s_1(x) \cdot g(x)$, for all $x \in U_1 \cap U_2$. The map g is called the *transition map* from s_1 to s_2. The local sections s_1 and s_2 are called *compatible* if the map g is smooth (this is the case, for instance, if $U_1 \cap U_2 = \emptyset$). An *atlas of local sections* of Π is a set $\mathcal{A}$ of local sections of Π such that:

- the union of the domains of the local sections belonging to $\mathcal{A}$ is the whole manifold M;
- any two local sections belonging to $\mathcal{A}$ are compatible.

It is easy to see that any atlas $\mathcal{A}$ of local sections of Π is contained in a unique maximal atlas $\mathcal{A}_{\max}$ of local sections of Π (see Exercise 1.41).

DEFINITION 1.3.1. A *principal fiber bundle* (or, more simply, a *principal bundle*) consists of:

- a set P, called the *total space*;
- a differentiable manifold M, called the *base space*;
- a map $\Pi : P \to M$, called the *projection*;
- a Lie group G, called the *structural group*;
- a right action (1.3.1) of G on P that makes the fiber $P_x = \Pi^{-1}(x)$ into a principal space with structural group G, for all $x \in M$;

- a maximal atlas $\mathcal{A}_{\max}$ of local sections of Π. The elements of $\mathcal{A}_{\max}$ are called the *admissible local sections* of the principal bundle.

When working with principal fiber bundles we will usually refer to the projection $\Pi : P \to M$ or to the total space P as if it were the collection of all the objects listed in Definition 1.3.1. We will also say that P is a principal bundle *over* M or that P (or $\Pi : P \to M$) is a *G-principal bundle*.

Let P be a G-principal bundle over M. For every admissible local section $s : U \to P$ the map:

$$\beta_s : U \times G \ni (x, g) \longmapsto s(x) \cdot g \in \Pi^{-1}(U) \subset P \tag{1.3.2}$$

is a bijection. It follows from the result of Exercise A.1 that there exists a unique differential structure on the set P such that for every admissible local section $s : U \to P$ the set $\Pi^{-1}(U)$ is open in P and the map β_s is a smooth diffeomorphism. *We will always regard the total space P of a principal bundle to be endowed with such differential structure*. The fact that the topologies of M and G are Hausdorff and second countable implies that the topology of P is also Hausdorff and second countable, so that P is a differentiable manifold. One can easily check the following facts:

- the right action (1.3.1) of G on P is a smooth map;
- the projection $\Pi : P \to M$ is a smooth submersion;
- for every $x \in M$ the fiber P_x is a smooth submanifold of P;
- for every $x \in M$ and every $p \in P_x$ the map $\beta_p : G \to P_x$ (recall (1.1.4)) is a smooth diffeomorphism;
- every admissible local section $s : U \to P$ is a smooth map;
- if a local section $s : U \to P$ is a smooth map then it is compatible with every admissible local section and therefore (by the maximality of $\mathcal{A}_{\max}$) it is itself an admissible local section.

Thus, the admissible local sections of P are precisely the same as the smooth local sections of P. Observe also that if $s : U \to P$ is a smooth local section of P and if $g : U \to G$ is a smooth map then, since the action (1.3.1) is smooth, it follows that:

$$s' : U \ni x \longmapsto s(x) \cdot g(x) \in P$$

is also a smooth local section of P.

EXAMPLE 1.3.2 (trivial principal bundle). Let M be a differentiable manifold and let P_0 be a principal space whose structural group G is a Lie group (for instance, we can take $P_0 = G$). Set $P = M \times P_0$. Let $\Pi : P \to M$ denote the projection onto the first coordinate and define a right action of G on P by setting $(x, p) \cdot g = (x, p \cdot g)$, for all $x \in M$, $p \in P_0$ and all $g \in G$. For every $p \in P_0$ the map $s^p : M \ni x \mapsto (x, p) \in P$ is a (globally defined) local section of Π and the set $\{s^p : p \in P_0\}$ is an atlas of local sections of Π. Thus P is a G-principal bundle over M which we call the *trivial principal bundle over M with typical fiber* P_0.indexprincipal bundle!trivial Let P_0 be endowed with the differential structure that makes the map $\beta_p : G \to P_0$ a smooth diffeomorphism, for every $p \in P$ (the existence of such differential structure follows from commutative diagram (1.2.1)).

Clearly the differential structure of $P = M \times P_0$ coincides with the standard differential structure defined on a cartesian product of differentiable manifolds.

EXAMPLE 1.3.3. Let $\Pi : P \to M$ be a G-principal bundle. If U is an open subset of M, we set:

$$P|_U = \Pi^{-1}(U) \subset P.$$

The right action of G on P restricts to a right action of G on $P|_U$ and the projection Π restricts to a map (also denoted by Π) from $P|_U$ to U. The set $P|_U$ is then a G-principal bundle over the manifold U endowed with the maximal atlas of local sections consisting of all the smooth local sections of P with domain contained in U. We call $P|_U$ the *restriction* of the principal bundle P to the open set U. Obviously, $P|_U$ is an open subset of P; moreover, the differential structure of $P|_U$ coincides with the differential structure it inherits from P as an open subset.

EXAMPLE 1.3.4. Let G be a Lie group and H a closed subgroup of G. Consider the quotient map $\Pi : G \to G/H$ and the action of H on G by right translations. For each $x \in G/H$, the fiber $\Pi^{-1}(x)$ is a left coset of H in G and it is therefore a principal space with structural group H (see Example 1.2.3). Since G is a manifold, we can talk about smooth local sections of Π. If $s_1 : U \to G$, $s_2 : V \to G$ are smooth local sections of Π then the transition map $h : U \cap V \to H$ is given by:

$$h(x) = s_1(x)^{-1} s_2(x),$$

for all $x \in U \cap V$, and therefore it is smooth. Hence the set of all smooth local sections of Π is an atlas of local sections of Π and $\Pi : G \to G/H$ is an H-principal bundle endowed with atlas of all smooth local sections of Π. It is easily seen that the differential structure on G induced by such atlas coincides with the original differential structure of G.

DEFINITION 1.3.5. Given $x \in M$ and $p \in P_x$, then the tangent space $T_p P_x$ is a subspace of $T_p P$ and it is called the *vertical space* of P at p; we write:

$$\mathrm{Ver}_p(P) = T_p P_x.$$

Clearly, $\mathrm{Ver}_p(P)$ is equal to the kernel of $\mathrm{d}\Pi(p)$, i.e.:

$$\mathrm{Ver}_p(P) = \mathrm{Ker}\big(\mathrm{d}\Pi(p)\big).$$

Since the map β_p is a smooth diffeomorphism from G onto the fiber containing p, its differential at the unit element $1 \in G$ is an isomorphism

$$\mathrm{d}\beta_p(1) : \mathfrak{g} \longrightarrow \mathrm{Ver}_p(P) \tag{1.3.3}$$

from the Lie algebra $\mathfrak{g}$ of the structural group G onto the vertical space $\mathrm{Ver}_p(P)$. We call (1.3.3) the *canonical isomorphism* from $\mathfrak{g}$ to $\mathrm{Ver}_p(P)$.

By differentiating the right action (1.3.1) of G on P with respect to the first variable we obtain a right action $TP \times G \to TP$ of G on the tangent bundle TP; more explicitly, for every $g \in G$ and every $\zeta \in TP$ we set:

$$\zeta \cdot g = \mathrm{d}\gamma_g(\zeta) \in TP,$$

where $\gamma_g : P \to P$ is the diffeomorphism given by the action of g on P. Since the diffeomorphism γ_g takes fibers to fibers, the action of G on TP takes vertical spaces to vertical spaces, i.e.:

$$\mathrm{d}\gamma_g\big(\mathrm{Ver}_p(P)\big) = \mathrm{Ver}_{p\cdot g}(P), \tag{1.3.4}$$

for all $p \in P$ and all $g \in G$. Let us look at the action of G on vertical spaces by identifying them with the Lie algebra $\mathfrak{g}$ via the canonical isomorphisms; for every $p \in P$, $g \in G$, we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Ver}_p(P) \arrow[r, "\text{action of } g"] & \mathrm{Ver}_{p\cdot g}(P) \\ \mathfrak{g} \arrow[u, "\mathrm{d}\beta_p(1)", "\cong"'] \arrow[r, "\mathrm{Ad}_{g^{-1}}"'] & \mathfrak{g} \arrow[u, "\cong", "\mathrm{d}\beta_{p\cdot g}(1)"'] \end{tikzcd} \tag{1.3.5}$$

where Ad denotes the *adjoint representation* of G on $\mathfrak{g}$ defined by (recall (1.1.3)):

$$\mathrm{Ad}_g = \mathrm{d}\mathcal{I}_g(1) : \mathfrak{g} \longrightarrow \mathfrak{g},$$

for all $g \in G$. The commutativity of diagram (1.3.5) follows from the commutativity of diagram (1.2.5) by differentiation.

DEFINITION 1.3.6. Let P be a G-principal bundle over M and let H be a Lie subgroup of G. A *principal subbundle* of P with structural group H is a subset Q of P satisfying the following conditions:

- for all $x \in M$, $Q_x = P_x \cap Q$ is a principal subspace of P_x with structural group H, i.e., Q_x is an H-orbit;
- for all $x \in M$, there exists a smooth local section $s : U \to P$ such that $x \in U$ and $s(U) \subset Q$.

We consider the restriction of the right action of G on P to a right action of H on Q and we consider the restriction of the projection $\Pi : P \to M$ to Q. Then Q is an H-principal bundle over M endowed with the maximal atlas consisting of all local sections $s : U \to Q$ of Q for which $i \circ s : U \to P$ is smooth[3], where $i : Q \to P$ denotes the inclusion map.

Being the total space of a principal bundle, the set Q is endowed with a differential structure. Let us take a look at the relation between the differential structure of Q and of P. If $s : U \to Q$ is a smooth local section of Q then $i \circ s : U \to P$ is

[3]To prove the compatibility between the local sections of Q the reader should recall the following important result from the theory of Lie groups: if G is a Lie group and H is a Lie subgroup of G then a smooth map having G as its counter-domain and having its image contained in H remains a smooth map if we replace its counter-domain by H.

a smooth local section of P and we have a commutative diagram:

$$\begin{tikzcd} U \times G \arrow[r, "\beta_{i\circ s}", "\cong"'] & P|_U \\ U \times H \arrow[u, "\text{inclusion}"] \arrow[r, "\cong", "\beta_s"'] & Q|_U \arrow[u, "\text{inclusion}"'] \end{tikzcd}$$

in which the horizontal arrows are smooth diffeomorphisms. It follows that the inclusion map $i : Q \to P$ is a smooth immersion. Unfortunately, it is not in general an embedding; in fact, the inclusion map $i : Q \to P$ is an embedding if and only if H is an embedded Lie subgroup of G (recall that a subgroup H of G is an embedded Lie subgroup of G if and only if H is closed in G). Although Q is in general just an immersed submanifold of P, it has the following *reduction of counter-domain property*: if X is a locally connected topological space (resp., a differentiable manifold) and if $\phi : X \to Q$ is a map such that $i \circ \phi : X \to P$ is continuous (resp., smooth) then the map $\phi : X \to Q$ is also continuous (resp., smooth). In fact, the principal subbundle Q is an almost embedded submanifold of P.

Let us now define the natural morphisms of the category of principal bundles with base space M.

DEFINITION 1.3.7. Let P, Q be principal bundles over the same differentiable manifold M, with structural groups G and H respectively. A map $\phi : P \to Q$ is called *fiber preserving* if $\phi(P_x) \subset Q_x$, for all $x \in M$. A *morphism of principal bundles* from P to Q is a smooth fiber preserving map $\phi : P \to Q$ for which there exists a group homomorphism $\phi_0 : G \to H$ such that for all $x \in M$, the map $\phi_x = \phi|_{P_x} : P_x \to Q_x$ is a morphism of principal spaces with subjacent group homomorphism ϕ_0.

The group homomorphism $\phi_0 : G \to H$ is uniquely determined from the morphism of principal bundles $\phi : P \to Q$; the commutativity of diagram (1.2) (with P and Q replaced by fibers P_x and Q_x, respectively) shows that the smoothness of ϕ implies the smoothness of the group homomorphism ϕ_0. Thus, ϕ_0 is indeed a Lie group homomorphism. We call it the *Lie group homomorphism subjacent to the morphism of principal bundles ϕ.*

The composition $\psi \circ \phi$ of morphisms of principal bundles ϕ and ψ with subjacent Lie group homomorphisms ϕ_0 and ψ_0 is a morphism of principal bundles with subjacent Lie group homomorphism $\psi_0 \circ \phi_0$ (see Exercise 1.43). A morphism of principal bundles ϕ is bijective if and only if its subjacent Lie group homomorphism ϕ_0 is bijective. A bijective morphism of principal bundles is called an *isomorphism of principal bundles*. If ϕ is an isomorphism of principal bundles with subjacent Lie group homomorphism ϕ_0 then ϕ is a smooth diffeomorphism and ϕ^{-1} is also an isomorphism of principal bundles with subjacent Lie group homomorphism ϕ_0^{-1} (see Exercise 1.46).

EXAMPLE 1.3.8. If P is a G-principal bundle, H is a Lie subgroup of G and Q is an H-principal subbundle of P then the inclusion map from Q to P is a morphism of principal bundles whose subjacent Lie group homomorphism is the inclusion map from H to G (compare with Example 1.2.22).

EXAMPLE 1.3.9. Let M be a differentiable manifold and P_0, Q_0 be principal spaces whose structural groups are Lie groups G, H, respectively; consider the trivial principal bundles $M \times P_0$ and $M \times Q_0$. Let $\phi : P_0 \to Q_0$ be a morphism of principal spaces whose subjacent group homomorphism $\phi_0 : G \to H$ is a Lie group homomorphism. Then $\mathrm{Id} \times \phi : M \times P_0 \to M \times Q_0$ is a morphism of principal bundles whose subjacent Lie group homomorphism is ϕ_0.

EXAMPLE 1.3.10. Let P be a G-principal bundle over a differentiable manifold M and let $s : U \to P$ be a smooth local section of P. The map β_s is an isomorphism of principal bundles from the trivial G-principal bundle $U \times G$ onto $P|_U$. The Lie group homomorphism subjacent to β_s is the identity map of G.

A fiber preserving map $\phi : P \to Q$ that is a morphism of principal spaces on each fiber can be used to push-forward the principal bundle structure of the domain P to the counter-domain Q; more precisely, we have the following:

LEMMA 1.3.11. *Let $\Pi : P \to M$ be a G-principal bundle over a differentiable manifold M. Let Q be a set, $\Pi' : Q \to M$ be a map, H be a Lie group and assume that it is given right action of H on Q that makes the fiber Q_x into a principal space with structural group H, for all $x \in M$. Let $\phi_0 : G \to H$ be a Lie group homomorphism and let $\phi : P \to Q$ be a fiber preserving map such that $\phi|_{P_x} : P_x \to Q_x$ is a morphism of principal spaces with subjacent group homomorphism ϕ_0, for all $x \in M$. Then there exists a unique maximal atlas of local sections of Π' that makes $\phi : P \to Q$ a morphism of principal bundles.*

PROOF. Consider the following set of local sections of Π':

$$\big\{\phi \circ s : s \text{ is a smooth local section of } P\big\}. \tag{1.3.6}$$

Let us show that (1.3.6) is an atlas of local sections of Π'. Obviously, the domains of the local sections belonging to (1.3.6) constitute a covering of M. Moreover, if $s_1 : U_1 \to P$, $s_2 : U_2 \to P$ are smooth local sections of P with transition map $g : U_1 \cap U_2 \to G$ then the transition map from $\phi \circ s_1$ to $\phi \circ s_2$ is $\phi_0 \circ g : U_1 \cap U_2 \to H$; thus $\phi \circ s_1$ and $\phi \circ s_2$ are compatible and (1.3.6) is an atlas of local sections of Π'. To conclude the proof, observe that a maximal atlas $\mathcal{A}_{\max}$ of local sections of Π' makes $\phi : P \to Q$ a morphism of principal bundles if and only if $\mathcal{A}_{\max}$ is the maximal atlas of local sections of Π' containing (1.3.6) (see Exercise 1.45). □

COROLLARY 1.3.12. *Let P, P', Q be principal bundles over a differentiable manifold M with structural groups G, G' and H respectively. Let $\phi : P \to Q$, $\psi : P \to P'$ be morphisms of principal bundles with subjacent Lie group homomorphisms $\phi_0 : G \to H$ and $\psi_0 : G \to G'$. Let $\phi'_0 : G' \to H$ be a Lie group homomorphism and let $\phi' : P' \to Q$ be a fiber preserving map such that*

$\phi|_{P'_x} : P'_x \to Q_x$ is a morphism of principal spaces with subjacent group homomorphism ϕ'_0, for all $x \in M$. Assume that the diagram:

$$\begin{tikzcd} P \arrow[d, "\psi"'] \arrow[dr, "\phi"] & \\ P' \arrow[r, "\phi'"'] & Q \end{tikzcd}$$

commutes. Then ϕ' is a morphism of principal bundles with subjacent Lie group homomorphism ϕ'_0.

PROOF. Let $\mathcal{A}_{\max}$ be the maximal atlas of local sections of the principal bundle Q and let $\mathcal{A}'_{\max}$ be the unique maximal atlas of local sections of Q that makes ϕ' a morphism of principal bundles. Both $\mathcal{A}_{\max}$ and $\mathcal{A}'_{\max}$ make $\phi = \phi' \circ \psi$ a morphism of principal bundles; by the uniqueness part of Lemma 1.3.11, we have $\mathcal{A}_{\max} = \mathcal{A}'_{\max}$. This concludes the proof. $\square$

1.3.1. Pull-back of principal bundles. A G-principal bundle over a differentiable manifold M can be though of as a "smoothly varying" family $(P_x)_{x \in M}$ of principal spaces P_x with structural group G parameterized by the points of M. If M' is another differentiable manifold and $f : M' \to M$ is a smooth map then it is natural to consider a reparametrization $(P_{f(y)})_{y \in M'}$ of the family $(P_x)_{x \in M}$ by the map f. This idea motivates the definition of the pull-back of a principal bundle. Let us now give the precise definitions.

Let $\Pi : P \to M$ be a G-principal bundle and let $f : M' \to M$ be a smooth map defined on a differentiable manifold M'. The *pull-back* of P by f is the set f^*P defined by:

$$f^*P = \bigcup_{y \in M'} \big(\{y\} \times P_{f(y)}\big).$$

Thus, the set f^*P is a subset of the cartesian product $M' \times P$. The restriction to f^*P of the projection onto the first coordinate is a map $\Pi_1 : f^*P \to M'$ and the restriction to f^*P of the projection onto the second coordinate is a map $\bar{f} : f^*P \to P$; the following diagram commutes:

$$\begin{tikzcd} f^*P \arrow[r, "\bar{f}"] \arrow[d, "\Pi_1"'] & P \arrow[d, "\Pi"] \\ M' \arrow[r, "f"'] & M \end{tikzcd} \tag{1.3.7}$$

We call $\bar{f} : f^*P \to P$ the *canonical map* associated to the pull-back f^*P; when it is necessary to make the principal bundle P explicit, we will also write $\bar{f}^P$ instead of just $\bar{f}$.

Notice that the pull-back f^*P is precisely the subset of $M' \times P$ where the maps $\Pi \circ \bar{f}$ and $f \circ \Pi_1$ coincide; moreover, the map $(\Pi_1, \bar{f}) : f^*P \to M' \times P$ is just the inclusion map. From this two simple observations, we get the following set-theoretical lemma:

LEMMA 1.3.13. *Let* $\Pi : P \to M$ *be a* G*-principal bundle,* M' *be a differentiable manifold and* $f : M' \to M$ *be a smooth map. Given a set* X *and maps* $\tau_1 : X \to M'$, $\tau_2 : X \to P$ *with* $\Pi \circ \tau_2 = f \circ \tau_1$ *then there exists a unique map* $\tau : X \to f^*P$ *such that* $\Pi_1 \circ \tau = \tau_1$ *and* $\bar{f} \circ \tau = \tau_2$.

PROOF. The condition $\Pi \circ \tau_2 = f \circ \tau_1$ means that the image of the map $(\tau_1, \tau_2) : X \to M' \times P$ is contained in f^*P; since $(\Pi_1, \bar{f})$ is the inclusion map of f^*P into $M' \times P$, there exists a unique map $\tau : X \to f^*P$ such that:

$$(\Pi_1, \bar{f}) \circ \tau = (\tau_1, \tau_2). \tag{1.3.8}$$

But this last equality is equivalent to $\Pi_1 \circ \tau = \tau_1$ and $\bar{f} \circ \tau = \tau_2$. □

The situation in Lemma 1.3.13 is illustrated by the following commutative diagram:

$$\begin{tikzcd}
X \arrow[drr, bend left, "\tau_2"] \arrow[dr, dotted, "\tau"] \arrow[ddr, bend right, "\tau_1"'] & & \\
& f^*P \arrow[r, "\bar{f}"'] \arrow[d, "\Pi_1"] & P \arrow[d, "\Pi"] \\
& M' \arrow[r, "f"'] & M
\end{tikzcd} \tag{1.3.9}$$

In Exercise 1.53 we define the general notion of pull-back in arbitrary categories and in Exercise 1.54 we ask the reader to generalize Lemma 1.3.13 by presenting the notion of pull-back in the category of sets and maps.

Our goal now is to make $\Pi_1 : f^*P \to M'$ into a G-principal bundle over M'. For each $y \in M'$, the fiber $(f^*P)_y$ is equal to $\{y\} \times P_{f(y)}$; *we will identify the fiber* $(f^*P)_y$ *of* f^*P *with the fiber* $P_{f(y)}$ *of* P. Under such identification, every fiber of f^*P is a fiber of P and thus each fiber of f^*P is endowed with a right action of G that makes it into a principal space with structural group G. Our next step is to define an atlas of local sections of Π_1.

DEFINITION 1.3.14. By a *local section of the principal bundle* P *along* f we mean a map $\sigma : U' \to P$ defined on an open subset U' of M' satisfying the condition $\Pi \circ \sigma = f|_{U'}$.

EXAMPLE 1.3.15. If $s : U \to P$ is a local section of P then the composition $s \circ f : f^{-1}(U) \to P$ is a local section of P along f.

Clearly, if we compose a local section of $\Pi_1 : f^*P \to M'$ on the left with $\bar{f}$, we obtain a local section of P along f; moreover, if $\sigma : U' \to P$ is a local section of P along f then there exists a unique local section $\overleftarrow{\sigma} : U' \to f^*P$ of $\Pi_1 : f^*P \to M'$ such that $\bar{f} \circ \overleftarrow{\sigma} = \sigma$. Namely, taking $X = U'$, τ_1 to be the inclusion map of U' in M' and $\tau_2 = \sigma$ then $\overleftarrow{\sigma}$ is the map τ given by the thesis of Lemma 1.3.13. The following commutative diagram illustrates the relation between $\overleftarrow{\sigma}$ and σ:

$$\begin{tikzcd}
f^*P \arrow[r, "\bar{f}"] & P \arrow[d, "\Pi"] \\
U' \arrow[u, dotted, "\overleftarrow{\sigma}"] \arrow[ur, "\sigma"] \arrow[r, "f|_{U'}"'] & M
\end{tikzcd}$$

We have thus established that composition on the left with $\bar{f}$ induces a bijection between the set of local sections of $\Pi_1 : f^*P \to M'$ and the set of local sections of P along f.

Let $s_1 : U_1 \to P$, $s_2 : U_2 \to P$ be smooth local sections of P with transition map $g : U_1 \cap U_2 \to G$. Set $\sigma_i = s_i \circ f$, $i = 1, 2$, and consider the local section $\overleftarrow{\sigma_i} : f^{-1}(U_i) \to f^*P$ of $\Pi_1 : f^*P \to M'$ such that $\bar{f} \circ \overleftarrow{\sigma_i} = \sigma_i$, $i = 1, 2$. Evidently, the transition map from $\overleftarrow{\sigma_1}$ to $\overleftarrow{\sigma_2}$ is $g \circ f : f^{-1}(U_1 \cap U_2) \to G$ and therefore the local sections $\overleftarrow{\sigma_1}$ and $\overleftarrow{\sigma_2}$ are compatible. This observation implies that the set:

$$\big\{\overleftarrow{\sigma} : \sigma = s \circ f \text{ and } s \text{ is a smooth local section of } P\big\} \tag{1.3.10}$$

is an atlas of local sections of $\Pi_1 : f^*P \to M'$. If we endow f^*P with the unique maximal atlas of local sections containing (1.3.10) then f^*P becomes a G-principal bundle over M'. *We will always consider the pull-back f^*P to be endowed with such maximal atlas of local sections.*

The following lemma allows us to understand better the manifold structure of the total space f^*P.

LEMMA 1.3.16. *Let $\Pi : P \to M$ be a G-principal bundle, M' be a differentiable manifold and $f : M' \to M$ be a smooth map. Let $\Pi_1 : f^*P \to M'$ denote the pull-back of P by f. Then the map $(\Pi_1, \bar{f}) : f^*P \to M' \times P$ is a smooth embedding; in particular, the canonical map $\bar{f} : f^*P \to P$ is smooth.*

PROOF. By the result of Exercise A.2, in order to prove that $(\Pi_1, \bar{f})$ is a smooth embedding, it suffices to show that for every smooth local section $s : U \to P$ of P the restriction of the map $(\Pi_1, \bar{f})$ to the open set $(\Pi_1, \bar{f})^{-1}\big(f^{-1}(U) \times P\big) = (f^*P)|_{f^{-1}(U)}$ is a smooth embedding. Set $\sigma = s \circ f$ and consider the local section $\overleftarrow{\sigma}$ of f^*P such that $\bar{f} \circ \overleftarrow{\sigma} = \sigma$. We have a commutative diagram:

$$\begin{tikzcd}
(f^*P)|_{f^{-1}(U)} \arrow[rr, "(\Pi_1, \bar{f})"] & & f^{-1}(U) \times P|_U \\
f^{-1}(U) \times G \arrow[u, "\beta_{\overleftarrow{\sigma}}", "\cong"'] \arrow[rr, "(y,g) \mapsto (y, f(y), g)"'] & & f^{-1}(U) \times (U \times G) \arrow[u, "\cong", "\mathrm{Id} \times \beta_s"']
\end{tikzcd}$$

in which the vertical arrows are smooth diffeomorphisms. The proof is concluded by observing that the bottom arrow of the diagram is a smooth embedding. $\square$

Lemma 1.3.16 says that the pull-back of principal bundles is a particular case of the notion of pull-back in the category of differentiable manifolds and smooth maps (see Exercise 1.55).

EXAMPLE 1.3.17. Let $\Pi : P \to M$ be a G-principal bundle. If U is an open subset of M and $i : U \to M$ denotes the inclusion map then the canonical map $\bar{\imath} : i^*P \to P$ is injective and its image is equal to $P|_U$. Moreover, the map $\bar{\imath} : i^*P \to P|_U$ is an isomorphism of principal bundles whose subjacent Lie group homomorphism is the identity map of G (the fact that $\bar{\imath}$ is smooth follows from Lemma 1.3.16). *We will use the map $\bar{\imath}$ to identify the pull-back i^*P with the restricted principal bundle $P|_U$.*

Using Lemma 1.3.16 we can prove the following important property of pull-backs.

PROPOSITION 1.3.18 (universal property of the pull-back). *Under the conditions of Lemma 1.3.13, if X is a differentiable manifold then the map τ is smooth if and only if both τ_1 and τ_2 are smooth.*

PROOF. Follows directly from the equality (1.3.8) and from the fact that the map $(\Pi_1, \bar{f})$ is a smooth embedding (Lemma 1.3.16). □

COROLLARY 1.3.19. *Let $\Pi : P \to M$ be a principal bundle, M' be a differentiable manifold and $f : M' \to M$ be a smooth map. A local section $\sigma : U' \to P$ of P along f is smooth if and only if the local section $\overleftarrow{\sigma} : U' \to f^*P$ of f^*P is smooth.*

PROOF. If we take $X = U'$, τ_1 to be the inclusion map of U' in M' and $\tau_2 = \sigma$ then $\overleftarrow{\sigma}$ is the map τ given by the thesis of Lemma 1.3.13. The conclusion follows from Proposition 1.3.18. □

Corollary 1.3.19 implies that composition on the left with $\bar{f}$ induces a bijection between the set of smooth local sections of f^*P and the set of smooth local sections of P along f.

DEFINITION 1.3.20. Let $\Pi : P \to M$, $\Pi' : P' \to M'$ be principal bundles with structural groups G and G', respectively and let $f : M' \to M$ be a smooth map. A map $\varphi : P' \to P$ is said to be *fiber preserving along* f if $\varphi(P'_y) \subset P_{f(y)}$, for all $y \in M'$. By a *morphism of principal bundles along* f from P' to P we mean a smooth map $\varphi : P' \to P$ such that:

- φ is fiber preserving along f;
- there exists a group homomorphism $\varphi_0 : G' \to G$ such that for all y in M' the map $\varphi_y = \varphi|_{P'_y} : P'_y \to P_{f(y)}$ is a morphism of principal spaces with subjacent group homomorphism φ_0.

As we have previously observed for morphisms of principal bundles (recall Definition 1.3.7), if φ is a morphism of principal bundles along f then the group homomorphism φ_0 is uniquely determined by φ and the smoothness of φ implies the smoothness of φ_0. We call φ_0 the *Lie group homomorphism subjacent to φ.*

Clearly a map $\varphi : P' \to P$ is fiber preserving along $f : M' \to M$ if and only if the diagram:

$$
\begin{tikzcd}
P' \arrow[r, "\varphi"] \arrow[d, "\Pi'"'] & P \arrow[d, "\Pi"] \\
M' \arrow[r, "f"'] & M
\end{tikzcd}
\tag{1.3.11}
$$

commutes.

EXAMPLE 1.3.21. If $\Pi : P \to M$ is a principal bundle with structural group G and if $f : M' \to M$ is a smooth map defined in a differentiable manifold M' then the canonical map $\bar{f} : f^*P \to P$ is fiber preserving along f (compare (1.3.7) with (1.3.11)); moreover, $\bar{f}$ is a morphism of principal bundles along f whose subjacent Lie group homomorphism is the identity map of G. If $\Pi' : P' \to M'$ is a G'-principal bundle over M' then the composition of $\bar{f}$ with a fiber preserving map from P' to f^*P is a fiber preserving map along f from P' to P. Conversely, if a map $\varphi : P' \to f^*P$ is fiber preserving along f then there exists a unique fiber preserving map $\overleftarrow{\varphi} : P' \to f^*P$ such that $\bar{f} \circ \overleftarrow{\varphi} = f$; namely, the map $\overleftarrow{\varphi}$ is the map τ given by the thesis of Lemma 1.3.13, if we take $X = P'$, $\tau_1 = \Pi'$ and $\tau_2 = \varphi$. The relation between φ and $\overleftarrow{\varphi}$ is illustrated by the following commutative diagram:

$$
\begin{tikzcd}
P' \arrow[dr, dotted, "\overleftarrow{\varphi}"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "\Pi'"'] & & \\
& f^*P \arrow[r, "\bar{f}"'] \arrow[d, "\Pi_1"] & P \arrow[d, "\Pi"] \\
& M' \arrow[r, "f"'] & M
\end{tikzcd}
$$

We can now state another corollary of Proposition 1.3.18.

COROLLARY 1.3.22. *Let* $\Pi : P \to M$*,* $\Pi' : P' \to M'$ *be principal bundles with structural groups* G *and* G'*, respectively,* $f : M' \to M$ *be a smooth map and* $\varphi : P' \to P$ *be a fiber preserving map along* f*. Then* φ *is smooth if and only if the fiber preserving map* $\overleftarrow{\varphi} : P' \to f^*P$ *is smooth. Moreover,* φ *is a morphism of principal bundles along* f *with subjacent Lie group homomorphism* $\varphi_0 : G' \to G$ *if and only if* $\overleftarrow{\varphi}$ *is a morphism of principal bundles with subjacent Lie group homomorphism* φ_0*.*

PROOF. The fact that φ is smooth if and only if $\overleftarrow{\varphi}$ is smooth follows from Proposition 1.3.18 and Example 1.3.21. The rest of the thesis follows from the observation that for all $y \in M'$ the maps:

$$\varphi_y : P'_y \longrightarrow P_{f(y)}, \quad \overleftarrow{\varphi}_y : P'_y \longrightarrow (f^*P)_y = P_{f(y)}$$

are the same. □

EXAMPLE 1.3.23. Let $\Pi : P \to M$, $\Pi' : Q \to M$ be principal bundles with structural groups G and H, respectively; let M' be a differentiable manifold and let $f : M' \to M$ be a smooth map. Given a morphism of principal bundles $\phi : P \to Q$ with subjacent Lie group homomorphism $\phi_0 : G \to H$ then:

$$\phi \circ \bar{f}^P : f^*P \longrightarrow Q$$

is a morphism of principal bundles along f with subjacent Lie group homomorphism ϕ_0; we set:

$$f^*\phi = \overleftarrow{\phi \circ \bar{f}^P},$$

so that $f^*\phi : f^*P \to f^*Q$ is the unique fiber preserving map such that the diagram:

(1.3.12)
$$\begin{tikzcd} P \arrow[r, "\phi"] & Q \\ f^*P \arrow[u, "\bar{f}^P"] \arrow[r, "f^*\phi"'] & f^*Q \arrow[u, "\bar{f}^Q"'] \end{tikzcd}$$

commutes. By Corollary 1.3.22, the map $f^*\phi$ is a morphism of principal bundles with subjacent Lie group homomorphism ϕ_0. We call $f^*\phi$ the *pull-back of the morphism ϕ by f*. As a particular case of this construction, notice that if P is a principal subbundle of Q and $i : P \to Q$ denotes the inclusion map then f^*P is a principal subbundle of f^*Q and $f^*i : f^*P \to f^*Q$ is the inclusion map.

EXAMPLE 1.3.24. Let M, M', M'' be differentiable manifolds, P be a G-principal bundle over M and let $f : M' \to M$, $g : M'' \to M'$ be smooth maps. The composition $\bar{f} \circ \bar{g}$ of the canonical maps:

$$\bar{f} : f^*P \longrightarrow P, \quad \bar{g} : g^*f^*P \longrightarrow f^*P$$

is a morphism of principal spaces along $f \circ g$ whose subjacent Lie group homomorphism is the identity map of G. Thus, by Corollary 1.3.22, the map:

$$\overleftarrow{\bar{f} \circ \bar{g}} : g^*f^*P \longrightarrow (f \circ g)^*P \tag{1.3.13}$$

characterized by the equality:

$$\overline{f \circ g} \circ \overleftarrow{\bar{f} \circ \bar{g}} = \bar{f} \circ \bar{g}$$

is an isomorphism of principal bundles whose subjacent Lie group homomorphism is the identity map of G. *We use the map* (1.3.13) *to identify the principal bundles g^*f^*P and $(f \circ g)^*P$*. Under such identification, we have:

$$\overline{f \circ g} = \bar{f} \circ \bar{g}.$$

1.3.2. The fiberwise product of principal bundles. Let $\Pi : P \to M$ and $\Pi' : Q \to M$ be principal fiber bundles with structural groups G and H, respectively. The *fiberwise product* of P with Q is the set $P \star Q$ defined by:

$$P \star Q = \bigcup_{x \in M} (P_x \times Q_x).$$

Thus the fiberwise product $P \star Q$ is the subset of the cartesian product $P \times Q$ consisting of all the pairs (p,q) such that $\Pi(p) = \Pi'(q)$. Consider the map $\Pi \star \Pi'$ defined by:

$$\Pi \star \Pi' : P \star Q \ni (p,q) \longmapsto \Pi(p) = \Pi'(q) \in M.$$

The fiber $(P \star Q)_x$ of the fiberwise product over a point $x \in M$ is the cartesian product $P_x \times Q_x$, which is a principal space with structural group $G \times H$ (recall Example 1.2.7). If $s : U \to P$, $s' : U \to Q$ are local sections of P and Q respectively then the map:

$$(s, s') : U \ni x \longmapsto \big(s(x), s'(x)\big) \in P \star Q$$

is a local section of $P \star Q$. The set:

$$\begin{aligned}\big\{(s,s') : s,\, s' \text{ are smooth local sections of } P,\, Q, \text{ respectively}\\ \text{and the domain of } s \text{ equals the domain of } s'\big\}\end{aligned} \tag{1.3.14}$$

is an atlas of local sections of $\Pi \star \Pi' : P \star Q \to M$. Thus, the fiberwise product $P \star Q$ is a $(G \times H)$-principal bundle over M endowed with the unique maximal atlas of local sections containing (1.3.14). *We will always consider the fiberwise product $P \star Q$ to be endowed with such maximal atlas of local sections.*

LEMMA 1.3.25. *Let $\Pi : P \to M$, $\Pi' : Q \to M$ be principal fiber bundles with structural groups G and H, respectively. The inclusion map of $P \star Q$ into the cartesian product $P \times Q$ is a smooth embedding.*

PROOF. By the result of Exercise A.2, in order to prove that the inclusion map $P \star Q \to P \times Q$ is a smooth embedding it suffices to show that given smooth local sections $s : U \to P$, $s' : U \to Q$ of P and Q respectively then the inclusion map from the open subset $(P \star Q) \cap (P|_U \times Q|_U) = (P \star Q)|_U$ of $P \star Q$ to $P \times Q$ is a smooth embedding. We have a commutative diagram:

$$\begin{tikzcd}
(P \star Q)|_U \arrow[r, "\text{inclusion}"] & (P|_U) \times (Q|_U) \\
U \times (G \times H) \arrow[u, "\beta_{(s,s')}", "\cong"'] \arrow[r, "(x,g,h)\mapsto(x,g,x,h)"'] & (U \times G) \times (U \times H) \arrow[u, "\cong", "\beta_s \times \beta_{s'}"']
\end{tikzcd}$$

in which the vertical arrows are smooth diffeomorphisms. Clearly the bottom arrow of the diagram is a smooth embedding and the conclusion follows. □

Let $\mathrm{pr}_1 : P \star Q \to P$, $\mathrm{pr}_2 : P \star Q \to Q$ denote the restrictions to $P \star Q$ of the projections of the cartesian product $P \times Q$. It follows from Lemma 1.3.25 that pr_1 and pr_2 are smooth; moreover, pr_1 and pr_2 are clearly morphisms of principal bundles whose subjacent Lie group homomorphisms are the correspondent projections of the cartesian product $G \times H$.

COROLLARY 1.3.26. *Under the conditions of Lemma 1.3.25, if $\phi_1 : X \to P$, $\phi_2 : X \to Q$ are smooth maps defined in a differentiable manifold X such that $\Pi \circ \phi_1 = \Pi' \circ \phi_2$ then there exists a unique map $\phi : X \to P \star Q$ such that $\mathrm{pr}_1 \circ \phi = \phi_1$ and $\mathrm{pr}_2 \circ \phi = \phi_2$. The map ϕ is smooth.*

PROOF. The hypothesis $\Pi \circ \phi_1 = \Pi' \circ \phi_2$ means that the image of the map $(\phi_1, \phi_2) : X \to P \times Q$ is contained in $P \star Q$. The map ϕ must therefore be the map obtained from (ϕ_1, ϕ_2) by replacing the counter-domain $P \times Q$ by $P \star Q$. $\square$

COROLLARY 1.3.27 (universal property of the fiberwise product). *Under the conditions of Lemma 1.3.25, let K be a Lie group R be a K-principal bundle over M. Given morphisms of principal bundles $\phi_1 : R \to P$, $\phi_2 : R \to Q$ then there exists a unique morphism of principal bundles $\phi : R \to P \star Q$ such that* $\mathrm{pr}_1 \circ \phi = \phi_1$ *and* $\mathrm{pr}_2 \circ \phi = \phi_2$.

PROOF. Apply Corollary 1.3.26 with $X = R$. If $\phi_1^0 : K \to G$, $\phi_2^0 : K \to H$ are the subjacent Lie group homomorphisms to ϕ_1 and ϕ_2 respectively then it is immediate that the smooth map $\phi : R \to P \star Q$ given by the thesis of Corollary 1.3.26 is a morphism of principal bundles with subjacent Lie group homomorphism $(\phi_1^0, \phi_2^0) : K \to G \times H$. $\square$

The commutative diagram below illustrates the statement of Corollary 1.3.27:

$$\begin{tikzcd}
& & P \\
R \arrow[urr, "\phi_1"] \arrow[r, dotted, "\phi"] \arrow[drr, "\phi_2"'] & P \star P' \arrow[ur, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
& & Q
\end{tikzcd}$$

If the reader feels that there is some relation between the notions of pull-back and of fiberwise product then he or she is right. See Exercise 1.57 for details.

EXAMPLE 1.3.28. Let P, P', Q, Q' be principal bundles over a differentiable manifold M and let $\phi : P \to P'$, $\psi : Q \to Q'$ be morphisms of principal bundles. Denote by:

$$\mathrm{pr}_1 : P \star Q \longrightarrow P, \quad \mathrm{pr}_2 : P \star Q \longrightarrow Q,$$
$$\mathrm{pr}_1' : P' \star Q' \longrightarrow P', \quad \mathrm{pr}_2' : P' \star Q' \longrightarrow Q',$$

the projections. By Corollary 1.3.27, there exists a unique morphism of principal bundles:

$$\phi \star \psi : P \star Q \longrightarrow P' \star Q'$$

such that $\mathrm{pr}_1' \circ (\phi \star \psi) = \phi \circ \mathrm{pr}_1$ and $\mathrm{pr}_2' \circ (\phi \star \psi) = \psi \circ \mathrm{pr}_2$.

LEMMA 1.3.29. *Let $\Pi : P \to M$ and $\Pi' : Q \to M$ be principal fiber bundles and let $f : M' \to M$ be a smooth map defined in a differentiable manifold M'. The map:*

$$f^*(P \star Q) \ni \big(y, (p, q)\big) \longmapsto \big((y, p), (y, q)\big) \in (f^*P) \star (f^*Q) \tag{1.3.15}$$

is an isomorphism of principal bundles whose subjacent Lie group homomorphism is the identity.

PROOF. The fact that (1.3.15) is a morphism of principal bundles follows by applying Corollary 1.3.27 with (recall Example 1.3.23):

$$\phi_1 = f^*\mathrm{pr}_1 : f^*(P \star Q) \longrightarrow f^*P, \quad \phi_2 = f^*\mathrm{pr}_2 : f^*(P \star Q) \longrightarrow f^*Q.$$

The fact that (1.3.15) is an isomorphism of principal bundles then follows from the result of Exercise 1.46. □

1.4. Associated bundles

Associated bundles are constructed from principal bundles by a fiberwise application of the notion of fiber product discussed in Subsection 1.2.1. We begin by stating a smooth version of Definition 1.2.24.

DEFINITION 1.4.1. By a *differentiable G-space* we mean a differentiable manifold N endowed with a smooth left action $G \times N \to N$ of a Lie group G.

Notice that the effective group G_{ef} of a differentiable G-space is a subgroup of the group $\mathrm{Diff}(N)$ of all smooth diffeomorphisms of N. The kernel of the homomorphism $G \ni g \mapsto \gamma_g \in \mathrm{Diff}(N)$ corresponding to the left action of G on N is a closed normal subgroup of G and G_{ef} is isomorphic to the quotient of G by such kernel; we can therefore endow G_{ef} with the structure of a Lie group.

Let $\Pi : P \to M$ be a G-principal bundle and let N be a differentiable G-space. For each $x \in M$ we consider the fiber product $P_x \times_G N$ of the principal space P_x by the G-space N and we set:

$$P \times_G N = \bigcup_{x \in M} (P_x \times_G N);$$

we have a *projection map*:

$$\pi : P \times_G N \longrightarrow M$$

that sends $P_x \times_G N$ to the point $x \in M$ and a *quotient map* $\mathfrak{q}$ defined by:

$$\mathfrak{q} : P \times N \ni (p, n) \longmapsto [p, n] \in P \times_G N.$$

The following commutative diagram illustrates the relation between the maps Π, π and $\mathfrak{q}$:

$$\begin{tikzcd} P \times N \arrow[d, "\text{first projection}"'] \arrow[dr, "\mathfrak{q}"] & \\ P \arrow[d, "\Pi"'] & P \times_G N \arrow[dl, "\pi"] \\ M & \end{tikzcd} \tag{1.4.1}$$

We call $\pi : P \times_G N \to M$ (or just $P \times_G N$) the *associated bundle* to the G-principal bundle P and to the differentiable G-space N. The set $P \times_G N$ is also called the *total space* of the associated bundle. For each $x \in M$, the set

$$P_x \times_G N = \pi^{-1}(x)$$

is called the *fiber* of $P \times_G N$ over x.

Notice that each fiber $P_x \times_G N$ is naturally endowed with the G_{ef}-structure $\widehat{P_x} = \big\{\hat{p} : p \in P_x\big\}$. Since G_{ef} is a subgroup of $\mathrm{Diff}(N)$, such G_{ef}-structure can be weakened to a $\mathrm{Diff}(N)$-structure that corresponds to the structure of a differentiable manifold smoothly diffeomorphic to N on the fiber $P_x \times_G N$ (recall Example 1.1.4).

Our goal now is to endow the entire total space $P \times_G N$ with the structure of a differentiable manifold. Given a smooth local section $s : U \to P$ of P then we have an associated bijective map:

$$\hat{s} : U \times N \ni (x, n) \longmapsto [s(x), n] = \widehat{s(x)}(n) \in \pi^{-1}(U) \subset P \times_G N, \tag{1.4.2}$$

which we call the *local trivialization* of the associated bundle $P \times_G N$ corresponding to the smooth local section s. If $s_1 : U_1 \to P$, $s_2 : U_2 \to P$ are smooth local sections of P and if $g : U_1 \cap U_2 \to G$ denotes the transition map from s_1 to s_2 then the *transition map* $\hat{s}_1^{-1} \circ \hat{s}_2$ from $\hat{s}_1$ to $\hat{s}_2$ is given by:

$$\hat{s}_1^{-1} \circ \hat{s}_2 : (U_1 \cap U_2) \times N \ni (x, n) \longmapsto \big(x, g(x) \cdot n\big) \in (U_1 \cap U_2) \times N$$

and is therefore a smooth diffeomorphism between open sets. It follows from the result of Exercise A.1 that there exists a unique differential structure on the set $P \times_G N$ such that for every smooth local section $s : U \to P$ of P the set $\pi^{-1}(U)$ is open in $P \times_G N$ and the local trivialization $\hat{s}$ is a smooth diffeomorphism. *We will always regard the total space $P \times_G N$ of an associated bundle to be endowed with such differential structure*. The fact that the topologies of M and N are Hausdorff and second countable implies that the topology of $P \times_G N$ is also Hausdorff and second countable, so that $P \times_G N$ is a differentiable manifold. One can easily check the following facts:

- the projection $\pi : P \times_G N \to M$ is a smooth submersion;
- the quotient map $\mathfrak{q} : P \times N \to P \times_G N$ is a smooth submersion;
- for every $x \in M$ the fiber $P_x \times_G N$ is a smooth submanifold of $P \times_G N$;
- for every $x \in M$ and every $p \in P_x$, if the fiber $P_x \times_G N$ is endowed with the differential structure inherited from $P \times_G N$ as a submanifold then the map $\hat{p} : N \to P_x \times_G N$ is a smooth diffeomorphism.

The last item above implies that the differential structure of $P_x \times_G N$ that is obtained by weakening the G_{ef}-structure $\widehat{P_x}$ coincides with the differential structure that $P_x \times_G N$ inherits from $P \times_G N$.

EXAMPLE 1.4.2 (the trivial associated bundle). Let M be a differentiable manifold, P_0 be a principal space whose structural group G is a Lie group and let N be a differentiable G-space. Consider the trivial principal bundle $P = M \times P_0$ (recall Example 1.3.2). The associated bundle $P \times_G N$ can be naturally identified with the cartesian product $M \times (P_0 \times_G N)$ of M by the fiber product $P_0 \times_G N$. The fiber product $P_0 \times_G N$ is endowed with a G_{ef}-structure that can be weakened into a $\mathrm{Diff}(N)$-structure that corresponds to the structure of a differentiable manifold smoothly diffeomorphic to N. Clearly the differential structure of $P \times_G N = M \times (P_0 \times_G N)$ coincides with the standard differential structure defined in a cartesian product of differentiable manifolds.

EXAMPLE 1.4.3. Let $\Pi : P \to M$ be a G-principal bundle and N be a differentiable G-space. If U is an open subset of M then the total space of the associated bundle $(P|_U) \times_G N$ is equal to the open subset $\pi^{-1}(U)$ of the total space of the associated bundle $\pi : P \times_G N \to M$. Clearly, the differential structure of the associated bundle $(P|_U) \times_G N$ coincides with the differential structure it inherits from $P \times_G N$ as an open subset.

DEFINITION 1.4.4. Given $x \in M$, $p \in P_x$ and $n \in N$ then the tangent space $T_{[p,n]}(P_x \times_G N)$ is a subspace of $T_{[p,n]}(P \times_G N)$ and it is called the *vertical space*f $P \times_G N$ at $[p, n]$; we write:

$$\mathrm{Ver}_{[p,n]}(P \times_G N) = T_{[p,n]}(P_x \times_G N).$$

Clearly:

$$\mathrm{Ver}_{[p,n]}(P \times_G N) = \mathrm{Ker}\big(\mathrm{d}\pi([p, n])\big).$$

Notice that, since $\hat{p}$ is a smooth diffeomorphism from N onto the fiber $P_x \times_G N$, its differential at $n \in N$ is an isomorphism:

$$\mathrm{d}\hat{p}(n) : T_n N \longrightarrow \mathrm{Ver}_{[p,n]}(P \times_G N) \tag{1.4.3}$$

from the tangent space $T_n N$ onto the vertical space.

In the example below we look at a case that is of particular interest to us.

EXAMPLE 1.4.5. Let E_0 be a real finite-dimensional vector space and assume that we are given a smooth representation $\rho : G \to \mathrm{GL}(E_0)$ of a Lie group G on E_0. Then E_0 is a differentiable G-space and the effective group G_{ef} is a Lie subgroup of the general linear group $\mathrm{GL}(E_0)$. If $\Pi : P \to M$ is a G-principal bundle then we can consider the associated bundle $P \times_G E_0$. For each $x \in M$, the fiber $P_x \times_G E_0$ has the structure of a real vector space such that for every $p \in P_x$ the map $\hat{p} : E_0 \to P_x \times_G E_0$ is a linear isomorphism (recall Example 1.2.26). Since each $\hat{p}$ is both a smooth diffeomorphism and a linear isomorphism, it follows that the differential structure of the fiber $P_x \times_G E_0$ (inherited from the total space $P \times_G E_0$) coincides with the differential structure that is determined by its real finite-dimensional vector space structure. We can therefore identify the vertical space at any point of the fiber $P_x \times_G E_0$ with the fiber itself, i.e.:

$$\mathrm{Ver}_{[p,e_0]}(P \times_G E_0) = T_{[p,e_0]}(P_x \times_G E_0) \cong P_x \times_G E_0, \tag{1.4.4}$$

for all $p \in P_x$ and all $e_0 \in E_0$. Moreover, the linear isomorphism (1.4.3) is just $\hat{p}$, i.e.:

$$\mathrm{d}\hat{p}(e_0) = \hat{p} : E_0 \longrightarrow P_x \times_G E_0, \tag{1.4.5}$$

for all $p \in P_x$ and all $e_0 \in E_0$.

1.4.1. Local sections of an associated bundle. Let $\Pi : P \to M$ be a G-principal bundle, N be a differentiable G-space and consider the associated bundle $\pi : P \times_G N \to M$ and the quotient map $\mathfrak{q} : P \times N \to P \times_G N$.

DEFINITION 1.4.6. By a *local section*of the associated bundle $P \times_G N$ we mean a map $\epsilon : U \to P \times_G N$ defined on an open subset U of M such that $\pi \circ \epsilon = \mathrm{Id}_U$, i.e., such that $\epsilon(x) \in P_x \times_G N$, for all $x \in U$.

If $\epsilon : U \to P \times_G N$ is a local section of $P \times_G N$ and if $s : U \to P$ is a smooth local section of P then there exists a unique map $\tilde{\epsilon} : U \to N$ such that $\epsilon = \mathfrak{q} \circ (s, \tilde{\epsilon})$, i.e., such that:

$$\epsilon(x) = [s(x), \tilde{\epsilon}(x)], \tag{1.4.6}$$

for all $x \in U$; namely, $\tilde{\epsilon}$ is just the second coordinate of the map $\hat{s}^{-1} \circ \epsilon$. We call $\tilde{\epsilon}$ the *representation* of ϵ with respect to s. Clearly ϵ is smooth if and only if $\tilde{\epsilon}$ is smooth.

1.4.2. The differential of the quotient map. Let $\Pi : P \to M$ be a G-principal bundle, N be a differentiable G-space and consider the associated bundle $\pi : P \times_G N \to M$ and the quotient map $\mathfrak{q} : P \times N \to P \times_G N$. For every $x \in M$, the map $\mathfrak{q}$ carries $P_x \times N$ onto the fiber $P_x \times_G N$ over x and therefore, for all $p \in P_x$ and all $n \in N$, the differential $\mathrm{d}\mathfrak{q}(p, n)$ carries $T_{(p,n)}(P_x \times N) = \mathrm{Ver}_p(P) \oplus T_n N$ to the vertical space $\mathrm{Ver}_{[p,n]}(P \times_G N)$. We wish to compute the restriction of the differential $\mathrm{d}\mathfrak{q}(p, n)$ to $\mathrm{Ver}_p(P) \oplus T_n N$. To this aim, we identify $\mathrm{Ver}_p(P)$ with the Lie algebra $\mathfrak{g}$ via the canonical isomorphism (1.3.3) and we identify $\mathrm{Ver}_{[p,n]}(P \times_G N)$ with $T_n N$ via the isomorphism (1.4.3). Recall from Definition A.2.3 that for every $X \in \mathfrak{g}$, we denote by X^N the induced vector field on the differentiable manifold N and by X^P the induced vector field on the differentiable manifold P.

LEMMA 1.4.7. *Let $\Pi : P \to M$ be a G-principal bundle, N be a differentiable G-space and consider the associated bundle $\pi : P \times_G N \to M$ and the quotient map $\mathfrak{q} : P \times N \to P \times_G N$. Given $p \in P$, $n \in N$ then the dotted arrow in the commutative diagram:*

$$\begin{tikzcd}
\mathrm{Ver}_p(P) \oplus T_n N \arrow[r, "\mathrm{d}\mathfrak{q}(p,n)"] & \mathrm{Ver}_{(p,n)}(P \times_G N) \\
\mathfrak{g} \oplus T_n N \arrow[u, "\mathrm{d}\beta_p(1)\oplus\mathrm{Id}", "\cong"'] \arrow[r, dotted] & T_n N \arrow[u, "\cong", "\mathrm{d}\hat{p}(n)"']
\end{tikzcd}$$

is given by:

$$\mathfrak{g} \oplus T_n N \ni (X, u) \longmapsto u + X^N(n) \in T_n N.$$

PROOF. Set $x = \Pi(p)$. The map $\mathfrak{q}(p, \cdot) : N \to P_x \times_G N$ is the same as $\hat{p}$ and therefore the dotted arrow on the diagram carries $(0, u)$ to u, for all $u \in T_n N$. To conclude the proof, we show that the dotted arrow on the diagram carries $(X, 0)$ to

$X^N(n)$, for all $X \in \mathfrak{g}$; this follows from the commutative diagram:

$$\begin{tikzcd} P_x \arrow[r, "\mathfrak{q}(\cdot,n)"] & P_x \times_G N \\ G \arrow[u, "\beta_p"] \arrow[r, "\beta_n"'] & N \arrow[u, "\hat{p}"'] \end{tikzcd}$$

by differentiation. $\square$

COROLLARY 1.4.8. *Let $\Pi : P \to M$ be a G-principal bundle, N be a differentiable G-space and consider the associated bundle $\pi : P \times_G N \to M$ and the quotient map $\mathfrak{q} : P \times N \to P \times_G N$. Given $p \in P$, $n \in N$ then the kernel of $\mathrm{d}\mathfrak{q}(p, n)$ is equal to the image under the isomorphism:*

$$\mathrm{d}\beta_p(1) \oplus \mathrm{Id} : \mathfrak{g} \oplus T_nN \longrightarrow \mathrm{Ver}_p(P) \oplus T_nN$$

of the space:

$$\big\{\big(X, -X^N(n)\big) : X \in \mathfrak{g}\big\} \subset \mathfrak{g} \oplus T_nN;$$

more explicitly:

$$\mathrm{Ker}\big(\mathrm{d}\mathfrak{q}(p, n)\big) = \big\{\big(X^P(p), -X^N(n)\big) : X \in \mathfrak{g}\big\} \subset \mathrm{Ver}_p(P) \oplus T_nN.$$

PROOF. By differentiating (1.4.1), we see that the diagram:

$$\begin{tikzcd} T_pP \oplus T_nN \arrow[rr, "\mathrm{d}\mathfrak{q}_{(p,n)}"] \arrow[dr, "\mathrm{d}\Pi_p \circ \mathrm{pr}_1"'] & & T_{[p,n]}(P \times_G N) \arrow[dl, "\mathrm{d}\pi_{[p,n]}"] \\ & T_xM & \end{tikzcd} \tag{1.4.7}$$

commutes. Thus the kernel of $\mathrm{d}\mathfrak{q}(p, n)$ is contained in $\mathrm{Ver}_p(P) \oplus T_nN$. The conclusion now follows easily from Lemma 1.4.7. $\square$

EXAMPLE 1.4.9. Let us go back to the context of Example 1.4.5 and let us take a closer look at the statement of Lemma 1.4.7. Denote by $\bar{\rho} : \mathfrak{g} \to \mathfrak{gl}(E_0)$ the differential at $1 \in G$ of the representation $\rho : G \to \mathrm{GL}(E_0)$; by $\mathfrak{gl}(E_0)$ we have denoted the Lie algebra of the general linear group $\mathrm{GL}(E_0)$, which is just the space of all linear endomorphisms of E_0, endowed with the standard Lie bracket of linear operators. Given $X \in \mathfrak{g}$ then the induced vector field X^{E_0} on the manifold E_0 is given by $X^{E_0}(e_0) = \bar{\rho}(X) \cdot e_0$, for all $e_0 \in E_0$; thus $X^{E_0} : E_0 \to E_0$ is just the linear map $\bar{\rho}(X)$. Keeping in mind (1.4.5), Lemma 1.4.7 tells us that the restriction to $\mathrm{Ver}_p(P) \oplus E_0$ of the differential of the quotient map $\mathfrak{q} : P \times E_0 \to P \times_G E_0$ at a point $(p, e_0) \in P \times E_0$ is given by:

$$\mathrm{d}\mathfrak{q}_{(p,e_0)}\big(\mathrm{d}\beta_p(1) \cdot X, u\big) = \hat{p}\big(u + \bar{\rho}(X) \cdot e_0\big) = [p, u + \bar{\rho}(X) \cdot e_0] \in P_x \times_G E_0,$$

for all $X \in \mathfrak{g}$ and all $u \in E_0$. For instance, if G is a Lie subgroup of $\mathrm{GL}(E_0)$ and $\rho : G \to \mathrm{GL}(E_0)$ is the inclusion map then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(E_0)$, $\bar{\rho} : \mathfrak{g} \to \mathfrak{gl}(E_0)$ is the inclusion map and thus $\bar{\rho}(X)$ is just X itself; hence:

$$\mathrm{d}\mathfrak{q}_{(p,e_0)}\big(\mathrm{d}\beta_p(1) \cdot X, u\big) = \hat{p}\big(u + X(e_0)\big) = [p, u + X(e_0)] \in P_x \times_G E_0,$$

for all $X \in \mathfrak{g}$ and all $u \in E_0$.

1.4.3. Induced maps on associated bundles. In Subsection 1.2.1 we have defined the notion of induced maps on fiber products. Such notion can be applied fiberwise to get a notion of induced map on an associated bundle. More precisely, let P, Q be principal bundles over a differentiable manifold M with structural groups G and H, respectively. Let $\phi : P \to Q$ be a morphism of principal bundles and let $\phi_0 : G \to H$ denote its subjacent Lie group homomorphism. Given a differentiable H-space N we consider the smooth left action of G on N defined in (1.2.20), so that N is also a differentiable G-space. For each $x \in M$, the morphism of principal spaces $\phi_x : P_x \to Q_x$ induces a map $\hat{\phi}_x : P_x \times_G N \to Q_x \times_H N$ and thus there is a map:

$$\hat{\phi} : P \times_G N \longrightarrow Q \times_H N$$

whose restriction to the fiber $P_x \times_G N$ is equal to $\hat{\phi}_x$, for all $x \in M$. More explicitly, we have:

$$\hat{\phi}\big([p,n]\big) = [\phi(p), n],$$

for all $p \in P$ and all $n \in N$. We call $\hat{\phi}$ the map *induced*indexassociated bundle!induced map on by ϕ on the associated bundles. Notice that the following diagram:

$$\begin{tikzcd} P \times N \arrow[r, "\phi\times\mathrm{Id}"] \arrow[d, "\mathfrak{q}"'] & Q \times N \arrow[d, "\mathfrak{q}'"] \\ P \times_G N \arrow[r, "\hat{\phi}"'] & Q \times_H N \end{tikzcd} \tag{1.4.8}$$

commutes, where $\mathfrak{q}$, $\mathfrak{q}'$ denote the quotient maps.

Since for each $x \in M$ the map $\hat{\phi}_x$ is bijective (Corollary 1.2.32) then clearly the map $\hat{\phi}$ is also bijective. Moreover, we have the following:

LEMMA 1.4.10. *Let P, Q be principal bundles over the same differentiable manifold M with structural groups G and H, respectively. Let $\phi : P \to Q$ be a morphism of principal bundles and let $\phi_0 : G \to H$ denote its subjacent Lie group homomorphism. Given a differentiable H-space N we consider the smooth left action of G on N defined by* (1.2.20). *The induced map $\hat{\phi} : P \times_G N \longrightarrow Q \times_H N$ is a smooth diffeomorphism.*

PROOF. Let $s : U \to P$ be a smooth local section of P and set $s' = \phi \circ s$, so that $s' : U \to Q$ is a smooth local section of Q. We have a commutative diagram analogous to diagram (1.2.23):

$$\begin{tikzcd} (P|_U) \times_G N \arrow[rr, "\hat{\phi}"] & & (Q|_U) \times_H N \\ & U \times N \arrow[ul, "\hat{s}", "\cong"'] \arrow[ur, "\widehat{s'}"', "\cong"] & \end{tikzcd} \tag{1.4.9}$$

Since $\hat{s}$ and $\widehat{s'}$ are smooth diffeomorphisms, we conclude that $\hat{\phi}$ is a smooth local diffeomorphism. Since $\hat{\phi}$ is bijective, it follows that $\hat{\phi}$ is indeed a global smooth diffeomorphism. $\square$

As in Subsection 1.2.1 we have also a more general notion of induced map. Let P, Q be principal bundles over a differentiable manifold M with structural groups G and H, respectively. Let $\phi : P \to Q$ be a morphism of principal bundles with subjacent Lie group homomorphism $\phi_0 : G \to H$. Let N be a differentiable G-space and N' be a differentiable H-space; assume that we are given a ϕ_0-equivariant map $\kappa : N \to N'$. For each $x \in M$ the map $\phi_x : P_x \to Q_x$ is a morphism of principal spaces with subjacent group homomorphism ϕ_0 and therefore we have an induced map:

$$\phi_x \underset{\sim}{\times} \kappa : P_x \times_G N \longrightarrow Q_x \times_H N'.$$

We can therefore consider the *induced map*:

$$\phi \underset{\sim}{\times} \kappa : P \times_G N \longrightarrow Q \times_H N'$$

whose restriction to the fiber $P_x \times_G N$ is equal to $\phi_x \underset{\sim}{\times} \kappa$, for all $x \in M$. Notice that the following diagram:

(1.4.10)
$$\begin{tikzcd} P \times N \arrow[r, "\phi\times\kappa"] \arrow[d, "\mathfrak{q}"'] & Q \times N' \arrow[d, "\mathfrak{q}'"] \\ P \times_G N \arrow[r, "\phi\underset{\sim}{\times}\kappa"'] & Q \times_H N' \end{tikzcd}$$

commutes, where $\mathfrak{q}$, $\mathfrak{q}'$ denote the quotient maps. If $N = N'$ and N is endowed with the action of G defined in (1.2.20) then the induced map $\phi \underset{\sim}{\times} \mathrm{Id}$ is the same as $\hat{\phi}$.

The induced map $\phi \underset{\sim}{\times} \kappa$ retains many properties of the map κ as is shown by the following:

LEMMA 1.4.11. *Let P, Q be principal bundles over a nonempty differentiable manifold M with structural groups G and H, respectively. Let $\phi : P \to Q$ be a morphism of principal bundles with subjacent Lie group homomorphism $\phi_0 : G \to H$, let N be a differentiable G-space and N' be a differentiable H-space; assume that we are given a ϕ_0-equivariant map $\kappa : N \to N'$. Consider the induced map $\phi \underset{\sim}{\times} \kappa : P \times_G N \to Q \times_H N'$. Then:*

(a) *$\phi \underset{\sim}{\times} \kappa$ is smooth if and only if κ is smooth;*
(b) *$\phi \underset{\sim}{\times} \kappa$ is injective (resp., surjective) if and only if κ is injective (resp., surjective);*
(c) *$\phi \underset{\sim}{\times} \kappa$ is a smooth immersion (resp., a smooth submersion) if and only if κ is a smooth immersion (resp., a smooth submersion);*
(d) *$\phi \underset{\sim}{\times} \kappa$ is a smooth embedding if and only if κ is a smooth embedding;*
(e) *$\phi \underset{\sim}{\times} \kappa$ is an open mapping if and only if κ is an open mapping.*

PROOF. Let $s : U \to P$ be a smooth local section of P and set $s' = \phi \circ s$, so that $s' : U \to Q$ is a smooth local section of Q. We have a commutative diagram similar to (1.2.22):

$$\begin{tikzcd}
(P|_U) \times_G N \arrow[r, "\phi \underset{\sim}{\times} \kappa"] & (Q|_U) \times_H N' \\
U \times N \arrow[u, "\hat{s}", "\cong"'] \arrow[r, "\mathrm{Id} \times \kappa"'] & U \times N' \arrow[u, "\cong", "\widehat{s'}"']
\end{tikzcd}$$

The conclusion follows then easily from the fact that the maps $\hat{s}$ and $\widehat{s'}$ are smooth diffeomorphisms (for the proof of item (d), use also the result of Exercise A.2). $\square$

Notice that if $\Pi : P \to M$ is a G-principal bundle, N is a differentiable G-space and N_0 is a smooth submanifold of N invariant by the action of G then the inclusion map $i : N_0 \to N$ is a smooth G-equivariant embedding and therefore, by Lemma 1.4.11, the induced map $\mathrm{Id} \underset{\sim}{\times} i : P \times_G N_0 \to P \times_G N$ is a smooth embedding. We use the map $\mathrm{Id} \underset{\sim}{\times} i$ to identify $P \times_G N_0$ with a smooth submanifold of $P \times_G N$. Notice that if N_0 is an open submanifold of N then $P \times_G N_0$ is an open submanifold of $P \times_G N$ (item (e) of Lemma 1.4.11).

1.4.4. The associated bundle to a pull-back. Let P be a G-principal bundle over a differentiable manifold M and let $f : M' \to M$ be a smooth map defined in a differentiable manifold M'. Given a differentiable G-space N then the associated bundle $(f^*P) \times_G N$ can be identified with the following subset of the cartesian product $M' \times (P \times_G N)$:

$$\bigcup_{y \in M'} \big(\{y\} \times (P_{f(y)} \times_G N)\big). \tag{1.4.11}$$

We have the following:

LEMMA 1.4.12. *Let P be a G-principal bundle over a differentiable manifold M and let $f : M' \to M$ be a smooth map defined in a differentiable manifold M'. Let N be a differentiable G-space. If we identify the associated bundle $(f^*P) \times_G N$ with the set* (1.4.11) *then the inclusion map of $(f^*P) \times_G N$ in $M' \times (P \times_G N)$ is a smooth embedding.*

PROOF. By the result of Exercise A.2, in order to prove that the inclusion map $(f^*P) \times_G N \to M' \times (P \times_G N)$ is a smooth embedding it suffices to verify that for every smooth local section $s : U \to P$ of P the inclusion map from the open subset $\big((f^*P) \times_G N\big) \cap \big(f^{-1}(U) \times (P \times_G N)\big) = \big((f^*P)|_{f^{-1}(U)}\big) \times_G N$ of $(f^*P) \times_G N$ to $M' \times (P \times_G N)$ is a smooth embedding. Set $\sigma = s \circ f$ and consider the smooth local section $\overleftarrow{\sigma} : f^{-1}(U) \to f^*P$ of f^*P such that $\bar{f} \circ \overleftarrow{\sigma} = \sigma$. We have a commutative diagram:

$$\begin{tikzcd}
(f^*P)|_{f^{-1}(U)} \times_G N \arrow[r, "\text{inclusion}"] & f^{-1}(U) \times \big((P|_U) \times_G N\big) \\
f^{-1}(U) \times N \arrow[u, "\widehat{\overleftarrow{\sigma}}", "\cong"'] \arrow[r, "(y,n)\mapsto(y,f(y),n)"'] & f^{-1}(U) \times (U \times N) \arrow[u, "\cong", "\mathrm{Id}\times\hat{s}"']
\end{tikzcd}$$

in which the vertical arrows are smooth diffeomorphisms. Clearly the bottom arrow of the diagram is a smooth embedding and the conclusion follows. $\square$

1.5. Vector bundles and the principal bundle of frames

Let M be a differentiable manifold, E_0 be a real finite-dimensional vector space, E be a set and $\pi : E \to M$ be a map; for each $x \in M$ we denote by E_x the subset $\pi^{-1}(x)$ of E and we call it the *fiber* of E over x. Assume that for each $x \in M$ we are given a real vector space structure on the fiber E_x such that E_0 and E_x have the same dimension. The set $\mathrm{FR}_{E_0}(E_x)$ of all E_0-frames of E_x is thus a principal space with structural group $\mathrm{GL}(E_0)$. We set:

$$\mathrm{FR}_{E_0}(E) = \bigcup_{x \in M} \mathrm{FR}_{E_0}(E_x)$$

and we consider the map $\Pi : \mathrm{FR}_{E_0}(E) \to M$ that sends $\mathrm{FR}_{E_0}(E_x)$ to x, for all $x \in M$.

DEFINITION 1.5.1. A *vector bundle* consists of:

- a set E, called the *total space*;
- a differentiable manifold M, called the *base space*;
- a map $\pi : E \to M$, called the *projection*;
- a real finite-dimensional vector space E_0, called the *typical fiber*;
- a real vector space structure on the fiber $E_x = \pi^{-1}(x)$ such that E_0 and E_x have the same dimension, for all $x \in M$;
- a maximal atlas $\mathcal{A}_{\max}$ of local sections of $\Pi : \mathrm{FR}_{E_0}(E) \to M$.

When working with vector bundles we will refer to the projection π or to the total space E as if it were the collection of all the objects listed in Definition 1.5.1. We will also say that E is a vector bundle *over* M. The maximal atlas $\mathcal{A}_{\max}$ makes $\Pi : \mathrm{FR}_{E_0}(E) \to M$ into a $\mathrm{GL}(E_0)$-principal bundle over M; we call it the *principal bundle of E_0-frames* (or simply *the principal bundle of frames*) of the vector bundle $\pi : E \to M$. A (smooth) local section of $\mathrm{FR}_{E_0}(E)$ will also be called a (smooth) *local E_0-frame* (or simply a *local frame*) of the vector bundle E. When $E_0 = \mathbb{R}^n$ we write $\mathrm{FR}(E)$ instead of $\mathrm{FR}_{E_0}(E)$.

Let us now define a differential structure on the total space of a vector bundle. This is done as follows. The typical fiber E_0 is a differentiable $\mathrm{GL}(E_0)$-space in a obvious way; since the frame bundle $\mathrm{FR}_{E_0}(E)$ is a $\mathrm{GL}(E_0)$-principal bundle we may thus consider the associated bundle $\mathrm{FR}_{E_0}(E) \times_{\sim} E_0$. The *contraction map* $\mathcal{C}^E$ defined by:

$$\mathcal{C}^E : \mathrm{FR}_{E_0}(E) \times_{\sim} E_0 \ni [p, e_0] \longmapsto p(e_0) \in E \tag{1.5.1}$$

is bijective and it restricts to a linear isomorphism from $\mathrm{FR}_{E_0}(E_x) \underset{\sim}{\times} E_0$ to E_x, for all $x \in M$ (recall Example 1.2.28). Thus, there is a unique differential structure on the set E that makes the contraction map $\mathcal{C}^E$ a smooth diffeomorphism. *We will always consider the total space E of a vector bundle to be endowed with such differential structure*. Clearly the topology of E is Hausdorff and second countable, so that E is a differentiable manifold. The following facts follow directly from the corresponding facts stated in Section 1.4 for general associated bundles and from the comments made in Example 1.4.5:

- the projection $\pi : E \to M$ is a smooth submersion;
- the map $\mathrm{FR}_{E_0}(E) \times E_0 \ni (p, e_0) \mapsto p(e_0) \in E$ is a smooth submersion;
- for every $x \in M$ the fiber E_x is a smooth submanifold of E;
- for every $x \in M$ the differential structure that the fiber E_x inherits from E as a submanifold coincides with the differential structure that is determined by its real finite-dimensional vector space structure.

Let $s : U \to \mathrm{FR}_{E_0}(E)$ be a smooth local section of $\mathrm{FR}_{E_0}(E)$ and set $\check{s} = \mathcal{C}^E \circ \hat{s}$; more explicitly, the map $\check{s}$ is given by:

$$\check{s} : U \times E_0 \ni (x, e_0) \longmapsto s(x) \cdot e_0 \in \pi^{-1}(U) \subset E.$$

The map $\check{s}$ is a smooth diffeomorphism and we will call it the *local trivialization* of E corresponding to the smooth local E_0-frame s. Notice that the differential structure of the total space E can also be characterized by the fact that for every smooth local E_0-frame $s : U \to \mathrm{FR}_{E_0}(E)$ the map $\check{s}$ is a smooth diffeomorphism onto the open subset $\pi^{-1}(U)$ of E.

EXAMPLE 1.5.2 (the trivial vector bundle). Let M be a differentiable manifold and E_0 be a real finite-dimensional vector space. Set $E = M \times E_0$ and consider the map $\pi : E \to M$ given by projection onto the first coordinate. For every $x \in M$ we identify the fiber $E_x = \{x\} \times E_0$ with E_0 so that E_x has the structure of a real vector space and:

$$\mathrm{FR}_{E_0}(M \times E_0) = M \times \mathrm{GL}(E_0).$$

The set $\mathrm{FR}_{E_0}(M \times E_0)$ is thus naturally endowed with the structure of a $\mathrm{GL}(E_0)$-principle bundle (see Example 1.3.2) and therefore E is a vector bundle over M which we call the *trivial vector bundle over M with typical fiber E_0*. Clearly the differential structure of $E = M \times E_0$ coincides with the standard differential structure given to a cartesian product of differentiable manifolds.

EXAMPLE 1.5.3. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0. If U is an open subset of M, we set:

$$E|_U = \pi^{-1}(U);$$

the projection $\pi : E \to M$ restricts to a map from $E|_U$ to M and for each $x \in U$ the fiber E_x of $E|_U$ over x is endowed with the structure of a real vector space. Clearly:

$$\mathrm{FR}_{E_0}(E|_U) = \mathrm{FR}_{E_0}(E)|_U,$$

so that $\mathrm{FR}_{E_0}(E|_U)$ is a $\mathrm{GL}(E_0)$-principal bundle over the differentiable manifold U (see Example 1.3.3). Thus $E|_U$ is a vector bundle over U which we call the *restriction* of the vector bundle E to the open set U. Clearly, the differential structure of $E|_U$ coincides with the differential structure it inherits from E as an open subset.

EXAMPLE 1.5.4 (the tangent bundle). Let M be an n-dimensional differentiable manifold, let

$$TM = \bigcup_{x \in M} T_x M$$

denote its tangent bundle and let $\pi : TM \to M$ denote the standard projection that sends $T_x M$ to x, for all $x \in M$. For every $x \in M$, the fiber $T_x M$ has the structure of a real vector space isomorphic to $\mathbb{R}^n$. Let $\varphi : U \to \widetilde{U}$ be a local chart of M, where U is an open subset of M and $\widetilde{U}$ is an open subset of $\mathbb{R}^n$. For every $x \in U$ the map $\mathrm{d}\varphi(x)^{-1} : \mathbb{R}^n \to T_x M$ is a linear isomorphism and the map:

$$s^\varphi : U \ni x \longmapsto \mathrm{d}\varphi(x)^{-1} \in \mathrm{FR}(TM)$$

is a local section of $\mathrm{FR}(TM) \to M$. If $\psi : V \to \widetilde{V}$ is another local chart of M and if $\alpha = \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ denotes the transition map from ψ to φ then the transition map from s^φ to s^ψ is given by $U \cap V \ni x \mapsto \mathrm{d}\alpha\big(\psi(x)\big) \in \mathrm{GL}(\mathbb{R}^n)$ and therefore the set:

$$\big\{ s^\varphi : \varphi \text{ is a local chart of } M \big\} \tag{1.5.2}$$

is an atlas of local sections of $\mathrm{FR}(TM) \to M$. We endow $\mathrm{FR}(TM)$ with the unique maximal atlas of local sections of $\mathrm{FR}(TM) \to M$ containing (1.5.2) and then $\pi : TM \to M$ is a vector bundle over M with typical fiber $\mathbb{R}^n$.

EXAMPLE 1.5.5. Let P be a G-principal bundle over a differentiable manifold M, E_0 be a real finite-dimensional vector space and $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation of G on E_0. As explained in Example 1.4.5, the fibers of the associated bundle $P \times_G E_0$ have the structure of a real vector space isomorphic to E_0. In order to make $P \times_G E_0$ into a vector bundle over M with typical fiber E_0, we have to describe a maximal atlas of local sections of $\mathrm{FR}_{E_0}(P \times_G E_0)$. This is done as follows. Consider the map (recall (1.2.16)):

$$\mathfrak{H} : P \ni p \longmapsto \hat{p} \in \mathrm{FR}_{E_0}(P \times_G E_0). \tag{1.5.3}$$

Clearly $\mathfrak{H}$ is fiber-preserving and for each $x \in M$ it restricts to a morphism of principal spaces from P_x to $\mathrm{FR}_{E_0}(P_x \times_G E_0)$ whose subjacent group homomorphism is the representation $\rho : G \to \mathrm{GL}(E_0)$. By Lemma 1.3.11, there exists a unique maximal atlas of local sections of $\mathrm{FR}_{E_0}(P \times_G E_0) \to M$ that makes $\mathfrak{H}$ into a morphism of principal bundles. *We will always regard the associated bundle $P \times_G E_0$ as a vector bundle with $\mathrm{FR}_{E_0}(P \times_G E_0)$ endowed with the maximal atlas of local sections that makes $\mathfrak{H}$ a morphism of principal bundles.*

Observe that $P \times_G E_0$ has, in principle, *two* distinct differential structures: one that was defined in Section 1.4 for arbitrary associated bundles and the other that is assigned to the total space of vector bundles, i.e., the one for which the contraction map:

$$\mathcal{C}^{P \times_G E_0} : \mathrm{FR}_{E_0}(P \times_G E_0) \times E_0 \ni [\varrho, e_0] \longmapsto \varrho(e_0) \in P \times_G E_0 \tag{1.5.4}$$

is a smooth diffeomorphism. In order to check that these two differential structures coincide we endow $P \times_G E_0$ with the differential structure that makes $\mathcal{C}^{P\times_G E_0}$ into a smooth diffeomorphism and we show that for every smooth local section $s : U \to P$ the map $\hat{s}$ is a smooth diffeomorphism onto an open subset of $P \times_G E_0$ (recall that this is precisely the characterization of the differential structure of the total space of an associated bundle introduced in Section 1.4). Set $s_1 = \mathfrak{H} \circ s$, so that $s_1 : U \to \mathrm{FR}_{E_0}(P \times_G E_0)$ is a smooth local E_0-frame of the vector bundle $P \times_G E_0$. We claim that:

$$\hat{s} = \check{s}_1, \tag{1.5.5}$$

i.e., the local trivialization of the associated bundle $P \times_G E_0$ corresponding to the smooth local section s of P is equal to the local trivialization of the vector bundle $P \times_G E_0$ corresponding to the smooth local E_0-frame s_1. Namely, given $x \in U$, $e_0 \in E_0$ then:

$$\check{s}_1(x, e_0) = s_1(x) \cdot e_0 = \mathfrak{H}\big(s(x)\big) \cdot e_0 = \widehat{s(x)}(e_0) = [s(x), e_0] = \hat{s}(x, e_0).$$

Since the trivialization $\check{s}_1$ is a smooth diffeomorphism onto an open subset of the total space of the vector bundle $P \times_G E_0$, it follows from (1.5.5) that $\hat{s}$ is also a smooth diffeomorphism onto an open subset of $P \times_G E_0$. This concludes the proof that the two natural differential structures of $P \times_G E_0$ coincide. An alternative argument to prove the coincidence of these two differential structures of $P \times_G E_0$ is the following: we endow $P \times_G E_0$ with the differential structure defined in Section 1.4 and we show that the contraction map $\mathcal{C}^{P\times_G E_0}$ is a smooth diffeomorphism. Since $\mathfrak{H}$ is a morphism of principal bundles, we have an induced map:

$$\widehat{\mathfrak{H}} : P \times_G E_0 \ni [p, e_0] \longmapsto [\mathfrak{H}(p), e_0] \in \mathrm{FR}_{E_0}(P \times_G E_0) \times E_0.$$

By Lemma 1.4.10, $\widehat{\mathfrak{H}}$ is a smooth diffeomorphism. To conclude the proof that the contraction map $\mathcal{C}^{P\times_G E_0}$ is a smooth diffeomorphism, we show that $\mathcal{C}^{P\times_G E_0}$ is equal to the inverse of $\widehat{\mathfrak{H}}$. Since both $\mathcal{C}^{P\times_G E_0}$ and $\widehat{\mathfrak{H}}$ are bijective, it suffices to check that $\mathcal{C}^{P\times_G E_0} \circ \widehat{\mathfrak{H}}$ is the identity map of $P \times_G E_0$; given $p \in P$, $e_0 \in E_0$, we compute:

$$\begin{aligned}\mathcal{C}^{P\times_G E_0}\Big(\widehat{\mathfrak{H}}\big([p, e_0]\big)\Big) = \mathcal{C}^{P\times_G E_0}\big([\mathfrak{H}(p), e_0]\big) &= \mathcal{C}^{P\times_G E_0}\big([\hat{p}, e_0]\big) \\ &= \hat{p}(e_0) = [p, e_0].\end{aligned}$$

DEFINITION 1.5.6. Given $x \in M$ and $e \in E_x$ then the tangent space $T_e E_x$ is a subspace of $T_e E$ and it is called the *vertical space* of the vector bundle E at e; we write:

$$\mathrm{Ver}_e(E) = T_e E_x.$$

Clearly:

$$\mathrm{Ver}_e(E) = \mathrm{Ker}\big(\mathrm{d}\pi(e)\big).$$

Since for every $x \in M$ the fiber E_x is a real finite-dimensional vector space, we identify the tangent space $T_e E_x$ at a point $e \in E_x$ with E_x itself, i.e.:

$$\mathrm{Ver}_e(E) = T_e E_x \cong E_x.$$

For each $x \in M$, the contraction map $\mathcal{C}^E$ restricts to a linear isomorphism from $\mathrm{FR}_{E_0}(E_x) \underset{\sim}{\times} E_0$ to E_x and thus its differential at a point $[p, e_0]$ of $\mathrm{FR}_{E_0}(E_x) \underset{\sim}{\times} E_0$ restricts to a linear isomorphism from the vertical space $\mathrm{Ver}_{[p,e_0]}\big(\mathrm{FR}_{E_0}(E) \underset{\sim}{\times} E_0\big)$ to $\mathrm{Ver}_{p(e_0)}(E)$; recalling from (1.4.4) that the vertical space of $\mathrm{FR}_{E_0}(E) \underset{\sim}{\times} E_0$ at $[p, e_0]$ is identified with the fiber product $\mathrm{FR}_{E_0}(E_x) \underset{\sim}{\times} E_0$ then the restriction to the vertical space of the differential of $\mathcal{C}^E$ at $[p, e_0]$ is given by:

$$\mathcal{C}^{E_x} : \mathrm{FR}_{E_0}(E_x) \underset{\sim}{\times} E_0 \ni [p, e_0] \longmapsto p(e_0) \in E_x.$$

1.5.1. Local sections of a vector bundle. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0.

DEFINITION 1.5.7. By a *local section* of the vector bundle E we mean a map $\epsilon : U \to E$ defined on an open subset U of M such that $\pi \circ \epsilon$ is the inclusion map of U in M, i.e., such that $\epsilon(x) \in E_x$, for all $x \in U$.

If $\epsilon : U \to E$ is a local section of E and if $s : U \to \mathrm{FR}_{E_0}(E)$ is a smooth local E_0-frame of E then the map $\tilde{\epsilon} : U \to E_0$ defined by:

$$\tilde{\epsilon}(x) = s(x)^{-1} \cdot \epsilon(x) \in E_0,$$

for all $x \in U$ is called the *representation* of the section ϵ with respect to the smooth local E_0-frame s. If $\check{s}$ is the local trivialization of E corresponding to s then:

$$\epsilon(x) = \check{s}\big(x, \tilde{\epsilon}(x)\big),$$

for all $x \in U$; therefore the local section ϵ is smooth if and only if its representation $\tilde{\epsilon}$ is smooth.

A globally defined local section $\epsilon : M \to E$ of a vector bundle E will be called a *global section* (or just a *section*) of E. Notice that a local section $\epsilon : U \to E$ of E is the same as a global section of the restricted vector bundle $E|_U$. We denote by $\overline{\mathbf{\Gamma}}(E)$ the set of all sections of E and by $\mathbf{\Gamma}(E)$ the set of all smooth sections of E. Clearly $\overline{\mathbf{\Gamma}}(E)$ is a real vector space endowed with the obvious operations of pointwise addition and multiplication by scalars; moreover, $\overline{\mathbf{\Gamma}}(E)$ is a module over the ring $\mathbb{R}^M$ of all maps $f : M \to \mathbb{R}$. If $s : U \to \mathrm{FR}_{E_0}(E)$ is a smooth local E_0-frame of E then the map $\epsilon \to \tilde{\epsilon}$ that assigns to each section $\epsilon \in \mathbf{\Gamma}(E|_U)$ its representation $\tilde{\epsilon} : U \to E_0$ with respect to s is a linear isomorphism of real vector spaces and also an isomorphism of $\mathbb{R}^M$-modules. Since ϵ is smooth if and only if $\tilde{\epsilon}$ is smooth, it follows that $\mathbf{\Gamma}(E)$ is a subspace of $\overline{\mathbf{\Gamma}}(E)$; but it is obviously *not* an $\mathbb{R}^M$-submodule in general. Let $C^\infty(M)$ denote the set of all smooth maps $f : M \to \mathbb{R}$; clearly $C^\infty(M)$ is a subring of $\mathbb{R}^M$, $\overline{\mathbf{\Gamma}}(E)$ is a $C^\infty(M)$-module and $\mathbf{\Gamma}(E)$ is a $C^\infty(M)$-submodule of $\mathbf{\Gamma}(E)$.

EXAMPLE 1.5.8. Let M be a differentiable manifold. A (smooth) section of the tangent bundle TM is the same as a (smooth) vector field on M.

EXAMPLE 1.5.9. Let $\Pi : P \to M$ be a G-principal bundle, E_0 be a real finite-dimensional vector space and $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation. The associated bundle $P \times_G E_0$ is a vector bundle over M with typical fiber E_0 (recall Example 1.5.5) and the map $\mathfrak{H} : P \to \mathrm{FR}_{E_0}(P \times_G E_0)$ defined by (1.5.3) is a morphism of principal bundles whose subjacent Lie group homomorphism is ρ. If $s : U \to P$ is a smooth local section of P then the composition $\mathfrak{H} \circ s$ is a smooth local E_0-frame of $P \times_G E_0$. Let $\epsilon : U \to P \times_G E_0$ be a local section of the associated bundle $P \times_G E_0$. In Subsection 1.4.1 we have defined the notion of representation of ϵ with respect to s (recall (1.4.6)). It is easily seen that the map $\tilde{\epsilon} : U \to E_0$ that represents the local section ϵ of the vector bundle $P \times_G E_0$ with respect to the smooth local E_0-frame $\mathfrak{H} \circ s$ is the same as the map that represents the local section ϵ of the associated bundle $P \times_G E_0$ with respect to the smooth local section s of P.

EXAMPLE 1.5.10. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and consider the contraction map $\mathcal{C}^E : \mathrm{FR}_{E_0}(E) \times E_0 \to E$. If $\epsilon : U \to E$ is a local section of E then then $(\mathcal{C}^E)^{-1} \circ \epsilon$ is a local section of $\mathrm{FR}_{E_0}(E) \times E_0$ and

$$\big((\mathcal{C}^E)^{-1} \circ \epsilon\big)(x) = [s(x), \tilde{\epsilon}(x)],$$

for all $x \in U$. Notice that the representation of the local section $(\mathcal{C}^E)^{-1} \circ \epsilon$ of the associated bundle $\mathrm{FR}_{E_0}(E) \times E_0$ with respect to the smooth local section s of $\mathrm{FR}_{E_0}(E)$ (in the sense of Subsection 1.4.1) coincides with the representation of the local section ϵ of the vector bundle E with respect to the smooth local E_0-frame s of E.

1.5.2. Morphisms of vector bundles. We now define the natural morphisms of the category of vector bundles.

DEFINITION 1.5.11. Let E, F be vector bundles over the same differentiable manifold M. A map $L : E \to F$ is called *fiber preserving* if $L(E_x) \subset F_x$ for all $x \in M$; we set $L_x = L|_{E_x} : E_x \to F_x$. The map L is called *fiberwise linear* if L is fiber preserving and if L_x is a linear map, for all $x \in M$. A smooth fiberwise linear map $L : E \to F$ is called a *vector bundle morphism*.

Denote by E_0, F_0 the typical fibers of E and F, respectively and let s, s' be smooth local sections of $\mathrm{FR}_{E_0}(E)$ and $\mathrm{FR}_{F_0}(F)$ respectively, both defined in the same open subset U of M. If $L : E \to F$ is a fiberwise linear map then we set:

$$\widetilde{L}(x) = s'(x)^{-1} \circ L_x \circ s(x) \in \mathrm{Lin}(E_0, F_0),$$

for all $x \in U$, where $\mathrm{Lin}(E_0, F_0)$ denotes the space of all linear maps from E_0 to F_0. We call $\widetilde{L} : U \to \mathrm{Lin}(E_0, F_0)$ the *representation* of L with respect to s and s'. We have a commutative diagram:

$$\begin{tikzcd}
E|_U \arrow[r, "L"] & F|_U \\
U \times E_0 \arrow[u, "\check{s}", "\cong"'] \arrow[r, "(x,e_0)\mapsto(x,\widetilde{L}(x)\cdot e_0)"'] & U \times F_0 \arrow[u, "\check{s}'"', "\cong"]
\end{tikzcd}$$

in which the vertical arrows are smooth diffeomorphisms. Clearly the bottom arrow of the diagram is smooth if and only if the map $\widetilde{L}$ is smooth. It follows that:

- if L is a morphism of vector bundles then its representation $\widetilde{L}$ with respect to arbitrary smooth local sections s and s' is smooth;
- if L is a fiberwise linear map and if every point of M is contained in the domain U of a pair s, s' of smooth local sections for which the corresponding representation $\widetilde{L}$ is smooth then L is a morphism of vector bundles.

Let $L : E \to F$ be a morphism of vector bundles. Obviously L is bijective if and only if $L_x : E_x \to F_x$ is a linear isomorphism, for all $x \in M$. A bijective morphism of vector bundles $L : E \to F$ will be called an *isomorphism of vector bundles*. If $L : E \to F$ is an isomorphism of vector bundles then L is a smooth diffeomorphism and the map $L^{-1} : F \to E$ is also an isomorphism of vector bundles; namely, L^{-1} is clearly fiberwise linear and if $\widetilde{L}$ is the representation of L with respect to local sections s and s' then $x \mapsto \widetilde{L}(x)^{-1}$ is the representation of L^{-1} with respect to s and s'.

EXAMPLE 1.5.12. For any vector bundle $\pi : E \to M$, the contraction map $\mathcal{C}^E$ is obviously an isomorphism of vector bundles from $\mathrm{FR}_{E_0}(E) \underset{\approx}{\times} E_0$ onto E.

EXAMPLE 1.5.13. If $s : U \to \mathrm{FR}_{E_0}(E)$ is a smooth local E_0-frame of the vector bundle E then the local trivialization $\check{s} : U \times E_0 \to E|_U$ is an isomorphism of vector bundles from the trivial bundle $U \times E_0$ onto $E|_U$.

EXAMPLE 1.5.14. Let P be a G-principal bundle over a differentiable manifold M, E_0 be a real finite-dimensional vector space and $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation. If $s : U \to P$ is a smooth local section then the map $\hat{s} : U \times E_0 \to (P|_U) \times_G E_0$ (recall (1.4.2)) is a vector bundle isomorphism.[4] Notice that this example can also be seen as a particular case of Example 1.5.13. Namely, by (1.5.5), $\hat{s} = \check{s}_1$, where $s_1 = \mathfrak{H} \circ s$ and $\mathfrak{H} : P \to \mathrm{FR}_{E_0}(P \times_G E_0)$ is the morphism of principal bundles defined in (1.5.3).

Let us particularize Lemma 1.4.10 to the context of vector bundles.

LEMMA 1.5.15. *Let P, Q be principal bundles over the same differentiable manifold M with structural groups G and H, respectively. Let $\phi : P \to Q$ be a morphism of principal bundles and let $\phi_0 : G \to H$ denote its subjacent Lie group homomorphism. Given a real finite-dimensional vector space E_0 and a smooth representation $\rho : H \to \mathrm{GL}(E_0)$, we consider the smooth representation of G in E_0 given by $\rho \circ \phi_0 : G \to \mathrm{GL}(E_0)$. Then the induced map $\hat{\phi} : P \times_G E_0 \to Q \times_H E_0$ is an isomorphism of vector bundles.*

PROOF. The restriction of $\hat{\phi}$ to each fiber of $P \times_G E_0$ is a linear isomorphism (Example 1.2.33). Moreover, Lemma 1.4.10 implies that $\hat{\phi}$ is smooth. □

We also have a version of Lemma 1.4.11 for vector bundles.

[4]Recall from Example 1.5.5 that the differential structure of $P \times_G E_0$ that makes the map $\hat{s}$ a smooth diffeomorphism coincides with the differential structure that $P \times_G E_0$ has as the total space of a vector bundle.

LEMMA 1.5.16. *Let P, Q be principal bundles over the same differentiable manifold M with structural groups G and H, respectively. Let $\phi : P \to Q$ be a morphism of principal bundles and let $\phi_0 : G \to H$ denote its subjacent Lie group homomorphism. Let E_0, F_0 be real finite-dimensional vector spaces and let $\rho : G \to \mathrm{GL}(E_0)$, $\rho' : H \to \mathrm{GL}(F_0)$ be smooth representations. Assume that we are given a linear map $T_0 : E_0 \to F_0$ such that $T_0 \circ \rho(g) = \rho'\big(\phi_0(g)\big) \circ T_0$, for all $g \in G$. Then the induced map $\phi \underset{\sim}{\times} T_0 : P \times_G E_0 \to Q \times_H F_0$ is a vector bundle morphism.*

PROOF. Clearly $\phi \underset{\sim}{\times} T_0$ is fiber preserving and, by Example 1.2.33, $\phi \underset{\sim}{\times} T_0$ is fiberwise linear. Finally, Lemma 1.4.11 implies that $\phi \underset{\sim}{\times} T_0$ is smooth. $\square$

DEFINITION 1.5.17. Let P be a G-principal bundle over a differentiable manifold M, E_0 be a real finite-dimensional vector space, $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation, E be a vector bundle over M with typical fiber E_0 and $\phi : P \to \mathrm{FR}_{E_0}(E)$ be a morphism of principal bundles whose subjacent Lie group homomorphism is the representation ρ. We set:

$$\mathcal{C}^\phi = \mathcal{C}^E \circ \hat{\phi} : P \times_G E_0 \ni [p, e_0] \longmapsto \phi(p) \cdot e_0 \in E,$$

and we call $\mathcal{C}^\phi$ the *ϕ-contraction map*.

It follows from Lemma 1.5.15 and Example 1.5.12 that $\mathcal{C}^E_\phi$ is an isomorphism of vector bundles.

There is a relation between isomorphisms of vector bundles and isomorphisms of the corresponding principal bundles of frames. Let E, E' be vector bundles over a differentiable manifold M, with the same typical fiber E_0. Given a bijective fiberwise linear map $L : E \to E'$ then the map:

$$L_* : \mathrm{FR}_{E_0}(E) \ni p \longmapsto L \circ p \in \mathrm{FR}_{E_0}(E')$$

is fiber preserving and its restriction to each fiber is a morphism of principal spaces whose subjacent Lie group homomorphism is the identity (recall Examples 1.2.17 and 1.2.23). We call L_* the map *induced* by L on the frame bundles. We have the following:

LEMMA 1.5.18. *Let E, E' be vector bundles over the same differentiable manifold M, with the same typical fiber E_0. If $L : E \to E'$ is a bijective fiberwise linear map then L is smooth if and only if the induced map L_* is smooth; in other words, L is an isomorphism of vector bundles if and only if L_* is an isomorphism of principal bundles whose subjacent Lie group homomorphism is the identity.*

PROOF. Let $s : U \to \mathrm{FR}_{E_0}(E)$, $s' : U \to \mathrm{FR}_{E_0}(E')$ be smooth local sections and denote by $\widetilde{L} : U \to \mathrm{Lin}(E_0, E_0)$ the representation of L with respect to s and s'. Since L is an isomorphism of vector bundles, the map $\widetilde{L}$ takes values on the general linear group $\mathrm{GL}(E_0)$; we have:

$$(L_* \circ s)(x) = L_x \circ s(x) = s'(x) \circ \widetilde{L}(x),$$

for all $x \in U$. Since both s' and $\widetilde{L}$ are smooth, it follows that $L_* \circ s$ is a smooth local section of $\mathrm{FR}_{E_0}(E')$. Hence, by the result of Exercise 1.45, L_* is a morphism of principal bundles whose subjacent Lie group homomorphism is the identity.

Conversely, assume that L_* is an isomorphism of principal bundles whose subjacent Lie group homomorphism is the identity. We have an induced map:

$$\widehat{L_*} : \mathrm{FR}_{E_0}(E) \times_{\approx} E_0 \longrightarrow \mathrm{FR}_{E_0}(E') \times_{\approx} E_0$$

which is a smooth diffeomorphism, by Lemma 1.4.10. It is easily seen that the diagram:

$$\begin{tikzcd} \mathrm{FR}_{E_0}(E) \times_{\approx} E_0 \arrow[r, "\widehat{L_*}"] \arrow[d, "\mathcal{C}^E"'] & \mathrm{FR}_{E_0}(E') \times_{\approx} E_0 \arrow[d, "\mathcal{C}^{E'}"] \\ E \arrow[r, "L"'] & E' \end{tikzcd} \tag{1.5.6}$$

commutes. This proves that L is smooth. $\square$

EXAMPLE 1.5.19. If $s : U \to \mathrm{FR}_{E_0}(E)$ is a smooth local E_0-frame of a vector bundle E then the map $\beta_s : U \times \mathrm{GL}(E_0) \to \mathrm{FR}_{E_0}(E|_U)$ (recall (1.3.2)) is an isomorphism of principal bundles whose subjacent Lie group homomorphism is the identity map of $\mathrm{GL}(E_0)$ (see Example 1.3.10). Clearly $\beta_s = (\check{s})_*$.

1.5.3. Pull-back of vector bundles. Let $\pi : E \to M$ be a vector bundle over a differentiable manifold M with typical fiber E_0 and let $f : M' \to M$ be a smooth map defined on a differentiable manifold M'. The *pull-back* of E by f is the set f^*E defined by:

$$f^*E = \bigcup_{y \in M'} \big(\{y\} \times E_{f(y)}\big).$$

The set f^*E is a subset of the cartesian product $M' \times E$. The restriction to f^*E of the projection onto the first coordinate is a map $\pi_1 : f^*E \to M'$ and the restriction to f^*E of the projection onto the second coordinate is a map $\bar{f} : f^*E \to E$; we have a commutative diagram:

$$\begin{tikzcd} f^*E \arrow[r, "\bar{f}"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"'] & M \end{tikzcd}$$

For each $y \in M'$, the fiber $(f^*E)_y$ is equal to $\{y\} \times E_{f(y)}$; *we will identify the fiber $(f^*E)_y$ of f^*E with the fiber $E_{f(y)}$ of E.* Since every fiber of f^*E is a fiber of E, each fiber of f^*E is endowed with the structure of a real vector space isomorphic to E_0. The set $\mathrm{FR}_{E_0}(f^*E)$ can be naturally identified with the pull-back $f^*\mathrm{FR}_{E_0}(E)$; this identification makes $\mathrm{FR}_{E_0}(f^*E)$ into a $\mathrm{GL}(E_0)$-principal bundle and thus f^*E into a vector bundle with typical fiber E_0.

EXAMPLE 1.5.20. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0. If U is an open subset of M and $i : U \to M$ denotes the inclusion map then the pull-back i^*E can be identified with the restriction $E|_U$ (see Example 1.5.3); namely, by Example 1.3.17, we have $i^*\mathrm{FR}_{E_0}(E) = \mathrm{FR}_{E_0}(E)|_U = \mathrm{FR}_{E_0}(E|_U)$.

EXAMPLE 1.5.21. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0, $f : M' \to M$, $g : M'' \to M'$ be smooth maps, where M', M'' are differentiable manifolds. Both g^*f^*E and $(f \circ g)^*E$ are vector bundles over M''; there exists an obvious map $L : g^*f^*E \to (f \circ g)^*E$, which is the identity on each fiber. The corresponding map:

$$L_* : \mathrm{FR}_{E_0}(g^*f^*E) \longrightarrow \mathrm{FR}_{E_0}\big((f \circ g)^*E\big)$$

is the isomorphism of principal bundles considered in Example 1.3.24; thus, by Lemma 1.5.18, L is an isomorphism of vector bundles. We use such isomorphism to identify the vector bundles g^*f^*E and $(f \circ g)^*E$.

The following lemma is the analogue of Lemma 1.3.16 for vector bundles.

LEMMA 1.5.22. *Let $\pi : E \to M$ be a vector bundle with typical fiber E_0, M' be a differentiable manifold and $f : M' \to M$ be a smooth map. Denote by $\pi_1 : f^*E \to M'$ the pull-back of E by f. The map $(\pi_1, \bar{f}) : f^*E \to M' \times E$ is a smooth embedding whose image is the set of pairs $(y, e) \in M' \times E$ such that $f(y) = \pi(e)$. In particular, the map $\bar{f} : f^*E \to E$ is smooth.*

Notice that the map $(\pi_1, \bar{f})$ is just the inclusion map of f^*E into the cartesian product $M' \times E$.

PROOF. Clearly the image of $(\pi_1, \bar{f})$ consists of the pairs $(y, e) \in M' \times E$ such that $f(y) = \pi(e)$. To prove that $(\pi_1, \bar{f})$ is an embedding, we consider the commutative diagram:

$$\begin{tikzcd}
\mathrm{FR}_{E_0}(f^*E) \times_{\sim} E_0 \arrow[r, "\text{inclusion}"] \arrow[d, "\mathcal{C}^{f^*E}"'] & M' \times \big(\mathrm{FR}_{E_0}(E) \times_{\sim} E_0\big) \arrow[d, "\mathrm{Id}\times\mathcal{C}^{E}"] \\
f^*E \arrow[r, "(\pi_1,\bar{f})"'] & M' \times E
\end{tikzcd}$$

The vertical arrows of the diagram are smooth diffeomorphisms and the top arrow of the diagram is a smooth embedding, by Lemma 1.4.12. Hence $(\pi_1, \bar{f})$ is a smooth embedding. $\square$

COROLLARY 1.5.23 (universal property of the pull-back). *Under the conditions of Lemma 1.5.22, let X be a differentiable manifold and let $\phi_1 : X \to M'$, $\phi_2 : X \to E$ be maps with $\pi \circ \phi_2 = f \circ \phi_1$. Then there exists a unique map $\phi : X \to f^*E$ such that $\pi_1 \circ \phi = \phi_1$ and $\bar{f} \circ \phi = \phi_2$. The map ϕ is smooth.*

PROOF. The hypothesis $\pi \circ \phi_2 = f \circ \phi_1$ means that the image of the map $(\phi_1, \phi_2) : X \to M' \times E$ is contained in the image of the injective map $(\pi_1, \bar{f})$; thus there exists a unique map $\phi : X \to f^*E$ such that $(\pi_1, \bar{f}) \circ \phi = (\phi_1, \phi_2)$. Since $(\pi_1, \bar{f})$ is an embedding and (ϕ_1, ϕ_2) is smooth, it follows that ϕ is smooth. $\square$

DEFINITION 1.5.24. By a *local section of the vector bundle E along f* we mean a map $\epsilon : U' \to P$ defined on an open subset U' of M' satisfying the condition $\pi \circ \epsilon = f|_{U'}$.

EXAMPLE 1.5.25. If $\epsilon : U \to E$ is a local section of E then the composition $\epsilon \circ f : f^{-1}(U) \to E$ is a local section of E along f.

Given a local section $\epsilon : U' \to E$ of E along f there exists a unique local section $\bar{\epsilon} : U' \to f^*E$ of f^*E such that $\bar{f} \circ \bar{\epsilon} = \epsilon$; the following commutative diagram illustrates this situation:

$$
\begin{tikzcd}
f^*E \arrow[r, "\bar{f}"] & E \arrow[d, "\pi"] \\
U' \arrow[u, "\bar{\epsilon}"] \arrow[ur, "\epsilon"] \arrow[r, "f|_{U'}"'] & M
\end{tikzcd}
\tag{1.5.7}
$$

Thus, composition on the left with $\bar{f}$ induces a bijection between the set of local sections of f^*E and the set of local sections of E along f.

COROLLARY 1.5.26. *Under the conditions of Lemma 1.5.22, if $\epsilon : U' \to E$ is a smooth local section of E along f then the unique local section $\bar{\epsilon} : U' \to f^*E$ of f^*E such that $\bar{f} \circ \bar{\epsilon} = \epsilon$ is also smooth.*

PROOF. Apply Corollary 1.5.23 with $X = U'$, ϕ_1 the inclusion map of U' in M' and $\phi_2 = \epsilon$. The map ϕ given by the thesis of Corollary 1.5.23 is precisely $\bar{\epsilon}$. $\square$

Corollary 1.5.26 tells us that composition on the left with $\bar{f}$ induces a bijection between the set of smooth local sections of f^*E and the set of smooth local sections of E along f.

EXAMPLE 1.5.27. Let M', M be differentiable manifolds and $f : M' \to M$ be a smooth map. Denote by $\pi : TM \to M$, $\pi' : TM' \to M$ the projections. Applying the universal property of pull-backs (Corollary 1.5.23) with $X = TM'$, $\phi_1 = \pi'$, $\phi_2 = \mathrm{d}f : TM \to TM'$ and $E = TM$, we obtain a smooth map $\overleftarrow{\mathrm{d}f} : TM' \to f^*TM$ such that $\bar{f} \circ \overleftarrow{\mathrm{d}f} = \mathrm{d}f$ and $\pi_1 \circ \overleftarrow{\mathrm{d}f} = \pi'$. Clearly, $\overleftarrow{\mathrm{d}f}$ is a morphism of vector bundles. More generally, given vector bundles $\pi : E \to M$, $\pi' : E' \to M'$ and smooth maps $L : E' \to E$, $f : M' \to M$ such that the diagram:

$$
\begin{tikzcd}
E' \arrow[r, "L"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\
M' \arrow[r, "f"'] & M
\end{tikzcd}
$$

commutes then the universal property of pull-backs gives us a smooth map $\overline{L} : E' \to f^*E$ such that $\bar{f} \circ \overline{L} = L$ and $\pi_1 \circ \overline{L} = \pi'$. If for all $y \in M'$, the restriction $L|_{E'_y} : E'_y \to E_{f(y)}$ is linear then $\overline{L}$ is a morphism of vector bundles.

1.5.4. Vector subbundles. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0, F_0 be a subspace of E_0 and F be a subset of E such that for every $x \in M$, the set $F_x = F \cap E_x$ is a subspace of the fiber E_x having the same dimension as F_0. Given $x \in M$, then an E_0-frame $p \in \mathrm{FR}_{E_0}(E_x)$ is said to be *adapted* to (F_0, F) if p is adapted to (F_0, F_x), i.e., if $p(F_0) = F_x$ (recall Example 1.1.10). Consider the set:

$$\mathrm{FR}_{E_0}(E; F_0, F) = \bigcup_{x \in M} \mathrm{FR}_{E_0}(E_x; F_0, F_x)$$

of all E_0-frames of E adapted to (F_0, F). For each $x \in M$, the set $\mathrm{FR}_{E_0}(E_x; F_0, F_x)$ is a principal subspace of $\mathrm{FR}_{E_0}(E_x)$ whose structural group is the Lie subgroup $\mathrm{GL}(E_0; F_0)$ of $\mathrm{GL}(E_0)$.

DEFINITION 1.5.28. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0. A subset $F \subset E$ is called a *vector subbundle* if there exists a subspace F_0 of E_0 such that:

(a) for each $x \in M$, the set $F_x = F \cap E_x$ is a subspace of E_x having the same dimension as F_0;
(b) $\mathrm{FR}_{E_0}(E; F_0, F)$ is a principal subbundle of $\mathrm{FR}_{E_0}(E)$ with structural group $\mathrm{GL}(E_0; F_0)$.

Condition (b) in Definition 1.5.28 means that every point of M belongs to the domain U of a smooth local E_0-frame $s : U \to \mathrm{FR}_{E_0}(E)$ of E such that $s(x)$ maps F_0 to F_x, for all $x \in U$.

REMARK 1.5.29. If F_0 is a subspace of E_0 such that conditions (a) and (b) in Definition 1.5.28 are satisfied then every subspace F_0' of E_0 having the same dimension as F_0 satisfies conditions (a) and (b). Namely, let $g \in \mathrm{GL}(E_0)$ be a linear isomorphism of E_0 such that $g(F_0') = F_0$. The map $\gamma_g : \mathrm{FR}_{E_0}(E) \to \mathrm{FR}_{E_0}(E)$ is an isomorphism of principal bundles whose subjacent Lie group isomorphism is the inner automorphism $\mathcal{I}_{g^{-1}}$ of $\mathrm{GL}(E_0)$ (Exercise 1.44); since:

$$\gamma_g\big(\mathrm{FR}_{E_0}(E; F_0, F)\big) = \mathrm{FR}_{E_0}(E; F_0', F),$$

it follows that if $\mathrm{FR}_{E_0}(E; F_0, F)$ is a principal subbundle of $\mathrm{FR}_{E_0}(E)$ with structural group $\mathrm{GL}(E_0; F_0)$ then $\mathrm{FR}_{E_0}(E; F_0', F)$ is a principal subbundle of $\mathrm{FR}_{E_0}(E)$ with structural group $\mathcal{I}_{g^{-1}}\big(\mathrm{GL}(E; F_0)\big) = \mathrm{GL}(E; F_0')$ (Exercise 1.47). It follows that if F is a vector subbundle of E and if F_0 is a subspace of E_0 such that condition (a) in Definition 1.5.28 is satisfied then also condition (b) is satisfied.

Let us now show how a vector subbundle F of a vector bundle $\pi : E \to M$ can be regarded as a vector bundle in its own right. Let F_0 be a subspace of the typical fiber E_0 of E such that condition (a) in Definition 1.5.28 is satisfied (by Remark 1.5.29, also condition (b) is then satisfied). First of all, the projection $\pi : E \to M$ restricts to a map $\pi|_F : F \to M$ and for all $x \in M$ the fiber F_x is endowed with the structure of a real vector space isomorphic to the real finite-dimensional vector space F_0. In order to make F a vector bundle over M with typical fiber F_0, we have to describe a maximal atlas of local sections of

$\mathrm{FR}_{F_0}(F) \to M$. If $p \in \mathrm{FR}_{E_0}(E; F_0, F)$ is an E_0-frame of E adapted to (F_0, F) then $p|_{F_0}$ is an F_0-frame of F; we have therefore a map:

$$\mathrm{FR}_{E_0}(E; F_0, F) \ni p \longmapsto p|_{F_0} \in \mathrm{FR}_{F_0}(F) \tag{1.5.8}$$

that is fiber preserving and whose restriction to each fiber is a morphism of principal spaces whose subjacent Lie group homomorphism is:

$$\mathrm{GL}(E_0; F_0) \ni T \longmapsto T|_{F_0} \in \mathrm{GL}(F_0). \tag{1.5.9}$$

Thus, by Lemma 1.3.11, there exists a unique maximal atlas of local sections of $\mathrm{FR}_{F_0}(F)$ such that (1.5.8) is a morphism of principal bundles. *We will always consider a vector subbundle to be endowed with the structure of vector bundle described above*.

PROPOSITION 1.5.30. *If E is a vector bundle and F is a vector subbundle of E then F is an embedded submanifold of E and the differential structure of F (as a total space of a vector bundle) coincides with the differential structure it inherits from E as an embedded submanifold. In particular, the inclusion map of F in E is smooth and hence a morphism of vector bundles.*

PROOF. Denote by F_0, E_0 the typical fibers of F and E, respectively. If $\mathfrak{r}$ denotes the morphism of principal bundles (1.5.8) then by Lemma 1.4.10 the map:

$$\hat{\mathfrak{r}} : \mathrm{FR}_{E_0}(E; F_0, F) \underset{\sim}{\times} F_0 \longrightarrow \mathrm{FR}_{F_0}(F) \underset{\sim}{\times} F_0$$

is a smooth diffeomorphism, where we consider the smooth representation of the structural group $\mathrm{GL}(E_0; F_0)$ of $\mathrm{FR}_{E_0}(E; F_0, F)$ on F_0 given by (1.5.9). If i denotes the inclusion map of $\mathrm{FR}_{E_0}(E; F_0, F)$ in $\mathrm{FR}_{E_0}(E)$ and if i_0 denotes the inclusion map of F_0 in E_0 then the map:

$$i \underset{\sim}{\times} i_0 : \mathrm{FR}_{E_0}(E; F_0, F) \underset{\sim}{\times} F_0 \longrightarrow \mathrm{FR}_{E_0}(E) \underset{\sim}{\times} E_0$$

is a smooth embedding, by Lemma 1.4.11. It is easy to see that the diagram:

$$\begin{tikzcd}
\mathrm{FR}_{F_0}(F) \underset{\sim}{\times} F_0 \arrow[r, "(i \underset{\sim}{\times} i_0) \circ \hat{\mathfrak{r}}^{-1}"] \arrow[d, "\mathcal{C}^F"', "\cong"] & \mathrm{FR}_{E_0}(E) \underset{\sim}{\times} E_0 \arrow[d, "\cong"', "\mathcal{C}^E"] \\
F \arrow[r, "\text{inclusion}"'] & E
\end{tikzcd}$$

commutes. Thus, the inclusion map of F in E is a smooth embedding. $\square$

PROPOSITION 1.5.31. *Let E, E' be vector bundles over the same differentiable manifold M and let $L : E \to E'$ be a morphism of vector bundles. Then:*

(a) *if L is injective then its image $L(E)$ is a vector subbundle of E';*
(b) *if L is surjective then its* kernel $\mathrm{Ker}(L) = \bigcup_{x \in M} \mathrm{Ker}(L_x)$ *is a vector subbundle of E.*

To prove Proposition 1.5.31, we need the following:

LEMMA 1.5.32. *Let E_0, E_0' be real finite-dimensional vector spaces, M be a differentiable manifold, $\widetilde{L} : M \to \mathrm{Lin}(E_0, E_0')$ be a smooth map and $x_0 \in M$ be a fixed point.*

(a) *Assume that $\widetilde{L}(x_0)$ is injective. Given a subspace F_0 of E'_0 having the same dimension as E_0 then there exists a smooth map $g : U \to \mathrm{GL}(E'_0)$ defined in an open neighborhood U of x_0 in M such that the linear isomorphism $g(x) : E'_0 \to E'_0$ carries F_0 to the image of $\widetilde{L}(x)$, for all $x \in U$.*

(b) *Assume that $\widetilde{L}(x_0)$ is surjective. Given a subspace F_0 of E_0 with $\dim(F_0) = \dim(E_0) - \dim(E'_0)$ then there exists a smooth map $g : U \to \mathrm{GL}(E_0)$ defined in an open neighborhood U of x_0 in M such that the linear isomorphism $g(x) : E_0 \to E_0$ carries F_0 to the kernel of $\widetilde{L}(x)$, for all $x \in U$.*

PROOF. Let us prove (a). Choose a subspace Z of E'_0 such that $E'_0 = \widetilde{L}(x_0)(E_0) \oplus Z$. For each $x \in M$, let $\bar{g}(x) : E_0 \oplus Z \to E'_0$ be the linear map such that $\bar{g}(x)|_{E_0}$ equals $\widetilde{L}(x)$ and $\bar{g}(x)|_Z$ equals the inclusion. Then $\bar{g} : M \to \mathrm{Lin}(E_0 \oplus Z, E'_0)$ is a smooth map and $\bar{g}(x_0)$ is a linear isomorphism; thus, there exists an open neighborhood U of x_0 in M such that $\bar{g}(x)$ is a linear isomorphism, for all $x \in U$. Since F_0 has the same dimension as E_0, there exists a linear isomorphism $T : E'_0 \to E_0 \oplus Z$ with $T(F_0) = E_0 \oplus \{0\}$. Setting $g(x) = \bar{g}(x) \circ T$, for all $x \in U$, then:

$$g(x)(F_0) = \bar{g}(x)\big(E_0 \oplus \{0\}\big) = \widetilde{L}(x)(E_0).$$

This concludes the proof of (a). Now let us prove (b). Choose a subspace Z of E_0 such that $E_0 = \mathrm{Ker}\big(\widetilde{L}(x_0)\big) \oplus Z$; denote by $\mathfrak{p} : E_0 \to \mathrm{Ker}\big(\widetilde{L}(x_0)\big)$ the projection onto the first coordinate corresponding to such direct sum decomposition. For each $x \in M$, let $\bar{g}(x) : E_0 \to E'_0 \oplus \mathrm{Ker}\big(\widetilde{L}(x_0)\big)$ be the linear map $\bar{g}(x) = \big(\widetilde{L}(x), \mathfrak{p}\big)$. Then:

$$\bar{g} : M \longrightarrow \mathrm{Lin}\Big(E_0, E'_0 \oplus \mathrm{Ker}\big(\widetilde{L}(x_0)\big)\Big)$$

is a smooth map and $\bar{g}(x_0)$ is a linear isomorphism; thus, there exists an open neighborhood U of x_0 in M such that $\bar{g}(x)$ is a linear isomorphism, for all $x \in U$. It is easy to see that:

$$\bar{g}(x)\big[\mathrm{Ker}\big(\widetilde{L}(x)\big)\big] = \{0\} \oplus \mathrm{Ker}\big(\widetilde{L}(x_0)\big),$$

for all $x \in U$. Now, since F_0 has the same dimension as $\mathrm{Ker}\big(\widetilde{L}(x_0)\big)$, there exists a linear isomorphism $T : E_0 \to E'_0 \oplus \mathrm{Ker}\big(\widetilde{L}(x_0)\big)$ such that $T(F_0) = \{0\} \oplus \mathrm{Ker}\big(\widetilde{L}(x_0)\big)$. Setting $g(x) = \bar{g}(x)^{-1} \circ T$ for all $x \in U$ then $g : U \to \mathrm{GL}(E_0)$ is a smooth map and:

$$g(x)(F_0) = \mathrm{Ker}\big(\widetilde{L}(x)\big),$$

for all $x \in U$. This concludes the proof. □

PROOF OF PROPOSITION 1.5.31. Denote by E_0, E'_0 respectively the typical fibers of E and E'. Let us prove (a). Assume that L is injective and let F_0 be a subspace of E'_0 having the same dimension as E_0. Given $x_0 \in M$, we have to find a smooth local section of $\mathrm{FR}_{E'_0}(E')$ defined in an open neighborhood of x_0 in M with image contained in $\mathrm{FR}_{E'_0}\big(E'; F_0, L(E)\big)$. Let $s : V \to \mathrm{FR}_{E_0}(E)$, $s' : V \to \mathrm{FR}_{E'_0}(E')$ be smooth local sections, defined in the same open neighborhood

V of x_0 in M; denote by $\widetilde{L} : V \to \mathrm{Lin}(E_0, E_0')$ the representation of L with respect to s and s'. Since $\widetilde{L}(x_0)$ is injective, Lemma 1.5.32 gives us a smooth map $g : U \to \mathrm{GL}(E_0')$ defined in an open neighborhood U of x_0 in V such that the linear isomorphism $g(x) : E_0' \to E_0'$ carries F_0 to the image of $\widetilde{L}(x)$, for all $x \in U$. Clearly, the smooth local section:

$$U \ni x \longmapsto s'(x) \circ g(x) \in \mathrm{FR}_{E_0'}(E')$$

of $\mathrm{FR}_{E_0'}(E')$ has image contained in $\mathrm{FR}_{E_0'}\big(E'; F_0, L(E)\big)$. This proves (a). Let us prove (b). Assume that L is surjective and let F_0 be a subspace of E_0 with $\dim(F_0) = \dim(E_0) - \dim(E_0')$. Given $x_0 \in M$, we have to find a smooth local section of $\mathrm{FR}_{E_0}(E)$ defined in an open neighborhood of x_0 in M with image contained in $\mathrm{FR}_{E_0}\big(E; F_0, \mathrm{Ker}(L)\big)$. As before, let $s : V \to \mathrm{FR}_{E_0}(E)$, $s' : V \to \mathrm{FR}_{E_0'}(E')$ be smooth local sections, defined in the same open neighborhood V of x_0 in M; denote by $\widetilde{L} : V \to \mathrm{Lin}(E_0, E_0')$ the representation of L with respect to s and s'. Since $\widetilde{L}(x_0)$ is surjective, Lemma 1.5.32 gives us a smooth map $g : U \to \mathrm{GL}(E_0')$ defined in an open neighborhood U of x_0 in V such that the linear isomorphism $g(x) : E_0 \to E_0$ carries F_0 to the kernel of $\widetilde{L}(x)$, for all $x \in U$. Clearly, the smooth local section:

$$U \ni x \longmapsto s(x) \circ g(x) \in \mathrm{FR}_{E_0}(E)$$

of $\mathrm{FR}_{E_0}(E)$ has image contained in $\mathrm{FR}_{E_0}\big(E; F_0, \mathrm{Ker}(L)\big)$. This concludes the proof of (b). $\square$

DEFINITION 1.5.33. Let M be a differentiable manifold. By a *distribution* on M we mean a subset $\mathcal{D}$ of TM such that for all $x \in M$, $\mathcal{D}_x = \mathcal{D} \cap T_xM$ is a subspace of the tangent space T_xM. By a *smooth distribution* on M we mean a vector subbundle $\mathcal{D}$ of the tangent bundle TM.

1.6. Functorial constructions with vector bundles

Given an integer $n \geq 1$, we denote by $\underline{\mathfrak{Vec}}^n$ the category whose objects are n-tuples $(V_i)_{i=1}^n$ of real finite-dimensional vector spaces and whose morphisms from $(V_i)_{i=1}^n$ to $(W_i)_{i=1}^n$ are n-tuples $(T_i)_{i=1}^n$ of linear isomorphisms $T_i : V_i \to W_i$. We set $\underline{\mathfrak{Vec}} = \underline{\mathfrak{Vec}}^1$. A functor $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ is called *smooth* if for any object $(V_i)_{i=1}^n$ of $\underline{\mathfrak{Vec}}^n$ the map:

$$\underline{\mathfrak{F}} : \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n) \longrightarrow \mathrm{GL}\big(\underline{\mathfrak{F}}(V_1, \ldots, V_n)\big) \tag{1.6.1}$$

is smooth. Observe that (1.6.1) is a Lie group homomorphism; its differential at the identity is a Lie algebra homomorphism that will be denoted by:

$$\underline{\mathfrak{f}} : \mathfrak{gl}(V_1) \oplus \cdots \oplus \mathfrak{gl}(V_n) \longrightarrow \mathfrak{gl}\big(\underline{\mathfrak{F}}(V_1, \ldots, V_n)\big). \tag{1.6.2}$$

We call $\underline{\mathfrak{f}}$ the *differential* of the smooth functor $\underline{\mathfrak{F}}$.

Let $\pi : E \to M$ be a vector bundle with typical fiber E_0. Given a smooth functor $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ we set:

$$\underline{\mathfrak{F}}(E) = \bigcup_{x \in M} \underline{\mathfrak{F}}(E_x), \tag{1.6.3}$$

where the union in (1.6.3) is understood to be disjoint[5]; we have an obviously defined projection map $\underline{\mathfrak{F}}(E) \to M$ that sends $\underline{\mathfrak{F}}(E_x)$ to x, for all $x \in M$. For each $x \in M$, the fiber $\underline{\mathfrak{F}}(E_x)$ of $\underline{\mathfrak{F}}(E)$ over x has the structure of a real vector space having the same dimension as $\underline{\mathfrak{F}}(E_0)$. In order to make $\underline{\mathfrak{F}}(E)$ into a vector bundle with typical fiber $\underline{\mathfrak{F}}(E_0)$, we will describe a maximal atlas of local sections of $\mathrm{FR}_{\underline{\mathfrak{F}}(E_0)}\big(\underline{\mathfrak{F}}(E)\big) \to M$. The map:

$$\underline{\mathfrak{F}} : \mathrm{FR}_{E_0}(E) \ni p \longmapsto \underline{\mathfrak{F}}(p) \in \mathrm{FR}_{\underline{\mathfrak{F}}(E_0)}\big(\underline{\mathfrak{F}}(E)\big) \tag{1.6.4}$$

is fiber preserving and its restriction to each fiber is a morphism of principal spaces whose subjacent Lie group homomorphism is:

$$\mathrm{GL}(E_0) \ni T \longmapsto \underline{\mathfrak{F}}(T) \in \mathrm{GL}\big(\underline{\mathfrak{F}}(E_0)\big). \tag{1.6.5}$$

Thus, Lemma 1.3.11 gives us a unique maximal atlas of local sections of $\mathrm{FR}_{\underline{\mathfrak{F}}(E_0)}\big(\underline{\mathfrak{F}}(E)\big)$ that makes (1.6.4) into a morphism of principal bundles. *We will always consider $\underline{\mathfrak{F}}(E)$ to be endowed with the vector bundle structure described above.*

Notice that if $s : U \to \mathrm{FR}_{E_0}(E)$ is a (smooth) local E_0-frame of E then $\underline{\mathfrak{F}} \circ s$ is a (smooth) local $\underline{\mathfrak{F}}(E_0)$-frame of $\underline{\mathfrak{F}}(E)$; we call $\underline{\mathfrak{F}} \circ s$ the local frame *induced* by s on $\underline{\mathfrak{F}}(E)$.

REMARK 1.6.1. Let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ be a smooth functor and $\pi : E \to M$ be a vector bundle with typical fiber E_0. Since (1.6.4) is a morphism of principal bundles whose subjacent Lie group homomorphism is the representation (1.6.5), we are in the situation described in Definition 1.5.17 and thus we have the following isomorphism of vector bundles:

$$\mathcal{C}^{\underline{\mathfrak{F}}} = \mathcal{C}^{\underline{\mathfrak{F}}(E)} \circ \widehat{\underline{\mathfrak{F}}} : \mathrm{FR}_{E_0}(E) \times_{\underline{\mathfrak{F}}} \underline{\mathfrak{F}}(E_0) \ni [p, \mathfrak{e}] \longmapsto \underline{\mathfrak{F}}(p) \cdot \mathfrak{e} \in \underline{\mathfrak{F}}(E).$$

EXAMPLE 1.6.2. If $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ is the identity functor then for every vector bundle E the vector bundle $\underline{\mathfrak{F}}(E)$ coincides with E itself. For any object V of $\underline{\mathfrak{Vec}}$, the map $\underline{\mathfrak{f}}$ is the identity map of $\mathfrak{gl}(V)$.

EXAMPLE 1.6.3. Let Z be a fixed real finite-dimensional vector space and consider the *constant functor* $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ that sends any object V of $\underline{\mathfrak{Vec}}$ to Z and any linear isomorphism $T : V \to W$ to the identity map of Z. For any object V of $\underline{\mathfrak{Vec}}$, the map $\underline{\mathfrak{f}} : \mathfrak{gl}(V) \to \mathfrak{gl}(Z)$ is the identically zero map. Given a vector bundle E over a differentiable manifold M with typical fiber E_0 then $\underline{\mathfrak{F}}(E)$ is the trivial vector bundle $M \times Z$ (recall Example 1.5.2); namely, if $\mathrm{FR}_Z(M \times Z) = M \times \mathrm{GL}(Z)$ is endowed with the structure of a trivial $\mathrm{GL}(Z)$-principal bundle (see Example 1.3.2) then the map:

$$\underline{\mathfrak{F}} : \mathrm{FR}_{E_0}(E) \ni p \longmapsto \big(\Pi(p), \mathrm{Id}\big) \in M \times \mathrm{GL}(Z) = \mathrm{FR}_Z(M \times Z)$$

is a morphism of principal bundles.

Now, a less trivial example.

[5] If the union is not disjoint, one can always replace $\underline{\mathfrak{F}}(E_x)$ with $\{x\} \times \underline{\mathfrak{F}}(E_x)$, or else modify the functor $\underline{\mathfrak{F}}$ so that $\underline{\mathfrak{F}}(V)$ is replaced with $\{V\} \times \underline{\mathfrak{F}}(V)$, for every real finite-dimensional vector space V.

EXAMPLE 1.6.4. Let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ be the functor that sends V to the dual space V^* and a linear isomorphism $T : V \to W$ to $T^{*-1} : V^* \to W^*$, where $T^* : W^* \to V^*$ denotes the transpose map of T. The functor $\underline{\mathfrak{F}}$ is clearly smooth and for any object V of $\underline{\mathfrak{Vec}}$, the map $\underline{\mathfrak{f}}$ is given by:

$$\underline{\mathfrak{f}} : \mathfrak{gl}(V) \ni X \longmapsto -X^* \in \mathfrak{gl}(V^*).$$

Given a vector bundle E then the vector bundle $\underline{\mathfrak{F}}(E)$ is denoted by E^* and it is called the *dual bundle* of E. If $E = TM$ is the tangent bundle of the differentiable manifold M then the dual bundle TM^* is also called the *cotangent bundle* of M.

EXAMPLE 1.6.5. Let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ be the functor that sends V to the space $\mathrm{Lin}(V)$ of linear endomorphisms of V and a linear isomorphism $T : V \to W$ to the linear isomorphism:

$$\mathcal{I}_T : \mathrm{Lin}(V) \ni L \longmapsto T \circ L \circ T^{-1} \in \mathrm{Lin}(W).$$

The functor $\underline{\mathfrak{F}}$ is clearly smooth and for any object V of $\underline{\mathfrak{Vec}}$, the map:

$$\underline{\mathfrak{f}} : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}\big(\mathrm{Lin}(V)\big)$$

is given by:

$$\underline{\mathfrak{f}}(X) \cdot L = [X, L] = X \circ L - L \circ X,$$

for all $X \in \mathfrak{gl}(V)$ and all $L \in \mathrm{Lin}(V)$. Given a vector bundle E then the vector bundle $\underline{\mathfrak{F}}(E)$ will be denoted by $\mathrm{Lin}(E)$.

Given vector spaces $V_1, \ldots, V_k, W$, we denote by $\mathrm{Lin}(V_1, \ldots, V_k; W)$ the space of k-linear maps $B : V_1 \times \cdots \times V_k \to W$; by $\mathrm{Lin}_k(V, W)$ we denote the space of k-linear maps $B : V \times \cdots \times V \to W$. By $\mathrm{Lin}_k^{\mathrm{s}}(V, W)$ (resp., $\mathrm{Lin}_k^{\mathrm{a}}(V, W)$) we denote the subspace of $\mathrm{Lin}_k(V, W)$ consisting of *symmetric* (resp., *alternating*) k-linear maps.

EXAMPLE 1.6.6. Let $k \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ be the functor that sends V to $\mathrm{Lin}_k(V, \mathbb{R})$ and a linear isomorphism $T : V \to W$ to the linear isomorphism:

$$\mathrm{Lin}_k(V, \mathbb{R}) \ni B \longmapsto B(T^{-1}\cdot, \ldots, T^{-1}\cdot) \in \mathrm{Lin}_k(W, \mathbb{R}).$$

The functor $\underline{\mathfrak{F}}$ is clearly smooth and for any object V of $\underline{\mathfrak{Vec}}$, the map:

$$\underline{\mathfrak{f}} : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}\big(\mathrm{Lin}_k(V, \mathbb{R})\big)$$

is given by:

$$\underline{\mathfrak{f}}(X) \cdot L = -L(X\cdot, \cdot, \ldots, \cdot) - L(\cdot, X\cdot, \ldots, \cdot) - \cdots - L(\cdot, \cdot, \ldots, X\cdot),$$

for all $X \in \mathfrak{gl}(V)$ and all $L \in \mathrm{Lin}_k(V, \mathbb{R})$. Given a vector bundle E then the vector bundle $\underline{\mathfrak{F}}(E)$ is denoted by $\mathrm{Lin}_k(E, \mathbb{R})$. If M is a differentiable manifold then a section of $\mathrm{Lin}_k(TM, \mathbb{R})$ is called a *covariant k-tensor field* on M.

EXAMPLE 1.6.7. By replacing Lin_k with $\mathrm{Lin}_k^{\mathrm{s}}$ or $\mathrm{Lin}_k^{\mathrm{a}}$ throughout Example 1.6.6 we obtain vector bundles $\mathrm{Lin}_k^{\mathrm{s}}(E, \mathbb{R})$, $\mathrm{Lin}_k^{\mathrm{a}}(E, \mathbb{R})$. The sections of $\mathrm{Lin}_k^{\mathrm{s}}(TM, \mathbb{R})$ are called *symmetric covariant k-tensor fields* on M and the sections of $\mathrm{Lin}_k^{\mathrm{a}}(TM, \mathbb{R})$ are called *k-forms* or *differential forms of degree k* on M.

EXAMPLE 1.6.8. Let Z be a fixed real finite-dimensional vector space. By replacing $\mathbb{R}$ with Z throughout Examples 1.6.6 and 1.6.7 we obtain vector bundles $\mathrm{Lin}_k(E,Z)$, $\mathrm{Lin}_k^{\mathrm{s}}(E,Z)$ and $\mathrm{Lin}_k^{\mathrm{a}}(E,Z)$. The sections of $\mathrm{Lin}_k(TM,Z)$ (resp., $\mathrm{Lin}_k^{\mathrm{s}}(TM,Z)$) are called *$Z$-valued covariant k-tensor fields* (resp., *symmetric Z-valued covariant k-tensor fields*) on M; the sections of $\mathrm{Lin}_k^{\mathrm{a}}(TM,Z)$ are called *Z-valued k-forms* on M.

Let us now generalize the construction described above to the case of smooth functors of several variables. Let $n \ge 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Given vector bundles $E^1, \ldots, E^n$ over a differentiable manifold M with typical fibers $E_0^1, \ldots, E_0^n$, respectively, we set:

$$\underline{\mathfrak{F}}(E^1,\ldots,E^n) = \bigcup_{x\in M} \underline{\mathfrak{F}}(E_x^1,\ldots,E_x^n),$$

where the union is understood to be disjoint. We have an obviously defined projection $\underline{\mathfrak{F}}(E^1,\ldots,E^n) \to M$ that sends $\underline{\mathfrak{F}}(E_x^1,\ldots,E_x^n)$ to x, for all $x \in M$; for each x in M the fiber $\underline{\mathfrak{F}}(E_x^1,\ldots,E_x^n)$ has the structure of a real finite-dimensional vector space having the same dimension as $\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)$. The fiberwise product $\mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^n}(E^n)$ is a principal bundle over M with structural group $\mathrm{GL}(E_0^1) \times \cdots \times \mathrm{GL}(E_0^n)$; the map:

$$\begin{aligned} \mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^n}(E^n) &\xrightarrow{\ \underline{\mathfrak{F}}\ } \mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)}\big(\underline{\mathfrak{F}}(E^1,\ldots,E^n)\big) \\ (p_1,\ldots,p_n) &\longmapsto \underline{\mathfrak{F}}(p_1,\ldots,p_n) \end{aligned} \tag{1.6.6}$$

is fiber preserving and its restriction to each fiber is a morphism of principal spaces whose subjacent Lie group homomorphism is:

$$\begin{aligned} \mathrm{GL}(E_0^1) \times \cdots \times \mathrm{GL}(E_0^n) &\longrightarrow \mathrm{GL}\big(\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)\big) \\ (T_1,\ldots,T_n) &\longmapsto \underline{\mathfrak{F}}(T_1,\ldots,T_n). \end{aligned} \tag{1.6.7}$$

Lemma 1.3.11 gives us a unique maximal atlas of local sections of

$$\mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)}\big(\underline{\mathfrak{F}}(E^1,\ldots,E^n)\big) \longrightarrow M$$

that makes (1.6.6) into a morphism of principal bundles. *We will always consider $\underline{\mathfrak{F}}(E^1,\ldots,E^n)$ to be endowed with the vector bundle structure described above.*

If $s^i : U \to \mathrm{FR}_{E_0^i}(E^i)$ is a (smooth) local E_0^i-frame of E^i, $i = 1,\ldots,n$, then $\underline{\mathfrak{F}} \circ (s^1,\ldots,s^n)$ is a (smooth) local $\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)$-frame of the vector bundle $\underline{\mathfrak{F}}(E^1,\ldots,E^n)$; we call $\underline{\mathfrak{F}} \circ (s^1,\ldots,s^n)$ the frame *induced* by $s^1, \ldots, s^n$ on $\underline{\mathfrak{F}}(E^1,\ldots,E^n)$.

REMARK 1.6.9. Let $E^1, \ldots, E^n$ be vector bundles over a differentiable manifold M with typical fibers $E_0^1, \ldots, E_0^n$ respectively, and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Since (1.6.6) is a morphism of principal bundles whose subjacent Lie group homomorphism is the representation (1.6.7), we are in the situation described in Definition 1.5.17 and thus we have the following isomorphism of vector bundles:

$$\begin{tikzcd}
\big(\mathrm{FR}_{E_0^1}(E^1)\star\cdots\star\mathrm{FR}_{E_0^n}(E^n)\big)\times\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n) \arrow[d, "\mathcal{C}^{\underline{\mathfrak{F}}}"] \\
\underline{\mathfrak{F}}(E^1,\ldots,E^n)
\end{tikzcd}\tag{1.6.8}$$

which is given by:

$$\mathcal{C}^{\underline{\mathfrak{F}}}=\mathcal{C}^{\underline{\mathfrak{F}}(E^1,\ldots,E^n)}\circ\widehat{\underline{\mathfrak{F}}}:[(p_1,\ldots,p_n),\mathfrak{e}]\longmapsto\underline{\mathfrak{F}}(p_1,\ldots,p_n)\cdot\mathfrak{e}.$$

EXAMPLE 1.6.10. Let M be a differentiable manifold, E_0^1, $\ldots$, E_0^n be real finite-dimensional vector spaces and consider the trivial vector bundles:

$$E^i=M\times E_0^i,\quad i=1,\ldots,n.$$

If $\underline{\mathfrak{F}}:\underline{\mathfrak{Vec}}^n\to\underline{\mathfrak{Vec}}$ is a smooth functor then $\underline{\mathfrak{F}}(E^1,\ldots,E^n)$ can be identified as a set with the trivial vector bundle $M\times\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)$. Let us show that $\underline{\mathfrak{F}}(E^1,\ldots,E^n)$ is a trivial vector bundle, i.e., such identification is a vector bundle isomorphism. To this aim, we look at the corresponding principal bundles of frames. The principal bundle of $\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)$-frames of $\underline{\mathfrak{F}}(E^1,\ldots,E^n)$ can be identified as a set with the trivial principal bundle:

$$M\times\mathrm{GL}\big(\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)\big).$$

We have to check that such identification is an isomorphism of principal bundles. This follows from the following two observations; first (see Exercise 1.56), the fiberwise product:

$$\mathrm{FR}_{E_0^1}(E^1)\star\cdots\star\mathrm{FR}_{E_0^n}(E^n)=\big(M\times\mathrm{GL}(E_0^1)\big)\star\cdots\star\big(M\times\mathrm{GL}(E_0^n)\big)$$

is identified as a principal bundle with the trivial principal bundle:

$$M\times\big(\mathrm{GL}(E_0^1)\times\cdots\times\mathrm{GL}(E_0^n)\big).$$

Second, the map (1.6.6) can be identified with the product of the identity map of M by the map (1.6.7) so that (1.6.6) is smooth when

$$\mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)}\big(\underline{\mathfrak{F}}(E^1,\ldots,E^n)\big)$$

is identified with the trivial principal bundle $M\times\mathrm{GL}\big(\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)\big)$.

EXAMPLE 1.6.11. Let $\underline{\mathfrak{F}}:\underline{\mathfrak{Vec}}^2\to\underline{\mathfrak{Vec}}$ be the functor that sends an object (V_1,V_2) to $V_1\oplus V_2$ and a morphism (T_1,T_2) to $T_1\oplus T_2$. The functor $\underline{\mathfrak{F}}$ is smooth and for any object (V_1,V_2) of $\underline{\mathfrak{Vec}}^2$, the map:

$$\underline{\mathfrak{f}}:\mathfrak{gl}(V_1)\oplus\mathfrak{gl}(V_2)\longrightarrow\mathfrak{gl}(V_1\oplus V_2)$$

is given by:

$$\underline{\mathfrak{f}}(X_1,X_2)=X_1\oplus X_2,$$

for all $X_1\in\mathfrak{gl}(V_1)$, $X_2\in\mathfrak{gl}(V_2)$. Given vector bundles E^1, E^2 over a differentiable manifold M then the vector bundle $\underline{\mathfrak{F}}(E^1,E^2)$ will be denoted by $E^1\oplus E^2$ is will be called the *Whitney sum* of E^1 and E^2.

EXAMPLE 1.6.12. Let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^2 \to \underline{\mathfrak{Vec}}$ be the functor that sends (V_1, V_2) to $\mathrm{Lin}(V_1, V_2)$ and (T_1, T_2) to:

$$\mathrm{Lin}(V_1, V_2) \ni L \longmapsto T_2 \circ L \circ T_1^{-1} \in \mathrm{Lin}(W_1, W_2),$$

where $T_1 : V_1 \to W_1$ and $T_2 : V_2 \to W_2$ are linear isomorphisms. The functor $\underline{\mathfrak{F}}$ is smooth and for any object (V_1, V_2) of $\underline{\mathfrak{Vec}}^2$, the map:

$$\underline{\mathfrak{f}} : \mathfrak{gl}(V_1) \oplus \mathfrak{gl}(V_2) \longrightarrow \mathfrak{gl}\big(\mathrm{Lin}(V_1, V_2)\big)$$

is given by:

$$\underline{\mathfrak{f}}(X_1, X_2) \cdot L = X_2 \circ L - L \circ X_1,$$

for all $X_1 \in \mathfrak{gl}(V_1)$, $X_2 \in \mathfrak{gl}(V_2)$ and all $L \in \mathrm{Lin}(V_1, V_2)$. Given vector bundles E^1, E^2 over M, the vector bundle $\underline{\mathfrak{F}}(E^1, E^2)$ will be denoted by $\mathrm{Lin}(E^1, E^2)$. A fiberwise linear map $L : E^1 \to E^2$ can be identified with a section $x \mapsto L_x$ of the vector bundle $\mathrm{Lin}(E^1, E^2)$. If $s^i : U \to \mathrm{FR}_{E_0^i}(E^i)$ is a smooth local E_0^i-frame of E^i, $i = 1, 2$, and if s denotes the frame of $\mathrm{Lin}(E^1, E^2)$ induced by s^1 and s^2 then the representation of the fiberwise linear map L with respect to s^1 and s^2 is equal to the representation of the section $x \mapsto L_x$ with respect to s. It follows that L is a vector bundle morphism if and only if $x \mapsto L_x$ is a smooth section of $\mathrm{Lin}(E^1, E^2)$. *From now on we will systematically identify vector bundle morphisms from E^1 to E^2 with smooth sections of* $\mathrm{Lin}(E^1, E^2)$.

EXAMPLE 1.6.13. Let $k \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^{k+1} \to \underline{\mathfrak{Vec}}$ be the functor that sends $(V_1, \ldots, V_k, W)$ to $\mathrm{Lin}(V_1, \ldots, V_k; W)$ and that sends linear isomorphisms $T_i : V_i \to V_i'$, $i = 1, \ldots, k$, $T : W \to W'$ to the linear isomorphism:

$$\begin{aligned} \mathrm{Lin}(V_1, \ldots, V_k; W) &\longrightarrow \mathrm{Lin}(V_1', \ldots, V_k'; W') \\ B &\longmapsto T \circ B(T_1^{-1}\cdot, \ldots, T_k^{-1}\cdot). \end{aligned}$$

The functor $\underline{\mathfrak{F}}$ is smooth and for any object $(V_1, \ldots, V_k, W)$ of $\underline{\mathfrak{Vec}}^{k+1}$ the map:

$$\underline{\mathfrak{f}} : \mathfrak{gl}(V_1) \oplus \cdots \oplus \mathfrak{gl}(V_k) \oplus \mathfrak{gl}(W) \longrightarrow \mathfrak{gl}\big(\mathrm{Lin}(V_1, \ldots, V_k; W)\big)$$

is given by:

$$\begin{aligned} \underline{\mathfrak{f}}(X_1, \ldots, X_k, X) \cdot L = X \circ L(\cdot, \ldots, \cdot) - L(X_1\cdot, \ldots, \cdot) - \cdots \\ - L(\cdot, \ldots, X_k\cdot), \end{aligned}$$

for all $X_i \in \mathfrak{gl}(V_i)$, $i = 1, \ldots, k$, $X \in \mathfrak{gl}(W)$, $L \in \mathrm{Lin}(V_1, \ldots, V_k; W)$. Given vector bundles $E^1, \ldots, E^k, F$ over M, we will denote the vector bundle $\underline{\mathfrak{F}}(E^1, \ldots, E^k, F)$ by $\mathrm{Lin}(E^1, \ldots, E^k; F)$. When $E^1 = \cdots = E^k = E$, we write $\mathrm{Lin}_k(E, F)$ rather than $\mathrm{Lin}(E^1, \ldots, E^k; F)$. Sections of the vector bundle $\mathrm{Lin}_k(TM, F)$ are called *F-valued covariant k-tensor fields* on M.

EXAMPLE 1.6.14. Let $k \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^2 \to \underline{\mathfrak{Vec}}$ be the functor that sends (V_1, V_2) to $\mathrm{Lin}_k^{\mathrm{s}}(V_1, V_2)$ and (T_1, T_2) to:

$$\mathrm{Lin}_k^{\mathrm{s}}(V_1, V_2) \ni B \longmapsto T_2 \circ B(T_1^{-1}\cdot, \ldots, T_1^{-1}\cdot) \in \mathrm{Lin}_k^{\mathrm{s}}(W_1, W_2),$$

where $T_1 : V_1 \to W_1$, $T_2 : V_2 \to W_2$ are linear isomorphisms. The functor $\underline{\mathfrak{F}}$ is smooth and for any object (V_1, V_2) of $\underline{\mathfrak{Vec}}^2$, the map:

$$\underline{\mathfrak{f}} : \mathfrak{gl}(V_1) \oplus \mathfrak{gl}(V_2) \longrightarrow \mathfrak{gl}\big(\mathrm{Lin}_k^{\mathrm{s}}(V_1, V_2)\big)$$

is given by:

$$\underline{\mathfrak{f}}(X_1, X_2) \cdot L = X_2 \circ L(\cdot, \ldots, \cdot) - L(X_1\cdot, \ldots, \cdot) - \cdots - L(\cdot, \ldots, X_1\cdot),$$

for all $X_1 \in \mathfrak{gl}(V_1)$, $X_2 \in \mathfrak{gl}(V_2)$ and all $L \in \mathrm{Lin}_k^{\mathrm{s}}(V_1, V_2)$. Given vector bundles E, F over M, the vector bundle $\underline{\mathfrak{F}}(E, F)$ will be denoted by $\mathrm{Lin}_k^{\mathrm{s}}(E, F)$. An analogous construction replacing $\mathrm{Lin}_k^{\mathrm{s}}$ with $\mathrm{Lin}_k^{\mathrm{a}}$ gives us the vector bundle $\mathrm{Lin}_k^{\mathrm{a}}(E, F)$. The sections of $\mathrm{Lin}_k^{\mathrm{s}}(TM, F)$ are called *symmetric F-valued covariant k-tensor fields* on M and the sections of $\mathrm{Lin}_k^{\mathrm{a}}(TM, F)$ are called *F-valued k-forms* on M.

CONVENTION. *From now on, when describing smooth functors we will only specify their actions on objects and leave as an exercise to the reader the task of describing their actions on morphisms.*

PROPOSITION 1.6.15. *Let $m, n \geq 1$ be fixed and let:*

$$\underline{\mathfrak{F}} = (\underline{\mathfrak{F}}^1, \ldots, \underline{\mathfrak{F}}^n) : \underline{\mathfrak{Vec}}^m \longrightarrow \underline{\mathfrak{Vec}}^n, \quad \underline{\mathfrak{G}} : \underline{\mathfrak{Vec}}^n \longrightarrow \underline{\mathfrak{Vec}}$$

be smooth functors[6]*; consider the smooth functor $\underline{\mathfrak{G}} \circ \underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^m \to \underline{\mathfrak{Vec}}$. Given vector bundles $E^1, \ldots, E^m$ over a differentiable manifold M then:*

$$(\underline{\mathfrak{G}} \circ \underline{\mathfrak{F}})(E^1, \ldots, E^m) = \underline{\mathfrak{G}}\big(\underline{\mathfrak{F}}^1(E^1, \ldots, E^m), \ldots, \underline{\mathfrak{F}}^n(E^1, \ldots, E^m)\big). \tag{1.6.9}$$

PROOF. Clearly both sides of (1.6.9) are equal as sets; we have to check that the principal bundle structure of their corresponding principal bundles of frames are also the same. Denote by E_0^i the typical fiber of the vector bundle E^i, $i = 1, \ldots, m$, by $F^j = \underline{\mathfrak{F}}^j(E_0^1, \ldots, E_0^m)$ the typical fiber of $\underline{\mathfrak{F}}^j(E^1, \ldots, E^m)$, $j = 1, \ldots, n$ and by $G = \underline{\mathfrak{G}}(F^1, \ldots, F^n)$ the typical fiber of $(\underline{\mathfrak{G}} \circ \underline{\mathfrak{F}})(E^1, \ldots, E^m)$. For each $j = 1, \ldots, n$, let $\mathrm{FR}_{F^j}\big(\underline{\mathfrak{F}}^j(E^1, \ldots, E^m)\big)$ be endowed with the unique principal fiber bundle structure that makes the map:

$$\underline{\mathfrak{F}}^j : \mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^m}(E^m) \longrightarrow \mathrm{FR}_{F^j}\big(\underline{\mathfrak{F}}^j(E^1, \ldots, E^m)\big) \tag{1.6.10}$$

a morphism of principal bundles and let $\mathrm{FR}_G\big((\underline{\mathfrak{G}} \circ \underline{\mathfrak{F}})(E^1, \ldots, E^m)\big)$ be endowed with the unique principal bundle structure that makes the map:

$$\begin{array}{c} \mathrm{FR}_{F^1}\big(\underline{\mathfrak{F}}^1(E^1, \ldots, E^m)\big) \star \cdots \star \mathrm{FR}_{F^n}\big(\underline{\mathfrak{F}}^n(E^1, \ldots, E^m)\big) \\ \Big\downarrow{\scriptstyle \underline{\mathfrak{G}}} \\ \mathrm{FR}_G\big((\underline{\mathfrak{G}} \circ \underline{\mathfrak{F}})(E^1, \ldots, E^m)\big) \end{array} \tag{1.6.11}$$

a morphism of principal bundles. To conclude the proof, we have to verify that the map:

$$\underline{\mathfrak{G}} \circ \underline{\mathfrak{F}} : \mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^m}(E^m) \longrightarrow \mathrm{FR}_G\big((\underline{\mathfrak{G}} \circ \underline{\mathfrak{F}})(E^1, \ldots, E^m)\big) \tag{1.6.12}$$

---

[6]The smoothness of $\underline{\mathfrak{F}}$ means that every $\underline{\mathfrak{F}}^i$ is smooth.

is a morphism of principal bundles. This follows from the universal property of the fiberwise product of principal bundles (Corollary 1.3.27) and from the fact that the composition of morphisms of principal bundles is a morphism of principal bundles (see Exercise 1.43). $\square$

PROPOSITION 1.6.16. *Let $n \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Let E^1, $\overline{E}^1$, …, E^n, $\overline{E}^n$ be vector bundles over a differentiable manifold M and $L^i : E^i \to \overline{E}^i$, $i = 1, \ldots, n$, be vector bundle isomorphisms. The map:*

$$\underline{\mathfrak{F}}(L^1, \ldots, L^n) : \underline{\mathfrak{F}}(E^1, \ldots, E^n) \longrightarrow \underline{\mathfrak{F}}(\overline{E}^1, \ldots, \overline{E}^n)$$

whose restriction to the fiber $\underline{\mathfrak{F}}(E^1_x, \ldots, E^n_x)$ is equal to $\underline{\mathfrak{F}}(L^1_x, \ldots, L^n_x)$, for all $x \in M$ is a vector bundle isomorphism.

PROOF. Clearly $\underline{\mathfrak{F}}(L^1, \ldots, L^n)$ is fiber preserving, fiberwise linear and bijective. For $i = 1, \ldots, n$, denote by E^i_0 (resp., by $\overline{E}^i_0$) the typical fiber of E^i (resp., of $\overline{E}^i$). Let s^1, $\bar{s}^1$, …, s^n, $\bar{s}^n$, be smooth local sections respectively of the principal bundles of frames $\mathrm{FR}_{E^1_0}(E^1)$, $\mathrm{FR}_{\overline{E}^1_0}(\overline{E}^1)$, …, $\mathrm{FR}_{E^n_0}(E^n)$, $\mathrm{FR}_{\overline{E}^n_0}(\overline{E}^n)$, all defined in the same open subset U of M. Set:

$$s = \underline{\mathfrak{F}} \circ (s^1, \ldots, s^n), \quad \bar{s} = \underline{\mathfrak{F}} \circ (\bar{s}^1, \ldots, \bar{s}^n),$$

so that s is a smooth local section of $\mathrm{FR}_{\underline{\mathfrak{F}}(E^1_0,\ldots,E^n_0)}\big(\underline{\mathfrak{F}}(E^1, \ldots, E^n)\big)$ and $\bar{s}$ is a smooth local section of $\mathrm{FR}_{\underline{\mathfrak{F}}(\overline{E}^1_0,\ldots,\overline{E}^n_0)}\big(\underline{\mathfrak{F}}(\overline{E}^1, \ldots, \overline{E}^n)\big)$. For $i = 1, \ldots, n$, let:

$$\widetilde{L}^i : U \longrightarrow \mathrm{Lin}(E^i_0, \overline{E}^i_0)$$

denote the representation of L^i with respect to s^i and $\bar{s}^i$ (see Subsection 1.5.2); since each L^i is a morphism of vector bundles, the maps $\widetilde{L}^i$ are smooth. Since each L^i is a vector bundle isomorphism, the map $\widetilde{L}^i$ actually takes values in the set $\mathrm{Iso}(E^i_0, \overline{E}^i_0)$ of linear isomorphisms from E^i_0 to $\overline{E}^i_0$. It is easy to see that the representation of L with respect to s and $\bar{s}$ is equal to the composition of the map $(\widetilde{L}^1, \ldots, \widetilde{L}^n)$ with the map:

$$\begin{array}{c} \mathrm{Iso}(E^1_0, \overline{E}^1_0) \times \cdots \times \mathrm{Iso}(E^n_0, \overline{E}^n_0) \\ \Big\downarrow \underline{\mathfrak{F}} \\ \mathrm{Iso}\big(\underline{\mathfrak{F}}(E^1_0, \ldots, E^n_0), \underline{\mathfrak{F}}(\overline{E}^1_0, \ldots, \overline{E}^n_0)\big) \end{array}$$

Such map is smooth (see Exercise 1.66) and hence the representation of L with respect to s and $\bar{s}$ is smooth. This concludes the proof. $\square$

EXAMPLE 1.6.17. Let $n \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Let $E^1, \ldots, E^n$ be vector bundles over the differentiable manifold M with typical fibers E^1_0, …, E^n_0, respectively. Let s^1, …, s^n be smooth local sections of the principal bundles $\mathrm{FR}_{E^1_0}(E^1)$, …, $\mathrm{FR}_{E^n_0}(E^n)$ respectively, defined in an open subset U of M. If $L^i = \check{s}^i : U \times E_0^i \to E^i|_U$ denotes the smooth local trivialization corresponding to s^i then L^i is a vector bundle isomorphism and:

$$\underline{\mathfrak{F}}(L^1, \ldots, L^n) = \check{s},$$

where $s = \underline{\mathfrak{F}} \circ (s^1, \ldots, s^n) : U \to \mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1, \ldots, E_0^n)}\big(\underline{\mathfrak{F}}(E^1, \ldots, E^n)\big)$.

Let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Given vector bundles $E^1, \ldots, E^n$ over a differentiable manifold M and a smooth map $f : M' \to M$ defined in a differentiable manifold M' then there exists an obvious bijective map:

$$f^*\underline{\mathfrak{F}}(E^1, \ldots, E^n) \longrightarrow \underline{\mathfrak{F}}(f^*E^1, \ldots, f^*E^n). \tag{1.6.13}$$

We have the following:

PROPOSITION 1.6.18. *Let $n \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Given vector bundles $E^1, \ldots, E^n$ over a differentiable manifold M and a smooth map $f : M' \to M$ defined in a differentiable manifold M' then the map* (1.6.13) *is an isomorphism of vector bundles.*

PROOF. The map (1.6.13) induces a map:

$$\begin{array}{c} \mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1, \ldots, E_0^n)}\big(f^*\underline{\mathfrak{F}}(E^1, \ldots, E^n)\big) \\ \big\downarrow \\ \mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1, \ldots, E_0^n)}\big(\underline{\mathfrak{F}}(f^*E^1, \ldots, f^*E^n)\big) \end{array} \tag{1.6.14}$$

as in the statement of Lemma 1.5.18; the map (1.6.14) is fiber preserving and its restriction to each fiber is a morphism of principal spaces whose subjacent group homomorphism is the identity. We have to show that (1.6.14) is an isomorphism of principal bundles; in fact, by the result of Exercise 1.46, it suffices to show that (1.6.14) is a morphism of principal bundles. Recall from Subsection 1.5.3 that:

$$\mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1, \ldots, E_0^n)}\big(f^*\underline{\mathfrak{F}}(E^1, \ldots, E^n)\big) = f^*\mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1, \ldots, E_0^n)}\big(\underline{\mathfrak{F}}(E^1, \ldots, E^n)\big).$$

By considering the pull-back by f of the morphism of principal bundles (1.6.6) (recall Example 1.3.23) we obtain a morphism of principal bundles:

$$\begin{array}{c} f^*\big(\mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^n}(E^n)\big) \\ \big\downarrow {\scriptstyle f^*\underline{\mathfrak{F}}} \\ f^*\mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1, \ldots, E_0^n)}\big(\underline{\mathfrak{F}}(E^1, \ldots, E^n)\big) \end{array}$$

Using the isomorphism of principal bundles described in Lemma 1.3.29 we identify the principal bundles:

$$f^*\big(\mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^n}(E^n)\big) \tag{1.6.15}$$

and:

$$\big(f^*\mathrm{FR}_{E_0^1}(E^1)\big)\star\cdots\star\big(f^*\mathrm{FR}_{E_0^n}(E^n)\big) = \mathrm{FR}_{E_0^1}(f^*E^1)\star\cdots\star\mathrm{FR}_{E_0^n}(f^*E^n). \tag{1.6.16}$$

We have a commutative diagram:

(1.6.17)

$$\begin{tikzcd}
\mathrm{FR}_{E_0^1}(f^*E^1)\star\cdots\star\mathrm{FR}_{E_0^n}(f^*E^n) \arrow[d, "f^*\underline{\mathfrak{F}}"'] \arrow[dr, "\underline{\mathfrak{F}}"] & \\
& \mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)}\big(\underline{\mathfrak{F}}(f^*E^1,\ldots,f^*E^n)\big) \\
\mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)}\big(f^*\underline{\mathfrak{F}}(E^1,\ldots,E^n)\big) \arrow[ur, "(1.6.14)"] &
\end{tikzcd}$$

To conclude that (1.6.14) is a morphism of principal bundles, simply apply Corollary 1.3.12 to such commutative diagram. □

1.6.1. Smooth natural transformations.

DEFINITION 1.6.19. Let $n\ge 1$ be fixed and let $\underline{\mathfrak{F}}$, $\underline{\mathfrak{G}}$ be smooth functors from $\underline{\mathfrak{Vec}}^n$ to $\underline{\mathfrak{Vec}}$. By a *smooth natural transformation* from $\underline{\mathfrak{F}}$ to $\underline{\mathfrak{G}}$ we mean a rule $\mathfrak{N}$ that associates to each object $(V_1,\ldots,V_n)$ of $\underline{\mathfrak{Vec}}^n$ an open subset $\mathrm{Dom}(\mathfrak{N}_{V_1,\ldots,V_n})$ of $\underline{\mathfrak{F}}(V_1,\ldots,V_n)$ and a smooth map:

$$\mathfrak{N}_{V_1,\ldots,V_n}:\mathrm{Dom}(\mathfrak{N}_{V_1,\ldots,V_n})\longrightarrow\underline{\mathfrak{G}}(V_1,\ldots,V_n)$$

in such a way that given objects $(V_1,\ldots,V_n)$, $(W_1,\ldots,W_n)$ of $\underline{\mathfrak{Vec}}^n$ and a morphism $(T_1,\ldots,T_n)$ from $(V_1,\ldots,V_n)$ to $(W_1,\ldots,W_n)$ then:

(a) $\underline{\mathfrak{F}}(T_1,\ldots,T_n)\big(\mathrm{Dom}(\mathfrak{N}_{V_1,\ldots,V_n})\big)=\mathrm{Dom}(\mathfrak{N}_{W_1,\ldots,W_n})$;
(b) the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Dom}(\mathfrak{N}_{V_1,\ldots,V_n}) \arrow[r, "\mathfrak{N}_{V_1,\ldots,V_n}"] \arrow[d, "\underline{\mathfrak{F}}(T_1,\ldots,T_n)"'] & \underline{\mathfrak{G}}(V_1,\ldots,V_n) \arrow[d, "\underline{\mathfrak{G}}(T_1,\ldots,T_n)"] \\
\mathrm{Dom}(\mathfrak{N}_{W_1,\ldots,W_n}) \arrow[r, "\mathfrak{N}_{W_1,\ldots,W_n}"'] & \underline{\mathfrak{G}}(W_1,\ldots,W_n)
\end{tikzcd}$$

A smooth natural transformation $\mathfrak{N}$ from $\underline{\mathfrak{F}}$ to $\underline{\mathfrak{G}}$ is said to be *linear* if for every object $(V_1,\ldots,V_n)$ of $\underline{\mathfrak{Vec}}^n$, we have:

$$\mathrm{Dom}(\mathfrak{N}_{V_1,\ldots,V_n})=\underline{\mathfrak{F}}(V_1,\ldots,V_n)$$

and the map $\mathfrak{N}_{V_1,\ldots,V_n}:\underline{\mathfrak{F}}(V_1,\ldots,V_n)\to\underline{\mathfrak{G}}(V_1,\ldots,V_n)$ is linear.

EXAMPLE 1.6.20. Consider the smooth functors $\underline{\mathfrak{F}}$, $\underline{\mathfrak{G}}^i$, $i=1,2$, from $\underline{\mathfrak{Vec}}^2$ to $\underline{\mathfrak{Vec}}$ defined by:

$$\underline{\mathfrak{F}}(V_1,V_2)=V_1\oplus V_2,\quad \underline{\mathfrak{G}}^i(V_1,V_2)=V_i,\ i=1,2.$$

The rule that assigns to each object (V_1, V_2) of $\underline{\mathfrak{Vec}}^2$ the map:

$$\mathfrak{N}^i_{V_1,V_2} : V_1 \oplus V_2 \ni (v_1, v_2) \longmapsto v_i \in V_i,$$

is a linear smooth natural transformation from $\underline{\mathfrak{F}}$ to $\underline{\mathfrak{G}}^i$, $i = 1, 2$.

EXAMPLE 1.6.21. If $\underline{\mathfrak{F}}$, $\underline{\mathfrak{G}}^i$ are as in Example 1.6.20 then the rules that assign to each object (V_1, V_2) of $\underline{\mathfrak{Vec}}^2$ the maps:

$$\mathfrak{N}^1_{V_1,V_2} : V_1 \ni v \longmapsto (v, 0) \in V_1 \oplus V_2,$$
$$\mathfrak{N}^2_{V_1,V_2} : V_2 \ni v \longmapsto (0, v) \in V_1 \oplus V_2,$$

are linear smooth natural transformations from $\underline{\mathfrak{G}}^1$ to $\underline{\mathfrak{F}}$ and from $\underline{\mathfrak{G}}^2$ to $\underline{\mathfrak{F}}$, respectively.

EXAMPLE 1.6.22. Consider the smooth functors $\underline{\mathfrak{F}}$, $\underline{\mathfrak{G}}$ from $\underline{\mathfrak{Vec}}^2$ to $\underline{\mathfrak{Vec}}$ defined by:

$$\underline{\mathfrak{F}}(V_1, V_2) = \mathrm{Lin}(V_1, V_2), \quad \underline{\mathfrak{G}}(V_1, V_2) = \mathrm{Lin}(V_2^*, V_1^*).$$

The rule that assigns to each object (V_1, V_2) of $\underline{\mathfrak{Vec}}^2$ the map:

$$\mathfrak{N}_{V_1,V_2} : \mathrm{Lin}(V_1, V_2) \ni T \longmapsto T^* \in \mathrm{Lin}(V_2^*, V_1^*)$$

is a linear smooth natural transformation from $\underline{\mathfrak{F}}$ to $\underline{\mathfrak{G}}$.

EXAMPLE 1.6.23. Consider the smooth functors $\underline{\mathfrak{F}}$, $\underline{\mathfrak{G}}$ from $\underline{\mathfrak{Vec}}^3$ to $\underline{\mathfrak{Vec}}$ defined by:

$$\underline{\mathfrak{F}}(V_1, V_2, V_3) = \mathrm{Lin}(V_2, V_3) \oplus \mathrm{Lin}(V_1, V_2),$$
$$\underline{\mathfrak{G}}(V_1, V_2, V_3) = \mathrm{Lin}(V_1, V_3).$$

The rule that assigns to each object (V_1, V_2, V_3) of $\underline{\mathfrak{Vec}}^3$ the map:

$$\mathfrak{N}_{V_1,V_2,V_3} : \mathrm{Lin}(V_2, V_3) \oplus \mathrm{Lin}(V_1, V_2) \ni (T, T') \longmapsto T \circ T' \in \mathrm{Lin}(V_1, V_3)$$

is a smooth natural transformation from $\underline{\mathfrak{F}}$ to $\underline{\mathfrak{G}}$.

EXAMPLE 1.6.24. Let $k \geq 1$ be fixed and consider the smooth functors $\underline{\mathfrak{F}}$, $\underline{\mathfrak{G}}$ from $\underline{\mathfrak{Vec}}^{k+1}$ to $\underline{\mathfrak{Vec}}$ defined by:

$$\underline{\mathfrak{F}}(V_1, \ldots, V_{k+1}) = \mathrm{Lin}(V_1, \ldots, V_k; V_{k+1}) \oplus V_1 \oplus \cdots \oplus V_k,$$
$$\underline{\mathfrak{G}}(V_1, \ldots, V_{k+1}) = V_{k+1}.$$

The rule that assigns to each object $(V_1, \ldots, V_{k+1})$ of $\underline{\mathfrak{Vec}}^{k+1}$ the map $\mathfrak{N}_{V_1,\ldots,V_{k+1}}$ defined by:

$$\begin{aligned} \mathrm{Lin}(V_1, \ldots, V_k; V_{k+1}) \oplus V_1 \oplus \cdots \oplus V_k &\longrightarrow V_{k+1} \\ (B, v_1, \ldots, v_k) &\longmapsto B(v_1, \ldots, v_k) \end{aligned}$$

is a smooth natural transformation from $\underline{\mathfrak{F}}$ to $\underline{\mathfrak{G}}$.

EXAMPLE 1.6.25. Let $k \geq 1$ be fixed and consider the smooth functors $\underline{\mathfrak{F}}$ from $\underline{\mathfrak{Vec}}^{k+2}$ to $\underline{\mathfrak{Vec}}$ defined by:

$$\underline{\mathfrak{F}}(V_1, \ldots, V_{k+2}) = \mathrm{Lin}(V_{k+1}, V_{k+2}) \oplus \mathrm{Lin}(V_1, \ldots, V_k; V_{k+1}),$$
$$\underline{\mathfrak{G}}(V_1, \ldots, V_{k+2}) = \mathrm{Lin}(V_1, \ldots, V_k; V_{k+2}).$$

The rule that assigns to each object $(V_1, \ldots, V_{k+2})$ of $\underline{\mathfrak{Vec}}^{k+2}$ the map $\mathfrak{N}_{V_1, \ldots, V_{k+2}}$ defined by:

$$\begin{aligned} \mathrm{Lin}(V_{k+1}, V_{k+2}) \oplus \mathrm{Lin}(V_1, \ldots, V_k; V_{k+1}) &\longrightarrow \mathrm{Lin}(V_1, \ldots, V_k; V_{k+2}) \\ (L, B) &\longmapsto L \circ B \end{aligned}$$

is a smooth natural transformation from $\underline{\mathfrak{F}}$ to $\underline{\mathfrak{G}}$.

EXAMPLE 1.6.26. Consider the smooth functors $\underline{\mathfrak{F}}$, $\underline{\mathfrak{G}}$ from $\underline{\mathfrak{Vec}}^2$ to $\underline{\mathfrak{Vec}}$ defined by:

$$\underline{\mathfrak{F}}(V_1, V_2) = \mathrm{Lin}(V_1, V_2), \quad \underline{\mathfrak{G}}(V_1, V_2) = \mathrm{Lin}(V_2, V_1).$$

Given real vector spaces V_1, V_2, we denote by $\mathrm{Iso}(V_1, V_2)$ the (possibly empty) subset of $\mathrm{Lin}(V_1, V_2)$ consisting of linear isomorphisms. The rule that assigns to each object (V_1, V_2) of $\underline{\mathfrak{Vec}}^2$ the map:

$$\mathfrak{N}_{V_1, V_2} : \mathrm{Iso}(V_1, V_2) \ni T \longmapsto T^{-1} \in \mathrm{Lin}(V_2, V_1)$$

is a smooth natural transformation from $\underline{\mathfrak{F}}$ to $\underline{\mathfrak{G}}$.

EXAMPLE 1.6.27. Consider the smooth functors $\underline{\mathfrak{F}}$, $\underline{\mathfrak{G}}$ from $\underline{\mathfrak{Vec}}$ to $\underline{\mathfrak{Vec}}$ defined by:

$$\underline{\mathfrak{F}}(V) = \mathrm{Lin}(V), \quad \underline{\mathfrak{G}}(V) = \mathbb{R}.$$

The rule that assigns to each object V of $\underline{\mathfrak{Vec}}$ the map:

$$\mathfrak{N}_V : \mathrm{Lin}(V) \ni T \longmapsto \det(T) \in \mathbb{R}$$

is a smooth natural transformation from $\underline{\mathfrak{F}}$ to $\underline{\mathfrak{G}}$.

Given smooth functors $\underline{\mathfrak{F}}$, $\underline{\mathfrak{G}}$ from $\underline{\mathfrak{Vec}}^n$ to $\underline{\mathfrak{Vec}}$, a smooth natural transformation $\mathfrak{N}$ from $\underline{\mathfrak{F}}$ to $\underline{\mathfrak{G}}$ and vector bundles E^1, ..., E^n over a differentiable manifold M then $\mathfrak{N}$ induces a map:

$$\mathfrak{N}_{E^1, \ldots, E^n} : \mathrm{Dom}(\mathfrak{N}_{E^1, \ldots, E^n}) \longrightarrow \underline{\mathfrak{G}}(E^1, \ldots, E^n), \tag{1.6.18}$$

where:

$$\mathrm{Dom}(\mathfrak{N}_{E^1, \ldots, E^n}) = \bigcup_{x \in M} \mathrm{Dom}(\mathfrak{N}_{E^1_x, \ldots, E^n_x}) \subset \underline{\mathfrak{F}}(E^1, \ldots, E^n).$$

The map $\mathfrak{N}_{E^1, \ldots, E^n}$ is defined by the requirement that for each $x \in M$, its restriction to $\mathrm{Dom}(\mathfrak{N}_{E^1_x, \ldots, E^n_x})$ is equal to $\mathfrak{N}_{E^1_x, \ldots, E^n_x}$.

PROPOSITION 1.6.28. *Let $n \geq 1$ be fixed. Given smooth functors $\underline{\mathfrak{F}}$, $\underline{\mathfrak{G}}$ from $\underline{\mathfrak{Vec}}^n$ to $\underline{\mathfrak{Vec}}$, a smooth natural transformation $\mathfrak{N}$ from $\underline{\mathfrak{F}}$ to $\underline{\mathfrak{G}}$ and vector bundles E^1, ..., E^n over a differentiable manifold M then $\mathrm{Dom}(\mathfrak{N}_{E^1, \ldots, E^n})$ is an open subset of the total space of the vector bundle $\underline{\mathfrak{F}}(E^1, \ldots, E^n)$ and the map $\mathfrak{N}_{E^1, \ldots, E^n}$ is smooth. In particular, if $\mathfrak{N}$ is linear then $\mathfrak{N}_{E^1, \ldots, E^n}$ is a vector bundle morphism.*

PROOF. Denote by E_0^i the typical fiber of E^i, $i = 1, \ldots, n$. The naturality of $\mathfrak{N}$ implies that the open subset $\mathrm{Dom}(\mathfrak{N}_{E_0^1,\ldots,E_0^n})$ of the vector space $\underline{\mathfrak{F}}(E_0^1, \ldots, E_0^n)$ is invariant under the representation (1.6.7) so that, by Lemma 1.4.11, the fiber product:

$$\big(\mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^n}(E^n)\big) \times_{\approx} \mathrm{Dom}(\mathfrak{N}_{E_0^1,\ldots,E_0^n}) \tag{1.6.19}$$

is an open submanifold of:

$$\big(\mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^n}(E^n)\big) \times_{\approx} \underline{\mathfrak{F}}(E_0^1, \ldots, E_0^n).$$

It follows easily from the naturality of $\mathfrak{N}$ that the vector bundle isomorphism (1.6.8) carries (1.6.19) to $\mathrm{Dom}(\mathfrak{N}_{E^1,\ldots,E^n})$, so $\mathrm{Dom}(\mathfrak{N}_{E^1,\ldots,E^n})$ is indeed an open subset of $\underline{\mathfrak{F}}(E^1, \ldots, E^n)$. The naturality of $\mathfrak{N}$ also implies that the diagram:

$$\begin{tikzcd}
P \times_{\approx} \mathrm{Dom}(\mathfrak{N}_{E_0^1,\ldots,E_0^n}) \arrow[r, "\mathrm{Id}_P \times_{\approx} \mathfrak{N}_{E_0^1,\ldots,E_0^n}"] \arrow[d, "\cong", "\mathcal{C}^{\underline{\mathfrak{F}}}"'] & P \times_{\approx} \underline{\mathfrak{G}}(E_0^1, \ldots, E_0^n) \arrow[d, "\mathcal{C}^{\underline{\mathfrak{G}}}", "\cong"'] \\
\mathrm{Dom}(\mathfrak{N}_{E^1,\ldots,E^n}) \arrow[r, "\mathfrak{N}_{E^1,\ldots,E^n}"'] & \underline{\mathfrak{G}}(E^1, \ldots, E^n)
\end{tikzcd} \tag{1.6.20}$$

commutes, where $P = \mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^n}(E^n)$. The fact that the map $\mathfrak{N}_{E^1,\ldots,E^n}$ is smooth now follows from the fact that the map $\mathrm{Id}_P \times_{\approx} \mathfrak{N}_{E_0^1,\ldots,E_0^n}$ is smooth (Lemma 1.4.11). $\square$

EXAMPLE 1.6.29. Let E^1, E^2 be vector bundles over a differentiable manifold M and consider the Whitney sum $E^1 \oplus E^2$. Applying Proposition 1.6.28 to the linear smooth natural transformations described in Examples 1.6.20 and 1.6.21 we conclude that the projections $\mathrm{pr}_i : E^1 \oplus E^2 \to E^i$ and the inclusions $\iota_i : E^i \to E^1 \oplus E^2$, $i = 1, 2$, are vector bundle morphisms. This implies the following property: if ϵ is a section of $E^1 \oplus E^2$ and $\epsilon^i = \mathrm{pr}_i \circ \epsilon$, $i = 1, 2$, are the *coordinates* of ϵ then ϵ is smooth if and only if ϵ^1 and ϵ^2 are smooth. Namely, if ϵ is smooth then obviously ϵ^1 and ϵ^2 are smooth, because the projections are smooth; conversely, if ϵ^1 and ϵ^2 are smooth then $\epsilon = \iota_1 \circ \epsilon^1 + \iota_2 \circ \epsilon^2$. See Exercises 1.70 and 1.71 for more basic results concerning Whitney sums.

REMARK 1.6.30. Let $\pi : E \to M$ be a vector bundle and E^1, E^2 be vector subbundles of E such that $E_x = E_x^1 \oplus E_x^2$, for all $x \in M$; denote by $\mathfrak{j}_i : E^i \to E$, $i = 1, 2$, the inclusion maps. Consider the Whitney sum $E^1 \oplus E^2$ and denote by $\iota_i : E^i \to E^1 \oplus E^2$, $i = 1, 2$, the inclusion maps. By the result of Exercise 1.70, there exists a unique vector bundle morphism $\mathfrak{j} : E^1 \oplus E^2 \to E$ such that $\mathfrak{j} \circ \iota_i = \mathfrak{j}_i$, $i = 1, 2$. Clearly $\mathfrak{j}$ is a vector bundle isomorphism. *We will use the isomorphism* $\mathfrak{j}$ *to identify the vector bundle E with the Whitney sum* $E^1 \oplus E^2$. Thus, if E^1, E^2 are vector subbundles of a vector bundle E such that $E_x = E_x^1 \oplus E_x^2$, for all $x \in M$, we will write $E = E^1 \oplus E^2$.

EXAMPLE 1.6.31. Let $E^1, \ldots, E^k$, F be vector bundles over a differentiable manifold M, B be a smooth section of $\mathrm{Lin}(E^1, \ldots, E^k; F)$ and ϵ^i be a smooth section of E^i, $i = 1, \ldots, k$. Applying Proposition 1.6.28 to the smooth natural transformation of Example 1.6.24 we obtain that the section $B(\epsilon^1, \ldots, \epsilon^k)$ of F defined by:

$$B(\epsilon^1, \ldots, \epsilon^k)(x) = B_x\big(\epsilon^1(x), \ldots, \epsilon^k(x)\big), \quad x \in M,$$

is smooth. Namely, the map:

$$\mathfrak{N}_{E^1,\ldots,E^{k+1}} : \mathrm{Lin}(E^1, \ldots, E^k; E^{k+1}) \oplus E^1 \oplus \cdots \oplus E^k \longrightarrow E^{k+1}$$

is smooth and:

$$B(\epsilon^1, \ldots, \epsilon^k) = \mathfrak{N}_{E^1,\ldots,E^k,F} \circ (B, \epsilon^1, \ldots, \epsilon^k).$$

Recall also that $(B, \epsilon^1, \ldots, \epsilon^k)$ is smooth (Example 1.6.29). Thus, every smooth section B of $\mathrm{Lin}(E^1, \ldots, E^k; F)$ defines a $C^\infty(M)$-multilinear map:

$$\boldsymbol{\Gamma}(E^1) \times \cdots \times \boldsymbol{\Gamma}(E^k) \ni (\epsilon^1, \ldots, \epsilon^k) \longmapsto B(\epsilon^1, \ldots, \epsilon^k) \in \boldsymbol{\Gamma}(F).$$

The result of Exercises 1.63 and 1.72 says that, conversely, every $C^\infty(M)$-multilinear map from $\boldsymbol{\Gamma}(E^1) \times \cdots \times \boldsymbol{\Gamma}(E^k)$ to $\boldsymbol{\Gamma}(F)$ is defined by a unique smooth section B of $\mathrm{Lin}(E^1, \ldots, E^k; F)$. In view of this correspondence we will be allowed to identify smooth sections of $\mathrm{Lin}(E^1, \ldots, E^k; F)$ with the corresponding $C^\infty(M)$-multilinear maps.

EXAMPLE 1.6.32. Let E^1, …, E^k, F, F' be vector bundles over a differentiable manifold M, B be a section of $\mathrm{Lin}(E^1, \ldots, E^k; F)$ and $L : F \to F'$ be a vector bundle morphism. Recall from Example 1.6.12 that we identify L with the smooth section $x \mapsto L_x$ of $\mathrm{Lin}(F, F')$. We will denote (with some abuse of notation) by $L \circ B$ the section of $\mathrm{Lin}(E^1, \ldots, E^k; F')$ defined by:

$$(L \circ B)(x) = L_x \circ B(x),$$

for all $x \in M$. We claim that if B is smooth then also $L \circ B$ is smooth. Namely, by Example 1.6.29, (L, B) is a smooth section of the Whitney sum:

$$\mathrm{Lin}(F, F') \oplus \mathrm{Lin}(E^1, \ldots, E^k; F).$$

If $\mathfrak{N}$ is the smooth natural transformation defined in Example 1.6.25 then:

$$L \circ B = \mathfrak{N}_{E^1,\ldots,E^k,F,F'} \circ (L, B),$$

and therefore $L \circ B$ is smooth by Proposition 1.6.28.

EXAMPLE 1.6.33. Given real finite-dimensional vector spaces V_1, …, V_k, $V_{k+1}, \ldots, V_{k+p}$, W then we have a linear isomorphism:

$$\begin{aligned} \mathrm{Lin}\big(V_1, \ldots, V_k; \mathrm{Lin}(V_{k+1}, \ldots, V_{k+p}; W)\big) &\longrightarrow \mathrm{Lin}(V_1, \ldots, V_{k+p}; W) \\ B &\longmapsto \widetilde{B} \end{aligned} \tag{1.6.21}$$

defined by:

$$\widetilde{B}(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+p}) = B(v_1, \ldots, v_k) \cdot (v_{k+1}, \ldots, v_{k+p}) \in W,$$

for all $v_1 \in V_1, \ldots, v_{k+p} \in V_{k+p}$. The linear isomorphism (1.6.21) defines a linear smooth natural transformation between smooth functors and therefore, given vector bundles $E^1, \ldots, E^{k+p}$, F over a differentiable manifold M, as an application of Proposition 1.6.28 we get an isomorphism of vector bundles:

$$\mathrm{Lin}\big(E^1, \ldots, E^k; \mathrm{Lin}(E^{k+1}, \ldots, E^{k+p}; F)\big) \longrightarrow \mathrm{Lin}(E^1, \ldots, E^{k+p}; F).$$

We will henceforth identify the vector bundles:

$$\mathrm{Lin}\big(E^1, \ldots, E^k; \mathrm{Lin}(E^{k+1}, \ldots, E^{k+p}; F)\big), \quad \mathrm{Lin}(E^1, \ldots, E^{k+p}; F)$$

using such isomorphism.

1.7. The group of left translations of the fiber

Let $\Pi : P \to M$ be a G-principal bundle. For each point $x \in M$, the fiber P_x is a principal space and thus we may consider the group $\mathrm{Left}(P_x)$ of all left translations of P_x (recall Definition 1.2.10). For each $p \in P_x$, we have an isomorphism $\mathcal{I}_p : G \to \mathrm{Left}(P_x)$ (recall (1.2.3)) and there exists a unique differential structure on $\mathrm{Left}(P_x)$ that makes $\mathcal{I}_p$ into a smooth diffeomorphism; the commutativity of diagram (1.2.4) shows that the differential structure on $\mathrm{Left}(P_x)$ does not depend on the choice of $p \in P_x$. Endowed with such differential structure, the group $\mathrm{Left}(P_x)$ is a Lie group and the map $\mathcal{I}_p$ is a Lie group isomorphism, for all $p \in P_x$. We know that the left action of $\mathrm{Left}(P_x)$ on P_x is free and transitive (recall Lemma 1.2.12). We claim that it is smooth. Namely, choose any $p \in P_x$; if we identify $\mathrm{Left}(P_x)$ with G via $\mathcal{I}_p$ and P_x with G via the smooth diffeomorphism β_p then the left action of $\mathrm{Left}(P_x)$ on P_x is identified with the action of G on itself by left-translations. More explicitly, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Left}(P_x) \times P_x \arrow[r, "\text{action}"] & P_x \\
G \times G \arrow[u, "\mathcal{I}_p \times \beta_p", "\cong"'] \arrow[r, "\text{multiplication}"'] & G \arrow[u, "\cong", "\beta_p"']
\end{tikzcd}$$

Since the vertical arrows of the diagram are smooth diffeomorphisms and the bottom arrow of the diagram is smooth, it follows that the top arrow of the diagram is also smooth.

Let us denote by $\mathfrak{left}(P_x)$ the Lie algebra of the Lie group $\mathrm{Left}(P_x)$. For each $p \in P_x$, the differential of the Lie group isomorphism $\mathcal{I}_p$ at the identity gives us a Lie algebra isomorphism Ad_p; more explicitly, we set:

$$\mathrm{Ad}_p = \mathrm{d}\mathcal{I}_p(1) : \mathfrak{g} \longrightarrow \mathfrak{left}(P_x),$$

where $\mathfrak{g}$ denotes the Lie algebra of G. By differentiating the commutative diagram (1.2.4) we obtain:

(1.7.1)
$$\begin{tikzcd}
\mathfrak{g} \arrow[dr, "\mathrm{Ad}_p"] \arrow[dd, "\mathrm{Ad}_{g^{-1}}"'] & \\
& \mathfrak{left}(P_x) \\
\mathfrak{g} \arrow[ur, "\mathrm{Ad}_{p'}"'] &
\end{tikzcd}$$

where $p, p' \in P_x$ and $g = p^{-1}p'$ is the element of G that carries p to p'.

EXAMPLE 1.7.1. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and consider its $\mathrm{GL}(E_0)$-principal bundle of frames $\mathrm{FR}_{E_0}(E)$. Let $x \in M$ be fixed. In Example 1.2.17 we made the convention of identifying the group $\mathrm{Left}\big(\mathrm{FR}_{E_0}(E_x)\big)$ of left translations of $\mathrm{FR}_{E_0}(E_x)$ with the general linear group $\mathrm{GL}(E_x)$ of E_x through the isomorphism $\mathrm{GL}(E_x) \ni T \mapsto T_* \in \mathrm{Left}\big(\mathrm{FR}_{E_0}(E_x)\big)$. Under such identification, the canonical left action of the group $\mathrm{Left}\big(\mathrm{FR}_{E_0}(E_x)\big)$ on $\mathrm{FR}_{E_0}(E_x)$ is identified with the action of $\mathrm{GL}(E_x)$ on $\mathrm{FR}_{E_0}(E_x)$ given by composition of linear isomorphisms. Moreover, for every $p \in \mathrm{FR}_{E_0}(E_x)$ the isomorphism $\mathcal{I}_p$ is given by $\mathcal{I}_p(g) = p \circ g \circ p^{-1}$ (recall (1.2.7)) and thus the differential structure of $\mathrm{GL}(E_x)$ that makes $\mathcal{I}_p$ into a smooth diffeomorphism is the standard one. The Lie algebra $\mathfrak{left}\big(\mathrm{FR}_{E_0}(E_x)\big)$ is therefore identified with the Lie algebra $\mathfrak{gl}(E_x)$ of $\mathrm{GL}(E_x)$; differentiating (1.2.7) we see that, for every $p \in \mathrm{FR}_{E_0}(E_x)$, the Lie algebra isomorphism Ad_p is given by:

$$\mathrm{Ad}_p(X) = p \circ X \circ p^{-1} \in \mathfrak{gl}(E_x), \tag{1.7.2}$$

for all $X \in \mathfrak{gl}(E_0)$.

REMARK 1.7.2. Let H be a Lie subgroup of G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ and let $Q \subset P$ be an H-principal subbundle of P. For each $x \in M$, the fiber Q_x is a principal subspace of the fiber P_x with structural group $H \subset G$. Recall that we have made the convention of identifying the group $\mathrm{Left}(Q_x)$ with the subgroup of $\mathrm{Left}(P_x)$ consisting of those left translations $t : P_x \to P_x$ such that $t(Q_x) \subset Q_x$ (see Lemma 1.2.19). The commutativity of diagram (1.2.12) implies that $\mathrm{Left}(Q_x)$ is a Lie subgroup of $\mathrm{Left}(P_x)$ and therefore we identify the Lie algebra $\mathfrak{left}(Q_x)$ with a Lie subalgebra of $\mathfrak{left}(P_x)$. For each $p \in Q_x$, we have Lie group isomorphisms $\mathcal{I}_p^P : G \to \mathrm{Left}(P_x)$ and $\mathcal{I}_p^Q : H \to \mathrm{Left}(Q_x)$ (see Remark 1.2.20) whose differentials at the identity are respectively the Lie algebra isomorphisms $\mathrm{Ad}_p^P : \mathfrak{g} \to \mathfrak{left}(P_x)$ and $\mathrm{Ad}_p^Q : \mathfrak{h} \to \mathfrak{left}(Q_x)$. By differentiating (1.2.12) we obtain a commutative diagram:

$$\begin{tikzcd} \mathfrak{left}(Q_x) \arrow[r, "\text{inclusion}"] & \mathfrak{left}(P_x) \\ \mathfrak{h} \arrow[u, "\mathrm{Ad}_p^Q", "\cong"'] \arrow[r, "\text{inclusion}"'] & \mathfrak{g} \arrow[u, "\cong", "\mathrm{Ad}_p^P"'] \end{tikzcd} \tag{1.7.3}$$

that shows that Ad_p^Q is just the restriction of Ad_p^P to $\mathfrak{h}$.

1.8. G-structures on vector bundles

Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and denote by $\mathrm{FR}_{E_0}(E)$ the $\mathrm{GL}(E_0)$-principal bundle of frames of E.

DEFINITION 1.8.1. Given a Lie subgroup G of $\mathrm{GL}(E_0)$ then by a *G-structure* on E we mean a G-principal subbundle P of $\mathrm{FR}_{E_0}(E)$. A local E_0-frame $s : U \to \mathrm{FR}_{E_0}(E)$ of E with $s(U) \subset P$ is said to be *compatible* with the G-structure P.

Observe that if P is a G-structure on E then for each $x \in M$, P_x is a G-structure on the vector space E_x (recall Definition 1.1.7). We may therefore think intuitively of a G-structure P on a vector bundle E as a family $(P_x)_{x\in M}$ of G-structures on the fibers E_x of E that "varies smoothly" with $x \in M$.

Let P be a G-structure on a vector bundle $\pi : E \to M$ with typical fiber E_0. Recall from Example 1.7.1 that for each $x \in M$ we identify the Lie group $\mathrm{Left}\big(\mathrm{FR}_{E_0}(E_x)\big)$ of left translations of $\mathrm{FR}_{E_0}(E_x)$ with the general linear group $\mathrm{GL}(E_x)$. We will denote by G_x the Lie group $\mathrm{Left}(P_x)$ of left translations of the fiber P_x and by $\mathfrak{g}_x$ the Lie algebra $\mathfrak{left}(P_x)$ of G_x. Recall from Remark 1.7.2 that we identify G_x with a Lie subgroup of $\mathrm{Left}\big(\mathrm{FR}_{E_0}(E_x)\big) \cong \mathrm{GL}(E_x)$ and $\mathfrak{g}_x$ with a Lie subalgebra of $\mathfrak{left}\big(\mathrm{FR}_{E_0}(E_x)\big) \cong \mathfrak{gl}(E_x)$. Also recall from Example 1.2.18 that the Lie group G_x is identified with the group $\mathrm{Iso}_G(E_x)$ of all G-structure preserving endomorphisms $T : E_x \to E_x$ of E_x. It should be noticed that the two identifications we have made regarding G_x are compatible (see (1.2.11)).

DEFINITION 1.8.2. Let E, F be vector bundles over the same differentiable manifold M, with the same typical fiber E_0. Let G be a Lie subgroup of $\mathrm{GL}(E_0)$ and assume that E and F are endowed with G-structures P and Q, respectively. A morphism of vector bundles $L : E \to F$ is said to be *G-structure preserving* if for every $x \in M$, the linear map $L_x : E_x \to F_x$ is G-structure preserving.

Clearly, every G-structure preserving morphism of vector bundles is in fact an isomorphism of vector bundles. Moreover, an isomorphism of vector bundles $L : E \to F$ is G-structure preserving if and only if $L_*(P) \subset Q$ (recall Lemma 1.5.18).

EXAMPLE 1.8.3. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0, G be a Lie subgroup of $\mathrm{GL}(E_0)$ and $P \subset \mathrm{FR}_{E_0}(E)$ be a G-structure on E. Given a differentiable manifold M' and a smooth map $f : M' \to M$, then the pull-back f^*P is a G-structure on the vector bundle f^*E (recall Example 1.3.23).

EXAMPLE 1.8.4. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0. By a *semi-Riemannian structure* on E we mean a smooth section g of $\mathrm{Lin}_2^{\mathrm{s}}(E, \mathbb{R})$ such that for all $x \in M$, $g_x : E_x \times E_x \to \mathbb{R}$ is an indefinite inner product on E_x. If g is a semi-Riemannian structure on E and if the index $\mathrm{n}_-(g_x)$ of g_x is independent of $x \in M$ then we call $\mathrm{n}_-(g_x)$ the *index* of the semi-Riemannian structure g and we write $\mathrm{n}_-(g) = \mathrm{n}_-(g_x)$, for all $x \in M$. A semi-Riemannian structure of index zero is also called a *Riemannian structure*. If g is a semi-Riemannian structure on E of index r and if an indefinite inner product $\langle\cdot,\cdot\rangle_0$ of index r is fixed on the typical fiber E_0 then the set:

$$\mathrm{FR}^{\mathrm{o}}_{E_0}(E) = \bigcup_{x\in M} \mathrm{FR}^{\mathrm{o}}_{E_0}(E_x)$$

of all orthonormal frames of E is a principal subbundle of $\mathrm{FR}_{E_0}(E)$ with structural group $\mathrm{O}(E_0)$. Thus, $\mathrm{FR}^{\mathrm{o}}_{E_0}(E)$ is an $\mathrm{O}(E_0)$-structure on the vector bundle E.

EXAMPLE 1.8.5. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and F be a vector subbundle of E. If F_0 is a subspace of E_0 such that $\dim(F_0) = \dim(F_x)$ for all $x \in M$ then the set $\mathrm{FR}_{E_0}(E; F_0, F)$ of all E_0-frames of E adapted to (F_0, F) is a principal subbundle of $\mathrm{FR}_{E_0}(E)$ with structural group $\mathrm{GL}(E_0; F_0)$. Thus $\mathrm{FR}_{E_0}(E; F_0, F)$ is a $\mathrm{GL}(E_0; F_0)$-structure on the vector bundle E. If g is a semi-Riemannian structure on E and if an indefinite inner product on E_0 is fixed then the set:

$$\mathrm{FR}^{\mathrm{o}}_{E_0}(E; F_0, F) \stackrel{\text{def}}{=} \mathrm{FR}_{E_0}(E; F_0, F) \cap \mathrm{FR}^{\mathrm{o}}_{E_0}(E)$$

is an $\mathrm{O}(E_0; F_0)$-structure on the vector bundle E if $\mathrm{FR}^{\mathrm{o}}_{E_0}(E_x; F_0, F_x) \neq \emptyset$ for all $x \in M$.

EXAMPLE 1.8.6. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and $\epsilon \in \mathbf{\Gamma}(E)$ be a smooth section of E with $\epsilon(x) \neq 0$, for all $x \in M$. If $e_0 \in E_0$ is a nonzero vector then the set:

$$\mathrm{FR}_{E_0}(E; e_0, \epsilon) \stackrel{\text{def}}{=} \bigcup_{x \in M} \mathrm{FR}_{E_0}\big(E_x; e_0, \epsilon(x)\big)$$

of all E_0-frames of E that are *adapted* to (e_0, ϵ) is a $\mathrm{GL}(E_0; e_0)$-structure on the vector bundle E. If g is a semi-Riemannian structure on E and if an indefinite inner product on E_0 is fixed then the set:

$$\mathrm{FR}^{\mathrm{o}}_{E_0}(E; e_0, \epsilon) \stackrel{\text{def}}{=} \mathrm{FR}_{E_0}(E; e_0, \epsilon) \cap \mathrm{FR}^{\mathrm{o}}_{E_0}(E)$$

is an $\mathrm{O}(E_0; e_0)$-structure on the vector bundle E if $\mathrm{FR}^{\mathrm{o}}_{E_0}\big(E_x; e_0, \epsilon(x)\big) \neq \emptyset$ for all $x \in M$.

EXAMPLE 1.8.7. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0. By an *almost complex* structure on E we mean a smooth section J of $\mathrm{Lin}(E)$ such that J_x is a complex structure on E_x, for all $x \in M$. If J_0 is a complex structure on E_0 then the set:

$$\mathrm{FR}^{\mathrm{c}}_{E_0}(E) \stackrel{\text{def}}{=} \bigcup_{x \in M} \mathrm{FR}^{\mathrm{c}}_{E_0}(E_x)$$

of all complex frames of E is a $\mathrm{GL}(E_0, J_0)$-structure on the vector bundle E. If g is a semi-Riemannian structure on E of index r, $\langle \cdot, \cdot \rangle_{E_0}$ is an indefinite inner product on E_0 of index r, J_0 is anti-symmetric with respect to $\langle \cdot, \cdot \rangle_{E_0}$ and J_x is anti-symmetric with respect to g_x for all $x \in M$ then:

$$\mathrm{FR}^{\mathrm{u}}_{E_0}(E) \stackrel{\text{def}}{=} \mathrm{FR}^{\mathrm{o}}_{E_0}(E) \cap \mathrm{FR}^{\mathrm{c}}_{E_0}(E)$$

is an $\mathrm{U}(E_0)$-structure on E.

REMARK 1.8.8. The reader might find odd the use of the name "*almost* complex" in Example 1.8.7. This choice comes from the fact that, in the literature, an *almost complex structure* in a manifold M is a smooth section J of $\mathrm{Lin}(TM)$ such that J_x is a complex structure in T_xM, for all $x \in M$. By a *complex structure* on M it is meant an almost complex structure J on M which is *integrable* in the sense that M can be covered with local charts $\varphi : U \to \widetilde{U} \subset \mathbb{R}^{2n}$ such that $\mathrm{d}\varphi_x \circ J_x = J_0 \circ \mathrm{d}\varphi_x$, for all $x \in U$, where J_0 denotes the canonical complex structure of $\mathbb{R}^{2n}$.

DEFINITION 1.8.9. Let M be an n-dimensional differentiable manifold and let G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$. By a *$G$-structure on M* we mean a G-structure $P \subset \mathrm{FR}(TM)$ on the tangent bundle TM.

DEFINITION 1.8.10. Let G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$ and M', M be n-dimensional differentiable manifolds endowed with G-structures P' and P, respectively. A smooth map $f : M' \to M$ is said to be *G-structure preserving* if the vector bundle morphism $\overleftarrow{\mathrm{d}f} : TM' \to f^*TM$ (recall Example 1.5.27) is G-structure preserving, where f^*TM is endowed with the G-structure f^*P.

Clearly, if a smooth map $f : M' \to M$ is G-structure preserving then f is a local diffeomorphism. Moreover, given a smooth local diffeomorphism $f : M' \to M$, if we define a map $(\mathrm{d}f)_* : \mathrm{FR}(TM') \to \mathrm{FR}(TM)$ by:

$$(\mathrm{d}f)_* : \mathrm{FR}(TM') \ni p \longmapsto \mathrm{d}f \circ p \in \mathrm{FR}(TM), \tag{1.8.1}$$

then f is G-structure preserving if and only if $(\mathrm{d}f)_*(P') \subset P$.

Clearly the composition of G-structure preserving maps is a G-structure preserving map and if f is a G-structure preserving diffeomorphism then also f^{-1} is a G-structure preserving diffeomorphism.

DEFINITION 1.8.11. Let M be a differentiable manifold. By a *Riemannian metric* (resp., *semi-Riemannian metric*) on M we mean a Riemannian structure (resp., semi-Riemannian structure) g on TM; the pair (M, g) is called a *Riemannian manifold* (resp., *semi-Riemannian manifold*).

Exercises

G-structures on sets.

EXERCISE 1.1. Let G be a group and assume that we are given a left (resp., right) action of G on a set A. A subset B of A is called *G-invariant* if $g \cdot a$ (resp., $a \cdot g$) is in B for all $a \in B$. Show that a subset B of A is G-invariant if and only if it is equal to a union of G-orbits.

EXERCISE 1.2. Let $\mathcal{G}$ be a group and assume that we are given a (left or right) action of $\mathcal{G}$ on a set A. If G is a subgroup of $\mathcal{G}$ then for every G-orbit $B \subset A$ there exists exactly one $\mathcal{G}$-orbit $\mathcal{B} \subset A$ containing B.

EXERCISE 1.3. Let $\mathcal{G}$ be a group and assume that we are given a left (resp., right) action of $\mathcal{G}$ on a nonempty set A. Assume that the action is free and transitive. Let G be a subgroup of $\mathcal{G}$. For any fixed $a \in A$, show that the bijective map $\beta_a : \mathcal{G} \to A$ induces a bijection between the set of right (resp., left) cosets of G in $\mathcal{G}$ and the set of orbits of the action of G on A. Conclude that if X_0 and X are sets having the same cardinality and if G is a subgroup of $\mathrm{Bij}(X_0)$ then the (possibly infinite) number of possible G-structures on X is equal to the index of G in $\mathrm{Bij}(X_0)$.

EXERCISE 1.4. Let R_0 be a ring and let G be the subgroup of $\mathrm{Bij}(R_0)$ consisting of all ring automorphisms of R_0. Show that:

- given a ring R isomorphic to R_0 then the set of all ring isomorphisms $p : R_0 \to R$ is a G-structure on R modeled upon R_0;
- if a G-structure P is given on a set R then there exists a unique ring structure on R such that P is the set of all ring isomorphisms from R_0 to R.

Repeat the exercise above replacing the word "ring" by "group", "field", "topological space" or any other standard mathematical structure.

EXERCISE 1.5. Let X_0 be a set and G be a subgroup of $\mathrm{Bij}(X_0)$. Show that:

- the composite of G-structure preserving maps is a G-structure preserving map;
- any G-structure preserving map is bijective and its inverse is also a G-structure preserving map.

Conclude that sets endowed with G-structures and G-structure preserving maps constitute a category in which all morphisms are isomorphisms and in which all objects are isomorphic.

EXERCISE 1.6. Let R_0 be a ring and let G be the subgroup of $\mathrm{Bij}(R_0)$ consisting of all ring automorphisms of R_0. Given a ring R isomorphic to R_0, let us regard R as a set endowed with the G-structure consisting of all ring isomorphisms $p : R_0 \to R$ (see Exercise 1.4). Show that:

- given rings R, S then a map $f : R \to S$ is G-structure preserving if and only if f is a ring isomorphism;
- the category of rings isomorphic to R_0 and ring isomorphisms is isomorphic to the category of sets endowed with G-structure and G-structure preserving maps.

Repeat the exercise above replacing the word "ring" by "group", "field", "topological space" or any other standard mathematical structure.

EXERCISE 1.7. Let $\underline{\mathfrak{C}}$ be a category in which all morphisms are isomorphisms and in which all objects are isomorphic[7]. Let $\mathcal{X}_0$ be a fixed object of $\underline{\mathfrak{C}}$ and let $\underline{\mathfrak{F}}$ be a functor from $\underline{\mathfrak{C}}$ to the category of sets and maps. Let $G = \mathrm{Iso}(\mathcal{X}_0)$ denote the group of isomorphisms of the object $\mathcal{X}_0$, $X_0 = \underline{\mathfrak{F}}(\mathcal{X}_0)$ be the set corresponding to the object $\mathcal{X}_0$ and G_{ef} denote the image of G under the group homomorphism $\underline{\mathfrak{F}} : \mathrm{Iso}(\mathcal{X}_0) \to \mathrm{Bij}(X_0)$. For every object $\mathcal{X}$ of $\underline{\mathfrak{C}}$, denote by $P^{\mathcal{X}}$ the image of the map $\underline{\mathfrak{F}} : \mathrm{Iso}(\mathcal{X}_0, \mathcal{X}) \to \mathrm{Bij}\big(X_0, \underline{\mathfrak{F}}(\mathcal{X})\big)$.

- Show that $P^{\mathcal{X}}$ is a G_{ef}-structure on the set $\underline{\mathfrak{F}}(\mathcal{X})$, for every object $\mathcal{X}$ of the category $\underline{\mathfrak{C}}$.
- If $\mathfrak{f} : \mathcal{X} \to \mathcal{Y}$ is a morphism of $\underline{\mathfrak{C}}$, show that $\underline{\mathfrak{F}}(\mathfrak{f}) : \underline{\mathfrak{F}}(\mathcal{X}) \to \underline{\mathfrak{F}}(\mathcal{Y})$ is a G_{ef}-structure preserving map.
- For every object $\mathcal{X}$ of $\underline{\mathfrak{C}}$, let $\underline{\mathfrak{F}}^{\bullet}(X)$ denote the set $\underline{\mathfrak{F}}(X)$ endowed with the G_{ef}-structure $P^{\mathcal{X}}$ and for each morphism $\mathfrak{f} : \mathcal{X} \to \mathcal{Y}$ of $\underline{\mathfrak{C}}$ let $\underline{\mathfrak{F}}^{\bullet}(\mathfrak{f})$

[7]For instance, one can start with an arbitrary category and then consider the subcategory whose objects are an isomorphism class of objects of the original category and whose morphisms are the morphisms of the original category that are isomorphisms.

be the same as $\underline{\mathfrak{F}}(\mathfrak{f})$. Show that $\underline{\mathfrak{F}}^{\bullet}$ is a functor from $\underline{\mathfrak{C}}$ to the category of sets endowed with G_{ef}-structure and G_{ef}-structure preserving maps.

- Assume that the functor $\underline{\mathfrak{F}}$ has the following additional property: given an object $\mathcal{X}$ of $\underline{\mathfrak{C}}$, a set Y and a bijection $f : \underline{\mathfrak{F}}(\mathcal{X}) \to Y$, there exists a unique pair $(\mathcal{Y}, \mathfrak{f})$, such that $\mathcal{Y}$ is an object of $\underline{\mathfrak{C}}$, $\mathfrak{f} : \mathcal{X} \to \mathcal{Y}$ is a morphism of $\underline{\mathfrak{C}}$, $\underline{\mathfrak{F}}(\mathcal{Y}) = Y$ and $\underline{\mathfrak{F}}(\mathfrak{f}) = f$. Under this assumption, show that the functor $\underline{\mathfrak{F}}^{\bullet}$ is an isomorphism of categories.
- Obtain the results of Exercises 1.4 and 1.6 as consequences of the previous items by considering appropriate categories $\underline{\mathfrak{C}}$ and by taking $\underline{\mathfrak{F}}$ to be a forgetful functor.

Principal spaces.

EXERCISE 1.8. Let $\underline{\mathfrak{C}}$ be an arbitrary category and let $\mathcal{X}_0$ be a fixed object of $\underline{\mathfrak{C}}$. Show that, for any object $\mathcal{X}$ of $\underline{\mathfrak{C}}$ that is isomorphic to $\mathcal{X}_0$, the set $\mathrm{Iso}(\mathcal{X}_0, \mathcal{X})$ of all isomorphisms from $\mathcal{X}_0$ to $\mathcal{X}$ is a principal space whose structural group is the group $\mathrm{Iso}(\mathcal{X}_0)$ of all isomorphisms of the object $\mathcal{X}_0$ (the right action of $\mathrm{Iso}(\mathcal{X}_0)$ on $\mathrm{Iso}(\mathcal{X}_0, \mathcal{X})$ is given by composition of morphisms).

EXERCISE 1.9. Given principal spaces P, Q, R with the same structural group G and left translations $t : P \to Q$, $s : Q \to R$, show that the composite $s \circ t$ is also a left translation. Show also that every left translation $t : P \to Q$ is bijective and that its inverse $t^{-1} : Q \to P$ is again a left translation.

EXERCISE 1.10. Let G be a group. There is a category whose objects are principal spaces with structural group G and whose morphisms are left translations. Show that in this category every morphism is an isomorphism and all objects are isomorphic.

EXERCISE 1.11 (*the functor* $\mathrm{Iso}(\mathcal{X}_0, \cdot)$). Let $\underline{\mathfrak{C}}$ be a category as in the statement of Exercise 1.7; let $\mathcal{X}_0$ be a fixed object of $\underline{\mathfrak{C}}$. Recall from Exercise 1.8 that for every object $\mathcal{X}$ of $\underline{\mathfrak{C}}$, the set $\mathrm{Iso}(\mathcal{X}_0, \mathcal{X})$ is a principal space with structural group $\mathrm{Iso}(\mathcal{X}_0)$. Given objects $\mathcal{X}$, $\mathcal{Y}$ of $\underline{\mathfrak{C}}$ and a morphism $\mathfrak{f}$ from $\mathcal{X}$ to $\mathcal{Y}$, show that the map:

$$\mathfrak{f}_* : \mathrm{Iso}(\mathcal{X}_0, \mathcal{X}) \longrightarrow \mathrm{Iso}(\mathcal{X}_0, \mathcal{Y})$$

given by composition with $\mathfrak{f}$ on the left is a left translation. Moreover, show that the rule:

$$\mathcal{X} \longmapsto \mathrm{Iso}(\mathcal{X}_0, \mathcal{X}), \quad \mathfrak{f} \longmapsto \mathfrak{f}_*$$

defines a functor from the category $\underline{\mathfrak{C}}$ to the category of principal spaces with structural group $\mathrm{Iso}(\mathcal{X}_0)$ and left translations.

EXERCISE 1.12. Prove that the functor $\mathrm{Iso}(\mathcal{X}_0, \cdot)$ defined in Exercise 1.11 is both *full* and *faithful*, i.e., given objects $\mathcal{X}$, $\mathcal{Y}$ of $\underline{\mathfrak{C}}$, show that the map:

$$\mathrm{Iso}(X, Y) \ni \mathfrak{f} \longmapsto \mathfrak{f}_* \in \mathrm{Left}\big(\mathrm{Iso}(\mathcal{X}_0, \mathcal{X}), \mathrm{Iso}(\mathcal{X}_0, \mathcal{Y})\big)$$

is a bijection[8].

EXERCISE 1.13. Let P be a principal space with structural group G. Recall from Example 1.2.14 that $\mathrm{Left}(G,P) = \{\beta_p : p \in P\}$ and from Example 1.2.13 that $\mathrm{Left}(G) = \{L_g : g \in G\}$. Show that $\mathrm{Left}(G,P) \subset \mathrm{Bij}(G,P)$ is a $\mathrm{Left}(G)$-structure on the set P. Moreover, given a $\mathrm{Left}(G)$-structure $\mathcal{P}$ on a set P, show that there exists a unique right action of G on P that makes P into a principal space with structural group G such that $\mathrm{Left}(G,P) = \mathcal{P}$. This means that *a* $\mathrm{Left}(G)$-*structure on a set P is the same as the structure of a principal space with structural group G on P* (compare with Exercise 1.4).

EXERCISE 1.14. Let P, Q be principal spaces with the same structural group G and let us regard P, Q as sets endowed with the $\mathrm{Left}(G)$-structures $\mathrm{Left}(G,P)$ and $\mathrm{Left}(G,Q)$ respectively (see Exercise 1.13). Show that a map $t : P \to Q$ is $\mathrm{Left}(G)$-structure preserving if and only if t is a left translation. Conclude that the category of principal spaces with structural group G and left translations is isomorphic to the category of sets endowed with $\mathrm{Left}(G)$-structure and $\mathrm{Left}(G)$-structure preserving maps (compare with Exercise 1.6).

EXERCISE 1.15. Let V, W be n-dimensional real vector spaces and consider the principal spaces $\mathrm{FR}(V)$ and $\mathrm{FR}(W)$ with structural group $\mathrm{GL}(\mathbb{R}^n)$. Let $T : V \to W$ be a linear isomorphism and consider the corresponding left translation $T_* : \mathrm{FR}(V) \to \mathrm{FR}(W)$. Given $p \in \mathrm{FR}(V)$, $q \in \mathrm{FR}(W)$ then, as in Example 1.2.15, the left translation T_* corresponds to an element g of $\mathrm{GL}(\mathbb{R}^n)$, which we can identify with an $n \times n$ invertible real matrix. Show that g is the matrix representation of the linear map $T : V \to W$ with respect to the bases p and q.

EXERCISE 1.16. Let P, Q, R be principal spaces whose structural groups are G, H and K respectively. Let $\phi : P \to Q$, $\psi : Q \to R$ be morphisms of principal spaces with subjacent group homomorphisms $\phi_0 : G \to H$ and $\psi_0 : H \to K$ respectively. Show that $\psi \circ \phi : P \to R$ is a morphism of principal spaces with subjacent group homomorphism $\psi_0 \circ \phi_0 : G \to K$.

EXERCISE 1.17. Let P, Q be principal spaces with structural groups G and H respectively and let $\phi : P \to Q$ be a morphism of principal spaces with subjacent group homomorphism $\phi_0 : G \to H$. Given $p \in P$ and setting $q = \phi(p)$, show that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "\phi"] & Q \\ G \arrow[u, "\beta_p", "\cong"'] \arrow[r, "\phi_0"'] & H \arrow[u, "\cong", "\beta_q"'] \end{tikzcd} \tag{1.2}$$

[8] Since the category of principal spaces with structural group $\mathrm{Iso}(\mathcal{X}_0)$ has only one isomorphism class of objects, it follows that the functor $\mathrm{Iso}(\mathcal{X}_0, \cdot)$ is a category equivalence. However, there is no natural choice of a category equivalence going in the opposite direction.

Conclude that ϕ is injective (resp., surjective) if and only if ϕ_0 is injective (resp., surjective).

EXERCISE 1.18. Let P, Q be principal spaces with structural groups G and H respectively. Let $\phi : P \to Q$ be an isomorphism of principal spaces with subjacent group homomorphism $\phi_0 : G \to H$. Show that $\phi^{-1} : Q \to P$ is an isomorphism of principal spaces with subjacent group homomorphism $\phi_0^{-1} : H \to G$.

EXERCISE 1.19. Let P, Q be principal spaces with structural groups G and H respectively and let $\phi : P \to Q$ be a morphism of principal spaces with subjacent group homomorphism $\phi_0 : G \to H$. Given principal subspaces $P' \subset P$, $Q' \subset Q$ with structural groups $G' \subset G$, $H' \subset H$ respectively, show that:

- $\phi(P')$ is a principal subspace of Q and its structural group is $\phi_0(G')$;
- $\phi^{-1}(Q')$ is a principal subspace of P and its structural group is $\phi_0^{-1}(H')$.

EXERCISE 1.20. Let P, Q be principal spaces with structural groups G and H respectively and let $\phi_0 : G \to H$ be a homomorphism. Given $p \in P$, $q \in Q$, show that there exists a unique morphisms of principal spaces $\phi : P \to Q$ with subjacent group homomorphism ϕ_0 such that $\phi(p) = q$.

EXERCISE 1.21. Let P be a principal space with structural group G and let K be a normal subgroup of G. Let P/K denote the quotient set of P consisting of all K-orbits. Show that:

$$(pK) \cdot (gK) \stackrel{\text{def}}{=} (p \cdot g)K, \quad p \in P,\ g \in G,$$

defines a right action of the quotient group G/K on the set P/K. Show that this action makes P/K into a principal space with structural group G/K and that the quotient map $P \to P/K$ is a morphism of principal spaces whose subjacent group homomorphism is the quotient map $G \to G/K$. We call P/K the *quotient of the principal space P by the action of the normal subgroup K of G.*

EXERCISE 1.22 (reduction of counter-domain). Let P, Q', Q be principal spaces with structural groups G, H' and H respectively and let $\phi : P \to Q$, $\iota : Q' \to Q$ be morphisms of principal spaces with subjacent group homomorphisms $\phi_0 : G \to H$ and $\iota_0 : H' \to H$ respectively. Assume that ι_0 is injective and that $\phi(P) \subset \iota(Q')$. Show that there exists a unique map $\phi' : P \to Q'$ such that the diagram:

(1.3)
$$\begin{tikzcd} & Q \\ P \arrow[ur, "\phi"] \arrow[r, "\phi'"'] & Q' \arrow[u, "\iota"'] \end{tikzcd}$$

commutes; moreover, show that $\phi_0(G) \subset \iota_0(H')$ and that ϕ' is a morphism of principal spaces whose subjacent group homomorphism is the unique map ϕ_0' for which the diagram

$$
\begin{tikzcd}
 & H \\
G \arrow[ur, "\phi_0"] \arrow[r, "\phi_0'"'] & H' \arrow[u, "\iota_0"']
\end{tikzcd}
\tag{1.4}
$$

commutes.

EXERCISE 1.23 (passing to the quotient). Let P, $\overline{P}$ and Q be principal spaces with structural groups G, $\overline{G}$ and H respectively and let $\mathfrak{q} : P \to \overline{P}$, $\phi : P \to Q$ be morphisms of principal spaces with subjacent group homomorphisms $\mathfrak{q}_0 : G \to \overline{G}$ and $\phi_0 : G \to H$ respectively. Assume that $\mathfrak{q}_0$ is surjective and that its kernel is contained in the kernel of ϕ_0. Show that:

- there exists a unique map $\bar{\phi} : \overline{P} \to Q$ for which the following diagram commutes:

$$
\begin{tikzcd}
P \arrow[d, "\mathfrak{q}"'] \arrow[dr, "\phi"] & \\
\overline{P} \arrow[r, "\bar{\phi}"'] & Q
\end{tikzcd}
\tag{1.5}
$$

- the map $\bar{\phi}$ is a morphism of principal spaces whose subjacent group homomorphism $\bar{\phi}_0 : \overline{G} \to H$ is the unique map for which the following diagram commutes:

$$
\begin{tikzcd}
G \arrow[d, "\mathfrak{q}_0"'] \arrow[dr, "\phi_0"] & \\
\overline{G} \arrow[r, "\bar{\phi}_0"'] & H
\end{tikzcd}
\tag{1.6}
$$

- set $K = \mathrm{Ker}(\phi_0)$, $\overline{P} = P/K$, $\overline{G} = G/K$ and take $\mathfrak{q} : P \to P/K$ to be the quotient map. Conclude that $\bar{\phi} : P/K \to \phi(P)$ is an isomorphism of principal spaces whose subjacent group homomorphism is the group isomorphism $\bar{\phi}_0 : G/K \to H$.

EXERCISE 1.24. Let G, H be groups, P, Q be principal spaces with structural group G, and P', Q' be principal spaces with structural group H. Let $\phi : P' \to P$, $\psi : Q' \to Q$ be morphisms of principal spaces with the same subjacent group homomorphism $\phi_0 : H \to G$. Show that for every left translation $t : P' \to Q'$ there exists a unique left translation $\bar{t} : P \to Q$ for which the following diagram commutes:

$$
\begin{tikzcd}
P \arrow[r, "\bar{t}"] & Q \\
P' \arrow[u, "\phi"] \arrow[r, "t"'] & Q' \arrow[u, "\psi"']
\end{tikzcd}
$$

We have therefore a map:

$$\mathrm{Left}(P',Q') \ni t \longmapsto \bar{t} \in \mathrm{Left}(P,Q). \tag{1.7}$$

Prove the following facts about the map (1.7):

- if ϕ_0 is injective then the map (1.7) is also injective;
- if ϕ_0 is surjective then the map (1.7) is also surjective;
- if $P = Q$, $P' = Q'$ and $\phi = \psi$ then (1.7) is a group homomorphism from $\mathrm{Left}(P')$ to $\mathrm{Left}(P)$.

EXERCISE 1.25. Let P, P' be principal spaces with structural groups G and H respectively. Let $\phi : P' \to P$ be a morphism of principal spaces with subjacent group homomorphism $\phi_0 : H \to G$. In Exercise 1.24 we have constructed a group homomorphism $\mathrm{Left}(P') \ni t \mapsto \bar{t} \in \mathrm{Left}(P)$. Given $p' \in P'$ and setting $p = \phi(p') \in P$, show that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Left}(P') \arrow[r, "t\mapsto\bar{t}"] & \mathrm{Left}(P) \\ H \arrow[u, "\mathcal{I}_{p'}", "\cong"'] \arrow[r, "\phi_0"'] & G \arrow[u, "\cong", "\mathcal{I}_p"'] \end{tikzcd}$$

EXERCISE 1.26. Let P, P' be principal spaces with structural groups G and G' respectively; let $\phi : P \to P'$ be an isomorphism of principal spaces whose subjacent group homomorphism is $\phi_0 : G \to G'$.

- Show that the map:

  $$\mathcal{I}_\phi : \mathrm{Left}(P) \ni t \longmapsto \phi \circ t \circ \phi^{-1} \in \mathrm{Left}(P')$$

  is a group isomorphism.
- Given $p \in P$ and setting $p' = \phi(p) \in P'$, show that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Left}(P) \arrow[r, "\mathcal{I}_\phi"] & \mathrm{Left}(P') \\ G \arrow[u, "\mathcal{I}_p"] \arrow[r, "\phi_0"'] & G' \arrow[u, "\mathcal{I}_{p'}"'] \end{tikzcd} \tag{1.8}$$

- Let $Q \subset P$ be a principal subspace; set $Q' = \phi(Q)$, $\psi = \phi|_Q : Q \to Q'$. Show that the diagram:

$$\begin{tikzcd} \mathrm{Left}(P) \arrow[r, "\mathcal{I}_\phi"] & \mathrm{Left}(P') \\ \mathrm{Left}(Q) \arrow[u, "\text{inclusion}"] \arrow[r, "\mathcal{I}_\psi"'] & \mathrm{Left}(Q') \arrow[u, "\text{inclusion}"'] \end{tikzcd} \tag{1.9}$$

  commutes.

EXERCISE 1.27. Let P be a principal space with structural group G. Show that, for any $g \in G$, the map $\gamma_g : P \to P$ is an isomorphism of principal spaces whose subjacent group homomorphism is $\mathcal{I}_{g^{-1}} : G \to G$.

EXERCISE 1.28. Let $\underline{\mathfrak{C}}$ be a category, $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{X}$ be isomorphic objects of $\underline{\mathfrak{C}}$ and let $\mathfrak{i} : \mathcal{X}_1 \to \mathcal{X}_0$ be an isomorphism. Show that the map:

$$\gamma_{\mathfrak{i}} : \mathrm{Iso}(\mathcal{X}_0, \mathcal{X}) \ni \mathfrak{f} \longmapsto \mathfrak{f} \circ \mathfrak{i} \in \mathrm{Iso}(\mathcal{X}_1, \mathcal{X})$$

is an isomorphism of principal spaces whose subjacent group homomorphism is $\mathcal{I}_{\mathfrak{i}}^{-1}$, where:

$$\mathcal{I}_{\mathfrak{i}} : \mathrm{Iso}(\mathcal{X}_1) \ni \mathfrak{f} \longmapsto \mathfrak{i} \circ \mathfrak{f} \circ \mathfrak{i}^{-1} \in \mathrm{Iso}(\mathcal{X}_0).$$

EXERCISE 1.29. Let $\underline{\mathfrak{C}}$ be a category, $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{X}$ be isomorphic objects of $\underline{\mathfrak{C}}$ and let $\mathfrak{i} : \mathcal{X}_1 \to \mathcal{X}_0$ be an isomorphism. Show that the following diagram commutes (see Exercises 1.12 and 1.28):

$$\begin{tikzcd}
\mathrm{Left}\big(\mathrm{Iso}(\mathcal{X}_0, \mathcal{X})\big) \arrow[rr, "\gamma_{\mathfrak{i}}"] & & \mathrm{Left}\big(\mathrm{Iso}(\mathcal{X}_1, \mathcal{X})\big) \\
& \mathrm{Iso}(X) \arrow[ul, "\cong", "\mathfrak{f} \mapsto \mathfrak{f}_*"'] \arrow[ur, "\cong"', "\mathfrak{f} \mapsto \mathfrak{f}_*"] &
\end{tikzcd}$$

Fiber products.

EXERCISE 1.30. Let P, Q be principal spaces with structural groups G and H respectively; let $\phi : P \to Q$ be a morphism of principal spaces with subjacent group homomorphism $\phi_0 : G \to H$. Let N be an H-space with effective group $H_{\mathrm{ef}} \subset \mathrm{Bij}(N)$. We regard N also as a G-space by considering the action of G on N defined by (1.2.20), so that the effective group G_{ef} of the G-space N is a subgroup of H_{ef}. The fiber product $Q \times_H N$ is endowed with an H_{ef}-structure and the fiber product $P \times_G N$ is endowed with a G_{ef}-structure; such G_{ef}-structure can be weakened into an H_{ef}-structure. Show that the induced map $\hat{\phi} : P \times_G N \to Q \times_H N$ is H_{ef}-structure preserving.

EXERCISE 1.31 (*the functor* $\bullet \times_G N$). Let P, Q be principal spaces with the same structural group G and let N be a G-space. Given a left translation $t : P \to Q$ then, since t is a morphism of principal spaces whose subjacent group homomorphism is the identity (Example 1.2.23), we have an induced map:

$$\hat{t} : P \times_G N \longrightarrow Q \times_G N,$$

which is G_{ef}-structure preserving (Exercise 1.30). Show that:

(a) the rule $P \mapsto P \times_G N$, $t \mapsto \hat{t}$ defines a functor from the category of principal spaces with structural group G and left translations to the category of sets endowed with G_{ef}-structures and G_{ef}-structure preserving maps;

(b) the functor $\bullet \times_G N$ defined in item (a) is *full*, i.e., given principal spaces P, Q with structural group G, the map:

$$\mathrm{Left}(P, Q) \ni t \longmapsto \hat{t} \in \mathrm{Iso}_{G_{\mathrm{ef}}}(P \times_G N, Q \times_G N) \tag{1.10}$$

is surjective;

(c) if the action of G on N is effective then the functor $\bullet \times_G N$ is *faithful*, i.e., given principal spaces P, Q with structural group G, the map (1.10) is injective.

EXERCISE 1.32 (*the functor $\bullet \times_G G$ is naturally isomorphic to the identity*). Let G be a group and let us regard G as a G-space by letting G act on itself by left translations; then $G_{\mathrm{ef}} = \mathrm{Left}(G)$. Given a principal space P with structural group G, show that the map:

$$P \ni p \longmapsto [p, 1] \in P \times_G G \tag{1.11}$$

is $\mathrm{Left}(G)$-structure preserving, where P is endowed with the $\mathrm{Left}(G)$-structure $\mathrm{Left}(G, P)$ (recall Exercise 1.13). Show that (1.11) gives a *natural isomorphism* from the identity functor to the functor $\bullet \times_G G$; more explicitly, given principal spaces P, Q with structural group G and a left translation $t : P \to Q$, show that the diagram:

$$\begin{tikzcd} P \arrow[r, "(1.11)"] \arrow[d, "t"'] & P \times_G G \arrow[d, "\hat{t}"] \\ Q \arrow[r, "(1.11)"'] & Q \times_G G \end{tikzcd}$$

commutes.

EXERCISE 1.33. Let G be a group and let us regard G as a G-space by letting G act on itself on the left by conjugation; then $G_{\mathrm{ef}} = \big\{\mathcal{I}_g : g \in G\big\}$ is the group of all inner automorphisms of G, which is a subgroup of $\mathrm{Aut}(G)$, the group of all group automorphisms of G. Let P be a principal space with structural group G. The fiber product $P \times_G G$ is endowed with a G_{ef}-structure (the reader should be aware that this fiber product is *not* the same considered in Exercise 1.32) that can be weakened to an $\mathrm{Aut}(G)$-structure. We can therefore regard $P \times_G G$ as a group (recall Exercise 1.4). Show that the map:

$$P \times_G G \ni [p, g] \longmapsto \mathcal{I}_p(g) \in \mathrm{Left}(P) \tag{1.12}$$

is a well-defined group isomorphism. Show that this isomorphism is *natural*, i.e., given principal spaces P and Q with structural group G and given a left translation $t : P \to Q$ then the following diagram commutes:

$$\begin{tikzcd} P \times_G G \arrow[r, "(1.12)"] \arrow[d, "\hat{t}"'] & \mathrm{Left}(P) \arrow[d, "\mathcal{I}_t"] \\ Q \times_G G \arrow[r, "(1.12)"'] & \mathrm{Left}(Q) \end{tikzcd} \tag{1.13}$$

where $\mathcal{I}_t : \mathrm{Left}(P) \to \mathrm{Left}(Q)$ is defined by $\mathcal{I}_t(s) = t \circ s \circ t^{-1}$.

EXERCISE 1.34. Generalize the naturality property described by the commutative diagram (1.13) to the following context: let P, Q be principal spaces with structural groups G and H, respectively. Let $\phi : P \to Q$ be a morphism of principal spaces with subjacent group homomorphism $\phi_0 : G \to H$. Let us regard G as a G-space (resp., H as an H-space) by letting G (resp., H) act on itself on the left by conjugation. Show that the following map is well-defined:

$$P \times_G G \ni [p, g] \longmapsto [\phi(p), \phi_0(g)] \in Q \times_H H. \tag{1.14}$$

Show also that the following diagram commutes:

$$\begin{tikzcd} P \times_G G \arrow[r, "(1.12)"] \arrow[d, "(1.14)"'] & \mathrm{Left}(P) \arrow[d, "t \mapsto \bar{t}"] \\ Q \times_H H \arrow[r, "(1.12)"'] & \mathrm{Left}(Q) \end{tikzcd}$$

where the map $t \mapsto \bar{t}$ is defined in Exercise 1.24.

EXERCISE 1.35. Let G be a group and let N and N' be G-spaces. A map $\kappa : N \to N'$ is called *G-equivariant* if $\kappa(g \cdot n) = g \cdot \kappa(n)$, for all $g \in G$ and all $n \in N$. For a fixed group G, show that:

- G-spaces and G-equivariant maps constitute a category;
- if $\kappa : N \to N'$ is a bijective G-equivariant map then $\kappa^{-1} : N' \to N$ is also G-equivariant.

EXERCISE 1.36 (*the functor $P \times_G \bullet$*). Let G be a group, N, N' be G-spaces and $\kappa : N \to N'$ be a G-equivariant map. Consider the induced map:

$$\mathrm{Id} \underset{\sim}{\times} \kappa : P \times_G N \ni [p, n] \longmapsto [p, \kappa(n)] \in P \times_G N'.$$

Show that the rule:

$$N \longmapsto P \times_G N, \quad \kappa \longmapsto \mathrm{Id} \underset{\sim}{\times} \kappa$$

defines a functor from the category of G-spaces and G-equivariant maps to the category of sets and maps.

EXERCISE 1.37 (*the functor $G \times_G \bullet$ is naturally isomorphic to the identity*). Let G be a group and N be a G-space with effective group G_{ef}. We regard G as a principal space with structural group G (recall Example 1.2.2) and the set N to be endowed with its canonical G_{ef}-structure (recall Example 1.1.5). Show that the map:

$$N \ni n \longmapsto [1, n] \in G \times_G N \tag{1.15}$$

is G_{ef}-structure preserving. Show also that the map (1.15) is *natural* in the following sense: given G-spaces N, N' and a G-equivariant map $\kappa : N \to N'$ then the diagram

$$\begin{tikzcd} N \arrow[r, "(1.15)"] \arrow[d, "\kappa"'] & G \times_G N \arrow[d, "\mathrm{Id} \underset{\sim}{\times} \kappa"] \\ N' \arrow[r, "(1.15)"'] & G \times_G N' \end{tikzcd}$$

commutes.

EXERCISE 1.38. Let P be a principal space with structural group G and let N be a G-space. Show that if N is identified with the fiber product $G \times_G N$ by (1.15) then, for all $p \in P$, the map $\hat{p} : N \to P \times_G N$ is identified with the map $\hat{\beta}_p : G \times_G N \to P \times_G N$ (recall Example 1.2.11 and Exercise 1.31).

EXERCISE 1.39. Let $\underline{\mathfrak{C}}$ be a category as in the statement of Exercise 1.7 and let $\mathcal{X}_0$ be a fixed object of $\underline{\mathfrak{C}}$; set $G = \mathrm{Iso}(\mathcal{X}_0)$. Let $\underline{\mathfrak{F}}$ be a functor from $\underline{\mathfrak{C}}$ to the category of sets and maps and set $X_0 = \underline{\mathfrak{F}}(\mathcal{X}_0)$. The functor $\underline{\mathfrak{F}}$ induces a homomorphism from $G = \mathrm{Iso}(\mathcal{X}_0)$ to $\mathrm{Bij}(X_0)$ and therefore we get a left action of G on X_0 that makes the set X_0 into a G-space with effective group G_{ef}. Recall that in Exercise 1.7 we have constructed a functor $\underline{\mathfrak{F}}^\bullet$ from $\underline{\mathfrak{C}}$ to the category of sets endowed with G_{ef}-structures and G_{ef}-structure preserving maps. Show that:

- for each object $\mathcal{X}$ of $\underline{\mathfrak{C}}$ the map:
$$\mathrm{Iso}(\mathcal{X}_0, \mathcal{X}) \times_G X_0 \ni [p, n] \longmapsto \underline{\mathfrak{F}}(p)(n) \in \underline{\mathfrak{F}}^\bullet(\mathcal{X}) \tag{1.16}$$
is G_{ef}-structure preserving;
- (1.16) defines a *natural isomorphism* from the composition of the functors $\mathrm{Iso}(\mathcal{X}_0, \cdot)$ and $\bullet \times_G X_0$ to the functor $\underline{\mathfrak{F}}^\bullet$, i.e., for every morphism $\mathfrak{f} : \mathcal{X} \to \mathcal{Y}$ of $\underline{\mathfrak{C}}$, the following diagram commutes:
$$\begin{tikzcd} \mathrm{Iso}(\mathcal{X}_0, \mathcal{X}) \times_G X_0 \arrow[r, "(1.16)"] \arrow[d, "\widehat{\mathfrak{f}_*}"'] & \underline{\mathfrak{F}}(\mathcal{X}) \arrow[d, "\underline{\mathfrak{F}}(\mathfrak{f})"] \\ \mathrm{Iso}(\mathcal{X}_0, \mathcal{Y}) \times_G X_0 \arrow[r, "(1.16)"'] & \underline{\mathfrak{F}}(\mathcal{Y}) \end{tikzcd}$$
- obtain Lemma 1.2.29 as a consequence of the previous items.

EXERCISE 1.40. The goal of this exercise is to prove a naturality property for the map (1.2.18). Let P, Q be principal spaces with structural groups G, H respectively and let $\phi : P \to Q$ be a morphism of principal spaces with subjacent group homomorphism $\phi_0 : G \to H$. Let N be an H-space; we regard N as a G-space by considering the action of G on N defined by $g \cdot n = \phi_0(g) \cdot n$, for all $g \in G$ and all $n \in N$. Consider the induced map $\hat{\phi} : P \times_G N \to Q \times_H N$ and let $(\hat{\phi})_* : \mathrm{Bij}(N, P \times_G N) \to \mathrm{Bij}(N, Q \times_H N)$ be the map given by composition with $\hat{\phi}$ on the left. Show that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[r, "\phi"] \arrow[dl, "p \mapsto \hat{p}"'] & Q \arrow[dr, "q \mapsto \hat{q}"] & \\ \mathrm{Bij}(N, P \times_G N) \arrow[rrr, "(\hat{\phi})_*"'] & & & \mathrm{Bij}(N, Q \times_H N) \end{tikzcd}$$

Principal fiber bundles.

EXERCISE 1.41. Let M be a differentiable manifold, G be a Lie group, P be a set and let $\Pi : P \to M$ be a map. Assume that for each $x \in M$ we are given a right action of G on the fiber P_x that makes it into a principal space with structural group G. Let $\mathcal{A}$ be an atlas of local sections of Π. Show that:

(a) if two local sections $s_1 : U_1 \to P$, $s_2 : U_2 \to P$ of Π are compatible with every local section that belongs to $\mathcal{A}$ then s_1 and s_2 are compatible with each other;
(b) the set $\mathcal{A}_{\max}$ of all local sections of Π that are compatible with every local section that belongs to $\mathcal{A}$ is the largest atlas of local sections of Π containing $\mathcal{A}$, i.e., $\mathcal{A}_{\max}$ is an atlas of local sections of Π containing $\mathcal{A}$ and $\mathcal{A}_{\max}$ contains every atlas of local sections of Π that contains $\mathcal{A}$;
(c) the set $\mathcal{A}_{\max}$ define on item (b) is a maximal atlas of local sections of Π in the sense that it is not properly contained in any atlas of local sections of Π.

EXERCISE 1.42. Let $\Pi : P \to M$ be a G-principal bundle. Show that:

$$\mathrm{d}\Pi_{p\cdot g}(\zeta \cdot g) = \mathrm{d}\Pi_p(\zeta),$$

for all $p \in P$, $\zeta \in T_pP$ and all $g \in G$.

EXERCISE 1.43. Let P, Q, R be principal bundles over a differentiable manifold M with structural groups G, H and K, respectively. Let $\phi : P \to Q$, $\psi : Q \to R$ be morphisms of principal bundles with subjacent Lie group homomorphisms $\phi_0 : G \to H$ and $\psi_0 : H \to K$ respectively. Show that the composition $\psi \circ \phi : P \to R$ is a morphism of principal bundles with subjacent Lie group homomorphism $\psi_0 \circ \phi_0 : G \to K$.

EXERCISE 1.44. Let $\Pi : P \to M$ be a G-principal bundle. Show that for every $g \in G$, the map $\gamma_g : P \to P$ is an isomorphism of principal bundles whose subjacent Lie group homomorphism is $\mathcal{I}_{g^{-1}} : G \to G$ (compare with Exercise 1.27).

EXERCISE 1.45. Let P, Q be principal bundles over the same differentiable manifold M with structural groups G and H, respectively. Let $\phi : P \to Q$ be a fiber-preserving map and let $\phi_0 : G \to H$ be a Lie group homomorphism such that for every $x \in M$, $\phi|_{P_x} : P_x \to Q_x$ is a morphism of principal spaces with subjacent group homomorphism ϕ_0. Show that if there exists an atlas $\mathcal{A}$ of smooth sections of P such that $\phi \circ s$ is smooth for all s in $\mathcal{A}$ then ϕ is a morphism of principal bundles with subjacent Lie group homomorphism ϕ_0.

EXERCISE 1.46. Let P, Q be principal bundles over the same differentiable manifold M with structural groups G and H, respectively. Let $\phi : P \to Q$ be a morphism of principal bundles and let $\phi_0 : G \to H$ be the Lie group homomorphism subjacent to ϕ. Show that:

- ϕ is injective (resp., surjective) if and only if ϕ_0 is injective (resp., surjective);
- ϕ_0 is injective (resp., surjective) if and only if ϕ is an immersion (resp., a submersion);
- ϕ is a map of constant rank (the rank of ϕ is equal to the dimension of M plus the rank of ϕ_0);
- if ϕ_0 is bijective then $\phi : P \to Q$ is a smooth diffeomorphism and the map $\phi^{-1} : Q \to P$ is a morphism of principal bundles whose subjacent Lie group homomorphism is $\phi_0^{-1} : H \to G$.

EXERCISE 1.47. Let P, Q be principal bundles over the same differentiable manifold M with structural groups G and H, respectively. Let $\phi : P \to Q$ be a morphism of principal bundles and let $\phi_0 : G \to H$ be its subjacent Lie group homomorphism. Given a principal subbundle P' of P with structural group G', show that $\phi(P')$ is a principal subbundle of Q with structural group $\phi_0(G')$.

EXERCISE 1.48. Let $\Pi : P \to M$ be a G-principal bundle and let K be a closed normal subgroup of G. Let P/K denote the quotient set of P consisting of all K-orbits. We have a map:

$$\overline{\Pi} : P/K \ni pK \longmapsto \Pi(p) \in M$$

such that for each $x \in M$ the fiber $(P/K)_x$ is equal to the quotient P_x/K of the principal space P_x by the action of the normal subgroup K of G; the quotient P_x/K is itself a principal space with structural group G/K (recall Exercise 1.21) and the quotient group G/K is a Lie group. Denote by $\mathfrak{q} : P \to P/K$ the quotient map. Show that there exists a unique maximal atlas of local sections of $\overline{\Pi} : P/K \to M$ that makes P/K a (G/K)-principal bundle and the quotient map $\mathfrak{q}$ a morphism of principal bundles whose subjacent Lie group homomorphism is the quotient map $G \to G/K$. We call P/K the *quotient of the principal bundle P by the action of the closed normal subgroup K of G*.

EXERCISE 1.49 (reduction of counter-domain). Let P, Q', Q be principal bundles over a differentiable manifold M with structural groups G, H' and H, respectively. Let $\phi : P \to Q$, $\iota : Q' \to Q$ be morphisms of principal bundles with subjacent Lie group homomorphisms $\phi_0 : G \to H$ and $\iota_0 : H' \to H$ respectively. Assume that ι_0 is injective and that $\phi(P) \subset \iota(Q')$. Show that there exists a unique map $\phi' : P \to Q'$ such that diagram (1.3) commutes; moreover, show that $\phi_0(G) \subset \iota_0(H')$ and that ϕ' is a morphism of principal bundles whose subjacent Lie group homomorphism is the unique map ϕ'_0 for which diagram (1.4) commutes.

EXERCISE 1.50 (passing to the quotient). Let P, $\overline{P}$ and Q be principal bundles over the same differentiable manifold M with structural groups G, $\overline{G}$ and H, respectively. Let $\mathfrak{q} : P \to \overline{P}$, $\phi : P \to Q$ be morphisms of principal bundles with subjacent Lie group homomorphisms $\mathfrak{q}_0 : G \to \overline{G}$ and $\phi_0 : G \to H$ respectively. Assume that $\mathfrak{q}_0$ is surjective and that its kernel is contained in the kernel of ϕ_0. Show that:

- there exists a unique map $\bar{\phi} : \overline{P} \to Q$ for which diagram (1.5) commutes;
- the map $\bar{\phi}$ is a morphism of principal bundles whose subjacent Lie group homomorphism $\bar{\phi}_0 : \overline{G} \to H$ is the unique map for which diagram (1.6) commutes:
- set $K = \mathrm{Ker}(\phi_0)$, $\overline{P} = P/K$, $\overline{G} = G/K$ and take $\mathfrak{q} : P \to P/K$ to be the quotient map. Conclude that $\bar{\phi} : P/K \to \phi(P)$ is an isomorphism of principal bundles whose subjacent Lie group homomorphism is the Lie group isomorphism $\bar{\phi}_0 : G/K \to H$.

EXERCISE 1.51. Let M, $\mathcal{E}$, $\mathcal{E}_0$ be differentiable manifolds and let $\pi : \mathcal{E} \to M$ be a smooth map. We call the quadruple $(M, \mathcal{E}, \pi, \mathcal{E}_0)$ a *fiber bundle* if every point of M has an open neighborhood $U \subset M$ for which there exists a smooth diffeomorphism $\alpha : \pi^{-1}(U) \to U \times \mathcal{E}_0$ such that the diagram:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & U \times \mathcal{E}_0 \arrow[dl, "\text{first projection}"] \\ & U & \end{tikzcd}$$

commutes. Such a map α is called a *local trivialization* of the fiber bundle. We call M the *base space*, $\mathcal{E}$ the *total space*, π the *projection* and $\mathcal{E}_0$ the *typical fiber*. For each $x \in M$, the set $\mathcal{E}_x = \pi^{-1}(x)$ is called the *fiber* over x. Show that the projection π is a surjective submersion and that for each $x \in M$ the fiber $\mathcal{E}_x$ is a smooth submanifold of $\mathcal{E}$ diffeomorphic to $\mathcal{E}_0$.

EXERCISE 1.52. Let $\Pi : P \to M$ be a principal fiber bundle with structural group G. Show that P is a fiber bundle over M with typical fiber G.

Pull-back of principal bundles.

EXERCISE 1.53. Let $\underline{\mathfrak{C}}$ be an arbitrary category, $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{S}$ be objects of $\underline{\mathfrak{C}}$ and $\mathfrak{f} : \mathcal{M}' \to \mathcal{M}$, $\pi : \mathcal{S} \to \mathcal{M}$ be morphisms of $\underline{\mathfrak{C}}$. A *pull-back* of the quintuple $(\mathfrak{f}, \pi, \mathcal{M}, \mathcal{M}', \mathcal{S})$ is a triple $(\mathfrak{f}^*\mathcal{S}, \pi_1, \bar{\mathfrak{f}})$ such that:

- $\mathfrak{f}^*\mathcal{S}$ is an object of $\underline{\mathfrak{C}}$, $\pi_1 : \mathfrak{f}^*\mathcal{S} \to \mathcal{M}'$, $\bar{\mathfrak{f}} : \mathfrak{f}^*\mathcal{S} \to \mathcal{S}$ are morphisms of $\underline{\mathfrak{C}}$ and $\pi \circ \bar{\mathfrak{f}} = \mathfrak{f} \circ \pi_1$;
- given an object $\mathcal{X}$ of $\underline{\mathfrak{C}}$ and morphisms $\tau_1 : \mathcal{X} \to \mathcal{M}'$, $\tau_2 : \mathcal{X} \to \mathcal{S}$ with $\pi \circ \tau_2 = \mathfrak{f} \circ \tau_1$ then there exists a unique morphism $\tau : \mathcal{X} \to \mathfrak{f}^*S$ of $\underline{\mathfrak{C}}$ such that $\pi_1 \circ \tau = \tau_1$ and $\bar{\mathfrak{f}} \circ \tau = \tau_2$.

The notion of pull-back is illustrated by the following commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[dr, dotted, "\tau"] \arrow[drr, bend left, "\tau_2"] \arrow[ddr, bend right, "\tau_1"'] & & \\ & \mathfrak{f}^*\mathcal{S} \arrow[r, "\bar{\mathfrak{f}}"'] \arrow[d, "\pi_1"] & \mathcal{S} \arrow[d, "\pi"] \\ & \mathcal{M}' \arrow[r, "\mathfrak{f}"'] & \mathcal{M} \end{tikzcd} \tag{1.17}$$

The morphism $\bar{\mathfrak{f}}$ is called the *canonical map* of the pull-back $\mathfrak{f}^*\mathcal{S}$. Show that a quintuple $(\mathfrak{f}, \pi, \mathcal{M}, \mathcal{M}', \mathcal{S})$ has at most one pull-back up to isomorphism; this means that if $(\mathfrak{f}^*\mathcal{S}, \pi_1, \bar{\mathfrak{f}})$ and $\big((\mathfrak{f}^*\mathcal{S})', \pi_1', \bar{\mathfrak{f}}'\big)$ are both pull-backs of $(\mathfrak{f}, \pi, \mathcal{M}, \mathcal{M}', \mathcal{S})$ then there exists a unique isomorphism $\phi : \mathfrak{f}^*\mathcal{S} \to (\mathfrak{f}^*\mathcal{S})'$ of $\underline{\mathfrak{C}}$ such that $\pi_1' \circ \phi = \pi_1$ and $\bar{\mathfrak{f}}' \circ \phi = \bar{\mathfrak{f}}$.

EXERCISE 1.54. Let M, M', S be sets and $f : M' \to M$, $\pi : S \to M$ be maps. Let f^*S denote the subset of the cartesian product $M' \times S$ defined by:

$$f^*S = \big\{(y, p) \in M' \times S : f(y) = \pi(p)\big\}. \tag{1.18}$$

Denote by $\pi_1 : f^*S \to M'$, $\bar{f} : f^*S \to S$ the restrictions to f^*S of the projections of $M' \times S$. Show that $(f^*S, \pi_1, \bar{f})$ is a pull-back of (f, π, M, M', S) in the category of sets and maps.

EXERCISE 1.55. Let M, M', S be differentiable manifolds and $f : M' \to M$, $\pi : S \to M$ be smooth maps such that at least one of them is a submersion[9]. Show that:

- the set (1.18) is a smooth submanifold of the cartesian product $M' \times S$;
- for all $(y, p) \in f^*S$, the tangent space $T_{(y,p)}(f^*S)$ is given by:

$$T_{(y,p)}(f^*S) = \big\{(v, \zeta) \in T_yM' \oplus T_\zeta S : \mathrm{d}f_y(v) = \mathrm{d}\pi_p(\zeta)\big\}; \tag{1.19}$$

- if $\pi_1 : f^*S \to M'$, $\bar{f} : f^*S \to S$ denote the restrictions of the projections of $M' \times S$ then the triple $(f^*S, \pi_1, \bar{f})$ is a pull-back of (f, π, M, M', S) in the category of differentiable manifolds and smooth maps.

The fiberwise product of principal bundles.

EXERCISE 1.56. Let M be a differentiable manifold and let P_0, Q_0 be principal spaces whose structural groups are Lie groups; consider the trivial principal bundles $P = M \times P_0$ and $Q = M \times Q_0$ (see Example 1.3.2). The fiberwise product $P \star Q$ can be naturally identified as a set with $M \times (P_0 \times Q_0)$. Show that $P \star Q$ is also a trivial principal bundle, i.e., if $M \times (P_0 \times Q_0)$ is regarded as a trivial principal bundle then the identification of $P \star Q$ with $M \times (P_0 \times Q_0)$ is an isomorphism of principal bundles whose subjacent Lie group homomorphism is the identity (recall from Example 1.2.7 that $P_0 \times Q_0$ is also a principal space whose structural group is a Lie group).

EXERCISE 1.57. Let $\Pi : P \to M$, $\Pi' : Q \to M$ be G-principal bundles. Consider the pull-back $\Pi_1 : \Pi^*Q \to P$ of the principal bundle Q by the map Π and the fiberwise product $\Pi \star \Pi' : P \star Q \to M$. Show that there exists a unique map $\Upsilon : \Pi^*Q \to P \star Q$ such that the diagram:

$$\begin{tikzcd}
 & & P \\
\Pi^*Q \arrow[urr, "\Pi_1"] \arrow[drr, "\overline{\Pi}"'] \arrow[r, dotted, "\Upsilon"] & P \star Q \arrow[ur, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\
 & & Q
\end{tikzcd}$$

[9]In fact, the following weaker hypothesis also works: for every $y \in M'$, $p \in S$ with $f(y) = \pi(p)$, the tangent space $T_{f(y)}(M)$ equals the sum of the images of the linear maps $\mathrm{d}f(y)$ and $\mathrm{d}\pi(p)$.

commutes. Moreover, show that Υ is a smooth diffeomorphism and that the diagram:

$$\begin{tikzcd} P\star Q \arrow[dr, dotted, "\Upsilon^{-1}"] \arrow[drr, bend left, "\mathrm{pr}_2"] \arrow[dd, bend right, "\mathrm{pr}_1"'] & & \\ & \Pi^*Q \arrow[r, "\overline{\Pi}"'] \arrow[d, "\Pi_1"] & Q \\ & P & \end{tikzcd}$$

commutes as well.

Associated bundles.

EXERCISE 1.58. Let $\Pi : P \to M$ be a G-principal bundle and consider the action of G on itself by left translations. Show that the map:

$$P \ni p \longmapsto [p, 1] \in P \times_G G$$

is a smooth fiber-preserving diffeomorphism.

EXERCISE 1.59. Let $\Pi : P \to M$ be a G-principal bundle and N be a differentiable G-space. For each $x \in M$, we have an action of $\mathrm{Left}(P_x)$ on $P_x \times_G N$ given by:

$$\mathrm{Left}(P_x) \times (P_x \times_G N) \ni \big(t, [p, n]\big) \longmapsto \hat{t}\big([p, n]\big) = [t(p), n] \in P_x \times_G N. \tag{1.20}$$

This action is effective if the action of G on N is effective (see Exercise 1.31). Show that, for fixed $p \in P_x$, if we identify $\mathrm{Left}(P_x)$ with G via $\mathcal{I}_p$ and $P_x \times_G N$ with N via $\hat{p}$ then the action (1.20) is identified with the action of G on N; more precisely, show that the diagram:

$$\begin{tikzcd} \mathrm{Left}(P_x) \times (P_x \times_G N) \arrow[r, "(1.20)"] & P_x \times_G N \\ G \times N \arrow[u, "\mathcal{I}_p \times \hat{p}", "\cong"'] \arrow[r, "\text{action}"'] & N \arrow[u, "\cong", "\hat{p}"'] \end{tikzcd}$$

commutes. Conclude that the action (1.20) is smooth.

EXERCISE 1.60. Let $\Pi : P \to M$ be a G-principal bundle and consider the union:

$$\mathrm{Left}(P) = \bigcup_{x \in M} \mathrm{Left}(P_x).$$

Let G act on itself on the left by conjugation. The result of Exercise 1.33 implies that the map:

$$P \times_G G \ni [p, g] \longmapsto \mathcal{I}_p(g) \in \mathrm{Left}(P) \tag{1.21}$$

is a fiber-preserving bijection. We endow $\mathrm{Left}(P)$ with the unique differential structure that makes (1.21) a smooth diffeomorphism. Show that:

- $\mathrm{Left}(P)$ is a fiber bundle over M with typical fiber G;

- the map $\mathrm{Left}(P) \star P \ni (t,p) \mapsto t(p) \in P$ is smooth;
- if N is a differentiable manifold and $(g,n) \mapsto g \cdot n$ is a smooth left action of G on N then the map:
$$\mathrm{Left}(P) \star (P \times_G N) \ni \big(t,[p,n]\big) \longmapsto [t(p),n] \in P \times_G N$$
is smooth.

Vector bundles.

EXERCISE 1.61 (change of typical fiber). Let $\pi : E \to M$ be a vector bundle with typical fiber E_0, let E_1 be a real vector space and let $\mathfrak{i} : E_1 \to E_0$ be a linear isomorphism. Consider the map:
$$\gamma_{\mathfrak{i}} : \mathrm{FR}_{E_0}(E) \ni p \longmapsto p \circ \mathfrak{i} \in \mathrm{FR}_{E_1}(E).$$

- Use Lemma 1.3.11 to show that there exists a unique maximal atlas of local sections of $\mathrm{FR}_{E_1}(E) \to M$ that makes $\gamma_{\mathfrak{i}}$ into an isomorphism of principal bundles whose subjacent Lie group homomorphism is $\mathcal{I}_{\mathfrak{i}}^{-1}$, where:
$$\mathcal{I}_{\mathfrak{i}} : \mathrm{GL}(E_1) \ni T \longmapsto \mathfrak{i} \circ T \circ \mathfrak{i}^{-1} \in \mathrm{GL}(E_0).$$
- Show that the maximal atlas of local sections of $\mathrm{FR}_{E_1}(E) \to M$ that makes $\gamma_{\mathfrak{i}}$ into an isomorphism of principal bundles does not depend on the choice of the linear isomorphism $\mathfrak{i} : E_1 \to E_0$.
- The construction above allows us to regard $\pi : E \to M$ as a vector bundle with typical fiber E_1. Show that the differential structure on the total space E does not change when the typical fiber is changed from E_0 to E_1.

EXERCISE 1.62. Let $\pi : E \to M$ be a vector bundle over a differentiable manifold M and let $\epsilon : U \to E$ be a smooth local section. Given $x \in U$, and an open neighborhood V of x in M with $\overline{V} \subset U$, show that there exists a smooth global section $\bar{\epsilon} \in \mathbf{\Gamma}(E)$ such that $\bar{\epsilon}|_V = \epsilon|_V$.

EXERCISE 1.63. Let $\pi : E \to M$, $\pi' : F \to M$ be vector bundles over a differentiable manifold M and let $L : \mathbf{\Gamma}(E) \to \mathbf{\Gamma}(F)$ be a $C^\infty(M)$-linear map. Given $x \in M$ show that there exists a linear map $L_x : E_x \to F_x$ such that $L(\epsilon)(x) = L_x\big(\epsilon(x)\big)$, for all $\epsilon \in \mathbf{\Gamma}(E)$.

More generally, given vector bundles $\pi^i : E^i \to M$, $i = 1, \ldots, n$ and a $C^\infty(M)$-multilinear map $B : \mathbf{\Gamma}(E^1) \times \cdots \times \mathbf{\Gamma}(E^n) \to \mathbf{\Gamma}(F)$, show that for every $x \in M$ there exists a multilinear map $B_x : E^1_x \times \cdots \times E^n_x \to F_x$ such that $B(\epsilon_1, \ldots, \epsilon_n)(x) = B_x\big(\epsilon_1(x), \ldots, \epsilon_n(x)\big)$, for all $\epsilon_i \in \mathbf{\Gamma}(E^i)$, $i = 1, \ldots, n$.

REMARK. The result of Exercise 1.63 does not hold for infinite-dimensional Hilbert vector bundles. See [**2**] for a counter-example.

EXERCISE 1.64. Let $\mathcal{E}$, M be differentiable manifolds and $\pi : \mathcal{E} \to M$ be a smooth submersion. Show that:
$$\bigcup_{e \in \mathcal{E}} \mathrm{Ker}\big(\mathrm{d}\pi(e)\big) \subset T\mathcal{E}$$

is a smooth distribution on $\mathcal{E}$.

Pull-back of vector bundles.

EXERCISE 1.65. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and let E_1 be a real vector space isomorphic to E_0. As we have seen in Exercise 1.61, the vector bundle E can also be regarded as a vector bundle with typical fiber E_1; denote such vector bundle with changed typical fiber by $\overline{E}$. Given a smooth map $f : M' \to M$ defined in a differentiable manifold M', show that the pull-backs f^*E and $f^*\overline{E}$ differ only by their typical fibers.

Functorial constructions with vector bundles.

EXERCISE 1.66. Let $n \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Given objects $(V_1, \ldots, V_n)$, $(W_1, \ldots, W_n)$ of $\underline{\mathfrak{Vec}}^n$, show that the map:

$$\begin{array}{c}\mathrm{Iso}(V_1, W_1) \times \cdots \times \mathrm{Iso}(V_n, W_n)\\ \Big\downarrow \underline{\mathfrak{F}}\\ \mathrm{Iso}\big(\underline{\mathfrak{F}}(V_1, \ldots, V_n), \underline{\mathfrak{F}}(W_1, \ldots, W_n)\big)\end{array}$$

is smooth.

EXERCISE 1.67. Let $n \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Given an isomorphism $(T_1, \ldots, T_n)$ from an object $(V_1, \ldots, V_n)$ of $\underline{\mathfrak{Vec}}^n$ to an object $(W_1, \ldots, W_n)$ of $\underline{\mathfrak{Vec}}^n$, show that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{gl}(V_1) \oplus \cdots \oplus \mathfrak{gl}(V_n) \arrow[r, "\underline{\mathfrak{f}}"] \arrow[d, "\mathcal{I}_{T_1} \oplus \cdots \oplus \mathcal{I}_{T_n}"'] & \mathfrak{gl}\big(\underline{\mathfrak{F}}(V_1, \ldots, V_n)\big) \arrow[d, "\mathcal{I}_{\underline{\mathfrak{F}}(T_1, \ldots, T_n)}"] \\
\mathfrak{gl}(W_1) \oplus \cdots \oplus \mathfrak{gl}(W_n) \arrow[r, "\underline{\mathfrak{f}}"'] & \mathfrak{gl}\big(\underline{\mathfrak{F}}(W_1, \ldots, W_n)\big)
\end{tikzcd}$$

where $\mathcal{I}_T$ denotes conjugation by an isomorphism T.

EXERCISE 1.68. Let $n \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Let $E^1, \ldots, E^n$ be vector bundles over a differentiable manifold M with typical fibers $E_0^1, \ldots, E_0^n$, respectively. For each $i = 1, \ldots, n$, let $\overline{E}_0^i$ be a real vector space isomorphic to E_0^i. As we have seen in Exercise 1.61, the vector bundle E^i can also be regarded as a vector bundle with typical fiber $\overline{E}_0^i$; denote such vector bundle with changed typical fiber by $\overline{E}^i$. Show that the vector bundles $\underline{\mathfrak{F}}(E^1, \ldots, E^n)$ and $\underline{\mathfrak{F}}(\overline{E}^1, \ldots, \overline{E}^n)$ differ only by their typical fibers.

EXERCISE 1.69. Let $k \geq 1$ be fixed and let Z be a fixed real finite-dimensional vector space. Consider the smooth functors $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^k \to \underline{\mathfrak{Vec}}$, $\underline{\mathfrak{G}} : \underline{\mathfrak{Vec}}^{k+1} \to \underline{\mathfrak{Vec}}$ defined by:

$$\underline{\mathfrak{F}}(V_1, \ldots, V_k) = \mathrm{Lin}(V_1, \ldots, V_k; Z),$$
$$\underline{\mathfrak{G}}(V_1, \ldots, V_k, W) = \mathrm{Lin}(V_1, \ldots, V_k; W);$$

the definitions of $\underline{\mathfrak{F}}$ and $\underline{\mathfrak{G}}$ on morphisms are as in Example 1.6.13. Given vector bundles $E^1, \ldots, E^k$ over a differentiable manifold M, show that:

$$\underline{\mathfrak{F}}(E^1, \ldots, E^k) = \underline{\mathfrak{G}}(E^1, \ldots, E^k, M \times Z).$$

EXERCISE 1.70. Let E^1, E^2, F be vector bundles over a differentiable manifold M and denote by $\mathrm{pr}_i : E^1 \oplus E^2 \to E^i$, $\iota_i : E^i \to E^1 \oplus E^2$, $i = 1, 2$, the projections and the inclusion maps, respectively.

(a) Given morphisms of vector bundles $L^i : F \to E^i$, $i = 1, 2$, show that there exists a unique morphism of vector bundles $L : F \to E$ such that $\mathrm{pr}_i \circ L = L^i$, for $i = 1, 2$.
(b) Given morphisms of vector bundles $L^i : E^i \to F$, $i = 1, 2$, show that there exists a unique morphism of vector bundles $L : E \to F$ such that $L \circ \iota_i = L^i$, for $i = 1, 2$.

EXERCISE 1.71. Let E^1, E^2 be vector bundles over a differentiable manifold M. Show that the natural inclusion map from the Whitney sum $E^1 \oplus E^2$ to the cartesian product $E^1 \times E^2$ is a smooth embedding. Prove analogues of Corollaries 1.3.26 and 1.3.27 to Whitney sums.

EXERCISE 1.72. Under the conditions of Exercise 1.63, show that the map $x \mapsto B_x$ is a smooth section of the vector bundle $\mathrm{Lin}(E^1, \ldots, E^n; F)$.

G-structures on vector bundles.

EXERCISE 1.73. Let A be a Lie group and M be a differentiable A-space. We define a smooth left action of A on $\mathrm{FR}(TM)$ by setting:

$$g \cdot p = \mathrm{d}\gamma_g(x) \circ p,$$

for all $x \in M$, $p \in \mathrm{FR}(T_xM)$ and all $g \in A$. Let $x_0 \in M$ be fixed and consider the *isotropic representation* ρ_{x_0} of A_{x_0} on $T_{x_0}M$ defined by:

$$\rho_{x_0} : A_{x_0} \ni g \longmapsto \mathrm{d}\gamma_g(x_0) \in \mathrm{GL}(T_{x_0}M).$$

Let $p_0 \in \mathrm{FR}(T_{x_0}M)$ be fixed and consider the group isomorphism

$$\mathcal{I}_{p_0} : \mathrm{GL}(\mathbb{R}^n) \longrightarrow \mathrm{GL}(T_{x_0}M)$$

defined by $\mathcal{I}_{p_0}(T) = p_0 \circ T \circ p_0^{-1}$, for all $T \in \mathrm{GL}(\mathbb{R}^n)$. Set $G = \mathcal{I}_{p_0}^{-1}\big(\rho_{x_0}(A_{x_0})\big) \subset \mathrm{GL}(\mathbb{R}^n)$. If the action of A on M is transitive, show that the A-orbit of p_0 in $\mathrm{FR}(TM)$ is a G-structure on M.

EXERCISE 1.74. Let M, M' be n-dimensional differentiable manifolds, $f : M \to M'$ be a smooth diffeomorphism, G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$ and P be a G-structure on M. Show that:

$$P' = \big\{\mathrm{d}f \circ p : p \in P\big\}$$

is the unique G-structure on M' that makes f into a G-structure preserving map.

EXERCISE 1.75. Let $\pi : E \to M$ be a vector bundle endowed with a semi-Riemannian structure g and let E' be a vector subbundle of E. The *orthogonal subbundle* of E' in E is defined by:

$$(E')^\perp = \bigcup_{x \in M} \big\{ e \in E_x : g_x(e, e') = 0, \text{ for all } e' \in E'_x \big\}.$$

(a) Show that $(E')^\perp$ is a vector subbundle of E.
(b) If E' is *nondegenerate* for g in the sense that the restriction of g_x to $E'_x \times E'_x$ is a nondegenerate symmetric bilinear form on E'_x, for all $x \in M$, show that $E = E' \oplus (E')^\perp$.

CHAPTER 2

The theory of connections

2.1. The general concept of connection

Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and let $\epsilon \in \mathbf{\Gamma}(E)$ be a smooth section of E. If $E = M \times E_0$ is the trivial vector bundle over M then ϵ is of the form $\epsilon(x) = \big(x, \tilde{\epsilon}(x)\big)$, where $\tilde{\epsilon} : M \to E_0$ is a smooth map; let us identity the smooth section ϵ of $E = M \times E_0$ with the smooth map $\tilde{\epsilon} : M \to E_0$. Given a point $x \in M$ and a tangent vector $v \in T_xM$, we can consider the directional derivative $\mathrm{d}\tilde{\epsilon}(x) \cdot v$ of $\tilde{\epsilon}$ at the point x, in the direction of v. In general, if E is an arbitrary vector bundle, what sense can be made of the directional derivative of a smooth section $\epsilon \in \mathbf{\Gamma}(E)$ at a point $x \in M$, in the direction of a vector $v \in T_xM$? Let us first approach the problem by considering a smooth local E_0-frame $s : U \to \mathrm{FR}_{E_0}(E)$ with $x \in U$. Let $\tilde{\epsilon} : U \to E_0$ denote the representation of $\epsilon|_U$ with respect to s. The directional derivative $\mathrm{d}\tilde{\epsilon}(x) \cdot v$ is an element of the typical fiber E_0 and it corresponds via the isomorphism $s(x) : E_0 \to E_x$ to a vector of the fiber E_x; an apparently reasonable attempt at defining the directional derivative of ϵ at the point x in the direction of v is:

$$\text{directional derivative of } \epsilon \text{ at the point } x \text{ in the direction of } v \\ = s(x)\big(\mathrm{d}\tilde{\epsilon}(x) \cdot v\big).$$

Of course, in order to check that such definition makes sense, one has to look at what happens when another smooth local E_0-frame $s' : V \to \mathrm{FR}_{E_0}(E)$ with $x \in V$ is chosen. Let $g : U \cap V \to \mathrm{GL}(E_0)$ denote the transition map from s' to s, so that $s(y) = s'(y) \circ g(y)$, for all $y \in U \cap V$; then:

$$\epsilon(y) = s(y) \cdot \tilde{\epsilon}(y) = s'(y) \cdot \big(g(y) \cdot \tilde{\epsilon}(y)\big),$$

for all $y \in U \cap V$, so that the representation $\tilde{\epsilon}'$ of $\epsilon|_V$ with respect to s' satisfies:

$$\tilde{\epsilon}'(y) = g(y) \cdot \tilde{\epsilon}(y),$$

for all $y \in U \cap V$. Then:

$$\mathrm{d}\tilde{\epsilon}'(x) \cdot v = \big(\mathrm{d}g(x) \cdot v\big) \cdot \tilde{\epsilon}(x) + g(x) \cdot \big(\mathrm{d}\tilde{\epsilon}(x) \cdot v\big),$$

and:

$$s'(x)\big(\mathrm{d}\tilde{\epsilon}'(x) \cdot v\big) = s(x)\Big(g(x)^{-1}\big[\big(\mathrm{d}g(x) \cdot v\big) \cdot \tilde{\epsilon}(x)\big]\Big) + s(x)\big(\mathrm{d}\tilde{\epsilon}(x) \cdot v\big).$$

The presence of the first term in the righthand side of the equality above shows that our plan for defining the directional derivative of a smooth section ϵ didn't work. Let us look at the problem from a different angle. If $\epsilon : M \to E$ is a smooth section of E then for every $x \in M$ we can consider the differential $\mathrm{d}\epsilon(x)$, which is a linear map from T_xM to $T_{\epsilon(x)}E$; for every $v \in T_xM$, we have therefore $\mathrm{d}\epsilon(x) \cdot v \in T_{\epsilon(x)}E$. In the case that $E = M \times E_0$ is the trivial bundle over M then ϵ is of the form $\epsilon(x) = \big(x, \tilde{\epsilon}(x)\big)$ and:

$$\mathrm{d}\epsilon(x) \cdot v = \big(v, \mathrm{d}\tilde{\epsilon}(x) \cdot v\big) \in T_{\epsilon(x)}E = T_xM \oplus E_0.$$

Hence, in the case of the trivial bundle, the object that we wish to call the directional derivative of ϵ at the point x in the direction of v is the second coordinate of the vector $\mathrm{d}\epsilon(x) \cdot v$. If $\pi : E \to M$ is a general vector bundle then $\mathrm{d}\epsilon(x) \cdot v$ is just an element of $T_{\epsilon(x)}E$ and it makes no sense to talk about the "second coordinate" of $\mathrm{d}\epsilon(x) \cdot v$. Notice that, since $\pi \circ \epsilon$ is the identity map of M, we have:

$$\mathrm{d}\pi_{\epsilon(x)}\big(\mathrm{d}\epsilon(x) \cdot v\big) = v,$$

so that, just in the case of the trivial bundle, the vector $\mathrm{d}\epsilon(x) \cdot v$ contains v as one of its components. The difficulty here is that there is no canonical way of extracting the "other component" from $\mathrm{d}\epsilon(x) \cdot v$. More precisely, the difficulty is that we don't have a direct sum decomposition $T_xM \oplus E_0$ of $T_{\epsilon(x)}E$ just like we had in the case of the trivial bundle $M \times E_0$. We have a canonical subspace $\mathrm{Ver}_{\epsilon(x)}E$ of $T_{\epsilon(x)}E$ (recall Definition 1.5.6) but such subspace has no canonical complement in the case of a general vector bundle E.

The problems we have encountered in the attempts to define a notion of directional derivative for sections of an arbitrary vector bundle indicate that indeed *no canonical notion of directional derivative for sections of general vector bundles exists*. In order to define such a notion, the vector bundle E has to be endowed with some additional structure. The additional structure on E that will allow us to define a notion of directional derivative for smooth sections of E is what we shall call a *connection* on E. In order to make this definition precise, we start by considering the problem of lack of a natural complement for the vertical space $\mathrm{Ver}_e(E)$ in the tangent space to the total space T_eE. Let us give some definitions.

DEFINITION 2.1.1. Let $\mathcal{E}$, M be differentiable manifolds and let $\pi : \mathcal{E} \to M$ be a smooth submersion. Given $e \in \mathcal{E}$ then the space $\mathrm{Ker}\big(\mathrm{d}\pi(e)\big)$ is called the *vertical subspace* of $T_e\mathcal{E}$ at the point e with respect to the submersion π; assuming that the submersion π is fixed by the context, we denote the vertical subspace by $\mathrm{Ver}_e(\mathcal{E})$. A subspace H of $T_e\mathcal{E}$ is called *horizontal* with respect to π if it is a complement of $\mathrm{Ver}_e(\mathcal{E})$ in $T_e\mathcal{E}$, i.e., if:

$$T_e\mathcal{E} = H \oplus \mathrm{Ver}_e(\mathcal{E}).$$

A distribution $\mathcal{H}$ on the manifold $\mathcal{E}$ is called *horizontal* with respect to π if $\mathcal{H}_e$ is a horizontal subspace of $T_e\mathcal{E}$ for every $e \in \mathcal{E}$. A smooth horizontal distribution on $\mathcal{E}$ will also be called a *generalized connection* on $\mathcal{E}$ (with respect to π).

Notice that for all $x \in M$, $\pi^{-1}(x)$ is a smooth submanifold of $\mathcal{E}$ and for every $e \in \pi^{-1}(x)$ we have:

$$\mathrm{Ver}_e(\mathcal{E}) = T_e\big(\pi^{-1}(x)\big).$$

We set:

$$\mathrm{Ver}(\mathcal{E}) = \bigcup_{e\in\mathcal{E}} \mathrm{Ver}_e(\mathcal{E}) \subset T\mathcal{E}.$$

The result of Exercise 1.64 says that $\mathrm{Ver}(\mathcal{E})$ is a smooth distribution on $\mathcal{E}$. We call it the *vertical distribution* on $\mathcal{E}$ or also the *vertical bundle* of $\mathcal{E}$ determined by π.

Notice that a subspace H of $T_e\mathcal{E}$ is horizontal with respect to π if and only if the restriction of $\mathrm{d}\pi(e)$ to H is an isomorphism onto $T_{\pi(e)}M$ (see Exercise 2.1).

When a horizontal distribution on $\mathcal{E}$ is fixed by the context we will usually denote it by $\mathrm{Hor}(\mathcal{E})$; then:

$$T_e\mathcal{E} = \mathrm{Hor}_e(\mathcal{E}) \oplus \mathrm{Ver}_e(\mathcal{E}), \tag{2.1.1}$$

for all $e \in \mathcal{E}$. We denote by $\mathfrak{p}_{\mathrm{ver}} : T\mathcal{E} \to \mathrm{Ver}(\mathcal{E})$ (resp., $\mathfrak{p}_{\mathrm{hor}} : T\mathcal{E} \to \mathrm{Hor}(\mathcal{E})$) the map whose restriction to $T_e\mathcal{E}$ is equal to the projection onto the second coordinate (resp., the first coordinate) corresponding to the direct sum decomposition (2.1.1), for all $e \in \mathcal{E}$. We call $\mathfrak{p}_{\mathrm{ver}}$ (resp., $\mathfrak{p}_{\mathrm{hor}}$) the *vertical projection* (resp., the *horizontal projection*) determined by the horizontal distribution $\mathrm{Hor}(\mathcal{E})$. Notice that if $\mathrm{Hor}(\mathcal{E})$ is a smooth distribution then the projections $\mathfrak{p}_{\mathrm{ver}}$ and $\mathfrak{p}_{\mathrm{hor}}$ are morphisms of vector bundles; in this case, we also call $\mathrm{Hor}(\mathcal{E})$ the *horizontal bundle* of $\mathcal{E}$.

DEFINITION 2.1.2. Let $\mathcal{E}$, M be differentiable manifolds and let $\pi : \mathcal{E} \to M$ be a smooth submersion. By a *local section* of π we mean a map $\epsilon : U \to \mathcal{E}$ defined on an open subset U of M such that $\pi \circ \epsilon$ is the inclusion map of U in M. Let $\mathrm{Hor}(\mathcal{E})$ be a generalized connection on $\mathcal{E}$. If $\epsilon : U \to \mathcal{E}$ is a smooth local section of π then, given $x \in U$, $v \in T_xM$, the *covariant derivative* of ϵ at the point x in the direction of v with respect to the generalized connection $\mathrm{Hor}(\mathcal{E})$ is denoted by $\nabla_v\epsilon$ and it is defined by:

$$\nabla_v\epsilon = \mathfrak{p}_{\mathrm{ver}}\big(\mathrm{d}\epsilon(x)\cdot v\big) \in \mathrm{Ver}_{\epsilon(x)}(\mathcal{E}); \tag{2.1.2}$$

we call ∇ the *covariant derivative operator* associated to the generalized connection $\mathrm{Hor}(\mathcal{E})$. Given $x \in U$, if $\nabla_v\epsilon = 0$, for all $v \in T_xM$ then the local section ϵ is said to be *parallel at* x with respect to $\mathrm{Hor}(\mathcal{E})$; if ϵ is parallel at every $x \in U$ we say simply that ϵ is *parallel* with respect to $\mathrm{Hor}(\mathcal{E})$.

Clearly the covariant derivative $\nabla_v\epsilon$ is linear in v. Moreover, ϵ is parallel at x with respect to $\mathrm{Hor}(\mathcal{E})$ if and only if:

$$\mathrm{d}\epsilon_x(T_xM) = \mathrm{Hor}_{\epsilon(x)}\mathcal{E}.$$

DEFINITION 2.1.3. Let $\pi : \mathcal{E} \to M$, $\pi' : \mathcal{E}' \to M$ be smooth submersions; a map $\phi : \mathcal{E} \to \mathcal{E}'$ is said to be *fiber preserving* if:

$$\pi' \circ \phi = \pi.$$

Let $\mathrm{Hor}(\mathcal{E})$, $\mathrm{Hor}(\mathcal{E}')$ be generalized connections on $\mathcal{E}$ and $\mathcal{E}'$ respectively. A smooth map $\phi : \mathcal{E} \to \mathcal{E}'$ is said to be *connection preserving* if it is fiber preserving and:

$$\mathrm{d}\phi_e\big(\mathrm{Hor}_e(\mathcal{E})\big) = \mathrm{Hor}_{\phi(e)}(\mathcal{E}'), \tag{2.1.3}$$

for all $e \in \mathcal{E}$.

Clearly the composition of fiber preserving (resp., connection preserving) maps is also fiber preserving (resp., connection preserving). Moreover, the inverse of a bijective fiber preserving map (resp., of a smooth connection preserving diffeomorphism) is also fiber preserving (resp., connection preserving).

Observe that if $\phi : \mathcal{E} \to \mathcal{E}'$ is fiber preserving and if $\epsilon : U \to \mathcal{E}$ is a local section of π then $\phi \circ \epsilon : U \to \mathcal{E}'$ is a local section of π'. If $\phi : \mathcal{E} \to \mathcal{E}'$ is a smooth fiber preserving map then for all $x \in M$ and all $e \in \pi^{-1}(x)$ the following diagram commutes:

(2.1.4)
$$\begin{tikzcd} T_e\mathcal{E} \arrow[rr, "\mathrm{d}\phi_e"] \arrow[dr, "\mathrm{d}\pi_e"'] & & T_{\phi(e)}\mathcal{E}' \arrow[dl, "\mathrm{d}\pi'_{\phi(e)}"] \\ & T_xM & \end{tikzcd}$$

In particular, we have:

$$\mathrm{d}\phi_e\big(\mathrm{Ver}_e(\mathcal{E})\big) \subset \mathrm{Ver}_{\phi(e)}(\mathcal{E}'). \tag{2.1.5}$$

DEFINITION 2.1.4. A smooth submersion $\pi : \mathcal{E} \to M$ is said to have the *global extension property* if for every smooth local section $\epsilon : U \to \mathcal{E}$ of π and every $x \in U$ there exists a smooth global section $\bar{\epsilon} : M \to \mathcal{E}$ such that ϵ and $\bar{\epsilon}$ are equal on some neighborhood of x contained in U.

The result of Exercise 1.62 shows that the projection of a vector bundle has the global extension property.

LEMMA 2.1.5. *Let $\pi : \mathcal{E} \to M$, $\pi' : \mathcal{E}' \to M$ be smooth submersions and let $\mathrm{Hor}(\mathcal{E})$, $\mathrm{Hor}(\mathcal{E}')$ be generalized connections on $\mathcal{E}$ and $\mathcal{E}'$ respectively. Denote by ∇ and ∇' respectively the covariant derivative operators corresponding to $\mathrm{Hor}(\mathcal{E})$ and $\mathrm{Hor}(\mathcal{E}')$. Given a smooth fiber preserving map $\phi : \mathcal{E} \to \mathcal{E}'$ then the following conditions are equivalent:*

(a) *ϕ is connection preserving;*
(b) *$\mathrm{d}\phi_e\big(\mathrm{Hor}_e(\mathcal{E})\big) \subset \mathrm{Hor}_{\phi(e)}(\mathcal{E}')$, for all $e \in \mathcal{E}$;*
(c) *for any smooth local section $\epsilon : U \to \mathcal{E}$ of π, it is:*

$$\nabla'_v(\phi \circ \epsilon) = \mathrm{d}\phi_{\epsilon(x)}(\nabla_v \epsilon), \tag{2.1.6}$$

for all $x \in U$ and all $v \in T_xM$.

If $\pi : \mathcal{E} \to M$ has the global extension property then conditions (a), (b) and (c) are also equivalent to:

(d) *for any smooth global section $\epsilon : M \to \mathcal{E}$ of π, equality* (2.1.6) *holds, for all $x \in M$ and all $v \in T_xM$.*

PROOF. The equivalence between (a) and (b) follows from the commutativity of diagram (2.1.4), applying the results of Exercises 2.2 and 2.3. Now assume (a) and let us prove (c). Denote by $\mathfrak{p}_{\mathrm{ver}}$ and $\mathfrak{p}'_{\mathrm{ver}}$ the vertical projections determined by $\mathrm{Hor}(\mathcal{E})$ and by $\mathrm{Hor}(\mathcal{E}')$, respectively. From (2.1.3) and (2.1.5) we get that:

$$\mathfrak{p}'_{\mathrm{ver}}\big(\mathrm{d}\phi_e(\zeta)\big) = \mathrm{d}\phi_e\big(\mathfrak{p}_{\mathrm{ver}}(\zeta)\big),$$

for all $e \in \mathcal{E}$ and all $\zeta \in T_e\mathcal{E}$. Thus, given a smooth local section $\epsilon : U \to \mathcal{E}$ of π, we have:

$$\nabla'_v(\phi \circ \epsilon) = \mathfrak{p}'_{\mathrm{ver}}\big[\mathrm{d}\phi_{\epsilon(x)}\big(\mathrm{d}\epsilon_x(v)\big)\big] = \mathrm{d}\phi_{\epsilon(x)}\big[\mathfrak{p}_{\mathrm{ver}}\big(\mathrm{d}\epsilon_x(v)\big)\big] \\ = \mathrm{d}\phi_{\epsilon(x)}(\nabla_v\epsilon),$$

for all $x \in U$ and all $v \in T_xM$. This proves (c). Conversely, assume (c) and let us prove (a). Let $e \in \mathcal{E}$ be fixed and set $\pi(e) = x \in M$. Choose an arbitrary submanifold S of $\mathcal{E}$ with $e \in S$ and $T_eS = \mathrm{Hor}_e(\mathcal{E})$. Since:

$$\mathrm{d}(\pi|_S)_e = \mathrm{d}\pi_e|_{T_eS} : T_eS \longrightarrow T_xM$$

is an isomorphism then, possibly taking a smaller S, we may assume that $\pi|_S$ is a smooth diffeomorphism onto an open neighborhood U of x in M. Then:

$$\epsilon = (\pi|_S)^{-1} : U \longrightarrow \mathcal{E}$$

is a smooth local section of π, $\epsilon(x) = e$ and ϵ is parallel at x with respect to $\mathrm{Hor}(\mathcal{E})$. Now (2.1.6) implies that $\phi \circ \epsilon$ is parallel at x with respect to $\mathrm{Hor}(\mathcal{E}')$ and hence:

$$\mathrm{d}\phi_e\big(\mathrm{Hor}_e(\mathcal{E})\big) = (\mathrm{d}\phi_e \circ \mathrm{d}\epsilon_x)(T_xM) = \mathrm{d}(\phi \circ \epsilon)_x(T_xM) = \mathrm{Hor}_{\phi(e)}(\mathcal{E}'),$$

proving (a). Finally, assume that $\pi : \mathcal{E} \to M$ has the global extension property and let us prove that (a), (b) and (c) are all equivalent to (d). It is obvious that (c) implies (d). The proof of the fact that (d) implies (a) can be done by repeating the same steps of our proof that (c) implies (a), keeping in mind that the smooth local section $\epsilon : U \to \mathcal{E}$ of π constructed in that proof can be replaced by a smooth global section $\bar{\epsilon} : M \to \mathcal{E}$. $\square$

COROLLARY 2.1.6. *Let $\pi : \mathcal{E} \to M$ be a smooth submersion endowed with generalized connections $\mathrm{Hor}(\mathcal{E})$ and $\mathrm{Hor}'(\mathcal{E})$; denote by ∇ and ∇' respectively the covariant derivative operators corresponding to $\mathrm{Hor}(\mathcal{E})$ and $\mathrm{Hor}'(\mathcal{E})$. If:*

$$\nabla_v\epsilon = \nabla'_v\epsilon, \tag{2.1.7}$$

for every smooth local section $\epsilon : U \to \mathcal{E}$ of π and for every $v \in TM|_U$ then $\mathrm{Hor}(\mathcal{E}) = \mathrm{Hor}'(\mathcal{E})$. Moreover, if π has the global extension property and if (2.1.7) holds for every smooth global section $\epsilon : M \to \mathcal{E}$ of π and for every $v \in TM$ then $\mathrm{Hor}(\mathcal{E}) = \mathrm{Hor}'(\mathcal{E})$.

PROOF. Apply Lemma 2.1.5 with ϕ the identity map of $\mathcal{E}$. $\square$

Let us go back to our discussion about directional derivatives of smooth sections of a vector bundle $\pi : E \to M$. The projection π of the vector bundle is a smooth submersion and the notions of vertical space and local section given in Definitions 2.1.1 and 2.1.2 are consistent with the ones given in Section 1.5. If $\mathrm{Hor}(E)$ is a generalized connection on E then for every smooth section $\epsilon \in \mathbf{\Gamma}(E)$, every point $x \in M$ and every vector $v \in T_xM$, the covariant derivative $\nabla_v\epsilon$ is an element of the vertical space $\mathrm{Ver}_{\epsilon(x)}E$, which is identified with the fiber E_x. Although the covariant derivative $\nabla_v\epsilon$ is linear in v, it doesn't have in general the other "nice" properties that one would expect from a notion of directional derivative; for instance, the covariant derivative $\nabla_v \epsilon$ is not in general linear in ϵ. It turns out that for some generalized connections $\mathrm{Hor}(E)$, the corresponding notion of covariant derivative of smooth sections of E satisfies all the desirable properties. The difficulty is that it is not so easy to give a direct description of the properties that the generalized connection $\mathrm{Hor}(E)$ should satisfy in order that the corresponding covariant derivative ∇ satisfies all the desirable properties.

Our plan for developing the theory of connections is the following: we first study the notion of connection on principal bundles. A principal connection on a principal bundle is just a generalized connection on the total space that is invariant under the action of the structural group. We show how a principal connection on a principal bundle induces a generalized connection in any of its associated bundles. In particular, if E is a vector bundle, a principal connection on the principal bundle of frames $\mathrm{FR}_{E_0}(E)$ induces a generalized connection $\mathrm{Hor}(E)$ on E (recall the isomorphism given by the contraction map (1.5.1)). Looking at the situation from a different perspective, we will define the notion of linear connection on a vector bundle E simply by stating that a linear connection on E is the same as a covariant derivative operator ∇ satisfying some natural properties. It will be seen that the covariant derivative operator determined by a generalized connection $\mathrm{Hor}(E)$ induced from a principal connection on $\mathrm{FR}_{E_0}(E)$ is indeed a linear connection on E; moreover, there is a one to one correspondence between the principal connections on the principal bundle $\mathrm{FR}_{E_0}(E)$ and the linear connections ∇ on the vector bundle E.

2.1.1. Pull-back of generalized connections and submersions.

DEFINITION 2.1.7. Let $\mathcal{E}$, M, M' be differentiable manifolds, $\pi : \mathcal{E} \to M$ be a smooth submersion and let $f : M' \to M$ be a smooth map. By a *local section of* π *along* f we mean a map $\varepsilon : U' \to \mathcal{E}$ with $\pi \circ \varepsilon = f|_{U'}$, where U' is an open subset of M'. If $\mathrm{Hor}(\mathcal{E})$ is a generalized connection on $\mathcal{E}$ with respect to π and if $\varepsilon : U' \to \mathcal{E}$ is a smooth local section of π along f then we set:

$$\nabla_v \varepsilon = \mathfrak{p}_{\mathrm{ver}}\big(\mathrm{d}\varepsilon(y) \cdot v\big) \in \mathrm{Ver}_{\varepsilon(y)}(\mathcal{E}),$$

for all $y \in U'$, $v \in T_y M'$ and we call $\nabla_v \varepsilon$ the *covariant derivative* of ε at the point y in the direction of v. Given $y \in U'$, if $\nabla_v \varepsilon = 0$, for all $v \in T_y M'$ then the local section ε is said to be *parallel at* y with respect to $\mathrm{Hor}(\mathcal{E})$; if ε is parallel at every $y \in U'$ we say simply that ε is *parallel* with respect to $\mathrm{Hor}(\mathcal{E})$.

Clearly the covariant derivative $\nabla_v \varepsilon$ is linear in v. Moreover, ε is parallel at y with respect to $\mathrm{Hor}(\mathcal{E})$ if and only if:

$$\mathrm{d}\varepsilon_y(T_y M') \subset \mathrm{Hor}_{\varepsilon(y)}\mathcal{E}.$$

If $\pi : \mathcal{E} \to M$, $\pi' : \mathcal{E}' \to M$ are smooth submersions, $\phi : \mathcal{E} \to \mathcal{E}'$ is fiber preserving and if $\varepsilon : U' \to \mathcal{E}$ is a local section of π along a map $f : M' \to M$ then obviously $\phi \circ \varepsilon : U \to \mathcal{E}'$ is a local section of π' along f. We have the following analogue of (2.1.6):

LEMMA 2.1.8. *Let $\pi : \mathcal{E} \to M$, $\pi' : \mathcal{E}' \to M$ be smooth submersions endowed with generalized connections $\mathrm{Hor}(\mathcal{E})$, $\mathrm{Hor}(\mathcal{E}')$, respectively, M' be a differentiable manifold, $f : M' \to M$ be a smooth map, $\phi : \mathcal{E} \to \mathcal{E}'$ be a smooth connection preserving map and $\varepsilon : U' \to \mathcal{E}$ be a local section of π along f. Then:*

$$\nabla'_v(\phi \circ \varepsilon) = \mathrm{d}\phi_{\varepsilon(y)}(\nabla_v \varepsilon),$$

for all $y \in U'$, $v \in T_y M'$, where ∇, ∇' denote respectively the covariant derivative operators with respect to $\mathrm{Hor}(\mathcal{E})$ and $\mathrm{Hor}(\mathcal{E}')$.

PROOF. It is analogous to the proof of (2.1.6) in Lemma 2.1.5. $\square$

Let $\mathcal{E}$, M, M' be differentiable manifolds, $\pi : \mathcal{E} \to M$ be a smooth submersion and let $f : M' \to M$ be a smooth map. We set:

$$f^*\mathcal{E} = \big\{(y, e) \in M' \times \mathcal{E} : f(y) = \pi(e)\big\}$$

and we denote by $\pi_1 : f^*\mathcal{E} \to M'$, $\bar{f} : f^*\mathcal{E} \to \mathcal{E}$ respectively the restriction to $f^*\mathcal{E}$ of the first and of the second projection of the cartesian product $M' \times \mathcal{E}$. Since π is a submersion, the result of Exercise 1.55 says that $f^*\mathcal{E}$ is a smooth submanifold of $M' \times \mathcal{E}$ and that the triple $(f^*\mathcal{E}, \pi_1, \bar{f})$ is the pull-back of $(f, \pi, M, M', \mathcal{E})$ in the category of differentiable manifolds and smooth maps. Since π is a submersion, it follows easily from (1.19) that also $\pi_1 : f^*\mathcal{E} \to M'$ is a submersion. We call the submersion $\pi_1 : f^*\mathcal{E} \to M'$ the *pull-back* of the submersion $\pi : \mathcal{E} \to M$ by f and we call $\bar{f} : f^*\mathcal{E} \to \mathcal{E}$ the *canonical map* of the pull-back $f^*\mathcal{E}$.

REMARK 2.1.9. Given $y \in M'$ then:

$$\pi_1^{-1}(y) = \{y\} \times \pi^{-1}\big(f(y)\big);$$

we thus identify $\pi_1^{-1}(y)$ with $\pi^{-1}\big(f(y)\big)$ in the obvious way. Under such identification, the restriction to $\pi_1^{-1}(y)$ of the canonical map $\bar{f}$ is the identity map of $\pi^{-1}\big(f(y)\big)$. In particular, for all $\mathfrak{e} \in f^*\mathcal{E}$, we identify the vertical space $\mathrm{Ver}_{\mathfrak{e}}(f^*\mathcal{E})$ with the vertical space $\mathrm{Ver}_{\bar{f}(\mathfrak{e})}(\mathcal{E})$ and the restriction to $\mathrm{Ver}_{\mathfrak{e}}(f^*\mathcal{E})$ of the differential $\mathrm{d}\bar{f}_{\mathfrak{e}}$ with the identity map of $\mathrm{Ver}_{\bar{f}(\mathfrak{e})}(\mathcal{E})$.

Clearly the composition on the left with $\bar{f}$ of a (smooth) local section of the submersion $\pi_1 : f^*\mathcal{E} \to M'$ is a (smooth) local section of $\pi : \mathcal{E} \to M$ along f. Conversely, using the property of pull-backs described in diagram (1.17), we see that if $\varepsilon : U' \to \mathcal{E}$ is a (smooth) local section of $\pi : \mathcal{E} \to M$ along f then there exists a unique (smooth) local section $\overleftarrow{\varepsilon} : U' \to f^*\mathcal{E}$ of $\pi_1 : f^*\mathcal{E} \to M'$ such that $\bar{f} \circ \overleftarrow{\varepsilon} = \varepsilon$. The situation is illustrated by the following commutative diagram:

$$\begin{tikzcd}
f^*\mathcal{E} \arrow[r, "\bar{f}"] & \mathcal{E} \arrow[d, "\pi"] \\
U' \arrow[u, dotted, "\overleftarrow{\epsilon}"] \arrow[ur, "\epsilon"'] \arrow[r, "f"'] & M
\end{tikzcd}$$

LEMMA 2.1.10. *Let $\pi : \mathcal{E} \to M$ be a smooth submersion, M' be a differentiable manifold and $f : M' \to M$ be a smooth map. If $\mathrm{Hor}(\mathcal{E})$ is a generalized connection on $\mathcal{E}$ with respect to $\pi : \mathcal{E} \to M$ then:*

$$\mathrm{Hor}_{\mathfrak{e}}(f^*\mathcal{E}) = \mathrm{d}\bar{f}_{\mathfrak{e}}^{-1}\big(\mathrm{Hor}_{f(\mathfrak{e})}(\mathcal{E})\big) \subset T_{\mathfrak{e}}(f^*\mathcal{E}), \quad \mathfrak{e} \in f^*\mathcal{E}, \tag{2.1.8}$$

is a generalized connection on $f^*\mathcal{E}$ with respect to $\pi_1 : f^*\mathcal{E} \to M'$.

PROOF. Keeping in mind Remark 2.1.9, it follows from the result of Exercise 2.4 that (2.1.8) defines an horizontal distribution on $f^*\mathcal{E}$ with respect to π_1. It is easy to see that $\mathrm{Hor}(f^*\mathcal{E})$ is indeed a smooth distribution on $f^*\mathcal{E}$. □

DEFINITION 2.1.11. The generalized connection $\mathrm{Hor}(f^*\mathcal{E})$ defined in (2.1.8) is called the *pull-back* of the generalized connection $\mathrm{Hor}(\mathcal{E})$ by f.

LEMMA 2.1.12. *Let $\pi : \mathcal{E} \to M$ be a smooth submersion endowed with a generalized connection $\mathrm{Hor}(\mathcal{E})$, M' be a differentiable manifold and $f : M' \to M$ be a smooth map; assume that $\pi_1 : f^*\mathcal{E} \to M'$ is endowed with the generalized connection $\mathrm{Hor}(f^*\mathcal{E})$ obtained from $\mathrm{Hor}(\mathcal{E})$ by pull-back. Then, given a smooth local section $\varepsilon : U' \to \mathcal{E}$ of π along f, we have:*

$$\nabla_v \overleftarrow{\varepsilon} = \nabla_v \varepsilon,$$

for all $v \in TM'|_{U'}$. In particular, ε is parallel at a point $y \in U'$ if and only if $\overleftarrow{\varepsilon}$ is parallel at y.

PROOF. It follows easily from the observation that, for all $\mathfrak{e} \in f^*\mathcal{E}$, the differential $\mathrm{d}\bar{f}_{\mathfrak{e}}$ maps $\mathrm{Hor}_{\mathfrak{e}}(f^*\mathcal{E})$ to $\mathrm{Hor}_{\bar{f}(\mathfrak{e})}(\mathcal{E})$ and from the observation that $\mathrm{d}\bar{f}_{\mathfrak{e}}$ is the identity on the vertical space $\mathrm{Ver}_{\bar{f}(\mathfrak{e})}(\mathcal{E})$ (see Remark 2.1.9). □

2.2. Connections on principal fiber bundles

Let $\Pi : P \to M$ be a G-principal bundle over a differentiable manifold M.

DEFINITION 2.2.1. A *principal connection* on P is a generalized connection $\mathrm{Hor}(P)$ on P that is *G-invariant*, i.e.:

$$\mathrm{d}\gamma_g\big(\mathrm{Hor}_p(P)\big) = \mathrm{Hor}_{p\cdot g}(P),$$

for all $p \in P$ and all $g \in G$, where $\gamma_g : P \to P$ denotes the diffeomorphism given by the action of g on P.

Recall from (1.3.4) that the vertical distribution $\mathrm{Ver}(P)$ is also G-invariant.

From now on, by a connection *on a principal bundle we will mean implicitly a principal connection*.

Let $\mathrm{Hor}(P)$ be a horizontal distribution on P. The existence of a canonical isomorphism between the vertical space $\mathrm{Ver}_p(P)$ and the Lie algebra $\mathfrak{g}$ of the structural group (recall (1.3.3)) allows us to canonically associate to the distribution $\mathrm{Hor}(P)$ a $\mathfrak{g}$-valued 1-form ω on P such that $\mathrm{Ker}(\omega_p) = \mathrm{Hor}_p(P)$, for all $p \in P$. Namely, we define ω by setting:

$$\omega_p(\zeta) = \begin{cases} \big(\mathrm{d}\beta_p(1)\big)^{-1}(\zeta) \in \mathfrak{g}, & \text{if } \zeta \in \mathrm{Ver}_p(P), \\ \qquad 0 \in \mathfrak{g}, & \text{if } \zeta \in \mathrm{Hor}_p(P), \end{cases} \tag{2.2.1}$$

for all $p \in P$, where $\big(\mathrm{d}\beta_p(1)\big)^{-1} : \mathrm{Ver}_p(P) \to \mathfrak{g}$ is the inverse of the linear isomorphism (1.3.3).

LEMMA 2.2.2. *Let* $\mathrm{Hor}(P)$ *be a horizontal distribution on* P *and let* ω *be the* $\mathfrak{g}$*-valued* 1*-form on* P *defined by* (2.2.1)*. Then* $\mathrm{Hor}(P)$ *is smooth if and only if* ω *is smooth.*

PROOF. Consider the map $L_\omega : TP \to P \times \mathfrak{g}$ whose restriction to T_pP is given by $\zeta \mapsto \big(p, \omega_p(\zeta)\big)$, for all $p \in P$. If ω is smooth then L_ω is smooth and therefore it is a morphism of vector bundles from the tangent bundle TP to the trivial vector bundle $P \times \mathfrak{g}$. Since L_ω is surjective, its kernel $\mathrm{Ker}(L_\omega) = \mathrm{Hor}(P)$ is a vector subbundle of TP, by Proposition 1.5.31; thus, $\mathrm{Hor}(P)$ is a smooth distribution on P. Conversely, assume that the horizontal distribution $\mathrm{Hor}(P)$ is smooth. Consider the map $L_\beta : P \times \mathfrak{g} \to \mathrm{Ver}(P)$ defined by

$$L_\beta(p, X) = \mathrm{d}\beta_p(1) \cdot X,$$

for all $p \in P$ and all $X \in \mathfrak{g}$. The map L_β is smooth, since it is the restriction to $P \times \mathfrak{g} \subset TP \times TG$ of the differential of the right action $P \times G \to P$ of G on P. Thus L_β is an isomorphism of vector bundles. Since $\mathrm{Hor}(P)$ is smooth, the vertical projection $\mathfrak{p}_{\mathrm{ver}} : TP \to \mathrm{Ver}(P)$ is a morphism of vector bundles and therefore $L_\omega = L_\beta^{-1} \circ \mathfrak{p}_{\mathrm{ver}} : TP \to P \times \mathfrak{g}$ is also a morphism of vector bundles. It follows that ω is smooth. $\square$

Let us determine what conditions on the $\mathfrak{g}$-valued 1-form ω defined by (2.2.1) correspond to the G-invariance of the horizontal distribution $\mathrm{Hor}(P)$. Assume that $\mathrm{Hor}(P)$ is G-invariant. From the commutativity of diagram (1.3.5), it follows that the diagram:

(2.2.2)
$$\begin{tikzcd} T_pP \arrow[r, "\omega_p"] \arrow[d, "\mathrm{d}\gamma_g(p)"'] & \mathfrak{g} \arrow[d, "\mathrm{Ad}_{g^{-1}}"] \\ T_{p\cdot g}P \arrow[r, "\omega_{p\cdot g}"'] & \mathfrak{g} \end{tikzcd}$$

commutes, for all $p \in P$ and all $g \in G$; namely, simply check that $\mathrm{Ad}_{g^{-1}} \circ \omega_p$ and $\omega_{p\cdot g} \circ \mathrm{d}\gamma_g(p)$ coincide both on $\mathrm{Hor}_p(P)$ and on $\mathrm{Ver}_p(P)$. The commutativity of diagram (2.2.2) for all $p \in P$, $g \in G$ is equivalent to the requirement that:

$$\gamma_g^*\,\omega = \mathrm{Ad}_{g^{-1}} \circ \omega, \tag{2.2.3}$$

for all $g \in G$. Motivated by this, we give the following:

DEFINITION 2.2.3. Let V be a real finite-dimensional vector space and let $\rho : G \to \mathrm{GL}(V)$ be a smooth representation of G on V. A V-valued differential form λ on the total space P is said to be *ρ-pseudo G-invariant* (or *pseudo G-invariant with respect to ρ*) if:

$$\gamma_g^*\,\lambda = \rho(g)^{-1} \circ \lambda,$$

for all $g \in G$.

Equality (2.2.3) says that ω is pseudo G-invariant with respect to the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ of the Lie group G on its Lie algebra $\mathfrak{g}$.

LEMMA 2.2.4. *Let* $\mathrm{Hor}(P)$ *be a horizontal distribution on* P *and let* ω *be the* $\mathfrak{g}$*-valued* 1*-form on* P *defined by* (2.2.1)*. Then* $\mathrm{Hor}(P)$ *is* G*-invariant if and only if* ω *is* Ad*-pseudo* G*-invariant.*

PROOF. We have already shown that if $\mathrm{Hor}(P)$ is G-invariant then ω is Ad-pseudo G-invariant. Conversely, if ω is Ad-pseudo G-invariant then diagram (2.2.2) commutes for all $p \in P$, $g \in G$ and therefore:

$$\mathrm{d}\gamma_g\big(\mathrm{Hor}_p(P)\big) \subset \mathrm{Hor}_{p\cdot g}(P), \tag{2.2.4}$$

for all $p \in P$, $g \in G$. Replacing p with $p \cdot g$ and g with g^{-1} in (2.2.4) we get the opposite inclusion $\mathrm{Hor}_{p\cdot g}(P) \subset \mathrm{d}\gamma_g\big(\mathrm{Hor}_p(P)\big)$ and hence $\mathrm{Hor}(P)$ is G-invariant. □

DEFINITION 2.2.5. Let $\Pi : P \to M$ be a G-principal bundle and for each $p \in P$ denote by $\big(\mathrm{d}\beta_p(1)\big)^{-1} : \mathrm{Ver}_p(P) \to \mathfrak{g}$ the inverse of the linear isomorphism (1.3.3). A smooth $\mathfrak{g}$-valued Ad-pseudo G-invariant 1-form ω on P satisfying the condition:

$$\omega_p|_{\mathrm{Ver}_p(P)} = \big(\mathrm{d}\beta_p(1)\big)^{-1} \tag{2.2.5}$$

for all $p \in P$ is called a *connection form* on P.

If ω is a $\mathfrak{g}$-valued 1-form on P satisfying condition (2.2.5) for all $p \in P$ then the distribution $\mathrm{Hor}(P)$ defined by:

$$\mathrm{Hor}_p(P) = \mathrm{Ker}(\omega_p), \tag{2.2.6}$$

for all $p \in P$ is horizontal (see Exercise 2.1). If ω is a connection form on P then Lemmas 2.2.2 and 2.2.4 imply that the horizontal distribution $\mathrm{Hor}(P)$ defined by (2.2.6) is a connection on P. Conversely, if $\mathrm{Hor}(P)$ is a connection on P then the $\mathfrak{g}$-valued 1-form ω on P defined by (2.2.1) is a connection form on P. Thus, we have the following:

THEOREM 2.2.6. *Let* $\Pi : P \to M$ *be a principal bundle. Equality* (2.2.6) *defines a one-to-one correspondence between connections* $\mathrm{Hor}(P)$ *on* P *and smooth connection forms* ω *on* P*.* □

EXAMPLE 2.2.7. Let M be a differentiable manifold and let P_0 be a principal space whose structural group is a Lie group G. There is a canonical connection on the trivial principal bundle $P = M \times P_0$ defined by:

$$\mathrm{Hor}_{(x,p)}(P) = T_xM \oplus \{0\} \subset T_xM \oplus T_pP_0 = T_{(x,p)}P,$$

for all $x \in M$, $p \in P_0$. If ω is the connection form associated to such connection then for all $(x,p) \in P$, $\omega_{(x,p)} : T_xM \oplus T_pP_0 \to \mathfrak{g}$ is the composition of the projection $T_xM \oplus T_pP_0 \to T_pP_0$ with $\big(\mathrm{d}\beta_p(1)\big)^{-1} : T_pP_0 \to \mathfrak{g}$.

EXAMPLE 2.2.8. Let $\Pi : P \to M$ be a G-principal bundle and let $\mathrm{Hor}(P)$ be a connection on P. If U is an open subset of M then clearly the intersection $\mathrm{Hor}(P) \cap T(P|_U)$ is a connection on the restricted principal bundle $P|_U$. Obviously the connection form associated to $\mathrm{Hor}(P) \cap T(P|_U)$ is just the restriction of the connection form associated to $\mathrm{Hor}(P)$. In Exercise 2.6 we ask the reader to show that a connection on P is determined by a family of pairwise compatible connections on restrictions of P to open subsets of M.

EXAMPLE 2.2.9. Let us understand better the notion of connection form by considering a trivial principal bundle $P = M \times G$. Let ω be a $\mathfrak{g}$-valued 1-form on P. By differentiating the action of G on itself by right translations we obtain a right action of G on its own tangent bundle TG given by:

$$Xg = \mathrm{d}R_g(X),$$

for all $g \in G$ and all $X \in TG$ (recall (1.1.2)). The right action of G on TP obtained by differentiating the right action of G on P is therefore given by:

$$(v, X) \cdot g \stackrel{\text{def}}{=} \mathrm{d}\gamma_g(v, X) = (v, Xg),$$

for all $v \in TM$, $X \in TG$ and all $g \in G$. Let us take a closer look at the condition of Ad-pseudo G-invariance. By the result of Exercise 2.7, ω is Ad-pseudo G-invariant if and only if the equality (2.2.3) holds at the point of $M \times \{1\} \subset P$, i.e., if and only if:

$$\omega_{(x,g)}(v, Xg) = \mathrm{Ad}_{g^{-1}}\big(\omega_{(x,1)}(v, X)\big), \tag{2.2.7}$$

for all $x \in M$, $v \in T_xM$, $g \in G$ and all $X \in \mathfrak{g}$. Let $\bar{\omega}$ be the $\mathfrak{g}$-valued 1-form on M which is the pull-back of ω by the local section $s^1 : M \ni x \mapsto (x, 1) \in P$ of P. The equality $(s^1)^*\omega = \bar{\omega}$ means that:

$$\omega_{(x,1)}(v, 0) = \bar{\omega}_x(v), \tag{2.2.8}$$

for all $x \in M$ and all $v \in T_xM$. Now let us consider condition (2.2.5). By the result of Exercise 2.9, under the assumption that ω is Ad-pseudo G-invariant, condition (2.2.5) holds for all $p \in P$ if and only if it holds for all $p \in M \times \{1\} \subset P$, i.e., if and only if:

$$\omega_{(x,1)}(0, X) = X, \tag{2.2.9}$$

for all $x \in M$ and all $X \in \mathfrak{g}$. Conditions (2.2.7), (2.2.8) and (2.2.9) together are equivalent to:

$$\omega_{(x,g)}(v, Xg) = \mathrm{Ad}_{g^{-1}}\big(\bar{\omega}_x(v) + X\big), \tag{2.2.10}$$

for all $x \in M$, $v \in T_xM$, $g \in G$ and all $X \in \mathfrak{g}$. We have shown that given a $\mathfrak{g}$-valued 1-form $\bar{\omega}$ on M then there exists a unique Ad-pseudo G-invariant $\mathfrak{g}$-valued 1-form ω on $P = M \times G$ satisfying condition (2.2.5) for all $p \in P$ with $(s^1)^*\omega = \bar{\omega}$; the 1-form ω is given by (2.2.10). Notice that equality (2.2.10) implies that ω is smooth if and only if $\bar{\omega}$ is smooth.

LEMMA 2.2.10. *Let $\Pi : P \to M$ be a G-principal bundle and let $s : U \to P$ be a smooth local section of P. If $\bar{\omega}$ is a $\mathfrak{g}$-valued 1-form on U then there exists a unique Ad-pseudo G-invariant $\mathfrak{g}$-valued 1-form ω on the principal bundle $P|_U$ satisfying condition (2.2.5) for all $p \in P|_U$ with $s^*\omega = \bar{\omega}$. Moreover, ω is smooth if and only if $\bar{\omega}$ is smooth.*

PROOF. The discussion presented in Example 2.2.9 shows that the lemma holds in the case that $P = M \times G$ is the trivial bundle and the local section s is equal to $s^1 : M \ni x \mapsto (x, 1) \in P$. To prove the general case, consider the following commutative diagram (recall (1.3.2)):

$$\begin{tikzcd} U \times G \arrow[rr, "\beta_s", "\cong"'] & & P|_U \\ & U \arrow[ul, "s^1"] \arrow[ur, "s"'] & \end{tikzcd}$$

The map β_s is an isomorphism from the trivial principal bundle $U \times G$ to $P|_U$ whose subjacent Lie group homomorphism is the identity map of G (recall Example 1.3.10). Given a $\mathfrak{g}$-valued 1-form ω on $P|_U$ then the result of Exercises 2.11 and 2.12 imply that ω is Ad-pseudo G-invariant and satisfies (2.2.5) for all $p \in P|_U$ if and only if $\beta_s^*\omega$ is Ad-pseudo G-invariant and satisfies (2.2.5) for all $p \in U \times G$. Moreover, $s^*\omega = \bar{\omega}$ if and only if $(s^1)^*(\beta_s^*\omega) = \bar{\omega}$. The conclusion follows. $\square$

If ω is a connection form on P and $s : U \to P$ is a smooth local section then the smooth $\mathfrak{g}$-valued 1-form $\bar{\omega} = s^*\omega$ on U is called the *representation* of ω with respect to the smooth local section s. Lemma 2.2.10 states that a connection form ω on $P|_U$ is uniquely determined by its representation $\bar{\omega}$ with respect to a given smooth local section $s : U \to P$.

Let us now discuss the notion of connection preserving maps in the context of principal bundles (recall Definition 2.1.3).

LEMMA 2.2.11. *Let $\Pi : P \to M$, $\Pi' : Q \to M$ be principal bundles with structural groups G and H, respectively; denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. Let $\phi : P \to Q$ be a morphism of principal bundles with subjacent Lie group homomorphism $\phi_0 : G \to H$; denote by $\bar{\phi}_0 : \mathfrak{g} \to \mathfrak{h}$ the differential of ϕ_0 at the identity. Let $\mathrm{Hor}(P)$, $\mathrm{Hor}(Q)$ be respectively a G-invariant horizontal distribution on P and an H-invariant horizontal distribution on Q. Denote by ω^P, ω^Q respectively the $\mathfrak{g}$-valued 1-form on P associated to $\mathrm{Hor}(P)$ and the $\mathfrak{h}$-valued 1-form on Q associated to $\mathrm{Hor}(Q)$, defined as in (2.2.1). The following conditions are equivalent:*

(a) *for every $x \in M$ there exists $p \in P_x$ such that*

$$\mathrm{d}\phi_p\big(\mathrm{Hor}_p(P)\big) \subset \mathrm{Hor}_{\phi(p)}(Q); \tag{2.2.11}$$

(b) *ϕ is connection preserving;*
(c) *$\phi^*\omega^Q = \bar{\phi}_0 \circ \omega^P$;*
(d) *every point of M is in the domain of a smooth local section $s : U \to P$ of P such that $(\phi \circ s)^*\omega^Q = \bar{\phi}_0 \circ (s^*\omega^P)$.*

PROOF. Assume (a) and let us prove (b). Given $g \in G$, $h \in H$, denote by $\gamma_g^P : P \to P$ and $\gamma_h^Q : Q \to Q$ respectively the diffeomorphism given by the action of g on P and the diffeomorphism given by the action of h on Q. Since ϕ_0 is the Lie group homomorphism subjacent to ϕ, setting $h = \phi_0(g)$, then the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\gamma_g^P"'] & Q \arrow[d, "\gamma_h^Q"] \\ P \arrow[r, "\phi"'] & Q \end{tikzcd}$$

by differentiation, we get another commutative diagram:

$$\begin{tikzcd} T_pP \arrow[r, "\mathrm{d}\phi(p)"] \arrow[d, "\mathrm{d}\gamma_g^P(p)"'] & T_{\phi(p)}Q \arrow[d, "\mathrm{d}\gamma_h^Q(\phi(p))"] \\ T_{p\cdot g}P \arrow[r, "\mathrm{d}\phi(p\cdot g)"'] & T_{\phi(p\cdot g)}Q \end{tikzcd} \tag{2.2.12}$$

Observing $\mathrm{d}\gamma_g^P(p)$ maps the space $\mathrm{Hor}_p(P)$ to the space $\mathrm{Hor}_{p\cdot g}(P)$ and $\mathrm{d}\gamma_h^Q\big(\phi(p)\big)$ maps the space $\mathrm{Hor}_{\phi(p)}(Q)$ to the space $\mathrm{Hor}_{\phi(p\cdot g)}(Q)$, the commutativity of diagram (2.2.12) and (2.2.11) imply that:

$$\mathrm{d}\phi_{p\cdot g}\big(\mathrm{Hor}_{p\cdot g}(P)\big) \subset \mathrm{Hor}_{\phi(p\cdot g)}(Q),$$

for all $g \in G$. Thus (2.2.11) holds for all $p \in P$. Now (b) follows directly from Lemma 2.1.5. Now assume (b) and let us prove (c). Given $p \in P$ then the linear maps $(\phi^*\omega^Q)_p = \omega^Q_{\phi(p)} \circ \mathrm{d}\phi_p$ and $\bar{\phi}_0 \circ \omega^P_p$ are both zero on $\mathrm{Hor}_p(P)$ and they coincide on $\mathrm{Ver}_p(P)$, by the result of Exercise 2.11. Therefore (c) holds. To prove that (c) implies (d), simply observe that the equality in (d) is equivalent to:

$$s^*(\phi^*\omega^Q) = s^*(\bar{\phi}_0 \circ \omega^P). \tag{2.2.13}$$

Finally, assume (d) and let us prove (a). Let $x \in M$ be fixed and choose a smooth local section $s : U \to P$ of P with $x \in U$ such that (2.2.13) holds. Set $p = s(x)$ and let us show that (2.2.11) holds. Equality (2.2.13) implies that the linear maps $(\phi^*\omega^Q)_p$ and $\bar{\phi}_0 \circ \omega^P_p$ coincide on the image of $\mathrm{d}s_x$; by the result of Exercise 2.11, they also coincide on $\mathrm{Ver}_p(P)$. Since $T_pP = \mathrm{d}s_x(T_xM) \oplus \mathrm{Ver}_p(P)$, it follows that:

$$\omega^Q_{\phi(p)} \circ \mathrm{d}\phi_p = (\phi^*\omega^Q)_p = \bar{\phi}_0 \circ \omega^P_p.$$

Hence $\mathrm{d}\phi_p$ maps the kernel of ω^P_p into the kernel of $\omega^Q_{\phi(p)}$, proving (2.2.11). □

EXAMPLE 2.2.12. The morphism of principal bundle $\mathrm{Id} \times \phi$ of Example 1.3.9 is obviously connection preserving if the trivial principal bundles $M \times P_0$ and $M \times Q_0$ are endowed with their canonical connections (see Example 2.2.7).

PROPOSITION 2.2.13. *Let P, Q be principal bundles over the same differentiable manifold M and let $\phi : P \to Q$ be a morphism of principal bundles. Given a connection* $\mathrm{Hor}(P)$ *on P then there exists a unique connection* $\mathrm{Hor}(Q)$ *on Q for which ϕ is connection preserving.*

PROOF. Let G, H denote respectively the structural groups of P and Q and let $\phi_0 : G \to H$ denote the Lie group homomorphism subjacent to ϕ. Let ω^P be the connection form corresponding to $\mathrm{Hor}(P)$. We first prove the proposition under the assumption that P admits a globally defined smooth local section $s : M \to P$. Let $\bar{\omega} = s^*\omega^P$ denote the representation of ω^P with respect to s. A connection $\mathrm{Hor}(Q)$ on Q makes ϕ connection preserving if and only if its connection form ω^Q satisfies:

$$(\phi \circ s)^*\omega^Q = \bar{\phi}_0 \circ \bar{\omega}, \tag{2.2.14}$$

where $\bar{\phi}_0$ denotes the differential of ϕ_0 at the identity (see item (d) on the statement of Lemma 2.2.11). Since $\phi \circ s : M \to Q$ is a smooth globally defined local section of Q, Lemma 2.2.10 implies that there exists a unique connection form ω^Q on Q such that (2.2.14) holds. This completes the proof in the case where P admits a globally defined smooth local section. To prove the general case, let $M = \bigcup_{i \in I} U_i$ be an open cover of M such that U_i is the domain of some smooth local section of P, for all $i \in I$. The case already proven therefore applies to the restriction of ϕ to $P|_{U_i}$. The conclusion is now easily obtained by applying the result of Exercise 2.6. □

DEFINITION 2.2.14. If $\phi : P \to Q$ is a morphism of principal bundles and $\mathrm{Hor}(P)$ is a connection on P then the unique connection $\mathrm{Hor}(Q)$ on Q that makes ϕ connection preserving is called the *push-forward* of $\mathrm{Hor}(P)$ by ϕ.

In analogy with Corollary 1.3.12, we have the following:

COROLLARY 2.2.15. *Let P, P', Q be principal bundles over a differentiable manifold M and let $\phi : P \to Q$, $\psi : P \to P'$, $\phi' : P' \to Q$ be morphisms of principal bundles such that the diagram:*

$$\begin{tikzcd} P \arrow[d, "\psi"'] \arrow[dr, "\phi"] & \\ P' \arrow[r, "\phi'"'] & Q \end{tikzcd}$$

commutes. Given connections $\mathrm{Hor}(P)$, $\mathrm{Hor}(P')$, $\mathrm{Hor}(Q)$ *on P, P' and Q respectively such that both ϕ and ψ are connection preserving then also ϕ' is connection preserving.*

PROOF. Let $\mathrm{Hor}'(Q)$ be the push-forward of $\mathrm{Hor}(P')$ by ϕ'. Both connections $\mathrm{Hor}(Q)$ and $\mathrm{Hor}'(Q)$ make $\phi = \phi' \circ \psi$ connection preserving. By the uniqueness part of Proposition 2.2.13, we have $\mathrm{Hor}(Q) = \mathrm{Hor}'(Q)$. This concludes the proof. □

2.2.1. The connection on the fiberwise product. Given connections on principal bundles P, Q over M then we have a naturally induced connection on the fiberwise product $P \star Q$.

PROPOSITION 2.2.16. *Let P, Q be principal bundles over the same differentiable manifold M, with structural groups G and H, respectively; denote by $\mathfrak{g}$, $\mathfrak{h}$ respectively the Lie algebras of G and H. If $\mathrm{Hor}(P)$ is a connection on P and $\mathrm{Hor}(Q)$ is a connection on Q then there exists a unique connection $\mathrm{Hor}(P \star Q)$ on $P \star Q$ such that $\mathrm{Hor}(P)$ is the push-forward of $\mathrm{Hor}(P \star Q)$ by the projection $\mathrm{pr}_1 : P \star Q \to P$ and such that $\mathrm{Hor}(Q)$ is the push-forward of $\mathrm{Hor}(P \star Q)$ by the projection $\mathrm{pr}_2 : P \star Q \to Q$. Moreover, if ω^P is the $\mathfrak{g}$-valued connection form associated to $\mathrm{Hor}(P)$ and ω^Q is the $\mathfrak{h}$-valued connection form associated to $\mathrm{Hor}(Q)$ then $(\mathrm{pr}_1^* \omega^P, \mathrm{pr}_2^* \omega^Q)$ is the $(\mathfrak{g} \oplus \mathfrak{h})$-valued connection form associated to $\mathrm{Hor}(P \star Q)$.*

PROOF. The maps pr_1 and pr_2 are both connection preserving if and only if the connection form associated to the connection on $P \star Q$ is equal to $(\mathrm{pr}_1^* \omega^P, \mathrm{pr}_2^* \omega^Q)$ (see item (c) on Lemma 2.2.11). To conclude the proof, we just have to show that $\omega = (\mathrm{pr}_1^* \omega^P, \mathrm{pr}_2^* \omega^Q)$ is indeed a connection form on $P \star Q$. Clearly, ω is a smooth $(\mathfrak{g} \oplus \mathfrak{h})$-valued 1-form on $P \star Q$. Given $(p, q) \in P \star Q$, the fact that the restriction of $\omega_{(p,q)}$ to $\mathrm{Ver}_{(p,q)}(P \star Q)$ is equal to $\big(\mathrm{d}\beta_{(p,q)}(1)\big)^{-1}$ follows from the result of Exercise 2.11 applied to the morphisms pr_1 and pr_2. Finally, the Ad-pseudo $(G \times H)$-invariance of ω follows from the result of Exercise 2.12, also applied to the morphisms pr_1 and pr_2. $\square$

DEFINITION 2.2.17. Let P, Q be principal bundles over the same differentiable manifold M. Given connections $\mathrm{Hor}(P)$ and $\mathrm{Hor}(Q)$ on P and Q respectively then the unique connection $\mathrm{Hor}(P \star Q)$ on $P \star Q$ that makes the projections $\mathrm{pr}_1 : P \star Q \to P$, $\mathrm{pr}_2 : P \star Q \to Q$ connection preserving is called the *fiberwise product connection* of $\mathrm{Hor}(P)$ by $\mathrm{Hor}(Q)$.

EXAMPLE 2.2.18. Let M be a differentiable manifold and P_0, Q_0 be principal spaces whose structural groups are Lie groups; consider the trivial principal bundles $P = M \times P_0$ and $Q = M \times Q_0$. In Exercise 1.56 we asked the reader to show that the fiberwise product $P \star Q$ is identified with the trivial principal bundle $M \times (P_0 \times Q_0)$. We claim that the canonical connection of the trivial principal bundle $P \star Q$ is equal to the fiberwise product of the canonical connections of the trivial principal bundles P and Q (see Example 2.2.7). Namely, if $P \star Q$ is endowed with its canonical connection then the projections $\mathrm{pr}_1 : P \star Q \to P$, $\mathrm{pr}_2 : P \star Q \to Q$ are connection preserving, which follows from what has been observed in Example 2.2.12.

LEMMA 2.2.19. *Let P, P', Q be principal bundles over a differentiable manifold M endowed with connections $\mathrm{Hor}(P)$, $\mathrm{Hor}(P')$ and $\mathrm{Hor}(Q)$, respectively. Let $P \star P'$ be endowed with the fiberwise product connection of $\mathrm{Hor}(P)$ by $\mathrm{Hor}(P')$. Denote by $\mathrm{pr}_1 : P \star P' \to P$, $\mathrm{pr}_2 : P \star P' \to P'$ the projections. A morphism of principal bundles $\phi : Q \to P \star P'$ is connection preserving if and only if both $\mathrm{pr}_1 \circ \phi$ and $\mathrm{pr}_2 \circ \phi$ are connection preserving.*

PROOF. Obviously $\mathrm{pr}_1 \circ \phi$ and $\mathrm{pr}_2 \circ \phi$ are connection preserving if ϕ is connection preserving. Conversely, assume that both $\mathrm{pr}_1 \circ \phi$ and $\mathrm{pr}_2 \circ \phi$ are connection preserving. Denote by $\mathrm{Hor}(P \star P')$ the push-forward of $\mathrm{Hor}(Q)$ by ϕ. The proof will be concluded if we show that $\mathrm{Hor}(P \star P')$ is the fiberwise product connection of $\mathrm{Hor}(P)$ by $\mathrm{Hor}(P')$; to this aim, it suffices to verify that both pr_1 and pr_2 are connection preserving when $P \star P'$ is endowed with the connection $\mathrm{Hor}(P \star P')$. This follows by applying Corollary 2.2.15 to the diagrams:

$$\begin{tikzcd} Q \arrow[d, "\phi"'] \arrow[dr, "\mathrm{pr}_1 \circ \phi"] & \\ P \star P' \arrow[r, "\mathrm{pr}_1"'] & P \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[d, "\phi"'] \arrow[dr, "\mathrm{pr}_2 \circ \phi"] & \\ P \star P' \arrow[r, "\mathrm{pr}_2"'] & P' \end{tikzcd}$$

□

COROLLARY 2.2.20. *Let P, P', Q, Q' be principal bundles over a differentiable manifold M endowed with connections and let $\phi : P \to P'$, $\psi : Q \to Q'$ be connection preserving morphisms of principal bundles. The morphism of principal bundles $\phi \star \psi : P \star Q \to P' \star Q'$ (see Example 1.3.28) is also connection preserving.* □

2.2.2. Pull-back of connections. Let $\Pi : P \to M$ be a G-principal bundle over a differentiable manifold M and let $f : M' \to M$ be a smooth map defined in a differentiable manifold M'. We will now show how a connection $\mathrm{Hor}(P)$ on P induces a connection on the pull-back bundle f^*P.

LEMMA 2.2.21. *Let $\Pi : P \to M$ be a G-principal bundle over a differentiable manifold M and let $f : M' \to M$ be a smooth map defined in a differentiable manifold M'; denote by $\bar{f} : f^*P \to P$ the canonical map of the pull-back f^*P. If ω is a connection form on P then $\bar{f}^*\omega$ is a connection form on f^*P.*

PROOF. We have to check the Ad-pseudo G-invariance of $\bar{f}^*\omega$ and the equality:

$$(\bar{f}^*\omega)_p|_{\mathrm{Ver}_p(f^*P)} = \big(\mathrm{d}\beta_p(1)\big)^{-1}, \tag{2.2.15}$$

for all $p \in f^*P$. Equality (2.2.15) follows from (2.2.5), observing that, for all $y \in M'$, the restriction of $\bar{f}$ to the fiber $(f^*P)_y = P_{f(y)}$ is just the identity map of $P_{f(y)}$. Let us check the Ad-pseudo G-invariant of $\bar{f}^*\omega$. For each $g \in G$, denote by $\gamma_g^P : P \to P$ and by $\gamma_g^{f^*P} : f^*P \to f^*P$ respectively the map given by the action of g on P and the map given by the action of g on f^*P. Clearly, $\gamma_g^P \circ \bar{f} = \bar{f} \circ \gamma_g^{f^*P}$, for all $g \in G$. We compute:

$$\begin{aligned} (\gamma_g^{f^*P})^*(\bar{f}^*\omega) = \big(\bar{f} \circ \gamma_g^{f^*P}\big)^*\omega = (\gamma_g^P \circ \bar{f})^*\omega = \bar{f}^*\big((\gamma_g^P)^*\omega\big) \\ = \bar{f}^*(\mathrm{Ad}_{g^{-1}} \circ \omega) = \mathrm{Ad}_{g^{-1}} \circ (\bar{f}^*\omega). \end{aligned}$$

This concludes the proof. □

Recalling from Theorem 2.2.6 that we have a one-to-one correspondence between smooth connection forms and connection on a principal bundle, we can give the following:

DEFINITION 2.2.22. Let $\Pi : P \to M$ be a principal bundle and $f : M' \to M$ be a smooth map defined on a differentiable manifold M'. Given a connection $\mathrm{Hor}(P)$ on P then the *pull-back* of $\mathrm{Hor}(P)$ by f is the connection $\mathrm{Hor}(f^*P)$ on f^*P corresponding to the connection form $\bar{f}^*\omega$, where ω is the connection form on P corresponding to $\mathrm{Hor}(P)$.

EXAMPLE 2.2.23. Let $\Pi : P \to M$ be a principal bundle and U be an open subset of M. In Example 1.3.17 we have identified the restriction $P|_U$ with the pull-back i^*P, where $i : U \to M$ denotes the inclusion map. If $\mathrm{Hor}(P)$ is a connection on P then clearly the pull-back of $\mathrm{Hor}(P)$ by i is equal to the connection $\mathrm{Hor}(P) \cap T(P|_U)$ on $P|_U$ (see Example 2.2.8).

EXAMPLE 2.2.24. Let $\Pi : P \to M$ be a principal bundle and $f : M' \to M$, $g : M'' \to M'$ be smooth maps, where M', M'' are differentiable manifolds. Let $\mathrm{Hor}(P)$ be a connection on P. In Example 1.3.24 we have identified the principal bundles g^*f^*P and $(f \circ g)^*P$. Using such identification, we have $\bar{f} \circ \bar{g} = \overline{f \circ g}$. We claim that the pull-back of $\mathrm{Hor}(P)$ by $f \circ g$ is equal to the pull-back by g of the pull-back by f of $\mathrm{Hor}(P)$. Namely, if ω denotes the connection form of $\mathrm{Hor}(P)$ then:

$$(\overline{f \circ g})^*\omega = (\bar{f} \circ \bar{g})^*\omega = \bar{g}^*(\bar{f}^*\omega).$$

LEMMA 2.2.25. *Let* $\Pi : P \to M$ *be a principal bundle and* $f : M' \to M$ *be a smooth map defined on a differentiable manifold* M'. *Let* $\mathrm{Hor}(P)$ *be a connection on* P *and let* $\mathrm{Hor}(f^*P)$ *denote the pull-back of* $\mathrm{Hor}(P)$ *by* f. *If* $\bar{f} : f^*P \to P$ *denotes the map defined in Subsection 1.3.1 then:*

$$\mathrm{Hor}_p(f^*P) = \mathrm{d}\bar{f}_p^{-1}\big(\mathrm{Hor}_{\bar{f}(p)}(P)\big),$$

for all $p \in f^*P$.

PROOF. Since $\bar{f}^*\omega$ is the connection form corresponding to the connection $\mathrm{Hor}(f^*P)$, then:

$$\begin{aligned}\mathrm{Hor}_p(f^*P) = \mathrm{Ker}\big((\bar{f}^*\omega)_p\big) &= \mathrm{Ker}(\omega_{\bar{f}(p)} \circ \mathrm{d}\bar{f}_p) \\ &= \mathrm{d}\bar{f}_p^{-1}\big(\mathrm{Ker}(\omega_{\bar{f}(p)})\big) = \mathrm{d}\bar{f}_p^{-1}\big(\mathrm{Hor}_{\bar{f}(p)}(P)\big),\end{aligned}$$

for all $p \in f^*P$. □

LEMMA 2.2.26. *Let* P, Q *be principal bundles over a differentiable manifold* M *endowed with connections* $\mathrm{Hor}(P)$, $\mathrm{Hor}(Q)$, *respectively and let* $\phi : P \to Q$ *be a connection preserving morphism of principal bundles. If* $f : M' \to M$ *is a smooth map defined on a differentiable manifold* M' *and* f^*P, f^*Q *are endowed respectively with the pull-back of* $\mathrm{Hor}(P)$, $\mathrm{Hor}(Q)$ *by* f *then the morphism of principal bundles* $f^*\phi : f^*P \to f^*Q$ *(recall Example 1.3.23) is also connection preserving.*

PROOF. If ω^P, ω^Q denote respectively the connection forms associated to $\mathrm{Hor}(P)$ and $\mathrm{Hor}(Q)$ then the connection form on f^*P is $(\bar{f}^P)^*\omega^P$ and the connection form on f^*Q is $(\bar{f}^Q)^*\omega^Q$. Denote by ϕ_0 the subjacent Lie group homomorphism of ϕ and by $\bar{\phi}_0$ its differential at the identity. Since ϕ is connection preserving, $\phi^*\omega^Q = \bar{\phi}_0 \circ \omega^P$ (recall part (c) of Lemma 2.2.11). Hence:

$$(f^*\phi)^*\big((\bar{f}^Q)^*\omega^Q\big) \overset{(1.3.12)}{=} (\bar{f}^P)^*(\phi^*\omega^Q) = (\bar{f}^P)^*(\bar{\phi}_0 \circ \omega^P) = \bar{\phi}_0 \circ \big((\bar{f}^P)^*\omega^P\big). \quad \square$$

LEMMA 2.2.27. *Let P, Q be principal bundles over a differentiable manifold M endowed with connections* $\mathrm{Hor}(P)$, $\mathrm{Hor}(Q)$, *respectively and denote by* $\mathrm{Hor}(P \star Q)$ *the fiberwise product connection. Let $f : M' \to M$ be a smooth map defined on a differentiable manifold M' and let f^*P, f^*Q, $f^*(P \star Q)$ be endowed respectively with the pull-back of* $\mathrm{Hor}(P)$, $\mathrm{Hor}(Q)$, $\mathrm{Hor}(P \star Q)$ *by f. If $(f^*P) \star (f^*Q)$ is endowed with the fiberwise product connection then the isomorphism of principal bundles* (1.3.15) *from $f^*(P\star Q)$ to $(f^*P)\star(f^*Q)$ is connection preserving.*

PROOF. Let $\lambda : f^*(P \star Q) \to (f^*P) \star (f^*Q)$ denote the isomorphism of principal bundles (1.3.15) and:

$$\mathrm{pr}_1 : P \star Q \longrightarrow P, \quad \mathrm{pr}_2 : P \star Q \longrightarrow Q,$$
$$\mathrm{pr}_1^f : (f^*P) \star (f^*Q) \longrightarrow f^*P, \quad \mathrm{pr}_2^f : (f^*P) \star (f^*Q) \longrightarrow f^*Q,$$

denote the projections. If ω^P, ω^Q denote respectively the connection form associated to $\mathrm{Hor}(P)$, $\mathrm{Hor}(Q)$ then the connection form of $f^*(P \star Q)$ is (recall Proposition 2.2.16):

$$(\bar{f}^{P\star Q})^*(\mathrm{pr}_1^*\omega^P, \mathrm{pr}_2^*\omega^Q)$$

and the connection form of $(f^*P) \star (f^*Q)$ is:

$$\Big((\mathrm{pr}_1^f)^*\big((\bar{f}^P)^*\omega^P\big), (\mathrm{pr}_2^f)^*\big((\bar{f}^Q)^*\omega^Q\big)\Big).$$

Since λ is an isomorphism of principal bundles whose subjacent Lie group homomorphism is the identity, the conclusion will follow if we show that (recall part (c) of Lemma 2.2.11):

(2.2.16)
$$\lambda^*\Big((\mathrm{pr}_1^f)^*\big((\bar{f}^P)^*\omega^P\big), (\mathrm{pr}_2^f)^*\big((\bar{f}^Q)^*\omega^Q\big)\Big) = (\bar{f}^{P\star Q})^*(\mathrm{pr}_1^*\omega^P, \mathrm{pr}_2^*\omega^Q).$$

But (2.2.16) follows directly from the equalities:

$$\bar{f}^P \circ \mathrm{pr}_1^f \circ \lambda = \mathrm{pr}_1 \circ \bar{f}^{P\star Q}, \quad \bar{f}^Q \circ \mathrm{pr}_2^f \circ \lambda = \mathrm{pr}_2 \circ \bar{f}^{P\star Q}.$$

This conclude the proof. $\square$

2.2.3. Parallel transport. Let $\Pi : P \to M$ be a G-principal bundle endowed with a connection $\mathrm{Hor}(P)$. Given a smooth curve $\gamma : I \to M$ then by a *parallel lifting* of γ we mean a smooth curve $\tilde{\gamma} : I \to P$ with $\Pi \circ \tilde{\gamma} = \gamma$ such that $\tilde{\gamma}'(t) \in \mathrm{Hor}_{\tilde{\gamma}(t)}(P)$, for all $t \in I$. Recall from Definition 2.1.7 that $\tilde{\gamma}$ is a parallel lifting of γ if and only if $\tilde{\gamma}$ is a parallel section of $\Pi : P \to M$ along γ. In the terminology of Definition A.5.1, we say that $\tilde{\gamma}$ is a *horizontal lifting* of γ.

We have the following:

PROPOSITION 2.2.28. *Let* $\Pi : P \to M$ *be a* G*-principal bundle endowed with a connection* $\mathrm{Hor}(P)$*. Given a smooth curve* $\gamma : I \to M$*,* $t_0 \in I$ *and* $p \in P_{\gamma(t_0)}$ *then there exists a unique parallel lifting* $\tilde{\gamma} : I \to P$ *of* γ *with* $\tilde{\gamma}(t_0) = p$*.*

PROOF. We have to show that M has the horizontal lifting property for paths (see Definition A.5.5); by Lemma A.5.6, it is sufficient to show that for every smooth local section $s : U \to P$, the open set U has the horizontal lifting property for paths. To this aim, it is enough to show that for every smooth curve $\gamma : I \to U$, every $t_0 \in I$ and every $g_0 \in G$, there exists a parallel lifting $(\gamma, g) : I \to U \times G$ of γ with $g(t_0) = g_0$, where the trivial principal bundle $U \times G$ is endowed with the connection that makes the isomorphism of principal bundles $\beta_s : U \times G \to P|_U$ connection preserving (recall Example 1.3.10). Let ω denote the connection form of $U \times G$ and let $\bar{\omega}$ be the $\mathfrak{g}$-valued 1-form on U that is equal to the pull-back of ω by the section $U \ni x \mapsto (x, 1) \in U \times G$; then (recall (2.2.10)):

$$\omega_{(x,g)}(v, Xg) = \mathrm{Ad}_{g^{-1}}\big(\bar{\omega}_x(v) + X\big),$$

for all $x \in U$, $g \in G$, $v \in T_xM$ and all $X \in \mathfrak{g}$. The curve (γ, g) is horizontal if and only if:

$$\omega_{(\gamma(t),g(t))}\big(\gamma'(t), g'(t)\big) = 0,$$

for all $t \in I$; this is equivalent to:

$$g'(t) = -\bar{\omega}_{\gamma(t)}\big(\gamma'(t)\big)g(t),$$

for all $t \in I$. The conclusion now follows from Corollary A.2.15 by setting $X(t) = -\bar{\omega}_{\gamma(t)}\big(\gamma'(t)\big) \in \mathfrak{g}$. $\square$

DEFINITION 2.2.29. Let $\Pi : P \to M$ be a G-principal bundle endowed with a connection $\mathrm{Hor}(P)$. Given a smooth curve $\gamma : [a, b] \to M$ and a point $p \in P_{\gamma(a)}$, if $\tilde{\gamma} : [a, b] \to P$ denotes the unique parallel lifting of γ with $\tilde{\gamma}(a) = p$ then $\tilde{\gamma}(b) \in P$ is called the *parallel transport of* p *along* γ.

We now prove (for later use) a result concerning the existence of local sections of a principal bundle that are parallel along "radial curves" issuing from a fixed point. More precisely, we have the following:

LEMMA 2.2.30. *Let* $\Pi : P \to M$ *be a* G*-principal bundle endowed with a connection* $\mathrm{Hor}(P)$*,* Z *be a real finite-dimensional vector space,* U_0 *be an open subset of* Z *which is star-shaped at the origin*[1]*,* U *be an open subset of* M *and* $f : U_0 \to U$ *be a smooth diffeomorphism. Then, for all* $p \in P_{f(0)}$ *there exists a smooth local section* $s : U \to P$ *such that* $s\big(f(0)\big) = p$ *and for all* $z \in Z$ *the curve:*

$$\big\{t \in \mathbb{R} : tz \in U_0\big\} \ni t \longmapsto s\big(f(tz)\big) \in P$$

is a parallel lifting of the "radial curve" $t \mapsto f(tz)$*.*

[1]This means that $0 \in U_0$ and $tz \in U_0$, for all $z \in U_0$ and all $t \in [0, 1]$.

PROOF. The map:

$$H : \big\{(z,t) \in Z \times \mathbb{R} : tz \in U_0\big\} \ni (z,t) \longmapsto f(tz) \in M$$

is a smooth Z-parametric family of curves (see Definition A.5.7). For each $z \in U_0$, let:

$$\big\{t \in \mathbb{R} : tz \in U_0\big\} \ni t \mapsto \widetilde{H}(z,t) \in P$$

be the parallel lifting of $t \mapsto H(z,t)$ with $\widetilde{H}(z,0) = p$, whose existence is guaranteed by Proposition 2.2.28. By Proposition A.5.9, the map $\widetilde{H}$ is smooth. We claim that:

$$\widetilde{H}(ct,z) = \widetilde{H}(t,cz), \tag{2.2.17}$$

for all $t \in \mathbb{R}$, $c \in \mathbb{R}$, $z \in Z$ with $ctz \in U_0$. Namely, for fixed $c \in \mathbb{R}$ and $z \in Z$, both curves:

$$t \longmapsto \widetilde{H}(ct,z), \quad t \longmapsto \widetilde{H}(t,cz)$$

are parallel liftings of $t \mapsto f(ctz) \in M$ and they assume the same value p at $t = 0$. The claim follows from the uniqueness part of Proposition 2.2.28. Now the desired smooth local section $s : U \to P$ is defined by:

$$s(x) = \widetilde{H}\big(1, f^{-1}(x)\big),$$

for all $x \in U$. We have:

$$\Pi\big(s(x)\big) = H\big(1, f^{-1}(x)\big) = f\big(f^{-1}(x)\big) = x,$$

and, for all $z \in Z$, the curve:

$$t \longmapsto s\big(f(tz)\big) = \widetilde{H}(1,tz) \overset{(2.2.17)}{=} \widetilde{H}(t,z) \in P$$

is a parallel lifting of $t \mapsto f(tz)$. This concludes the proof. $\square$

2.3. The generalized connection on the associated bundle

A connection on a principal bundle P induces a generalized connection on all the associated bundles of P. More precisely, we have the following:

LEMMA 2.3.1. *Let $\Pi : P \to M$ be a G-principal bundle and let N be a differentiable G-space. Consider the associated bundle $P \times_G N$ and denote by $\mathfrak{q} : P \times N \to P \times_G N$ the quotient map. Given a connection $\mathrm{Hor}(P)$ on P then there is a unique distribution $\mathrm{Hor}(P \times_G N)$ on $P \times_G N$ such that:*

$$\mathrm{Hor}_{[p,n]}(P \times_G N) = \mathrm{d}\mathfrak{q}_{(p,n)}\big(\mathrm{Hor}_p(P) \oplus \{0\}\big), \tag{2.3.1}$$

for all $p \in P$, $n \in N$. Moreover, the distribution $\mathrm{Hor}(P \times_G N)$ is smooth and horizontal with respect to the projection $\pi : P \times_G N \to M$, i.e., $\mathrm{Hor}(P \times_G N)$ is a generalized connection on $P \times_G N$.

PROOF. Given $g \in G$, denote by $\gamma_g^P : P \to P$ and by $\gamma_g^N : N \to N$ the diffeomorphisms given by the action of g on P and on N, respectively. The action of g on the product $P \times N$ (recall (1.2.14)) is given by $\gamma^P_{g^{-1}} \times \gamma^N_g$. We have a commutative diagram:

$$\begin{tikzcd} P \times N \arrow[d, "\gamma^P_{g^{-1}} \times \gamma^N_g"'] \arrow[dr, "\mathfrak{q}"] & \\ & P \times_G N \\ P \times N \arrow[ur, "\mathfrak{q}"'] & \end{tikzcd} \tag{2.3.2}$$

Given $p \in P$, $n \in N$, then the differential of $\gamma^P_{g^{-1}}$ takes $\mathrm{Hor}_p(P)$ to $\mathrm{Hor}_{p \cdot g^{-1}}(P)$ and thus the differential of $\gamma^P_{g^{-1}} \times \gamma^N_g$ takes $\mathrm{Hor}_p(P) \oplus \{0\}$ to $\mathrm{Hor}_{p\cdot g^{-1}}(P) \oplus \{0\}$. Differentiating diagram (2.3.2) we therefore obtain:

$$\mathrm{d}\mathfrak{q}_{(p,n)}\big(\mathrm{Hor}_p(P) \oplus \{0\}\big) = \mathrm{d}\mathfrak{q}_{(p\cdot g^{-1}, g\cdot n)}\big(\mathrm{Hor}_{p\cdot g^{-1}}(P) \oplus \{0\}\big),$$

proving that $\mathrm{Hor}(P \times_G N)$ is well-defined by equality (2.3.1). The uniqueness of the distribution $\mathrm{Hor}(P \times_G N)$ satisfying (2.3.1) is obvious. The fact that the distribution $\mathrm{Hor}(P \times_G N)$ is horizontal follows from the result of Exercise 2.2 and from the commutativity of diagram (1.4.7). Finally, let us prove that $\mathrm{Hor}(P \times_G N)$ is smooth. Consider the morphism of vector bundles:

$$\overline{\mathrm{d}\mathfrak{q}} : T(P \times N) \longrightarrow \mathfrak{q}^* T(P \times_G N),$$

defined as in Example 1.5.27. Notice that the result of Exercise 2.2 also says that the restriction of $\overline{\mathrm{d}\mathfrak{q}}$ to the vector subbundle $\mathrm{Hor}(P) \oplus \{0\}$ of $T(P \times N)$ is injective; thus, by Proposition 1.5.31, $\overline{\mathrm{d}\mathfrak{q}}\big(\mathrm{Hor}(P) \oplus \{0\}\big)$ is a vector subbundle of $\mathfrak{q}^* T(P \times_G N)$. Let $f : A \to P \times N$ be a smooth local section of the submersion $\mathfrak{q}$, where A is an open subset of $P \times_G N$. Since $\mathfrak{q} \circ f$ is the identity map of A, we can identify $T(P \times_G N)|_A$ with the pull-back $f^* \mathfrak{q}^* T(P \times_G N)$. Then:

$$\mathrm{Hor}(P \times_G N) \cap T(P \times_G N)|_A = f^* \overline{\mathrm{d}\mathfrak{q}}\big(\mathrm{Hor}(P) \oplus \{0\}\big)$$

and hence $\mathrm{Hor}(P \times_G N)$ is a subbundle of $T(P \times_G N)$. □

DEFINITION 2.3.2. The generalized connection $\mathrm{Hor}(P \times_G N)$ whose existence is given by Lemma 2.3.1 is called the generalized connection *associated* to the principal connection $\mathrm{Hor}(P)$ on P.

LEMMA 2.3.3. *Under the conditions of Lemma 1.4.11, assume that P, Q are endowed with connections* $\mathrm{Hor}(P)$ *and* $\mathrm{Hor}(Q)$*, respectively and that the associated bundles $P \times_G N$, $Q \times_H N'$ are endowed with the corresponding associated connections. If ϕ is connection preserving then also $\phi \underset{\sim}{\times} \kappa$ is connection preserving; conversely, if $\phi \underset{\sim}{\times} \kappa$ is connection preserving and if the action of H is effective on $\kappa(N)$ then also ϕ is connection preserving.*

PROOF. Denote by $\mathfrak{q} : P \times N \to P \times_G N$, $\mathfrak{q}' : Q \times N' \to Q \times_H N'$ the quotient maps. Let $p \in P$, $n \in N$ be fixed and set $q = \phi(p)$, $n' = \kappa(n)$. Differentiating

(1.4.10), we obtain a commutative diagram:

$$
\begin{tikzcd}
T_pP \oplus T_nN \arrow[rr, "\mathrm{d}\phi_p \oplus \mathrm{d}\kappa_n"] \arrow[d, "\mathrm{d}\mathfrak{q}(p,n)"'] & & T_qQ \oplus T_{n'}N' \arrow[d, "\mathrm{d}\mathfrak{q}'(q,n')"] \\
T_{[p,n]}(P \times_G N) \arrow[rr, "\mathrm{d}(\phi \underset{\sim}{\times} \kappa)([p,n])"'] & & T_{[q,n']}(Q \times_H N')
\end{tikzcd}
\tag{2.3.3}
$$

If ϕ is connection preserving then the top arrow of (2.3.3) carries the space $\mathrm{Hor}_p(P) \oplus \{0\}$ to $\mathrm{Hor}_q(Q) \oplus \{0\}$, so that the bottom arrow of (2.3.3) carries the space $\mathrm{Hor}_{[p,n]}(P \times_G N)$ to $\mathrm{Hor}_{[q,n']}(Q \times_H N')$; thus, $\phi \underset{\sim}{\times} \kappa$ is connection preserving. Conversely, assume that $\phi \underset{\sim}{\times} \kappa$ is connection preserving and that the action of H on N' is effective on $\kappa(N)$. Let $p \in P$ be fixed and set $q = \phi(p)$. The commutativity of diagram (2.3.3) and the fact that $\phi \underset{\sim}{\times} \kappa$ is connection preserving imply that:

$$\mathrm{d}\mathfrak{q}'_{[q,n']}\big[\mathrm{d}\phi_p\big(\mathrm{Hor}_p(P)\big) \oplus \{0\}\big] = \mathrm{d}\mathfrak{q}'_{[q,n']}\big(\mathrm{Hor}_q(Q) \oplus \{0\}\big),$$

for all $n \in N$, where $n' = \kappa(n)$. This is equivalent to (see Exercise 2.13):

$$\Big(\mathrm{d}\phi_p\big(\mathrm{Hor}_p(P)\big) \oplus \{0\}\Big) + \mathrm{Ker}\big(\mathrm{d}\mathfrak{q}'(q,n')\big) = \big(\mathrm{Hor}_q(Q) \oplus \{0\}\big) + \mathrm{Ker}\big(\mathrm{d}\mathfrak{q}'(q,n')\big). \tag{2.3.4}$$

Recall from Corollary 1.4.8 that:

$$\mathrm{Ker}\big(\mathrm{d}\mathfrak{q}'(q,n')\big) = \big\{\big(X^Q(q), -X^{N'}(n')\big) : X \in \mathfrak{h}\big\}. \tag{2.3.5}$$

Choose any $\zeta \in \mathrm{d}\phi_p\big(\mathrm{Hor}_p(P)\big)$ and let us show that ζ is in $\mathrm{Hor}_q(Q)$. Write $\zeta = \zeta_{\mathrm{h}} + \zeta_{\mathrm{v}}$, with $\zeta_{\mathrm{h}} \in \mathrm{Hor}_q(Q)$ and $\zeta_{\mathrm{v}} \in \mathrm{Ver}_q(Q)$; let $X \in \mathfrak{h}$ be such that $\zeta_{\mathrm{v}} = X^Q(q)$. We have to show that $X = 0$. Given $n \in N$ then $(\zeta, 0)$ belongs to the lefthand side of (2.3.4) and thus it belongs also to the righthand side (2.3.4); thus, by (2.3.5), there exists $\zeta' \in \mathrm{Hor}_q(Q)$ and $Y \in \mathfrak{h}$ with:

$$(\zeta, 0) = (\zeta', 0) + \big(Y^Q(q), -Y^{N'}(n')\big),$$

so that:

$$\zeta = \zeta_{\mathrm{h}} + X^Q(q) = \zeta' + Y^Q(q) \quad \text{and} \quad Y^{N'}(n') = 0.$$

Since $\zeta_{\mathrm{h}}, \zeta' \in \mathrm{Hor}_q(Q)$ and $X^Q(q), Y^Q(q) \in \mathrm{Ver}_q(Q)$, we have $X^Q(q) = Y^Q(q)$ and therefore $X = Y$; thus:

$$X^{N'}(n') = X^{N'}\big(\kappa(n)\big) = 0,$$

for all $n \in N$. It now follows from Lemma A.2.4 that $X = 0$. □

COROLLARY 2.3.4. *Under the conditions of Lemma 1.4.10, assume that P, Q are endowed with connections $\mathrm{Hor}(P)$ and $\mathrm{Hor}(Q)$, respectively and that the associated bundles $P \times_G N$, $Q \times_H N$ are endowed with the corresponding associated connections. If ϕ is connection preserving then also $\hat{\phi}$ is connection preserving; conversely, if the action of H on N is effective then if $\hat{\phi}$ is connection preserving then also ϕ is connection preserving.*

PROOF. Apply Lemma 2.3.3 to the case where κ is the identity map of N. □

COROLLARY 2.3.5. *Let P be a G-principal bundle endowed with connections $\mathrm{Hor}(P)$, $\mathrm{Hor}'(P)$ and let N be a differentiable G-space; assume that the action of G on N is effective. If both connections $\mathrm{Hor}(P)$ and $\mathrm{Hor}'(P)$ are associated to the same connection on $P \times_G N$ then $\mathrm{Hor}(P) = \mathrm{Hor}'(P)$.*

PROOF. Apply Corollary 2.3.4 with ϕ the identity map of P. $\square$

Let us show how the covariant derivative of a local section of $P \times_G N$ can be computed using its representation with respect to a local section of P (recall (1.4.6)).

LEMMA 2.3.6. *Let $\Pi : P \to M$ be a G-principal bundle with connection $\mathrm{Hor}(P)$; denote by ω its connection form. Let N be a differentiable G-space and $P \times_G N$ be the corresponding associated bundle of P, endowed with the generalized connection associated to $\mathrm{Hor}(P)$. Let $s : U \to P$, $\epsilon : U \to P \times_G N$ be smooth local sections; denote by $\tilde{\epsilon}$ and $\bar{\omega}$ respectively the representations of ϵ and of ω with respect to s. Given $x \in U$, $v \in T_xM$ and setting $p = s(x)$, $n = \tilde{\epsilon}(x)$ then the covariant derivative $\nabla_v \epsilon$ is given by (recall Definition A.2.3):*

$$\nabla_v \epsilon = \mathrm{d}\hat{p}_n\big[\mathrm{d}\tilde{\epsilon}_x(v) + \big(\bar{\omega}_x(v)\big)^N(n)\big] \in \mathrm{Ver}_{[p,n]}(P \times_G N).$$

PROOF. Since $\epsilon = \mathfrak{q} \circ (s, \tilde{\epsilon})$, we have:

$$\mathrm{d}\epsilon_x(v) = \mathrm{d}\mathfrak{q}_{(p,n)}\big(\mathrm{d}s_x(v), \mathrm{d}\tilde{\epsilon}_x(v)\big);$$

writing $\mathrm{d}s_x(v) = \zeta_{\mathrm{hor}} + \zeta_{\mathrm{ver}}$ with $\zeta_{\mathrm{hor}} \in \mathrm{Hor}_p(P)$ and $\zeta_{\mathrm{ver}} \in \mathrm{Ver}_p(P)$ then:

$$\mathrm{d}\epsilon_x(v) = \mathrm{d}\mathfrak{q}_{(p,n)}(\zeta_{\mathrm{hor}}, 0) + \mathrm{d}\mathfrak{q}_{(p,n)}\big(\zeta_{\mathrm{ver}}, \mathrm{d}\tilde{\epsilon}_x(v)\big), \tag{2.3.6}$$

and $\mathrm{d}\mathfrak{q}_{(p,n)}(\zeta_{\mathrm{hor}}, 0) \in \mathrm{Hor}_{[p,n]}(P \times_G N)$. Lemma 1.4.7 then implies that the second term on the righthand side of (2.3.6) is equal to $\mathfrak{p}_{\mathrm{ver}}\big(\mathrm{d}\epsilon_x(v)\big)$ and that:

$$\mathrm{d}\mathfrak{q}_{(p,n)}\big(\zeta_{\mathrm{ver}}, \mathrm{d}\tilde{\epsilon}_x(v)\big) = \mathrm{d}\hat{p}_n\big[\mathrm{d}\tilde{\epsilon}_x(v) + X^N(n)\big],$$

where $X \in \mathfrak{g}$ satisfies $\zeta_{\mathrm{ver}} = \mathrm{d}\beta_p(1) \cdot X$. Clearly, $X = \omega_p\big(\mathrm{d}s_x(v)\big) = \bar{\omega}_x(v)$. The conclusion follows. $\square$

COROLLARY 2.3.7. *Let $\Pi : P \to M$ be a G-principal bundle with connection $\mathrm{Hor}(P)$ and denote by ω its connection form. Let E_0 be a real finite-dimensional vector space and let $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation of G on E_0; consider the corresponding associated bundle $P \times_G E_0$, endowed with the generalized connection associated to $\mathrm{Hor}(P)$. Let $s : U \to P$, $\epsilon : U \to P \times_G E_0$ be smooth local sections and denote by $\tilde{\epsilon}$ and $\bar{\omega}$ respectively the representations of ϵ and of ω with respect to s. Given $x \in U$, $v \in T_xM$ and setting $p = s(x)$, then the covariant derivative $\nabla_v \epsilon$ is given by:*

$$\nabla_v \epsilon = \hat{p}\big[\mathrm{d}\tilde{\epsilon}_x(v) + \bar{\rho}\big(\bar{\omega}_x(v)\big) \cdot \tilde{\epsilon}(x)\big] = \big[p, \mathrm{d}\tilde{\epsilon}_x(v) + \bar{\rho}\big(\bar{\omega}_x(v)\big) \cdot \tilde{\epsilon}(x)\big],$$

where $\bar{\rho} = \mathrm{d}\rho(1) : \mathfrak{g} \to \mathfrak{gl}(E_0)$. In particular, if G is a Lie subgroup of $\mathrm{GL}(E_0)$ and ρ is the inclusion then:

$$\nabla_v \epsilon = \hat{p}\big[\mathrm{d}\tilde{\epsilon}_x(v) + \bar{\omega}_x(v) \cdot \tilde{\epsilon}(x)\big] = [p, \mathrm{d}\tilde{\epsilon}_x(v) + \bar{\omega}_x(v) \cdot \tilde{\epsilon}(x)]. \tag{2.3.7}$$

PROOF. Follows directly from Lemma 2.3.6 (recall also (1.4.5) and Example 1.4.9). □

2.4. Connections on vector bundles

Let $\pi : E \to M$ be a vector bundle with typical fiber E_0. Recall (see Subsection 1.5.1) that the set $\mathbf{\Gamma}(E)$ of all smooth sections of E is a real vector space and also a module over the ring $C^\infty(M)$ of all smooth real-valued functions on M.

Let $X \in \mathbf{\Gamma}(TM)$ be a smooth vector field on M and $f \in C^\infty(M)$ be a smooth real valued function on M (or, more generally, f can be a smooth map on M taking values on a fixed real finite-dimensional vector space). We denote by $X(f)$ the map defined by $X(f)(x) = \mathrm{d}f(x) \cdot X(x)$, for all $x \in M$.

DEFINITION 2.4.1. A *connection* on the vector bundle E is an $\mathbb{R}$-bilinear map

$$\nabla : \mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(E) \ni (X, \epsilon) \longmapsto \nabla_X \epsilon \in \mathbf{\Gamma}(E)$$

that is $C^\infty(M)$-linear in X and satisfies the *Leibnitz rule*:

$$\nabla_X(f\epsilon) = X(f)\epsilon + f\nabla_X\epsilon, \tag{2.4.1}$$

for all $X \in \mathbf{\Gamma}(TM)$, $\epsilon \in \mathbf{\Gamma}(E)$ and all $f \in C^\infty(M)$.

Given $\epsilon \in \mathbf{\Gamma}(E)$, it follows from the $C^\infty(M)$-linearity of the map $X \mapsto \nabla_X\epsilon$ (see Exercise 1.63) that for each $x \in M$ there exists a linear map:

$$T_xM \ni v \longmapsto \nabla_v\epsilon \in E_x$$

such that:

$$\nabla_v\epsilon = (\nabla_X\epsilon)(x),$$

for all $X \in \mathbf{\Gamma}(TM)$ with $X(x) = v$. Notice that (2.4.1) implies:

$$\nabla_v(f\epsilon) = \mathrm{d}f_x(v)\epsilon(x) + f(x)\nabla_v\epsilon, \tag{2.4.2}$$

for all $\epsilon \in \mathbf{\Gamma}(E)$, $f \in C^\infty(M)$, $x \in M$ and all $v \in T_xM$.

REMARK 2.4.2. If for every $x \in M$ we are given an $\mathbb{R}$-bilinear map:

$$T_xM \times \mathbf{\Gamma}(E) \ni (v, \epsilon) \longmapsto \nabla_v\epsilon \in E_x$$

such that (2.4.2) holds for all $\epsilon \in \mathbf{\Gamma}(E)$, $f \in C^\infty(M)$, $x \in M$, $v \in T_xM$ and such that for all $X \in \mathbf{\Gamma}(TM)$, $\epsilon \in \mathbf{\Gamma}(E)$ the map:

$$M \ni x \longmapsto \nabla_{X(x)}\epsilon \in E$$

is smooth then clearly there exists a unique connection ∇ on E such that:

$$(\nabla_X\epsilon)(x) = \nabla_{X(x)}\epsilon,$$

for all $X \in \mathbf{\Gamma}(TM)$, $\epsilon \in \mathbf{\Gamma}(E)$ and all $x \in M$.

Using the result of Exercise 1.72 we can give the following:

DEFINITION 2.4.3. Given a vector bundle $\pi : E \to M$ endowed with a connection ∇ and a smooth section $\epsilon \in \mathbf{\Gamma}(E)$ we define the *covariant derivative* of ϵ to be the smooth section $\nabla\epsilon$ of the vector bundle $\mathrm{Lin}(TM, E)$ that carries each $x \in M$ to the linear map $T_xM \ni v \mapsto \nabla_v\epsilon \in E_x$.

A connection on a vector bundle E induces a connection on every restriction of E to open subsets of the base space. This is the content of the following:

LEMMA 2.4.4. *Let $\pi : E \to M$ be a vector bundle and ∇ be a connection on E. Given an open subset U of M then there exists a unique connection ∇^U on the restricted vector bundle $E|_U$ such that:*

$$\nabla^U_v(\epsilon|_U) = \nabla_v\epsilon, \tag{2.4.3}$$

for all $\epsilon \in \mathbf{\Gamma}(E)$ and all $v \in TM|_U$.

PROOF. Let $\epsilon' \in \mathbf{\Gamma}(E|_U)$, $x \in U$ be given and choose $\epsilon \in \mathbf{\Gamma}(E)$ such that ϵ and ϵ' are equal on an open neighborhood of x in U (for instance, multiply ϵ' by a smooth real-valued map on M with support contained in U and that is equal to 1 in a neighborhood of x). If ∇^U is a connection on $E|_U$ satisfying (2.4.3) then the result of Exercise 2.14 implies that:

$$\nabla^U_v\epsilon' = \nabla_v\epsilon, \quad v \in T_xM; \tag{2.4.4}$$

this proves the uniqueness of ∇^U. Notice that the result of Exercise 2.14 also implies that the righthand side of (2.4.4) does not depend on the choice of the smooth section $\epsilon \in \mathbf{\Gamma}(E)$ that is equal to ϵ' on a open neighborhood of x in U. Thus, we can use (2.4.4) as a definition for $\nabla^U_v\epsilon'$. It is easily checked that ∇^U is indeed a connection on $E|_U$ (see Remark 2.4.2). $\square$

From now on, *we make the convention of denoting the connection ∇^U defined by Lemma 2.4.4 by the same symbol ∇ used to denote the connection of E, unless an explicit reference to the open subset U is needed.*

EXAMPLE 2.4.5. Let M be a differentiable manifold, E_0 be a real finite-dimensional vector space. In the trivial vector bundle $M \times E_0$ there exists a canonically defined connection that will be denoted by the symbol $\mathrm{d\!I}$; namely, identifying $\mathbf{\Gamma}(M \times E_0)$ with the space of E_0-valued smooth maps on M then $\mathrm{d\!I} : \mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(M \times E_0) \to \mathbf{\Gamma}(M \times E_0)$ is defined by:

$$(\mathrm{d\!I}_X\epsilon)(x) = \mathrm{d}\epsilon(x) \cdot X(x),$$

for all $x \in M$, all $X \in \mathbf{\Gamma}(TM)$ and every smooth map $\epsilon : M \to E_0$.

EXAMPLE 2.4.6. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and let $s : U \to \mathrm{FR}_{E_0}(E)$ be a smooth local E_0-frame of E. We define a connection $\mathrm{d\!I}^s$ associated to s on the vector bundle $E|_U$ by setting:

$$(\mathrm{d\!I}^s_X\epsilon)(x) = s(x)\big[\mathrm{d}\tilde{\epsilon}_x\big(X(x)\big)\big], \tag{2.4.5}$$

for all $\epsilon \in \mathbf{\Gamma}(E|_U)$, $X \in \mathbf{\Gamma}(TM|_U)$ and all $x \in U$, where $\tilde{\epsilon} : U \to E_0$ denotes the representation of ϵ with respect to s. The connection $\mathrm{d\!I}^s$ on $E|_U$ is related by the vector bundle isomorphism $\check{s} : U \times E_0 \to E|_U$ to the canonical connection $\mathrm{d\!I}$ on the trivial principal bundle $U \times E_0$ (this will be formalized later on, see Example 2.5.11).

REMARK 2.4.7. Given connections ∇ and ∇' on the vector bundle E then the map $\mathfrak{t} : \mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(E) \to \mathbf{\Gamma}(E)$ defined by:

$$\mathfrak{t}(X, \epsilon) = \nabla_X \epsilon - \nabla'_X \epsilon \in \mathbf{\Gamma}(E),$$

for all $X \in \mathbf{\Gamma}(TM)$, $\epsilon \in \mathbf{\Gamma}(E)$ is $C^\infty(M)$-bilinear. Thus, for each $x \in M$ there exists a bilinear map $\mathfrak{t}_x : T_xM \times E_x \to E_x$ such that:

$$\mathfrak{t}_x\big(X(x), \epsilon(x)\big) = \mathfrak{t}(X, \epsilon)(x),$$

for all $X \in \mathbf{\Gamma}(TM)$, $\epsilon \in \mathbf{\Gamma}(E)$ (see Exercise 1.63); in view of Example 1.6.31 we can identify $\mathfrak{t}$ with the smooth section $x \mapsto \mathfrak{t}_x$ of the vector bundle $\mathrm{Lin}(TM, E; E)$. Recall from Example 1.6.33 that the vector bundle $\mathrm{Lin}(TM, E; E)$ is identified with $\mathrm{Lin}\big(TM, \mathrm{Lin}(E)\big)$ and therefore $\mathfrak{t}$ is identified with a smooth $\mathrm{Lin}(E)$-valued covariant 1-tensor field on M. Notice that if ∇ is an arbitrary connection on E and $\mathfrak{t} : \mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(E) \to \mathbf{\Gamma}(E)$ is an arbitrary $C^\infty(M)$-bilinear map then $\nabla + \mathfrak{t}$ is also a connection on E.

DEFINITION 2.4.8. Let ∇ be a connection on E and let $s : U \to \mathrm{FR}_{E_0}(E)$ be a smooth local E_0-frame of E. The *Christoffel tensor* of the connection ∇ with respect to s is the $C^\infty(M)$-bilinear map

$$\Gamma : \mathbf{\Gamma}(TM|_U) \times \mathbf{\Gamma}(E|_U) \longrightarrow \mathbf{\Gamma}(E|_U)$$

defined by $\Gamma = \nabla - \mathrm{d}\!\mathrm{I}^s$ (recall (2.4.5)).

As in Remark 2.4.7, we can identify the Christoffel tensor Γ with a smooth section of the vector bundle $\mathrm{Lin}(TM|_U, E|_U; E|_U)$ or with a smooth $\mathrm{Lin}(E)$-valued covariant 1-tensor field on $U \subset M$.

Let us make more explicit the meaning of the Christoffel tensor Γ of the connection ∇ with respect to a smooth local E_0-frame $s : U \to \mathrm{FR}_{E_0}(E)$. For all $\epsilon \in \mathbf{\Gamma}(E|_U)$, $x \in U$ and all $v \in T_xM$, we have:

$$\nabla_v \epsilon = s(x)\big(\mathrm{d}\tilde{\epsilon}_x(v)\big) + \Gamma_x\big(v, \epsilon(x)\big), \tag{2.4.6}$$

where $\tilde{\epsilon} : U \to E_0$ denotes the representation of ϵ with respect to s.

DEFINITION 2.4.9. The *curvature tensor* of a connection ∇ is the map

$$R : \mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(E) \longrightarrow \mathbf{\Gamma}(E)$$

defined by:

$$R(X, Y)\epsilon = \nabla_X \nabla_Y \epsilon - \nabla_Y \nabla_X \epsilon - \nabla_{[X,Y]}\epsilon,$$

for all $X, Y \in \mathbf{\Gamma}(TM)$, $\epsilon \in \mathbf{\Gamma}(E)$, where $[X, Y] \in \mathbf{\Gamma}(TM)$ denotes the Lie bracket of X and Y.

It is easy to check that the curvature tensor R is $C^\infty(M)$-trilinear and thus, for each $x \in M$, it defines a trilinear map $R_x : T_xM \times T_xM \times E_x \to E_x$ (see Exercise 1.63). Obviously the curvature tensor is anti-symmetric with respect to its two first variables.

DEFINITION 2.4.10. Given a connection ∇ on the tangent bundle TM, the *torsion tensor* of ∇ is the map $T : \mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(TM) \to \mathbf{\Gamma}(TM)$ defined by:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

for all $X, Y \in \mathbf{\Gamma}(TM)$. More generally, if ∇ is a connection on an arbitrary vector bundle $\pi : E \to M$ and if $\iota : TM \to E$ is a vector bundle morphism then the *ι-torsion tensor* of ∇ is the map $T^\iota : \mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(TM) \to \mathbf{\Gamma}(E)$ defined by:

$$T^\iota(X, Y) = \nabla_X\big(\iota(Y)\big) - \nabla_Y\big(\iota(X)\big) - \iota\big([X, Y]\big),$$

for all $X, Y \in \mathbf{\Gamma}(TM)$. A connection ∇ on TM whose torsion tensor T is identically zero is said to be *symmetric*.

Clearly, if $E = TM$ and $\iota : TM \to TM$ is the identity then $T^\iota = T$. It is easy to check that the ι-torsion tensor T^ι is $C^\infty(M)$-bilinear and thus, for each $x \in M$, it defines a bilinear map $T^\iota_x : T_xM \times T_xM \to E_x$ (see Exercise 1.63). Obviously the ι-torsion tensor is anti-symmetric.

2.5. Relating linear connections with principal connections

Let $\pi : E \to M$ be a vector bundle over a differentiable manifold M with typical fiber E_0 and let $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ be a principal connection on the frame bundle $\mathrm{FR}_{E_0}(E)$. Such principal connection induces an associated connection $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E) \times E_0\big)$ on the associated bundle $\mathrm{FR}_{E_0}(E) \times E_0$ (see Section 2.3). The contraction map $\mathcal{C}^E$ (recall (1.5.1)) carries $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E) \times E_0\big)$ to a generalized connection $\mathrm{Hor}(E)$ on E, i.e., $\mathrm{Hor}(E)$ is the unique generalized connection on E that makes the smooth diffeomorphism $\mathcal{C}^E$ connection preserving. More explicitly, $\mathrm{Hor}(E)$ is defined by:

$$\mathrm{Hor}_{p(e_0)}(E) = \mathrm{d}\mathcal{C}^E_{[p,e_0]}\big[\mathrm{Hor}_{[p,e_0]}\big(\mathrm{FR}_{E_0}(E) \times E_0\big)\big], \tag{2.5.1}$$

for all $[p, e_0] \in \mathrm{FR}_{E_0}(E) \times E_0$.

DEFINITION 2.5.1. Let $\pi : E \to M$ be a vector bundle over a differentiable manifold M with typical fiber E_0 and let $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ be a principal connection on the frame bundle $\mathrm{FR}_{E_0}(E)$. The generalized connection $\mathrm{Hor}(E)$ on E defined by (2.5.1) is called the generalized connection *induced* by $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$.

The generalized connections on the vector bundle E and on the associated bundle $\mathrm{FR}_{E_0}(E) \times E_0$ define covariant derivative operators for smooth local sections of E and of $\mathrm{FR}_{E_0}(E) \times E_0$, respectively; let us use the symbol ∇ to denote both of them. Since $\mathcal{C}^E$ is connection preserving, by Lemma 2.1.5 we have:

$$\nabla_v \epsilon = \mathrm{d}\mathcal{C}^E\big[\nabla_v\big((\mathcal{C}^E)^{-1} \circ \epsilon\big)\big],$$

for every smooth local section $\epsilon : U \to E$ and for all $v \in TM|_U$. Since $\mathcal{C}^E$ is linear on the fibers, its differential restricted to a vertical space is just the restriction of the contraction map $\mathcal{C}^E$ itself; thus:

$$\nabla_v \epsilon = \mathcal{C}^E\big[\nabla_v\big((\mathcal{C}^E)^{-1} \circ \epsilon\big)\big], \tag{2.5.2}$$

for all $v \in TM|_U$ and all $\epsilon \in \mathbf{\Gamma}(E|_U)$.

Now let $s : U \to \mathrm{FR}_{E_0}(E)$ be a smooth local E_0-frame of the vector bundle E and let $\tilde{\epsilon} : U \to E_0$ denote the representation of a smooth local section $\epsilon : U \to E$ of E with respect to s; then:

$$(\mathcal{C}^E)^{-1} \circ \epsilon = \mathfrak{q} \circ (s, \tilde{\epsilon}),$$

where $\mathfrak{q} : \mathrm{FR}_{E_0}(E) \times E_0 \to \mathrm{FR}_{E_0}(E) \underset{\sim}{\times} E_0$ denotes the quotient map. The representation of $(\mathcal{C}^E)^{-1} \circ \epsilon$ with respect to s is also equal to $\tilde{\epsilon}$ (see Example 1.5.10). Using equality (2.3.7) we obtain:

$$\nabla_v \big((\mathcal{C}^E)^{-1} \circ \epsilon \big) = [s(x), \mathrm{d}\tilde{\epsilon}_x(v) + \bar{\omega}_x(v) \cdot \tilde{\epsilon}(x)], \tag{2.5.3}$$

for all $x \in U$ and all $v \in T_x M$, where $\bar{\omega}$ denotes the representation with respect to s of the connection form ω corresponding to $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$. From (2.5.2) and (2.5.3) we get:

$$\nabla_v \epsilon = s(x) \big[\mathrm{d}\tilde{\epsilon}_x(v) + \bar{\omega}_x(v) \cdot \tilde{\epsilon}(x) \big], \tag{2.5.4}$$

for all $\epsilon \in \mathbf{\Gamma}(E|_U)$, all $x \in U$ and all $v \in T_x M$. If we set:

$$\Gamma_x(v) = \mathcal{I}_{s(x)}\big(\bar{\omega}_x(v)\big) = s(x) \circ \bar{\omega}_x(v) \circ s(x)^{-1} \in \mathfrak{gl}(E_x), \tag{2.5.5}$$

for all $x \in U$, $v \in T_x M$, then formula (2.5.4) becomes (recall (2.4.5)):

$$\nabla_v \epsilon = \mathrm{d}\mathbb{I}^s_v \epsilon + \Gamma_x(v) \cdot \epsilon(x).$$

If follows that ∇ is indeed a connection on the vector bundle E and that the Christoffel tensor Γ of ∇ with respect to the smooth local E_0-frame s is given by (2.5.5). We have proven:

PROPOSITION 2.5.2. *Let $\pi : E \to M$ be a vector bundle over a differentiable manifold M with typical fiber E_0 and let $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ be a principal connection on the frame bundle $\mathrm{FR}_{E_0}(E)$; denote by $\mathrm{Hor}(E)$ the induced generalized connection on E. The covariant derivative operator ∇ corresponding to $\mathrm{Hor}(E)$ is a linear connection on the vector bundle E; moreover, if $s : U \to \mathrm{FR}_{E_0}(E)$ is a smooth local E_0-frame of E then the Christoffel tensor of ∇ with respect to the smooth local E_0-frame s and the representation $\bar{\omega} = s^*\omega$ of the connection form ω of $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ with respect to s are related by equality* (2.5.5). $\square$

As a converse to Proposition 2.5.2, we will now show that every linear connection ∇ on E is induced by a unique principal connection on the principal bundle of frames of E.

REMARK 2.5.3. If U is an open subset of M and $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ is a connection on $\mathrm{FR}_{E_0}(E)$ then we have a corresponding connection $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)|_U\big)$ on $\mathrm{FR}_{E_0}(E)|_U = \mathrm{FR}_{E_0}(E|_U)$ (see Example 2.2.8). Clearly, if ∇ is the connection on E associated to $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ then the connection on $E|_U$ associated to $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)|_U\big)$ is just ∇^U (recall Lemma 2.4.4).

PROPOSITION 2.5.4. *Let $\pi : E \to M$ be a vector bundle with typical fiber E_0. For every linear connection ∇ on the vector bundle E there exists a unique principal connection $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ on the principal bundle of frames $\mathrm{FR}_{E_0}(E)$*

such that ∇ *is the covariant derivative operator corresponding to the induced generalized connection* $\mathrm{Hor}(E)$ *on* E.

PROOF. Using the results of Exercises 2.6, 2.15 and Remark 2.5.3, it is easy to see that it suffices to prove the proposition in the case where the frame bundle $\mathrm{FR}_{E_0}(E)$ admits a globally defined smooth local section $s : M \to \mathrm{FR}_{E_0}(E)$. Let us therefore assume that such globally defined smooth local section s exists. Let Γ denote the Christoffel tensor of ∇ with respect to s. Given a connection $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ on $\mathrm{FR}_{E_0}(E)$ with connection form ω, let us denote by $\bar{\omega}$ the representation of ω with respect to s. Then ∇ is associated with $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ if and only if (2.5.5) holds, for all $x \in M$. But (2.5.5) defines a unique smooth $\mathfrak{gl}(E_0)$-valued 1-form on M and Lemma 2.2.10 says that there exists a unique connection form ω on $\mathrm{FR}_{E_0}(E)$ with $\bar{\omega} = s^*\omega$. The conclusion follows. $\square$

COROLLARY 2.5.5. *Let* $\pi : E \to M$ *be a vector bundle with typical fiber* E_0. *Given a linear connection* ∇ *on* E *then there exists a unique generalized connection* $\mathrm{Hor}(E)$ *on* E *whose covariant derivative operator is* ∇.

PROOF. The existence follows from Proposition 2.5.4 and the uniqueness follows from Corollary 2.1.6, keeping in mind the fact that the submersion π has the global extension property (see Exercise 1.62). $\square$

COROLLARY 2.5.6. *Let* $\pi : E \to M$ *be a vector bundle with typical fiber* E_0. *If* $\mathrm{Hor}(E)$ *is a generalized connection on* E *whose covariant derivative operator* ∇ *is a linear connection on* E *then there exists a unique principal connection* $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ *on the principal bundle of frames* $\mathrm{FR}_{E_0}(E)$ *such that* $\mathrm{Hor}(E)$ *is induced by* $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$. $\square$

The result of Propositions 2.5.2, 2.5.4 and of Corollaries 2.5.5 and 2.5.6 can be summarized as follows: the set of linear connections on a vector bundle E is in one-to-one correspondence with a subset of the set of all generalized connections on E. Such subset of the set of generalized connections on E is precisely the set of generalized connections that are induced by principal connections on $\mathrm{FR}_{E_0}(E)$. Moreover, there is also a one-to-one correspondence between the set of principal connections on $\mathrm{FR}_{E_0}(E)$ and the set of generalized connections on E whose covariant derivative operator is a linear connection; in particular, there is a one-to-one correspondence between the set of principal connections on $\mathrm{FR}_{E_0}(E)$ and the set of linear connections on E. *From now on, we use such one-to-one correspondence to identify the set of linear connections on* E *with a subset of the set of generalized connections on* E.

EXAMPLE 2.5.7. Let M be a differentiable manifold, E_0 be a real finite-dimensional vector space and consider the trivial vector bundle $E = M \times E_0$. Its principal bundle of E_0-frames is the trivial principal bundle:

$$P = M \times \mathrm{GL}(E_0).$$

We claim that the canonical connection $\mathrm{d}\!\mathrm{I}$ of E (see Example 2.4.5) is induced by the canonical connection of P (see Example 2.2.7). To prove the claim, let

$s : M \to P$ be the smooth section defined by $s(x) = (x, \mathrm{Id})$, where $\mathrm{Id} \in \mathrm{GL}(E_0)$ denotes the identity map of E_0. Obviously the connection dI^s (Example 2.4.6) on E is the canonical connection of the trivial bundle E. If ∇ is the connection on E induced by the trivial connection on P then $\nabla = \mathrm{dI}^s + \Gamma$, where Γ denotes the Christoffel tensor of ∇ with respect to s. We have to check that $\Gamma = 0$. If ω is the connection form of the trivial connection of P then it is easy to see that $\bar{\omega} = s^*\omega = 0$ and hence $\Gamma = 0$, by formula (2.5.5).

LEMMA 2.5.8. *Let $\pi : E \to M$ be a vector bundle with typical fiber E_0, ∇, ∇' be connections on E and ω, ω' respectively be the connections forms of the corresponding connections on the principal bundle $\mathrm{FR}_{E_0}(E)$. Set $\mathfrak{t} = \nabla - \nabla'$. If $s : U \to \mathrm{FR}_{E_0}(E)$ is a smooth local section then the diagram:*

$$\begin{tikzcd}
 & \mathfrak{gl}(E_x) \\
T_xM \arrow[ur, "\mathfrak{t}_x"] \arrow[dr, "(s^*(\omega-\omega'))_x"'] & \\
 & \mathfrak{gl}(E_0) \arrow[uu, "\mathcal{I}_{s(x)}"']
\end{tikzcd}$$

commutes, for all $x \in U$, where $\mathfrak{t}_x : T_xM \to \mathfrak{gl}(E_x)$ denotes the map $v \mapsto \mathfrak{t}_x(v, \cdot)$ and $\mathcal{I}_{s(x)}$ denotes conjugation by the linear isomorphism $s(x) : E_0 \to E_x$.

PROOF. Let Γ, Γ' denote the Christoffel tensors with respect to s of ∇ and ∇', respectively. Clearly, $\mathfrak{t} = \Gamma - \Gamma'$. The conclusion is obtained immediately using (2.5.5). $\square$

EXAMPLE 2.5.9 (linear connection induced on an associated vector bundle). Let $\Pi : P \to M$ be a G-principal bundle, E_0 be a real finite-dimensional vector space and $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation of G on E_0, so that the associated bundle $P \times_G E_0$ is a vector bundle (recall Example 1.5.5). Given a principal connection $\mathrm{Hor}(P)$ on the principal bundle P, we obtain a principal connection $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(P \times_G E_0)\big)$ on $\mathrm{FR}_{E_0}(P \times_G E_0)$ by taking the push-forward (recall Definition 2.2.14) of $\mathrm{Hor}(P)$ by the morphism of principal bundles $\mathfrak{H} : P \to \mathrm{FR}_{E_0}(P \times_G E_0)$ defined in (1.5.3). The principal connection $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(P \times_G E_0)\big)$ therefore induces a linear connection ∇ on the vector bundle $P \times_G E_0$.

Notice that the principal connection $\mathrm{Hor}(P)$ of P induces an associated generalized connection $\mathrm{Hor}(P \times_G E_0)$ on the associated bundle $P \times_G E_0$ as explained in Section 2.3. We claim that ∇ is precisely the covariant derivative operator of such generalized connection. To prove the claim, we have to check that if $\mathrm{FR}_{E_0}(P \times_G E_0) \mathbin{\underset{\sim}{\times}} E_0$ is endowed with the generalized connection associated to $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(P \times_G E_0)\big)$ and if $P \times_G E_0$ is endowed with the generalized connection associated to $\mathrm{Hor}(P)$ then the contraction map (1.5.4) is connection preserving. But this follows from the observation that the contraction map (1.5.4) is the inverse of the induced map $\widehat{\mathfrak{H}}$ (recall Example 1.5.5) and from the fact that $\widehat{\mathfrak{H}}$ is connection preserving (recall Corollary 2.3.4).

2.5.1. Connection preserving morphisms of vector bundles. Since linear connections on vector bundles are particular cases of generalized connections then it makes sense to talk about connection preserving maps between vector bundles endowed with linear connections (recall Definition 2.1.3).

LEMMA 2.5.10. *Let E, E' be vector bundles endowed with linear connections ∇ and ∇', respectively. Let $L : E \to E'$ be a morphism of vector bundles. The following conditions are equivalent:*

(a) *L is connection preserving;*
(b) *$\nabla'_v(L \circ \epsilon) = L(\nabla_v \epsilon)$, for all $v \in TM$ and all $\epsilon \in \mathbf{\Gamma}(E)$.*

Moreover, if E and E' have the same typical fiber E_0 and if L is an isomorphism of vector bundles then (a), (b) are also equivalent to:

(c) *the morphism of principal bundles $L_* : \mathrm{FR}_{E_0}(E) \to \mathrm{FR}_{E_0}(E')$ is connection preserving, where $\mathrm{FR}_{E_0}(E)$ and $\mathrm{FR}_{E_0}(E')$ are endowed with the unique principal connections that induces the linear connections ∇ and ∇', respectively.*

PROOF. The equivalence between (a) and (b) follows from the equivalence between (a) and (d) in Lemma 2.1.5, by observing that the projection of a vector bundle has the global extension property (see Exercise 1.62). To prove the equivalence between (a) and (c), consider the commutative diagram (1.5.6). Since the contraction maps $\mathcal{C}^E$ and $\mathcal{C}^{E'}$ are connection preserving diffeomorphisms, it follows that L is connection preserving if and only if $\widehat{L_*}$ is. Finally, since the action of $\mathrm{GL}(E_0)$ on E_0 is effective, it follows from Corollary 2.3.4 that $\widehat{L_*}$ is connection preserving if and only if L_* is. $\square$

EXAMPLE 2.5.11. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and $s : U \to \mathrm{FR}_{E_0}(E)$ be a smooth local E_0-frame of E. The corresponding trivialization $\check{s} : U \times E_0 \to E|_U$ is a vector bundle isomorphism (see Example 1.5.13); if the trivial vector bundle $U \times E_0$ is endowed with its canonical connection $\mathrm{d\!I}$ (see Example 2.4.5) and $E|_U$ is endowed with the connection $\mathrm{d\!I}^s$ (see Example 2.4.6) then $\check{s}$ is connection preserving.

EXAMPLE 2.5.12. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 endowed with a connection ∇ and let $s : U \to \mathrm{FR}_{E_0}(E)$ be a smooth local E_0-frame of E. Denote by ω the connection form of the connection on $\mathrm{FR}_{E_0}(E)$ associated to ∇ and set $\bar{\omega} = s^*\omega$. Then $\bar{\omega}$ is a smooth $\mathrm{Lin}(E_0)$-valued covariant 1-tensor field on U that can be identified with a $C^\infty(U)$-bilinear map from $\mathbf{\Gamma}(TM|_U) \times \mathbf{\Gamma}(U \times E_0)$ to $\mathbf{\Gamma}(U \times E_0)$ (recall Examples 1.6.31 and 1.6.33). If $E|_U$ is endowed with the connection ∇^U and the trivial vector bundle $U \times E_0$ is endowed with the connection $\mathrm{d\!I} + \bar{\omega}$ then it follows from (2.5.4) that the local trivialization $\check{s} : U \times E_0 \to E|_U$ is a connection preserving vector bundle isomorphism.

EXAMPLE 2.5.13. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 endowed with a linear connection ∇; denote by $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ the principal connection on the frame bundle $\mathrm{FR}_{E_0}(E)$ that induces ∇. As explained in Example 2.5.9, the principal connection $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ induces a linear connection on the fiber product $\mathrm{FR}_{E_0}(E) \underset{\approx}{\times} E_0$; moreover, such linear connection is the covariant derivative operator of the generalized connection induced by $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ on the fiber product $\mathrm{FR}_{E_0}(E) \underset{\approx}{\times} E_0$ (as explained in Section 2.3). The contraction map $\mathcal{C}^E : \mathrm{FR}_{E_0}(E) \underset{\approx}{\times} E_0 \to E$ is therefore a connection preserving isomorphism of vector bundles (by the very definition of the relation between $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ and ∇).

EXAMPLE 2.5.14. Let P be a G-principal bundle over a differentiable manifold M, E_0 be a real finite-dimensional vector space, $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation, E be a vector bundle over M with typical fiber E_0 and $\phi : P \to \mathrm{FR}_{E_0}(E)$ be a morphism of principal bundles whose subjacent Lie group homomorphism is the representation ρ. If P is endowed with a principal connection $\mathrm{Hor}(P)$, $\mathrm{FR}_{E_0}(E)$ is endowed with the principal connection $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ obtained by push-forward of $\mathrm{Hor}(P)$ by ϕ, E is endowed with the linear connection induced by $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ and if $P \times_G E_0$ is endowed with the linear connection induced by $\mathrm{Hor}(P)$ (as explained in Example 2.5.9) then the ϕ-contraction map $\mathcal{C}^\phi : P \times_G E_0 \to E$ (see Definition 1.5.17) is a connection preserving isomorphism of principal bundles. Namely, $\mathcal{C}^\phi$ is the composition of $\hat{\phi}$ and $\mathcal{C}^E$ and both of them are connection preserving isomorphisms of vector bundles (see Corollary 2.3.4 and Example 2.5.13).

2.6. Pull-back of connections on vector bundles

Let $\pi : E \to M$ be a vector bundle with typical fiber E_0. Let $f : M' \to M$ be a smooth map defined in a differentiable manifold M'. Given a connection ∇ on E, then the *pull-back* of ∇ by f is a connection $f^*\nabla$ on the pull-back vector bundle f^*E defined as follows; let $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ be the connection on the principal bundle of E_0-frames of E associated to ∇ (Proposition 2.5.4). Consider the pull-back $\mathrm{Hor}\big(f^*\mathrm{FR}_{E_0}(E)\big)$ of the connection $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ by f (recall Definition 2.2.22). Since the principal bundles $f^*\mathrm{FR}_{E_0}(E)$ and $\mathrm{FR}_{E_0}(f^*E)$ are identified with each other, $\mathrm{Hor}\big(f^*\mathrm{FR}_{E_0}(E)\big)$ is a connection on $\mathrm{FR}_{E_0}(f^*E)$; the connection $f^*\nabla$ on f^*E is the connection associated to $\mathrm{Hor}\big(f^*\mathrm{FR}_{E_0}(E)\big)$.

EXAMPLE 2.6.1. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 endowed with a connection ∇. If U is an open subset of M and $i : U \to M$ denotes the inclusion map then, identifying i^*E with $E|_U$ as in Example 1.5.20, the pull-back $i^*\nabla$ is equal to the connection ∇^U (see Lemma 2.4.4). This follows from Remark 2.5.3 and Example 2.2.23.

EXAMPLE 2.6.2. Let $\pi : E \to M$ be a vector bundle endowed with a connection ∇ and let $f : M' \to M$, $g : M'' \to M'$ be smooth maps, where M', M'' are differentiable manifolds. Recall from Example 1.5.21 that we have identified the vector bundles g^*f^*E and $(f \circ g)^*E$. It follows directly from Example 2.2.24 that:

$$(f \circ g)^*\nabla = g^*(f^*\nabla).$$

The connection $f^*\nabla$ defines a horizontal distribution $\mathrm{Hor}(f^*E)$ on f^*E. The horizontal distributions $\mathrm{Hor}(f^*E)$ and $\mathrm{Hor}(E)$ are related by the following:

LEMMA 2.6.3. *Let $\pi : E \to M$ be a vector bundle and $f : M' \to M$ be a smooth map defined in a differentiable manifold M'. Let ∇ be a connection on E and let $\mathrm{Hor}(E)$, $\mathrm{Hor}(f^*E)$ be the horizontal distributions defined by ∇ and $f^*\nabla$ respectively. If $\bar{f} : f^*E \to E$ denotes the map defined in Subsection 1.5.3, then:*

$$\mathrm{Hor}_e(f^*E) = \mathrm{d}\bar{f}_e^{-1}\big(\mathrm{Hor}_{\bar{f}(e)}(E)\big),$$

for all $e \in f^*E$.

PROOF. Consider the quotient maps:

$$\mathfrak{q} : \mathrm{FR}_{E_0}(E) \times E_0 \longrightarrow \mathrm{FR}_{E_0}(E) \times_{\approx} E_0,$$
$$\mathfrak{q}_f : \mathrm{FR}_{E_0}(f^*E) \times E_0 \longrightarrow \mathrm{FR}_{E_0}(f^*E) \times_{\approx} E_0.$$

We have a commutative diagram:

$$\begin{tikzcd} \mathrm{FR}_{E_0}(f^*E) \times E_0 \arrow[rr, "\bar{f}\times\mathrm{Id}"] \arrow[dr, "\mathcal{C}^{f^*E}\circ\mathfrak{q}_f"'] & & \mathrm{FR}_{E_0}(E) \times E_0 \arrow[d, "\mathcal{C}^{E}\circ\mathfrak{q}"] \\ & f^*E \arrow[r, "\bar{f}"'] & E \end{tikzcd} \tag{2.6.1}$$

where we have denoted by $\bar{f}$ also the map from:

$$\mathrm{FR}_{E_0}(f^*E) = f^*\mathrm{FR}_{E_0}(E)$$

to $\mathrm{FR}_{E_0}(E)$ defined in Subsection 1.3.1. Let $e \in f^*E$ be fixed. Choose a pair (p, e_0) in $\mathrm{FR}_{E_0}(f^*E) \times E_0$ with $(\mathcal{C}^{f^*E} \circ \mathfrak{q}_f)(p, e_0) = e$, so that $(\mathcal{C}^E \circ \mathfrak{q})\big(\bar{f}(p), e_0\big) = \bar{f}(e)$. Recall from (2.3.1) and (2.5.1) that:

$$\begin{aligned} \mathrm{Hor}_e(f^*E) &= \mathrm{d}(\mathcal{C}^{f^*E} \circ \mathfrak{q}_f)_{(p,e_0)}\Big(\mathrm{Hor}_p\big(\mathrm{FR}_{E_0}(f^*E)\big) \oplus \{0\}\Big), \\ \mathrm{Hor}_{\bar{f}(e)}(E) &= \mathrm{d}(\mathcal{C}^{E} \circ \mathfrak{q})_{(\bar{f}(p),e_0)}\Big(\mathrm{Hor}_{\bar{f}(p)}\big(\mathrm{FR}_{E_0}(E)\big) \oplus \{0\}\Big). \end{aligned} \tag{2.6.2}$$

Differentiating diagram (2.6.1), we obtain:

$$\begin{tikzcd} T_p\mathrm{FR}_{E_0}(f^*E) \oplus E_0 \arrow[r, "\mathrm{d}\bar{f}_p\oplus\mathrm{Id}"] \arrow[d, "\mathrm{d}(\mathcal{C}^{f^*E}\circ\mathfrak{q}_f)_{(p,e_0)}"'] & T_{\bar{f}(p)}\mathrm{FR}_{E_0}(E) \oplus E_0 \arrow[d, "\mathrm{d}(\mathcal{C}^{E}\circ\mathfrak{q})_{(\bar{f}(p),e_0)}"] \\ T_e(f^*E) \arrow[r, "\mathrm{d}\bar{f}_e"'] & T_{\bar{f}(e)}E \end{tikzcd} \tag{2.6.3}$$

By Lemma 2.2.25, we have:

$$(\mathrm{d}\bar{f}_p \oplus \mathrm{Id})^{-1}\Big(\mathrm{Hor}_{\bar{f}(p)}\big(\mathrm{FR}_{E_0}(E)\big) \oplus \{0\}\Big) = \mathrm{Hor}_p\big(\mathrm{FR}_{E_0}(f^*E)\big) \oplus \{0\}.$$

Applying the result of Exercise 2.18 to diagram (2.6.3), keeping in mind (2.6.2), we obtain:

$$\mathrm{Hor}_e(f^*E) \subset \mathrm{d}\bar{f}_e^{-1}\big(\mathrm{Hor}_{\bar{f}(e)}(E)\big).$$

Since $\mathrm{d}\bar{f}_e$ carries $\mathrm{Ver}_e(f^*E)$ isomorphically onto $\mathrm{Ver}_{\bar{f}(e)}(E)$, we have:

$$\mathrm{d}\bar{f}_e^{-1}\big(\mathrm{Hor}_{\bar{f}(e)}(E)\big) \cap \mathrm{Ver}_e(f^*E) = \{0\}.$$

The conclusion now follows from the result of Exercise 2.3. □

COROLLARY 2.6.4. *Under the hypotheses of Lemma 2.6.3, let $\epsilon : U' \to E$ be a smooth local section of E along f defined in an open subset U' of M' and let $\bar{\epsilon} : U' \to f^*E$ be the smooth local section of f^*E such that $\epsilon = \bar{f} \circ \bar{\epsilon}$ (recall diagram* (1.5.7)*). Then, for every $y \in U'$, $v \in T_yM'$, we have:*

$$(f^*\nabla)_v\, \bar{\epsilon} = \mathfrak{p}_{\mathrm{ver}}\big(\mathrm{d}\epsilon(y) \cdot v\big) \in E_{f(y)},$$

where $\mathfrak{p}_{\mathrm{ver}} : TE \to \mathrm{Ver}(E)$ denotes the vertical projection determined by the horizontal distribution $\mathrm{Hor}(E)$.

PROOF. Let $\mathfrak{p}^f_{\mathrm{ver}} : T(f^*E) \to \mathrm{Ver}(f^*E)$ denote the vertical projection determined by the horizontal distribution $\mathrm{Hor}(f^*E)$. We have (recall (2.1.2)):

$$(f^*\nabla)_v\, \bar{\epsilon} = \mathfrak{p}^f_{\mathrm{ver}}\big(\mathrm{d}\bar{\epsilon}(y) \cdot v\big).$$

Lemma 2.6.3 implies easily that:

$$\mathrm{d}\bar{f} \circ \mathfrak{p}^f_{\mathrm{ver}} = \mathfrak{p}_{\mathrm{ver}} \circ \mathrm{d}\bar{f}.$$

The conclusion follows by observing that for all $e \in (f^*E)_y \cong E_{f(y)}$, the restriction to $\mathrm{Ver}_e(f^*E) \cong E_{f(y)}$ of $\mathrm{d}\bar{f}_e$ is just the identity map of $E_{f(y)}$. □

Motivated by Corollary 2.6.4 we give the following:

DEFINITION 2.6.5. Let $\pi : E \to M$ be a vector bundle and $f : M' \to M$ be a smooth map defined in a differentiable manifold M'. Let ∇ be a connection on E. Given a smooth local section $\epsilon : U' \to E$ of E along f defined in an open subset U' of M', we set:

$$\nabla_v \epsilon = \mathfrak{p}_{\mathrm{ver}}\big(\mathrm{d}\epsilon(y) \cdot v\big) \in E_{f(y)},$$

for all $y \in U'$, $v \in T_yM'$, where $\mathfrak{p}_{\mathrm{ver}} : TE \to \mathrm{Ver}(E)$ denotes the vertical projection determined by the horizontal distribution defined by ∇.

Corollary 2.6.4 says that if $\bar{\epsilon} : U' \to f^*E$ is the smooth local section of f^*E such that $\bar{f} \circ \bar{\epsilon} = \epsilon$ then:

$$\nabla_v \epsilon = (f^*\nabla)_v\, \bar{\epsilon},$$

for all $v \in TM'|_{U'}$.

LEMMA 2.6.6. *Let $\pi : E \to M$ be a vector bundle and $f : M' \to M$ be a smooth map defined in a differentiable manifold M'. Let $\epsilon : U \to E$ be a smooth local section of E defined in an open subset U of M and consider the smooth local section $\epsilon \circ f : f^{-1}(U) \to E$ of E along f. For all $y \in f^{-1}(U)$ and all $v \in T_yM$, we have:*

$$\nabla_v(\epsilon \circ f) = \nabla_{\mathrm{d}f_y(v)}\epsilon.$$

PROOF. We compute:

$$\nabla_v(\epsilon \circ f) = \mathfrak{p}_{\mathrm{ver}}\big(\mathrm{d}(\epsilon \circ f)(y) \cdot v\big) = \mathfrak{p}_{\mathrm{ver}}\Big(\mathrm{d}\epsilon\big(f(y)\big) \cdot \big(\mathrm{d}f_y(v)\big)\Big) = \nabla_{\mathrm{d}f_y(v)}\epsilon. \ \square$$

REMARK 2.6.7. Lemma 2.6.6 says that if $\epsilon : U \to E$ is a smooth local section of E and if $\bar\epsilon : f^{-1}(U) \to f^*E$ is the smooth local section of f^*E such that $\bar f \circ \bar\epsilon = \epsilon \circ f$ then:

$$(f^*\nabla)_v \, \bar\epsilon = \nabla_{\mathrm{d}f_y(v)}\epsilon,$$

for all $y \in f^{-1}(U)$ and all $v \in T_yM'$. Such property actually completely characterizes the connection $f^*\nabla$. This follows from the result of Exercise 2.16 by observing that for all $y \in M'$ and all $e \in (f^*E)_y$ there exists smooth local sections $\epsilon : U \to E$, $\bar\epsilon : f^{-1}(U) \to f^*E$ with $\bar f \circ \bar\epsilon = \epsilon \circ f$ and $\bar\epsilon(y) = e$.

2.7. Functorial constructions with connections on vector bundles

Let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Let $E^1, \ldots, E^n$ be vector bundles over a differentiable manifold M with typical fibers $E^1_0, \ldots, E^n_0$; recall from Section 1.6 that we have defined a vector bundle $\underline{\mathfrak{F}}(E^1, \ldots, E^n)$ over M with typical fiber $\underline{\mathfrak{F}}(E^1_0, \ldots, E^n_0)$. Given connections $\nabla^1, \ldots, \nabla^n$ on $E^1, \ldots, E^n$ respectively, we will now define a naturally induced connection $\nabla = \underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n)$ on $\underline{\mathfrak{F}}(E^1, \ldots, E^n)$.

For $i = 1, \ldots, n$, let $\mathrm{Hor}\big(\mathrm{FR}_{E^i_0}(E^i)\big)$ be the connection on the principal bundle of E^i_0-frames of E^i associated to ∇^i (Proposition 2.5.4). Consider the fiberwise product:

$$\mathrm{FR}_{E^1_0}(E^1) \star \cdots \star \mathrm{FR}_{E^n_0}(E^n). \tag{2.7.1}$$

endowed with the fiberwise product of the connections $\mathrm{Hor}\big(\mathrm{FR}_{E^i_0}(E^i)\big)$, $i = 1, \ldots, n$ (recall Definition 2.2.17). We define $\nabla = \underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n)$ to be the connection on $\underline{\mathfrak{F}}(E^1, \ldots, E^n)$ induced by the push-forward of the connection on (2.7.1) by the morphism of principal bundles (1.6.6) (recall Definition 2.2.14). If ω^i is the $\mathfrak{gl}(E^i_0)$-valued connection form of $\mathrm{Hor}\big(\mathrm{FR}_{E^i_0}(E^i)\big)$ and ω is the $\mathfrak{gl}\big(\underline{\mathfrak{F}}(E^1_0, \ldots, E^n_0)\big)$-valued connection form of the connection on the principal bundle $\mathrm{FR}_{\underline{\mathfrak{F}}(E^1_0,\ldots,E^n_0)}\big(\underline{\mathfrak{F}}(E^1, \ldots, E^n)\big)$ associated to $\underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n)$ then:

$$\underline{\mathfrak{F}}^*\omega = \underline{\mathfrak{f}} \circ (\mathrm{pr}_1^*\omega^1, \ldots, \mathrm{pr}_n^*\omega^n), \tag{2.7.2}$$

where $\underline{\mathfrak{f}}$ denotes the differential of the smooth functor $\underline{\mathfrak{F}}$ (recall (1.6.2)), pr_i, $i = 1, \ldots, n$, denote the projections of the fiberwise product (2.7.1) and the map $\underline{\mathfrak{F}}$ that appears in (2.7.2) is the morphism of principal bundles (1.6.6). Formula (2.7.2) follows immediately from Lemma 2.2.11 part (c) and from Proposition 2.2.16.

EXAMPLE 2.7.1. Let M be a differentiable manifold and let $E^1_0, \ldots, E^n_0$ be real finite-dimensional vector spaces; consider the trivial vector bundles:

$$E^i = M \times E^i_0, \quad i = 1, \ldots, n,$$

endowed with their canonical connections ∇^i (see Example 2.4.5). Let $\underline{\mathfrak{F}}$ be a smooth functor from $\underline{\mathfrak{Vec}}^n$ to $\underline{\mathfrak{Vec}}$. Recall from Example 1.6.10 that $\underline{\mathfrak{F}}(E^1, \ldots, E^n)$ is identified with the trivial vector bundle $E = M \times \underline{\mathfrak{F}}(E^1_0, \ldots, E^n_0)$. We claim that $\underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n)$ is the canonical connection of the trivial vector bundle E. Namely, the connection ∇^i is associated to the trivial connection on the trivial principal bundle $\mathrm{FR}_{E_0^i}(E^i) = M \times \mathrm{GL}(E_0^i)$, $i = 1, \ldots, n$ (see Example 2.5.7); also, the fiberwise product connection on (2.7.1) is the canonical connection of the trivial principal bundle $M \times \big(\mathrm{GL}(E_0^1) \times \cdots \times \mathrm{GL}(E_0^n)\big)$ (see Example 2.2.18). The map (1.6.6) is identified with the product of the identity map of M by the map (1.6.7); thus, as observed in Example 2.2.12, when $M \times \mathrm{GL}\big(\underline{\mathfrak{F}}(E_0^1, \ldots, E_0^n)\big)$ is endowed with its canonical connection, the map (1.6.6) is connection preserving. This proves the claim.

PROPOSITION 2.7.2. *Under the hypotheses of Proposition 1.6.16, if the vector bundles E^1, $\overline{E}^1$, …, E^n, $\overline{E}^n$ are endowed respectively with connections ∇^1, $\overline{\nabla}^1$, …, ∇^n, $\overline{\nabla}^n$ and if the isomorphisms of vector bundles L^i are connection preserving then the isomorphism of vector bundles $\underline{\mathfrak{F}}(L^1, \ldots, L^n)$ is also connection preserving, when the vector bundles $\underline{\mathfrak{F}}(E^1, \ldots, E^n)$, $\underline{\mathfrak{F}}(\overline{E}^1, \ldots, \overline{E}^n)$ are endowed respectively with the connections $\underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n)$ and $\underline{\mathfrak{F}}(\overline{\nabla}^1, \ldots, \overline{\nabla}^n)$.*

PROOF. We can assume without loss of generality that $E_0^i = \overline{E}_0^i$, for all $i = 1, \ldots, n$; the formal justification of this claim is obtained from the results of Exercises 1.61, 1.68 and 2.19. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)}\big(\underline{\mathfrak{F}}(E^1, \ldots, E^n)\big) \arrow[r, "\underline{\mathfrak{F}}(L^1,\ldots,L^n)_*"] & \mathrm{FR}_{\underline{\mathfrak{F}}(E_0^1,\ldots,E_0^n)}\big(\underline{\mathfrak{F}}(\overline{E}^1, \ldots, \overline{E}^n)\big) \\
\mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^n}(E^n) \arrow[u, "\underline{\mathfrak{F}}"] \arrow[ur, dotted] \arrow[r, "L^1_* \star \cdots \star L^n_*"'] & \mathrm{FR}_{E_0^1}(\overline{E}^1) \star \cdots \star \mathrm{FR}_{E_0^n}(\overline{E}^n) \arrow[u, "\underline{\mathfrak{F}}"']
\end{tikzcd}$$

The vertical arrows of the diagram are connection preserving, by the definition of $\underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n)$ and $\underline{\mathfrak{F}}(\overline{\nabla}^1, \ldots, \overline{\nabla}^n)$. Since the vector bundle isomorphisms L^i are connection preserving, then also the bottom horizontal arrow of the diagram is connection preserving, by Lemma 2.5.10 and by Corollary 2.2.20. Then the dotted arrow is connection preserving and hence, by Corollary 2.2.15, also the top horizontal arrow is connection preserving. □

COROLLARY 2.7.3. *Let $n \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Let E^1, …, E^n be vector bundles over the differentiable manifold M with typical fibers E_0^1, …, E_0^n. Let s^1, …, s^n be smooth local sections of $\mathrm{FR}_{E_0^1}(E^1)$, …, $\mathrm{FR}_{E_0^n}(E^n)$ respectively, defined in an open subset U of M. If, for $i = 1, \ldots, n$, ∇^{s^i} denotes the connection associated to s^i as in Example 2.4.6 then $\underline{\mathfrak{F}}(\nabla^{s^1}, \ldots, \nabla^{s^n})$ is the connection associated to $s = \underline{\mathfrak{F}} \circ (s^1, \ldots, s^n)$.*

PROOF. For $i = 1, \ldots, n$, the local trivialization $\check{s}^i : U \times E_0^i \to E^i|_U$ is a connection preserving vector bundle isomorphism, when $U \times E_0^i$ is endowed with its canonical connection and $E^i|_U$ is endowed with ∇^{s^i} (see Example 2.5.11). We have $\check{s} = \underline{\mathfrak{F}}(\check{s}^1, \ldots, \check{s}^n)$ (Example 1.6.17) and thus $\check{s}$ is connection preserving when $U \times \underline{\mathfrak{F}}(E_0^1, \ldots, E_0^n)$ is endowed with its canonical connection and:

$$\underline{\mathfrak{F}}(E^1, \ldots, E^n)|_U = \underline{\mathfrak{F}}(E^1|_U, \ldots, E^n|_U)$$

is endowed with the connection $\underline{\mathfrak{F}}(\nabla^{s^1}, \ldots, \nabla^{s^n})$ (see Example 2.7.1 and Proposition 2.7.2). The conclusion follows. □

PROPOSITION 2.7.4. *Under the hypotheses of Proposition 1.6.15, if ∇^1, ..., ∇^m are connections on E^1, ..., E^m respectively then:*

$$(\underline{\mathfrak{G}} \circ \underline{\mathfrak{F}})(\nabla^1, \ldots, \nabla^m) = \underline{\mathfrak{G}}\big(\underline{\mathfrak{F}}^1(\nabla^1, \ldots, \nabla^m), \ldots, \underline{\mathfrak{F}}^n(\nabla^1, \ldots, \nabla^m)\big).$$

PROOF. For $i = 1, \ldots, m$, let $\mathrm{FR}_{E_0^i}(E^i)$ be endowed with the connection associated to ∇^i and let:

$$\mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^m}(E^m) \tag{2.7.3}$$

be endowed with the fiberwise product connection. As in the proof of Proposition 1.6.15, we set:

$$F^j = \underline{\mathfrak{F}}^j(E_0^1, \ldots, E_0^m), \quad j = 1, \ldots, n, \qquad G = \underline{\mathfrak{G}}(F^1, \ldots, F^n).$$

If $\mathrm{FR}_{F^j}\big(\underline{\mathfrak{F}}(E_0^1, \ldots, E_0^m)\big)$ is endowed with the connection associated to the connection $\underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^m)$ then the morphism of principal bundles (1.6.10) is connection preserving. Thus, if:

$$\mathrm{FR}_{F^1}\big(\underline{\mathfrak{F}}^1(E^1, \ldots, E^m)\big) \star \cdots \star \mathrm{FR}_{F^n}\big(\underline{\mathfrak{F}}^n(E^1, \ldots, E^m)\big)$$

is endowed with the fiberwise product connection then the morphism of principal bundles:

$$\begin{tikzcd}
\mathrm{FR}_{E_0^1}(E^1) \star \cdots \star \mathrm{FR}_{E_0^m}(E^m) \arrow[d, "\underline{\mathfrak{F}} = (\underline{\mathfrak{F}}^1, \ldots, \underline{\mathfrak{F}}^n)"] \\
\mathrm{FR}_{F^1}\big(\underline{\mathfrak{F}}^1(E^1, \ldots, E^m)\big) \star \cdots \star \mathrm{FR}_{F^n}\big(\underline{\mathfrak{F}}^n(E^1, \ldots, E^m)\big)
\end{tikzcd}$$

is connection preserving, by Lemma 2.2.19. If $\mathrm{FR}_G\big((\underline{\mathfrak{G}} \circ \underline{\mathfrak{F}})(E^1, \ldots, E^m)\big)$ is endowed with the connection associated to:

$$\underline{\mathfrak{G}}\big(\underline{\mathfrak{F}}^1(\nabla^1, \ldots, \nabla^m), \ldots, \underline{\mathfrak{F}}^n(\nabla^1, \ldots, \nabla^m)\big) \tag{2.7.4}$$

then the morphism of principal bundles (1.6.11) is also connection preserving. This implies that the composition (1.6.12) is connection preserving, which shows that the connection on $\mathrm{FR}_G\big((\underline{\mathfrak{G}} \circ \underline{\mathfrak{F}})(E^1, \ldots, E^m)\big)$ associated to (2.7.4) is the push-forward by $\underline{\mathfrak{G}} \circ \underline{\mathfrak{F}}$ of the connection on (2.7.3). This concludes the proof. □

PROPOSITION 2.7.5. *Under the hypotheses of Proposition 1.6.18, let ∇^i be a connection on E^i, $i = 1, \ldots, n$. If $f^*\underline{\mathfrak{F}}(E^1, \ldots, E^n)$ is endowed with the connection $f^*\underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n)$ and $\underline{\mathfrak{F}}(f^*E^1, \ldots, f^*E^n)$ is endowed with the connection $\underline{\mathfrak{F}}(f^*\nabla^1, \ldots, f^*\nabla^n)$ then the vector bundle isomorphism* (1.6.13) *is connection preserving.*

PROOF. We will show that the arrows $\underline{\mathfrak{F}}$ and $f^*\underline{\mathfrak{F}}$ in the commutative diagram (1.6.17) are connection preserving and this will imply (see Corollary 2.2.15) that also (1.6.14) is connection preserving. By Lemma 2.5.10, (1.6.13) is connection preserving if and only if (1.6.14) is connection preserving. The fact that the arrow $\underline{\mathfrak{F}}$ in (1.6.17) is connection preserving is just the definition of the connection $\underline{\mathfrak{F}}(f^*\nabla^1, \ldots, f^*\nabla^n)$. The fact that the arrow $f^*\underline{\mathfrak{F}}$ in (1.6.17) is connection preserving follows from the fact that (1.6.6) is connection preserving and from Lemma 2.2.26. The reader should observe that in this argument we have implicitly used that the identification between the principal bundles (1.6.15) and (1.6.16) made in the proof of Proposition 1.6.18 is also connection preserving; this is proved in Lemma 2.2.27. $\square$

COROLLARY 2.7.6. *Let $n \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Let $E^1, \ldots, E^n$ be vector bundles over the differentiable manifold M endowed with connections $\nabla^1, \ldots, \nabla^n$, respectively. If U is an open subset of M then (recall Lemma 2.4.4):*

$$\underline{\mathfrak{F}}\big((\nabla^1)^U, \ldots, (\nabla^n)^U\big) = \underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n)^U.$$

PROOF. Simply apply Proposition 2.7.5 to the inclusion map $f : U \to M$ of U in M. $\square$

PROPOSITION 2.7.7. *Let $n \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Let $E^1, \ldots, E^n$ be vector bundles over a differentiable manifold M with typical fibers $E^1_0, \ldots, E^n_0$. For $i = 1, \ldots, n$, let ∇^i, $\widetilde{\nabla}^i$ be connections on E^i and consider the $C^\infty(M)$-bilinear map $\mathfrak{t}^i = \nabla^i - \widetilde{\nabla}^i : \mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(E^i) \to \mathbf{\Gamma}(E^i)$ as in Remark 2.4.7. For each $i = 1, \ldots, n$, each $x \in M$ and each $v \in T_xM$, denote by $\mathfrak{t}^i_x(v, \cdot) \in \mathfrak{gl}(E^i_x)$ the linear map given by $e \mapsto \mathfrak{t}^i_x(v, e)$. Set:*

$$\nabla = \underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n), \quad \widetilde{\nabla} = \underline{\mathfrak{F}}(\widetilde{\nabla}^1, \ldots, \widetilde{\nabla}^n),$$

and $\mathfrak{t} = \nabla - \widetilde{\nabla}$. Then:

$$\mathfrak{t}_x(v, \cdot) = \underline{\mathfrak{f}}\big(\mathfrak{t}^1_x(v, \cdot), \ldots, \mathfrak{t}^n_x(v, \cdot)\big) \in \mathfrak{gl}\big(\underline{\mathfrak{F}}(E^1_x, \ldots, E^n_x)\big),$$

for all $x \in M$ and all $v \in T_xM$, where $\underline{\mathfrak{f}}$ denotes the differential of the smooth functor $\underline{\mathfrak{F}}$ (recall (1.6.2)*).*

PROOF. Let ω^i (resp., $\tilde{\omega}^i$) denote the connection form of the connection in $\mathrm{FR}_{E^i_0}(E^i)$ associated to ∇^i (resp., to $\widetilde{\nabla}^i$), $i = 1, \ldots, n$, and let ω (resp., $\tilde{\omega}$) denote the connection form of the connection in:

$$\mathrm{FR}_{\underline{\mathfrak{F}}(E^1_0, \ldots, E^n_0)}\big(\underline{\mathfrak{F}}(E^1, \ldots, E^n)\big) \tag{2.7.5}$$

associated to ∇ (resp., to $\widetilde{\nabla}$). Let $x \in M$ be fixed and choose smooth local sections $s^i : U \to \mathrm{FR}_{E^i_0}(E^i)$, $i = 1, \ldots, n$, where U is an open neighborhood of x in M. Then $(s^1, \ldots, s^n)$ is a smooth local section of (2.7.1) and $s = \underline{\mathfrak{F}} \circ (s^1, \ldots, s^n)$ is a smooth local section of (2.7.5). Using (2.7.2), we compute:

$$\begin{aligned} s^*\omega = (s^1, \ldots, s^n)^*(\underline{\mathfrak{F}}^*\omega) &= (s^1, \ldots, s^n)^*\big(\underline{\mathfrak{f}} \circ (\mathrm{pr}_1^*\omega^1, \ldots, \mathrm{pr}_n^*\omega^n)\big) \\ &= \underline{\mathfrak{f}} \circ \big((s^1, \ldots, s^n)^*(\mathrm{pr}_1^*\omega^1), \ldots, (s^1, \ldots, s^n)^*(\mathrm{pr}_n^*\omega^n)\big) \\ &= \underline{\mathfrak{f}} \circ \big((s^1)^*\omega^1, \ldots, (s^n)^*\omega^n\big); \end{aligned}$$

similarly:

$$s^*\tilde{\omega} = \underline{\mathfrak{f}} \circ \big((s^1)^*\tilde{\omega}^1, \ldots, (s^n)^*\tilde{\omega}^n\big),$$

so that:

$$s^*(\omega - \tilde{\omega}) = \underline{\mathfrak{f}} \circ \big((s^1)^*(\omega^1 - \tilde{\omega}^1), \ldots, (s^n)^*(\omega^n - \tilde{\omega}^n)\big). \tag{2.7.6}$$

Lemma 2.5.8 implies that:

$$\mathcal{I}_{s^i(x)}\big[\big((s^i)^*(\omega^i - \tilde{\omega}^i)\big)_x(v)\big] = \mathfrak{t}^i_x(v,\cdot) \in \mathfrak{gl}(E^i_x), \tag{2.7.7}$$

for all $v \in T_xM$, $i = 1, \ldots, n$; similarly:

$$\mathcal{I}_{s(x)}\big[\big(s^*(\omega - \tilde{\omega})\big)_x(v)\big] = \mathfrak{t}_x(v,\cdot) \in \mathfrak{gl}\big(\underline{\mathfrak{F}}(E^1_x, \ldots, E^n_x)\big), \tag{2.7.8}$$

for all $v \in T_xM$. The conclusion follows from (2.7.6), (2.7.7) and (2.7.8) by applying the result of Exercise 1.67 to the isomorphisms $s^i(x) : E^i_0 \to E^i_x$, $i = 1, \ldots, n$. □

COROLLARY 2.7.8. *Let $n \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Let E^1, ..., E^n be vector bundles over the differentiable manifold M with typical fibers E^1_0, ..., E^n_0. Let s^1, ..., s^n be smooth local sections of the principal bundles $\mathrm{FR}_{E^1_0}(E^1)$, ..., $\mathrm{FR}_{E^n_0}(E^n)$ respectively, defined in an open subset U of M. Let ∇^1, ..., ∇^n be connections on E^1, ..., E^n, respectively and denote by Γ^i the Christoffel tensor of ∇^i with respect to s^i, $i = 1, \ldots, n$. If $s = \underline{\mathfrak{F}} \circ (s^1, \ldots, s^n)$ and Γ is the Christoffel tensor of $\underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n)$ with respect to s then:*

$$\Gamma_x(v,\cdot) = \underline{\mathfrak{f}}\big(\Gamma^1_x(v,\cdot), \ldots, \Gamma^n_x(v,\cdot)\big),$$

for all $x \in U$ and all $v \in T_xM$, where $\underline{\mathfrak{f}}$ denotes the differential of the smooth functor $\underline{\mathfrak{F}}$ (recall (1.6.2)*).*

PROOF. Simply observe that $\Gamma^i = \nabla^i - \nabla^{s^i}$, $i = 1, \ldots, n$, and that:

$$\underline{\mathfrak{F}}(\nabla^{s^1}, \ldots, \nabla^{s^n}) = \nabla^s,$$

by Corollary 2.7.3. □

Recall that we have shown in Proposition 1.6.28 that smooth natural transformations between smooth functors induce smooth fiber-preserving maps between vector bundles; now we show how to compute the covariant derivative of such maps.

PROPOSITION 2.7.9. *Under the hypotheses of Proposition 1.6.28, let the vector bundles E^1, ..., E^n be endowed with connections ∇^1, ..., ∇^n; set:*

$$\nabla = \underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n), \quad \nabla' = \underline{\mathfrak{G}}(\nabla^1, \ldots, \nabla^n).$$

If the vector bundles $\underline{\mathfrak{F}}(E^1, \ldots, E^n)$ and $\underline{\mathfrak{G}}(E^1, \ldots, E^n)$ are endowed respectively with the connections ∇ and ∇' then the map $\mathfrak{N}_{E^1,\ldots,E^n}$ is connection preserving. In particular, by Lemma 2.1.5, given a smooth local section $\epsilon : U \to \underline{\mathfrak{F}}(E^1, \ldots, E^n)$ with image contained in $\mathrm{Dom}(\mathfrak{N}_{E^1,\ldots,E^n})$ then for all $x \in U$, $v \in T_xM$, we have:

$$\nabla'_v(\mathfrak{N}_{E^1,\ldots,E^n} \circ \epsilon) = \mathrm{d}\mathfrak{N}_{E^1_x,\ldots,E^n_x}\big(\epsilon(x)\big) \cdot \nabla_v\epsilon. \tag{2.7.9}$$

Moreover, if $\mathfrak{N}$ is linear then $\mathfrak{N}_{E^1,\ldots,E^n}$ is a connection preserving morphism of vector bundles, i.e.:

$$\nabla'_v(\mathfrak{N}_{E^1,\ldots,E^n} \circ \epsilon) = \mathfrak{N}_{E^1_x,\ldots,E^n_x}(\nabla_v \epsilon),$$

for all $x \in U$ and all $v \in T_x M$.

PROOF. Follows directly from the commutativity of diagram (1.6.20), keeping in mind that $\mathcal{C}^{\underline{\mathfrak{F}}}$, $\mathcal{C}^{\underline{\mathfrak{G}}}$ are connection preserving isomorphisms of vector bundles (Example 2.5.14) and that $\mathrm{Id}_P \underset{\sim}{\times} \mathfrak{N}_{E^1_0,\ldots,E^n_0}$ is connection preserving (Lemma 2.3.3). $\square$

EXAMPLE 2.7.10. Let E^1, E^2 be vector bundles over a differentiable manifold M endowed with connections ∇^1, ∇^2. If $\underline{\mathfrak{F}}$ denotes the functor defined in Example 1.6.11 then $\underline{\mathfrak{F}}(E^1, E^2) = E^1 \oplus E^2$ and

$$\nabla = \nabla^1 \oplus \nabla^2 \overset{\text{def}}{=} \underline{\mathfrak{F}}(\nabla^1, \nabla^2)$$

is a connection on $E^1 \oplus E^2$ called the *direct sum* of the connections ∇^1 and ∇^2. Given a smooth local section:

$$\epsilon = (\epsilon_1, \epsilon_2) : U \longrightarrow E^1 \oplus E^2$$

then:

$$\nabla_v \epsilon = (\nabla^1_v \epsilon_1, \nabla^2_v \epsilon_2),$$

for all $v \in TM$. Namely, by applying Proposition 2.7.9 to the linear smooth natural transformations $\mathfrak{N}^1$, $\mathfrak{N}^2$ defined in Example 1.6.20 we conclude that the projections $E^1 \oplus E^2 \to E^i$, $i = 1, 2$ are connection preserving vector bundle morphisms; thus, the i-th coordinate of $\nabla_v \epsilon$ is equal to $\nabla^i_v \epsilon_i$.

EXAMPLE 2.7.11. Let E, E' be vector bundles endowed with connections ∇, ∇' and consider the connections $\mathrm{Lin}_k(\nabla, \nabla')$ and $\mathrm{Lin}^{\mathrm{s}}_k(\nabla, \nabla')$ induced respectively on the vector bundles $\mathrm{Lin}_k(E, E')$, $\mathrm{Lin}^{\mathrm{s}}_k(E, E')$. If B is a smooth local section of $\mathrm{Lin}^{\mathrm{s}}_k(E, E')$ then the covariant derivatives of B with respect to $\mathrm{Lin}_k(\nabla, \nabla')$ and $\mathrm{Lin}^{\mathrm{s}}_k(\nabla, \nabla')$ coincide. Namely, the inclusion map:

$$\mathrm{Lin}^{\mathrm{s}}_k(V, V') \longrightarrow \mathrm{Lin}_k(V, V')$$

is a linear smooth natural transformation and thus, by Proposition 2.7.9, the inclusion map of $\mathrm{Lin}_k(E, E')$ into $\mathrm{Lin}^{\mathrm{s}}_k(E, E')$ is a connection preserving vector bundle morphism. A similar statement holds with $\mathrm{Lin}^{\mathrm{s}}_k$ replaced with $\mathrm{Lin}^{\mathrm{a}}_k$.

EXAMPLE 2.7.12. Let $E^1, \ldots, E^k, F$ be vector bundles over a differentiable manifold M endowed with connections $\nabla^1, \ldots, \nabla^k, \nabla^F$, respectively. Consider the induced connection:

$$\nabla = \mathrm{Lin}(\nabla^1, \ldots, \nabla^k; \nabla^F)$$

on the vector bundle $\mathrm{Lin}(E^1, \ldots, E^k; F)$. Given a smooth section B of the vector bundle $\mathrm{Lin}(E^1, \ldots, E^k; F)$ and a smooth section ϵ^i of the vector bundle E^i for

$i = 1, \ldots, k$ then:

$$\begin{aligned}(2.7.10)\quad \nabla^F_v\big(B(\epsilon^1, \ldots, \epsilon^k)\big) &= (\nabla_v B)(\epsilon^1, \ldots, \epsilon^k)\\ &\quad + B(\nabla^1_v \epsilon^1, \ldots, \epsilon^k) + \cdots + B(\epsilon^1, \ldots, \nabla^k_v \epsilon^k),\end{aligned}$$

for all $v \in TM$. Namely, consider the natural transformation $\mathfrak{N}$ defined in Example 1.6.31; we have:

$$B(\epsilon^1, \ldots, \epsilon^k) = \mathfrak{N}_{E^1,\ldots,E^k,F} \circ (B, \epsilon^1, \ldots, \epsilon^k),$$

and therefore, using Proposition 2.7.9 we get:

$$\begin{aligned}\nabla^F_v\big(B(\epsilon^1, \ldots, \epsilon^k)\big) &= \nabla^F_v\big(\mathfrak{N}_{E^1,\ldots,E^k,F} \circ (B, \epsilon^1, \ldots, \epsilon^k)\big)\\ &= \mathrm{d}\mathfrak{N}_{E^1_x,\ldots,E^k_x,F_x}\big(B(x), \epsilon^1(x), \ldots, \epsilon^k(x)\big) \cdot (\nabla_v B, \nabla^1_v \epsilon^1, \ldots, \nabla^k_v \epsilon^k)\\ &= (\nabla_v B)(\epsilon^1, \ldots, \epsilon^k) + B(\nabla^1_v \epsilon^1, \ldots, \epsilon^k) + \cdots + B(\epsilon^1, \ldots, \nabla^k_v \epsilon^k).\end{aligned}$$

Observe that (2.7.10) can be used as a formula to compute $\nabla_v B$.

REMARK 2.7.13. Let E, F be vector bundles over a differentiable manifold M endowed with connections ∇^E, ∇^F, respectively. If L is identified with a smooth section of $\mathrm{Lin}(E, F)$ then it follows directly from formula (2.7.10) that:

$$(2.7.11)\qquad (\nabla_X L)(\epsilon) = \nabla^F_X\big(L(\epsilon)\big) - L(\nabla^E_X \epsilon),$$

for all $X \in \mathbf{\Gamma}(TM)$, $\epsilon \in \mathbf{\Gamma}(E)$, where $\nabla = \mathrm{Lin}(\nabla^E, \nabla^F)$. It follows that L is connection preserving if and only if the section $x \mapsto L_x$ of $\mathrm{Lin}(E, F)$ is parallel.

REMARK 2.7.14. Let E be a vector bundle over a differentiable manifold M and let ∇^1, ∇^2 be connections on E with $\nabla^2 - \nabla^1 = \mathfrak{t}$. If $L : E \to E$ is the identity map and $\nabla = \mathrm{Lin}(\nabla^1, \nabla^2)$ then formula (2.7.11) implies that the covariant derivative of L (seen as a section of $\mathrm{Lin}(E)$) is given by:

$$\nabla L = \mathfrak{t}.$$

EXAMPLE 2.7.15. Let $E^1, \ldots, E^k, F, F'$ be vector bundles over a differentiable manifold M endowed with connections $\nabla^1, \ldots, \nabla^k$, ∇^F and $\nabla^{F'}$ respectively. Consider the induced connections:

$$\nabla = \mathrm{Lin}(\nabla^1, \ldots, \nabla^k; \nabla^F), \quad \nabla' = \mathrm{Lin}(\nabla^1, \ldots, \nabla^k; \nabla^{F'}),$$
$$\nabla'' = \mathrm{Lin}(\nabla^F, \nabla^{F'}),$$

on $\mathrm{Lin}(E^1, \ldots, E^k; F)$, $\mathrm{Lin}(E^1, \ldots, E^k; F')$ and $\mathrm{Lin}(F, F')$ respectively. Given a smooth section $B : M \to \mathrm{Lin}(E^1, \ldots, E^k; F)$ and a vector bundle morphism $L : F \to F'$ then:

$$\nabla'_v(L \circ B) = (\nabla''_v L) \circ B(x) + L_x \circ \nabla_v B,$$

for all $x \in M$, $v \in T_x M$, where, as usual, L is identified with the smooth section $x \mapsto L_x$ of $\mathrm{Lin}(F, F')$. Namely, consider the natural transformation $\mathfrak{N}$ defined in Example 1.6.32; we have:

$$L \circ B = \mathfrak{N}_{E^1,\ldots,E^k,F,F'} \circ (L, B),$$

and therefore, using Proposition 2.7.9 we get:

$$\begin{aligned}\nabla'_v(L \circ B) &= \nabla'_v\big(\mathfrak{N}_{E^1,\ldots,E^k,F,F'} \circ (L, B)\big)\\ &= \mathrm{d}\mathfrak{N}_{E^1_x,\ldots,E^k_x,F_x,F'_x}\big(L_x, B(x)\big) \cdot (\nabla''_v L, \nabla_v B)\\ &= (\nabla''_v L) \circ B(x) + L_x \circ \nabla_v B,\end{aligned}$$

for all $x \in M$ and all $v \in T_x M$.

REMARK 2.7.16. In Example 2.7.15, if the vector bundle morphism L is connection preserving then:

$$\nabla'_v(L \circ B) = L_x \circ \nabla_v B,$$

for all $x \in M$ and all $v \in T_x M$ (see Remark 2.7.13).

2.7.1. Covariant exterior differentiation of vector bundle valued forms. If $E^1, \ldots, E^k, F$ are vector bundles over a differentiable manifold M endowed with connections and if B is a smooth section of the vector bundle $\mathrm{Lin}(E^1, \ldots, E^k; F)$ then the covariant derivative ∇B is a smooth section of the vector bundle:

$$\mathrm{Lin}\big(TM, \mathrm{Lin}(E^1, \ldots, E^k; F)\big). \tag{2.7.12}$$

Recall from Example 1.6.33 that we identify the vector bundle (2.7.12) with the vector bundle $\mathrm{Lin}(TM, E^1, \ldots, E^k; F)$. Notice that, given a smooth section B of $\mathrm{Lin}(E^1, \ldots, E^k; F)$ then:

$$\nabla B(X, \epsilon_1, \ldots, \epsilon_k) = (\nabla_X B)(\epsilon_1, \ldots, \epsilon_k),$$

for all $X \in \mathbf{\Gamma}(TM)$, $\epsilon_1 \in \mathbf{\Gamma}(E^1)$, …, $\epsilon_k \in \mathbf{\Gamma}(E^k)$. In particular, if E is a vector bundle over M and if both E and TM are endowed with connections then for every smooth E-valued covariant k-tensor field B on M the covariant derivative ∇B is a smooth E-valued covariant $(k+1)$-tensor field on M.

LEMMA 2.7.17. *Let $\pi : E \to M$ be a vector bundle endowed with a connection ∇^E and let ∇, ∇' be symmetric connections on TM; we also denote by ∇ and ∇' the induced connections on the vector bundle* $\mathrm{Lin}^{\mathrm{a}}_k(TM, E)$. *Given a smooth E-valued k-form ℓ on M then:*

$$\mathrm{Alt}\big((\nabla\ell)(x)\big) = \mathrm{Alt}\big((\nabla'\ell)(x)\big),$$

for all $x \in M$.

PROOF. Set $\mathfrak{t} = \nabla - \nabla'$; since both ∇ and ∇' are symmetric, then also $\mathfrak{t}$ is symmetric. Using Proposition 2.7.7 and the computation of $\underline{\mathfrak{f}}$ done in Example 1.6.14, we obtain:

$$\begin{aligned}(\nabla_v\ell - \nabla'_v\ell)(v_1, \ldots, v_k) &= \big[\underline{\mathfrak{f}}\big(\mathfrak{t}(v, \cdot), 0\big) \cdot \ell_x\big](v_1, \ldots, v_k)\\ &= -\ell_x\big(\mathfrak{t}(v, v_1), v_2, \ldots, v_k\big) - \cdots - \ell_x\big(v_1, v_2, \ldots, \mathfrak{t}(v, v_k)\big),\end{aligned} \tag{2.7.13}$$

for all $v, v_1, \ldots, v_k \in T_x M$ and all $x \in M$. Since the i-th summand in (2.7.13) is symmetric in v and v_i, its alternator vanishes (see Remark A.3.2). The conclusion follows. □

In view of Lemma 2.7.17, we can give the following:

DEFINITION 2.7.18. Let $\pi : E \to M$ be a vector bundle endowed with a connection and let ℓ be a smooth E-valued k-form on M with values on E. The *covariant exterior differential* of ℓ is the smooth E-valued $(k+1)$-form on M defined by:

$$(\mathrm{D}\ell)(x) = \tfrac{1}{k!}\,\mathrm{Alt}\big((\nabla\ell)(x)\big),$$

for all $x \in M$, where ∇ denotes the connection induced on $\mathrm{Lin}_k^{\mathrm{a}}(TM, E)$ by the given connection on E and by an arbitrarily chosen symmetric connection on TM.

EXAMPLE 2.7.19. Let M be a differentiable manifold and E_0 be a real finite-dimensional vector space. A (smooth) k-form on M taking values in the trivial vector bundle $M \times E_0$ is the same as a (smooth) E_0-valued k-form on M. If $M \times E_0$ is endowed with the canonical connection $\mathrm{d}\mathrm{I}$ then the exterior covariant derivative of a smooth k-form on M taking values in $M \times E_0$ coincides with its standard exterior derivative.

PROPOSITION 2.7.20. *Let E, F be vector bundles over a differentiable manifold M endowed with connections, $L : E \to F$ be a vector bundle morphism and ℓ be a smooth E-valued k-form on M. Then:*

$$\mathrm{D}(L \circ \ell) = \nabla L \wedge \ell + L \circ \mathrm{D}\ell,$$

where ∇L is seen as a $\mathrm{Lin}(E, F)$-valued 1-form on M and the wedge product $\nabla L \wedge \ell$ is taken with respect to the obvious bilinear pairing:

$$\mathrm{Lin}(E_x, F_x) \times E_x \longrightarrow F_x.$$

In particular, if L is connection preserving then (see Remark 2.7.13):

$$\mathrm{D}(L \circ \ell) = L \circ \mathrm{D}\ell.$$

PROOF. We compute (see Example 2.7.15):

$$\begin{aligned}&\big(\nabla_{v_1}(L \circ \ell)\big)(v_2, \ldots, v_{k+1})\\ &\qquad\qquad = (\nabla_{v_1} L) \circ \ell_x(v_2, \ldots, v_{k+1}) + L_x \circ (\nabla_{v_1}\ell)(v_2, \ldots, v_{k+1}),\end{aligned}$$

for all $x \in M$ and all $v_1, \ldots, v_{k+1} \in T_xM$. The conclusion follows by taking alternators on both sides of the above equality. □

COROLLARY 2.7.21. *Let E be a vector bundle over a differentiable manifold M endowed with connections ∇^1, ∇^2; set $\mathfrak{t} = \nabla^2 - \nabla^1$. Given a smooth E-valued k-form ℓ on M, we denote by $\mathrm{D}^i\ell$ the exterior covariant derivative of ℓ associated to ∇^i, $i = 1, 2$. Then:*

$$\mathrm{D}^2\ell = \mathrm{D}^1\ell + \mathfrak{t} \wedge \ell,$$

where $\mathfrak{t}$ is seen as a $\mathrm{Lin}(E)$-valued 1-form.

PROOF. Apply Proposition 2.7.20 with L the identity morphism of E, keeping in mind Remark 2.7.14. □

EXAMPLE 2.7.22. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and let $\iota : TM \to E$ be a morphism of vector bundles. We identify ι with a smooth section of $\mathrm{Lin}(TM, E)$, which is a smooth E-valued 1-form on M. Given a connection ∇^E on E and an arbitrary connection ∇^M on TM then, by (2.7.11), we have:

$$(\nabla\iota)(X,Y) = (\nabla_X\iota)(Y) = \nabla^E_X\big(\iota(Y)\big) - \iota(\nabla^M_X Y).$$

If ∇^M is symmetric then:

$$\begin{aligned} \mathrm{D}\iota(X,Y) &= (\nabla\iota)(X,Y) - (\nabla\iota)(Y,X) \\ &= \nabla^E_X\big(\iota(Y)\big) - \nabla^E_Y\big(\iota(X)\big) - \iota(\nabla^M_X Y - \nabla^M_Y X) \\ &= \nabla^E_X\big(\iota(Y)\big) - \nabla^E_Y\big(\iota(X)\big) - \iota\big([X,Y]\big), \end{aligned}$$

proving that the covariant exterior differential $\mathrm{D}\iota$ is the ι-torsion tensor of ∇.

DEFINITION 2.7.23. Let $\pi : E \to M$ be a vector bundle and ℓ be an E-valued k-form on M. Given a differentiable manifold M' and a smooth map $f : M' \to M$ then the *pull-back* of ℓ by f, denoted by $f^*\ell$, is the f^*E-valued k-form on M' defined by:

$$(f^*\ell)_y = \mathrm{d}f_y^*\ell_{f(y)},$$

for all $y \in M'$.

More explicitly, $f^*\ell$ is given by:

$$(f^*\ell)_y(v_1,\ldots,v_k) = \ell_{f(y)}\big(\mathrm{d}f_y(v_1),\ldots,\mathrm{d}f_y(v_k)\big) \in E_{f(y)} = (f^*E)_y,$$

for all $y \in M$ and all $v_1,\ldots,v_k \in T_yM'$.

Clearly $f^*\ell$ is smooth if ℓ is smooth.

We have the following:

LEMMA 2.7.24. *Let $\pi : E \to M$ be a vector bundle endowed with a connection ∇ and ℓ be an E-valued k-form on M. Let $f : M' \to M$ be a smooth map defined in a differentiable manifold M' and let the vector bundle f^*E be endowed with the pull-back connection $f^*\nabla$. Then:*

$$\mathrm{D}(f^*\ell) = f^*\mathrm{D}\ell.$$

PROOF. □

COROLLARY 2.7.25. *Let $\pi : E \to M$ be a vector bundle endowed with a connection ∇, $\iota : TM \to E$ be a vector bundle morphism, M' be a differentiable manifold and $f : M' \to M$ be a smooth map. Consider the vector bundle morphism $\iota' : TM' \to f^*E$ defined by:*

$$\iota' = \overleftarrow{\iota\circ\mathrm{d}f} = (f^*\iota)\circ\overleftarrow{\mathrm{d}f}.$$

If T^ι denotes the ι-torsion of the connection ∇ and if $T^{\iota'}$ denotes the ι'-torsion of the connection $f^*\nabla$ then:

$$T^{\iota'}_y = \mathrm{d}f_y^* T^\iota_{f(y)},$$

for all $y \in M'$; more explicitly:

$$T^{\iota'}_y(v,w) = T^\iota_{f(y)}\big(\mathrm{d}f_y(v),\mathrm{d}f_y(w)\big) \in E_{f(y)} = (f^*E)_y,$$

for all $y \in M'$ and all $v,w \in T_yM'$.

PROOF. If the vector bundle morphism is identified with a E-valued 1-form on M and the vector bundle morphism ι' is identified with a f^*E-valued 1-form on M' then:

$$\iota' = f^*\iota.$$

The conclusion follows from Lemma 2.7.24 and from Example 2.7.22. $\square$

2.8. The components of a linear connection

Let $\pi : E \to M$ be a vector bundle and let E^1, E^2 be vector subbundles of E such that $E = E^1 \oplus E^2$ (see Remark 1.6.30); denote by $\mathrm{pr}_1 : E \to E^1$, $\mathrm{pr}_2 : E \to E^2$ the corresponding projections. If ∇^1, ∇^2 are connections on E^1 and E^2 respectively then the direct sum of ∇^1 and ∇^2 (recall Example 2.7.10) is the unique connection ∇ on E such that:

$$\nabla_X \epsilon = \nabla^1_X(\mathrm{pr}_1 \circ \epsilon) + \nabla^2_X(\mathrm{pr}_2 \circ \epsilon), \tag{2.8.1}$$

for all $X \in \mathbf{\Gamma}(TM)$ and all $\epsilon \in \mathbf{\Gamma}(E)$. Not every connection ∇ on E is a direct sum of connections on E^1 and E^2. Given a connection ∇ on E, we set:

$$\begin{aligned}
\nabla^1_X \epsilon_1 &= \mathrm{pr}_1 \circ \nabla_X \epsilon_1 \in \mathbf{\Gamma}(E^1),\\
\nabla^2_X \epsilon_2 &= \mathrm{pr}_2 \circ \nabla_X \epsilon_2 \in \mathbf{\Gamma}(E^2),\\
\alpha^1(X, \epsilon_2) &= \mathrm{pr}_1 \circ \nabla_X \epsilon_2 \in \mathbf{\Gamma}(E^1),\\
\alpha^2(X, \epsilon_1) &= \mathrm{pr}_2 \circ \nabla_X \epsilon_1 \in \mathbf{\Gamma}(E^2),
\end{aligned} \tag{2.8.2}$$

for all $X \in \mathbf{\Gamma}(TM)$ and all $\epsilon_1 \in \mathbf{\Gamma}(E^1)$, $\epsilon^2 \in \mathbf{\Gamma}(E^2)$. Clearly ∇^1 and ∇^2 are connections on E^1 and E^2, respectively. Moreover, α^1, α^2 are $C^\infty(M)$-bilinear and therefore (see Exercises 1.63 and 1.72) they can be identified respectively with smooth sections:

$$\alpha^1 \in \mathbf{\Gamma}\big(\mathrm{Lin}(TM, E^2; E^1)\big), \quad \alpha^2 \in \mathbf{\Gamma}\big(\mathrm{Lin}(TM, E^1; E^2)\big).$$

The maps ∇^1, ∇^2, α^1, α^2 defined in (2.8.2) are collectively called the *components* of the connection ∇ relatively to the direct sum decomposition $E = E^1 \oplus E^2$. Conversely, given connections ∇^1, ∇^2 respectively on E^1 and E^2 and given smooth sections $\alpha^1 \in \mathbf{\Gamma}\big(\mathrm{Lin}(TM, E^2; E^1)\big)$, $\alpha^2 \in \mathbf{\Gamma}\big(\mathrm{Lin}(TM, E^1; E^2)\big)$ then there exists a unique connection ∇ on E whose components are ∇^1, ∇^2, α^1 and α^2; namely, ∇ is given by:

$$\nabla_X \epsilon = \nabla^1_X \epsilon_1 + \alpha^1(X, \epsilon_2) + \nabla^2_X \epsilon_2 + \alpha^2(X, \epsilon_1), \tag{2.8.3}$$

for all $X \in \mathbf{\Gamma}(TM)$, $\epsilon \in \mathbf{\Gamma}(E)$, where $\epsilon_1 = \mathrm{pr}_1 \circ \epsilon$ and $\epsilon_2 = \mathrm{pr}_2 \circ \epsilon$.

PROPOSITION 2.8.1 (generalized Gauss, Codazzi, Ricci equations). *Let $\pi : E \to M$ be a vector bundle, E^1, E^2 be vector subbundles of E with $E = E^1 \oplus E^2$ and ∇ be a connection on E; denote by ∇^1, ∇^2, α^1 and α^2 the components of ∇.*

If R, R^1, R^2 denote respectively the curvature tensors of ∇, ∇^1 and ∇^2 then:

$$\mathrm{pr}_1\big(R_x(v,w)e_1\big) = R^1_x(v,w)e_1 + \alpha^1_x\big(v,\alpha^2_x(w,e_1)\big) - \alpha^1_x\big(w,\alpha^2_x(v,e_1)\big), \tag{2.8.4}$$

$$\mathrm{pr}_2\big(R_x(v,w)e_2\big) = R^2_x(v,w)e_2 + \alpha^2_x\big(v,\alpha^1_x(w,e_2)\big) - \alpha^2_x\big(w,\alpha^1_x(v,e_2)\big), \tag{2.8.5}$$

for all $x \in M$, $e_1 \in E^1_x$, $e_2 \in E^2_x$ and all $v, w \in T_xM$. Moreover, given a connection ∇^M on TM with torsion T and denoting by $\nabla^{\otimes}$ the induced connections on $\mathrm{Lin}(TM, E^2; E^1)$ *and on* $\mathrm{Lin}(TM, E^1; E^2)$ *then:*

$$\begin{aligned}\mathrm{pr}_2\big(R_x(v,w)e_1\big) = (\nabla^{\otimes}\alpha^2)_x(v,w,e_1) - (\nabla^{\otimes}\alpha^2)_x(w,v,e_1)\\ + \alpha^2_x\big(T_x(v,w),e_1\big),\end{aligned} \tag{2.8.6}$$

$$\begin{aligned}\mathrm{pr}_1\big(R_x(v,w)e_2\big) = (\nabla^{\otimes}\alpha^1)_x(v,w,e_2) - (\nabla^{\otimes}\alpha^1)_x(w,v,e_2)\\ + \alpha^1_x\big(T_x(v,w),e_2\big),\end{aligned} \tag{2.8.7}$$

for all $x \in M$, $e_1 \in E^1_x$, $e_2 \in E^2_x$ and all $v, w \in T_xM$.

PROOF. A straightforward computation. $\square$

Equation (2.8.4) is called the *generalized Gauss equation*, (2.8.5) is the *generalized Ricci equation* and (2.8.6) and (2.8.7) are the *generalized Codazzi equations*.

EXAMPLE 2.8.2. Let $\pi : E \to M$ be a vector bundle, E^1, E^2 be vector subbundles of E with $E = E^1 \oplus E^2$ and ∇ be a connection on E; denote by ∇^1, ∇^2, α^1 and α^2 the components of ∇. Assume that we are given a vector bundle morphism $\iota_1 : TM \to E^1$ and denote by $\iota : TM \to E$ the composition of ι_1 with the inclusion map of E^1 in E. The ι-torsion T^{ι} of ∇ is easily computed as:

$$T^{\iota}_x(v,w) = T^{\iota_1}_x(v,w) + \alpha^2_x\big(v,\iota_1(w)\big) - \alpha^2_x\big(w,\iota_1(v)\big), \tag{2.8.8}$$

for all $x \in M$, $v, w \in T_xM$, where T^{ι_1} denotes the ι_1-torsion of ∇^1. Notice that $T^{\iota} = 0$ if and only if $T^{\iota_1} = 0$ and for all $x \in M$, the E^2_x-valued bilinear map $\alpha^2_x(\cdot, \iota_1\cdot)$ on T_xM is symmetric.

EXAMPLE 2.8.3. Let $\pi : E \to M$ be a vector bundle and consider the Whitney sum $\widehat{E} = TM \oplus E$. Let $\widehat{\nabla}$ be a connection on $\widehat{E}$ with components ∇^M, ∇^E, $\alpha \in \mathbf{\Gamma}\big(\mathrm{Lin}(TM,TM;E)\big)$, $\alpha' \in \mathbf{\Gamma}\big(\mathrm{Lin}(TM,E;TM)\big)$, where ∇^M is a connection on TM and ∇^E is a connection on E. Denoting by $\iota : TM \to \widehat{E}$ the inclusion map, formula (2.8.8) becomes:

$$T^{\iota}_x(v,w) = \big(T^M_x(v,w), \alpha_x(v,w) - \alpha_x(w,v)\big),$$

for all $x \in M$, $v, w \in T_xM$, where T^{ι} denotes the ι-torsion of $\widehat{\nabla}$ and T^M denotes the torsion of ∇^M. Notice that $T^{\iota} = 0$ if and only if both the connection ∇^M and α are symmetric.

2.8.1. Connections compatible with a semi-Riemannian structure. Let $\pi : E \to M$ be a vector bundle endowed with a semi-Riemannian structure $g \in \mathbf{\Gamma}\big(\mathrm{Lin}_2^{\mathrm{s}}(E,\mathbb{R})\big)$. A connection ∇ on E is said to be *compatible* with g if $\nabla g = 0$. Using (2.7.10) the condition $\nabla g = 0$ is equivalent to:

$$X\big(g(\epsilon_1,\epsilon_2)\big) = g(\nabla_X\epsilon_1,\epsilon_2) + g(\epsilon_1,\nabla_X\epsilon_2), \tag{2.8.9}$$

for all $X \in \mathbf{\Gamma}(TM)$ and all $\epsilon_1,\epsilon_2 \in \mathbf{\Gamma}(E)$.

If g is a semi-Riemannian structure on a vector bundle E then two subbundles E^1, E^2 of E are said to be *orthogonal* with respect to g if $g_x(e_1,e_2) = 0$, for all $x \in M$, $e_1 \in E^1_x$, $e_2 \in E^2_x$. Observe that if $E = E^1 \oplus E^2$ with E^1, E^2 orthogonal subbundles of E then, for $i = 1,2$, the restriction of g to E^i is a semi-Riemannian structure on E^i.

LEMMA 2.8.4. *Let $\pi : E \to M$ be a vector bundle endowed with a semi-Riemannian structure g, E^1, E^2 be orthogonal vector subbundles of E with $E = E^1 \oplus E^2$ and ∇ be a connection on E; denote by ∇^1, ∇^2, α^1 and α^2 the components of ∇. Then ∇ is compatible with g if and only if the following conditions hold:*

1. *∇^i is compatible with g^i, $i = 1,2$, where g^i denotes the semi-Riemannian structure on E^i obtained by restriction of g;*
2. *$g_x\big(\alpha^2_x(v,e_1),e_2\big) + g_x\big(e_1,\alpha^1_x(v,e_2)\big) = 0$, for all $x \in M$, $v \in T_xM$, $e_1 \in E^1_x$, $e_2 \in E^2_x$.*

PROOF. It is a straightforward computation using (2.8.9), (2.8.2) and (2.8.3). $\square$

Condition (2) on the statement of Lemma 2.8.4 can be written as:

$$\alpha^1_x(v) = -\alpha^2_x(v)^*, \tag{2.8.10}$$

for all $x \in M$, $v \in T_xM$, where $\alpha^1_x(v) \in \mathrm{Lin}(E^2_x,E^1_x)$ is the linear map $e_2 \mapsto \alpha^1_x(v,e_2)$, $\alpha^2_x(v) \in \mathrm{Lin}(E^1_x,E^2_x)$ is the linear map $e_1 \mapsto \alpha^2_x(v,e_1)$ and the star denotes transposition with respect to the nondegenerate bilinear forms g^1_x and g^2_x.

Thus, if $E = E^1 \oplus E^2$ is a g-orthogonal direct sum decomposition, in order to describe the components of a connection ∇ on E which is compatible with g, one has only to specify connections ∇^1, ∇^2 on E^1, E^2 respectively compatible with g^1, g^2 and a smooth section α^2 of $\mathrm{Lin}(TM,E^1;E^2)$. The components α^1 of ∇ is then obtained by (2.8.10). Thus, when dealing with a connection ∇ compatible with a semi-Riemannian structure, we call ∇^1, ∇^2 and α^2 *the components* of ∇ with respect to the decomposition $E = E^1 \oplus E^2$.

Let us take a look at the generalized Gauss, Codazzi and Ricci equations for a connection ∇ compatible with a semi-Riemannian structure g. First, we observe that the Codazzi equations (2.8.6) and (2.8.7) are equivalent to each other. Namely, by the result of Exercise 2.21, for all $x \in M$, $v,w \in T_xM$, the linear operator $R_x(v,w)$ on E_x is anti-symmetric; thus, the linear map:

$$\mathrm{pr}_2\big(R_x(v,w)|_{E^1_x}\big) : E^1_x \longrightarrow E^2_x$$

is equal to minus the transpose of the linear map:

$$\mathrm{pr}_1\big(R_x(v,w)|_{E_x^2}\big) : E_x^2 \longrightarrow E_x^1.$$

Moreover, using (2.8.10), it follows that for all $x \in M$, $v, w \in T_xM$, the linear map:

$$(\nabla^{\otimes}\alpha^1)_x(v,w) : E_x^2 \longrightarrow E_x^1$$

is equal to minus the transpose of the linear map:

$$(\nabla^{\otimes}\alpha^2)_x(v,w) : E_x^1 \longrightarrow E_x^2.$$

Thus equation (2.8.7) is obtained from (2.8.6) by taking transpositions on both sides. Observe also that the generalized Ricci equation (2.8.5) can be rewritten as:

$$\mathrm{pr}_2 \circ R_x(v,w)|_{E_x^2} = R_x^2(v,w) + \alpha^2\big(w)\alpha^2(v)^* - \alpha^2(v)\alpha^2(w)^*. \tag{2.8.11}$$

Notice that both sides of (2.8.11) are anti-symmetric linear operators on E_x^2. Thus, if the fibers of E^2 are one-dimensional, it follows that the generalized Ricci equation is trivial in the case of connections compatible with a semi-Riemannian metric.

DEFINITION 2.8.5. Let (M,g), $(\overline{M},\bar{g})$ be semi-Riemannian manifolds. By an *isometric immersion* of (M,g) into $(\overline{M},\bar{g})$ we mean a smooth map $f : M \to \overline{M}$ such that:

$$\bar{g}_{f(x)}\big(\mathrm{d}f_x(v), \mathrm{d}f_x(w)\big) = g_x(v,w), \tag{2.8.12}$$

for all $x \in M$, $v, w \in T_xM$.

Clearly every isometric immersion is a smooth immersion.

EXAMPLE 2.8.6. Let (M,g), $(\overline{M},\bar{g})$ be semi-Riemannian manifolds and $f : M \to \overline{M}$ be an isometric immersion. The map $\overleftarrow{\mathrm{d}f} : TM \to f^*T\overline{M}$ is an injective morphism of vector bundles and therefore its image $\overleftarrow{\mathrm{d}f}(TM)$ is a vector subbundle of $f^*T\overline{M}$ that is isomorphic to TM. We denote by $f^{\perp}$ the orthogonal subbundle of $\overleftarrow{\mathrm{d}f}(TM)$ in $f^*T\overline{M}$ (see Exercise 1.75) and we call it the *normal bundle* of the isometric immersion f. It follows from (2.8.12) that $\overleftarrow{\mathrm{d}f}(TM)$ is nondegenerate for $\bar{g}$ and therefore:

$$f^*T\overline{M} = \overleftarrow{\mathrm{d}f}(TM) \oplus f^{\perp}. \tag{2.8.13}$$

Let $\overline{\nabla}$ denote the Levi-Civita connection of $(\overline{M},\bar{g})$ (see Exercise 2.22) and consider the pull-back connection $f^*\overline{\nabla}$. The components of $f^*\overline{\nabla}$ relatively to the direct sum decomposition (2.8.13) are denoted by ∇, $\nabla^{\perp}$, α, where ∇ is a connection on $\overleftarrow{\mathrm{d}f}(TM)$, $\nabla^{\perp}$ is a connection on $f^{\perp}$ and α is a smooth section of $\mathrm{Lin}(TM, \overleftarrow{\mathrm{d}f}(TM); f^{\perp})$. By Lemma 2.8.4, ∇ is compatible with the semi-Riemannian structure of $\overleftarrow{\mathrm{d}f}(TM)$ obtained by restricting $\bar{g}$; moreover, setting $\iota_1 = \overleftarrow{\mathrm{d}f} : TM \to \overleftarrow{\mathrm{d}f}(TM)$, $\iota = \overleftarrow{\mathrm{d}f} : TM \to f^*T\overline{M}$ then, since the ι-torsion of $f^*\overline{\nabla}$ is zero (Corollary 2.7.25), it follows from Example 2.8.2 that the ι_1-torsion of ∇ is zero. Thus, using ι_1 to identify TM with $\overleftarrow{\mathrm{d}f}(TM)$, it follows that ∇ is precisely the Levi-Civita connection of (M,g); namely, ∇ is symmetric and compatible with g. Again using ι_1 to identify TM with $\overleftarrow{\mathrm{d}f}(TM)$, we see that the component α of

$f^*\overline{\nabla}$ is identified with a smooth section of $\mathrm{Lin}_2(TM, f^\perp)$. Since the ι-torsion of $f^*\overline{\nabla}$ is zero, it follows from Example 2.8.2 that α is actually a smooth section of $\mathrm{Lin}_2^{\mathrm{s}}(TM, f^\perp)$, i.e., for every $x \in M$, $\alpha_x : T_xM \times T_xM \to f_x^\perp$ is a symmetric bilinear form. We call α the *second fundamental form* and $\nabla^\perp$ the *normal connection* of the isometric immersion f.

2.9. Differential forms in a principal bundle

Let $\Pi : P \to M$ be a G-principal bundle endowed with a connection $\mathrm{Hor}(P)$; denote by ω the connection form of $\mathrm{Hor}(P)$, which is a 1-form on P taking values in the Lie algebra $\mathfrak{g}$ of G. Given a (possibly vector-valued) smooth differential k-form λ on P, we denote by $\mathrm{d}\lambda$ its (standard) exterior differential, which is a smooth $(k+1)$-form on P (taking values in the same vector space as λ).

DEFINITION 2.9.1. Let λ be a (possibly vector-valued) smooth k-form on P. The *covariant exterior differential* of λ is the $(k+1)$-form $\mathrm{D}\lambda$ on P (taking values in the same vector space as λ) defined by:

$$\mathrm{D}\lambda_p(\zeta_1, \ldots, \zeta_{k+1}) = \mathrm{d}\lambda_p\big(\mathfrak{p}_{\mathrm{hor}}(\zeta_1), \ldots, \mathfrak{p}_{\mathrm{hor}}(\zeta_{k+1})\big),$$

for all $p \in P$ and all $\zeta_1, \ldots, \zeta_{k+1} \in T_pP$.

Clearly the covariant exterior differential of a smooth k-form on P is a smooth $(k+1)$-form on P.

LEMMA 2.9.2. *Let P be a G-principal bundle endowed with a connection* $\mathrm{Hor}(P)$*, E_0 be a real finite-dimensional vector space and $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation of G on E_0. If λ is a smooth ρ-pseudo G-invariant differential form on P then its covariant exterior differential* $\mathrm{D}\lambda$ *is also ρ-pseudo G-invariant.*

PROOF. Let $g \in G$, $p \in P$ and $\zeta_1, \ldots, \zeta_{k+1} \in T_pP$ be fixed. Using the result of Exercise 2.8, we compute:

$$\begin{aligned}
(\gamma_g^* \mathrm{D}\lambda)_p(\zeta_1, \ldots, \zeta_{k+1}) &= \mathrm{D}\lambda_{p\cdot g}(\zeta_1 \cdot g, \ldots, \zeta_{k+1} \cdot g)\\
&= \mathrm{d}\lambda_{p\cdot g}\big(\mathfrak{p}_{\mathrm{hor}}(\zeta_1 \cdot g), \ldots, \mathfrak{p}_{\mathrm{hor}}(\zeta_{k+1} \cdot g)\big)\\
&= \mathrm{d}\lambda_{p\cdot g}\big(\mathfrak{p}_{\mathrm{hor}}(\zeta_1) \cdot g, \ldots, \mathfrak{p}_{\mathrm{hor}}(\zeta_{k+1}) \cdot g\big)\\
&= (\gamma_g^* \mathrm{d}\lambda)_p\big(\mathfrak{p}_{\mathrm{hor}}(\zeta_1), \ldots, \mathfrak{p}_{\mathrm{hor}}(\zeta_{k+1})\big)\\
&= (\mathrm{d}\gamma_g^* \lambda)_p\big(\mathfrak{p}_{\mathrm{hor}}(\zeta_1), \ldots, \mathfrak{p}_{\mathrm{hor}}(\zeta_{k+1})\big)\\
&= \mathrm{d}\big(\rho(g)^{-1} \circ \lambda\big)_p\big(\mathfrak{p}_{\mathrm{hor}}(\zeta_1), \ldots, \mathfrak{p}_{\mathrm{hor}}(\zeta_{k+1})\big)\\
&= \rho(g)^{-1} \cdot \mathrm{d}\lambda_p\big(\mathfrak{p}_{\mathrm{hor}}(\zeta_1), \ldots, \mathfrak{p}_{\mathrm{hor}}(\zeta_{k+1})\big)\\
&= \rho(g)^{-1} \cdot \mathrm{D}\lambda_p(\zeta_1, \ldots, \zeta_{k+1}).
\end{aligned}$$

□

DEFINITION 2.9.3. The *curvature form* of the connection $\mathrm{Hor}(P)$ is the $\mathfrak{g}$-valued smooth 2-form on P defined by:

$$\Omega = \mathrm{D}\omega.$$

DEFINITION 2.9.4. A (possibly vector-valued) differential k-form λ on P is said to be *horizontal* if:

$$\lambda_p(\zeta_1, \ldots, \zeta_k) = 0,$$

for all $p \in P$, $\zeta_1, \ldots, \zeta_k \in T_pP$, provided that at least one of the vectors ζ_i is in $\mathrm{Ver}_p(P)$.

EXAMPLE 2.9.5. The covariant exterior differential of a smooth differential form λ on P is always horizontal, even if λ is not horizontal. In particular, the curvature form of a connection is always horizontal.

Given a G-principal bundle $\Pi : P \to M$ then, since we are given a smooth right action of G on P, one can define for every A in the Lie algebra $\mathfrak{g}$ the smooth vector field $A^P \in \mathbf{\Gamma}(TP)$ on P induced by A (recall Definition A.2.3). Clearly $A^P_p \in \mathrm{Ver}_p(P)$ and:

$$\omega_p(A^P_p) = A, \tag{2.9.1}$$

for all $p \in P$.

LEMMA 2.9.6. *Let* $\Pi : P \to M$ *be a principal bundle endowed with a connection* $\mathrm{Hor}(P)$. *Given a vector field* X *on* M *then for every* $g \in G$ *the horizontal lift* X^{hor} *is* γ_g*-related to itself, i.e.:*

$$X^{\mathrm{hor}}(p \cdot g) = X^{\mathrm{hor}}(p) \cdot g,$$

for all $p \in P$.

PROOF. Since $\mathrm{Hor}(P)$ is G-invariant $X^{\mathrm{hor}}(p) \cdot g$ is in $\mathrm{Hor}_{p\cdot g}(P)$; moreover, the result of Exercise 1.42 implies that:

$$\mathrm{d}\Pi_{p\cdot g}\big(X^{\mathrm{hor}}(p) \cdot g\big) = \mathrm{d}\Pi_p\big(X^{\mathrm{hor}}(p)\big) = X\big(\Pi(p)\big).$$

This proves that $X^{\mathrm{hor}}(p \cdot g) = X^{\mathrm{hor}}(p) \cdot g$. □

COROLLARY 2.9.7. *Let* $\Pi : P \to M$ *be a* G*-principal bundle endowed with a connection* $\mathrm{Hor}(P)$. *Given a vector field* X *on* M *and* $A \in \mathfrak{g}$ *then:*

$$[A^P, X^{\mathrm{hor}}] = 0.$$

PROOF. The flow of A^P at time t is equal to $\gamma_{\exp(tA)}$, for all $t \in \mathbb{R}$. The conclusion follows. □

PROPOSITION 2.9.8. *The curvature form* Ω *is given by:*

$$\Omega = \mathrm{d}\omega + \tfrac{1}{2}\,\omega \wedge \omega, \tag{2.9.2}$$

where the wedge product is considered with respect to the Lie bracket of $\mathfrak{g}$. *More explicitly,* (2.9.2) *means that:*

$$\Omega_p(\zeta_1, \zeta_2) = \mathrm{d}\omega_p(\zeta_1, \zeta_2) + [\omega_p(\zeta_1), \omega_p(\zeta_2)], \tag{2.9.3}$$

for all $p \in P$ *and all* $\zeta_1, \zeta_2 \in T_pP$.

PROOF. Since both sides of equality (2.9.3) are bilinear and antisymmetric in (ζ_1, ζ_2), it suffices to verify the equality in the cases:

(a) $\zeta_1, \zeta_2 \in \mathrm{Hor}_p(P)$;
(b) $\zeta_1 \in \mathrm{Hor}_p(P)$, $\zeta_2 \in \mathrm{Ver}_p(P)$;
(c) $\zeta_1, \zeta_2 \in \mathrm{Ver}_p(P)$.

Equality (2.9.3) is obvious in case (a). To prove the equality in case (b), let X be an arbitrary smooth vector field on M such that $X\big(\Pi(p)\big) = \mathrm{d}\Pi_p(\zeta_1)$ and set $A = \omega(\zeta_2) \in \mathfrak{g}$; clearly $X^{\mathrm{hor}}(p) = \zeta_1$ and $A^P(p) = \zeta_2$. Using Cartan's formula for exterior differentiation we compute:

$$\mathrm{d}\omega(X^{\mathrm{hor}}, A^P) = X^{\mathrm{hor}}\big(\omega(A^P)\big) - A^P\big(\omega(X^{\mathrm{hor}})\big) - \omega\big([X^{\mathrm{hor}}, A^P]\big).$$

Since $\omega(A^P)$ is a constant map (see (2.9.1)) and $\omega(X^{\mathrm{hor}}) \equiv 0$, then:

$$\mathrm{d}\omega(X^{\mathrm{hor}}, A^P) = -\omega\big([X^{\mathrm{hor}}, A^P]\big).$$

Moreover, by Corollary 2.9.7, $[X^{\mathrm{hor}}, A^P] = 0$ and thus $\mathrm{d}\omega_p(\zeta_1, \zeta_2) = 0$. Clearly all the other terms in equality (2.9.3) are also equal to zero, proving the equality in case (b). To prove the equality in case (c), set $A_i = \omega(\zeta_i) \in \mathfrak{g}$, so that $A_i^P(p) = \zeta_i$, $i = 1, 2$. Using again Cartan's formula for exterior differentiation, we obtain:

$$\mathrm{d}\omega_p(\zeta_1, \zeta_2) = -\omega_p\big([A_1^P, A_2^P]_p\big).$$

By the result of Exercise A.4, $[A_1^P, A_2^P] = [A_1, A_2]^P$, so that:

$$\mathrm{d}\omega_p(\zeta_1, \zeta_2) = -[A_1, A_2] = -[\omega(\zeta_1), \omega(\zeta_2)],$$

proving equality (2.9.3) in case (c). □

Let E_0 be a real finite-dimensional vector space and let $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation of G on E_0. Consider the associated bundle $P \times_G E_0$. Let ℓ be a k-form on M with values on the vector bundle $P \times_G E_0$. We define an E_0-valued k-form λ on P by setting:

$$\lambda_p(\zeta_1, \ldots, \zeta_k) = \hat{p}^{\,-1} \cdot \ell_x\big(\mathrm{d}\Pi_p(\zeta_1), \ldots, \mathrm{d}\Pi_p(\zeta_k)\big) \in E_0, \tag{2.9.4}$$

for all $x \in M$ and all $p \in P_x$. Clearly λ is horizontal and it is smooth if ℓ is smooth. We claim that λ is ρ-pseudo G-invariant. Let $p \in P$ and $g \in G$ be given and set $q = p \cdot g$. We compute:

$$\begin{aligned}(\gamma_g^* \lambda)_p(\zeta_1, \ldots, \zeta_k) = \lambda_q(\zeta_1 \cdot g, \ldots, \zeta_k \cdot g) = \hat{q}^{\,-1} \cdot \ell_x(\zeta_1, \ldots, \zeta_k)& \\ = \rho(g)^{-1} \cdot \lambda_p(\zeta_1, \ldots, \zeta_k)&,\end{aligned}$$

where in the second equality we have used the result of Exercise 1.42 and in the last equality we have used that $\hat{q} = \hat{p} \circ \rho(g)$ (recall (1.2.17)).

DEFINITION 2.9.9. Let ℓ be a $P \times_G E_0$-valued k-form on M. The E_0-valued k-form λ defined by (2.9.4), for all $p \in P$ and all $\zeta_1, \ldots, \zeta_k \in T_pP$, is called the *differential form associated to* ℓ.

LEMMA 2.9.10. *Let* $\Pi : P \to M$ *be a* G*-principal bundle,* E_0 *be a real finite-dimensional vector space and* $\rho : G \to \mathrm{GL}(E_0)$ *be a smooth representation of* G *on* E_0*. Let* λ *be an* E_0*-valued horizontal* ρ*-pseudo* G*-invariant* k*-form on* P*. Then* *there exists a unique $P \times_G E_0$-valued k-form ℓ on M such that λ is associated to ℓ. If $s : U \to P$ is a smooth local section then the following equality holds:*

$$[s(x), (s^*\lambda)_x(v_1, \ldots, v_k)] = \ell_x(v_1, \ldots, v_k), \tag{2.9.5}$$

for all $x \in U$ and all $v_1, \ldots, v_k \in T_xM$. Moreover, ℓ is smooth if λ is smooth.

PROOF. Given $x \in M$, $v_1, \ldots, v_k \in T_xM$, we set:

$$\ell_x(v_1, \ldots, v_k) = \hat{p} \cdot \lambda_p(\zeta_1, \ldots, \zeta_k) = [p, \lambda_p(\zeta_1, \ldots, \zeta_k)] \in P_x \times_G E_0, \tag{2.9.6}$$

where p is arbitrarily chosen in P_x and the vectors $\zeta_1, \ldots, \zeta_k \in T_pP$ are chosen with $\mathrm{d}\Pi_p(\zeta_i) = v_i$, $i = 1, \ldots, k$. We have to check that the righthand side of (2.9.6) does not depend on the choices of p and $\zeta_1, \ldots, \zeta_k$. Independence of the choice of the ζ_i's amounts to proving that:

$$\lambda_p(\zeta_1, \ldots, \zeta_k) = \lambda_p(\zeta_1 + A_1, \ldots, \zeta_k + A_k),$$

where $A_1, \ldots, A_k \in \mathrm{Ver}_p(P)$ are vertical; this follows immediately from the multilinearity of λ_p and from the horizontality of λ. Once the independence of the ζ_i's has been established, the independence of the p will follow once we prove the equality:

$$\hat{q} \cdot \lambda_q(\zeta_1 \cdot g, \ldots, \zeta_k \cdot g) = \hat{p} \cdot \lambda_p(\zeta_1, \ldots, \zeta_k), \tag{2.9.7}$$

where $q = p \cdot g$ (recall from Exercise 1.42 that $\mathrm{d}\Pi_q(\zeta_i \cdot g) = \mathrm{d}\Pi_p(\zeta_i) = v_i$, for $i = 1, \ldots, k$). To prove (2.9.7) we use $\hat{q} = \hat{p} \circ \rho(g)$ (recall (1.2.17)) and the ρ-pseudo G-invariance of λ as follows:

$$\begin{aligned}\hat{q} \cdot \lambda_q(\zeta_1 \cdot g, \ldots, \zeta_k \cdot g) &= \hat{q} \cdot (\gamma_g^* \lambda_p)(\zeta_1, \ldots, \zeta_k) \\ &= \big(\hat{q} \circ \rho(g)^{-1}\big) \cdot \lambda_p(\zeta_1, \ldots, \zeta_k) = \hat{p} \cdot \lambda_p(\zeta_1, \ldots, \zeta_k).\end{aligned}$$

Obviously equality (2.9.6) is equivalent to λ being associated to ℓ (equality (2.9.4)), so that ℓ is indeed the unique $P \times_G E_0$-valued k-form on M such that λ is associated to ℓ. If $s : U \to P$ is a smooth local section then equality (2.9.5) is proven by taking $p = s(x)$ and $\zeta_i = \mathrm{d}s_x(v_i)$, $i = 1, \ldots, k$, in (2.9.6), keeping in mind that $\mathrm{d}\Pi_{s(x)}\big(\mathrm{d}s_x(v_i)\big) = v_i$. Now assume that λ is smooth. The map $\hat{s} : U \times E_0 \to (P|_U) \times_G E_0$ defined in (1.4.2) is an isomorphism of vector bundles (see Example 1.5.14) and therefore ℓ is smooth if and only if $\hat{s}^{-1} \circ \ell$ is smooth (see Example 1.6.32). The smoothness of $\hat{s}^{-1} \circ \ell$ is proven by observing that equality (2.9.5) is the same as:

$$\hat{s}^{-1} \circ \ell = s^*\lambda. \qquad \square$$

REMARK 2.9.11. Let $\Pi : P \to M$ be a G-principal bundle and $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation. Let λ be a horizontal ρ-pseudo G-invariant k-form on P and ℓ be a $P \times_G E_0$-valued k-form on M. If every point of M is in the domain of a smooth local section $s : U \to P$ such that equality (2.9.5) holds then ℓ is associated to λ. Namely, if ℓ' is the $P \times_G E_0$-valued k-form on M associated to λ then equality (2.9.5) holds with ℓ replaced with ℓ'. Thus, $\ell|_U = \ell'|_U$.

LEMMA 2.9.12. *Let P be a G-principal bundle endowed with a connection $\mathrm{Hor}(P)$, E_0 be a real finite-dimensional vector space and $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation of G on E_0. If λ is a smooth horizontal ρ-pseudo G-invariant differential form on P then its exterior covariant derivative is given by:*

$$\mathrm{D}\lambda = \mathrm{d}\lambda + \omega \wedge \lambda, \tag{2.9.8}$$

where ω denotes the $\mathfrak{g}$-valued connection form of $\mathrm{Hor}(P)$ and the wedge product is taken with respect to the bilinear pairing:

$$\mathfrak{g} \times E_0 \ni (X, e) \longmapsto \bar{\rho}(X) \cdot e,$$

and $\bar{\rho} = \mathrm{d}\rho(1) : \mathfrak{g} \to \mathfrak{gl}(E_0)$.

PROOF. Formula (2.9.8) is equivalent to:

$$\begin{aligned} \mathrm{D}\lambda_p(\zeta_0, \ldots, \zeta_k) &= \mathrm{d}\lambda_p(\zeta_0, \zeta_1, \ldots, \zeta_k) \\ &\quad + \frac{1}{k!} \sum_{\sigma \in S_{k+1}} \mathrm{sgn}(\sigma)\, \bar{\rho}\big(\omega_p(\zeta_{\sigma(0)})\big) \cdot \lambda_p(\zeta_{\sigma(1)}, \ldots, \zeta_{\sigma(k)}), \end{aligned} \tag{2.9.9}$$

for all $p \in P$ and all $\zeta_0, \ldots, \zeta_k \in T_pP$. By multilinearity, it suffices to prove formula (2.9.9) when each ζ_i is either horizontal or vertical. If all the ζ_i's are horizontal, the equality is obvious. Assume that at least two of the ζ_i's are vertical, say ζ_0 and ζ_1. Clearly, both the lefthand side and the sum on the righthand side of (2.9.9) vanish; we have to check that, in this case, also the term with $\mathrm{d}\lambda$ vanishes. Set $A_i = \omega_p(\zeta_i) \in \mathfrak{g}$ and $Z_i = A_i^P$, so that $Z_i(p) = \zeta_i$, for $i = 0, 1$. Choose arbitrary smooth vector fields Z_i on P with $Z_i(p) = \zeta_i$, for $i = 2, \ldots, k$. Using Cartan's formula for exterior differentiation (A.3.2), it is clear that $\mathrm{d}\lambda(Z_0, \ldots, Z_k)$ vanishes; namely, since Z_0, Z_1 and $[Z_0, Z_1]$ are vertical and λ is horizontal, all the terms in the righthand side of Cartan's formula vanish. Now assume that exactly one of the ζ_i's is vertical; by antisymmetry, we may assume that ζ_0 is vertical. Set $A_0 = \omega_p(\zeta_0) \in \mathfrak{g}$, $Z_0 = A_0^P$; for $i = 1, \ldots, k$, let X_i be a smooth vector field on M with $X_i\big(\Pi(p)\big) = \mathrm{d}\Pi_p(\zeta_i)$ and set $Z_i = X_i^{\mathrm{hor}}$. Then $Z_i(p) = \zeta_i$, for all $i = 0, \ldots, k$. Using again Cartan's formula for exterior differentiation, keeping in mind Corollary 2.9.7 and the fact that λ is horizontal, we obtain:

$$\mathrm{d}\lambda(Z_0, \ldots, Z_k) = Z_0\big(\lambda(Z_1, \ldots, Z_k)\big).$$

Since Z_0 is vertical, in order to compute $Z_0\big(\lambda(Z_1, \ldots, Z_k)\big)(p)$, it suffices to consider the restriction of $\lambda(Z_1, \ldots, Z_k)$ to the fiber P_x, where $x = \Pi(p)$. Denoting by $f : P_x \to E_0$ such restriction, we obtain:

$$\mathrm{d}\lambda_p(\zeta_0, \ldots, \zeta_k) = \mathrm{d}f_p(\zeta_0) = \mathrm{d}f_p\big(\mathrm{d}\beta_p(1) \cdot A_0\big) = \mathrm{d}(f \circ \beta_p)(1) \cdot A_0.$$

But:

$$\begin{aligned}(f\circ\beta_p)(g)=f(p\cdot g)&=\lambda_{p\cdot g}\big(X_1^{\mathrm{hor}}(p\cdot g),\ldots,X_k^{\mathrm{hor}}(p\cdot g)\big)\\&=\lambda_{p\cdot g}\big(X_1^{\mathrm{hor}}(p)\cdot g,\ldots,X_k^{\mathrm{hor}}(p)\cdot g\big)\\&=\lambda_{p\cdot g}(\zeta_1\cdot g,\ldots,\zeta_k\cdot g)\\&=(\gamma_g^*\lambda)_p(\zeta_1,\ldots,\zeta_k)\\&=\rho(g)^{-1}\cdot\lambda_p(\zeta_1,\ldots,\zeta_k),\end{aligned}$$

for all $g\in G$; therefore:

$$\mathrm{d}\lambda_p(\zeta_0,\ldots,\zeta_k)=\mathrm{d}(f\circ\beta_p)(1)\cdot A_0=-\bar\rho(A_0)\cdot\lambda_p(\zeta_1,\ldots,\zeta_k). \tag{2.9.10}$$

Now let us compute the sum on the righthand side of (2.9.9); clearly, all the terms of that sum vanish, except for those with $\sigma(0)=0$. Such terms are all equal to $\bar\rho(A_0)\cdot\lambda_p(\zeta_1,\ldots,\zeta_k)$ and therefore their sum is equal to $k!\bar\rho(A_0)\cdot\lambda_p(\zeta_1,\ldots,\zeta_k)$. Using (2.9.10), we conclude that the righthand side of (2.9.9) vanishes. Obviously also the lefthand side of (2.9.9) vanishes and the proof is complete. □

REMARK 2.9.13. Let $\pi:E\to M$ be a vector bundle with typical fiber E_0 endowed with a connection ∇ and let $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ be the corresponding connection on the principal bundle of frames $\mathrm{FR}_{E_0}(E)$. Let $\rho:\mathrm{GL}(E_0)\to\mathrm{GL}(E_0)$ be the identity map. A horizontal ρ-pseudo $\mathrm{GL}(E_0)$-invariant differential form λ on $\mathrm{FR}_{E_0}(E)$ is associated to a unique differential form ℓ on M with values in $\mathrm{FR}_{E_0}(E)\times E_0$. By composing ℓ with the contraction map $\mathcal{C}^E$, we obtain a differential form $\mathcal{C}^E\circ\ell$ on M with values on E. In this situation, we will also say that λ and $\mathcal{C}^E\circ\ell$ are associated.

More generally, let P be a G-principal bundle over a differentiable manifold M, let E_0 be a real finite-dimensional vector space, let $\rho:G\to\mathrm{GL}(E_0)$ be a smooth representation and let $\phi:P\to\mathrm{FR}_{E_0}(E)$ be a morphism of principal bundles whose subjacent Lie group homomorphism is the representation ρ. We have then an isomorphism of vector bundles (recall Definition 1.5.17) $\mathcal{C}^\phi$ from $P\times_G E_0$ to E. A horizontal ρ-pseudo invariant differential form λ on P is associated to a unique differential form ℓ on M with values in $P\times_G E_0$. By composing ℓ with the ϕ-contraction map $\mathcal{C}^\phi$, we obtain a differential form $\mathcal{C}^\phi\circ\ell$ on M with values on E. In this situation, we will also say that λ and $\mathcal{C}^\phi\circ\ell$ are associated. A few particular situations where this occurs are presented in Remarks 1.6.1 and 1.6.9.

DEFINITION 2.9.14. Let M be a differentiable manifold and consider the $\mathrm{GL}(\mathbb{R}^n)$-principal bundle $\mathrm{FR}(TM)$ of frames of TM. The identity map of TM can be identified with a TM-valued smooth 1-form on M; the *canonical form* θ of $\mathrm{FR}(TM)$ is the $\mathbb{R}^n$-valued smooth 1-form on $\mathrm{FR}(TM)$ that is associated to the identity map of TM. More generally, let $\pi:E\to M$ be a vector bundle with typical fiber E_0 and let $\iota:TM\to E$ be a morphism of vector bundles. We can identify ι with a smooth E-valued 1-form on M. The *ι-canonical form* θ^ι of $\mathrm{FR}_{E_0}(E)$ is the E_0-valued smooth 1-form on $\mathrm{FR}_{E_0}(E)$ that is associated to ι.

More explicitly, we have:

$$\theta_p(\zeta)=p^{-1}\big(\mathrm{d}\Pi_p(\zeta)\big)\in\mathbb{R}^n, \tag{2.9.11}$$

for all $p \in \mathrm{FR}(TM)$, $\zeta \in T_p\mathrm{FR}(TM)$ and, more generally:

$$\theta^\iota_p(\zeta) = p^{-1}\big(\iota_x \cdot \mathrm{d}\Pi_p(\zeta)\big) \in E_0,$$

for all $x \in M$, $p \in \mathrm{FR}_{E_0}(E_x)$, $\zeta \in T_p\mathrm{FR}_{E_0}(E)$. Notice that if $s : U \to \mathrm{FR}_{E_0}(E)$ is a smooth local section then:

$$(s^*\theta^\iota)_x = s(x)^{-1} \circ \iota_x : T_xM \longrightarrow E_0, \tag{2.9.12}$$

for all $x \in U$; namely, if $\Pi : \mathrm{FR}_{E_0}(E) \to M$ denotes the projection then the composition $\mathrm{d}\Pi_{s(x)} \circ \mathrm{d}s_x$ is the identity map of T_xM.

DEFINITION 2.9.15. Let M be a differentiable manifold and consider the $\mathrm{GL}(\mathbb{R}^n)$-principal bundle $\mathrm{FR}(TM)$ of frames of TM. The *torsion form* Θ of $\mathrm{FR}(TM)$ is defined by:

$$\Theta = \mathrm{D}\theta.$$

More generally, let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and let $\iota : TM \to E$ be a morphism of vector bundles. The *ι-torsion form* Θ^ι of $\mathrm{FR}_{E_0}(E)$ is defined by:

$$\Theta^\iota = \mathrm{D}\theta^\iota.$$

Observe that by (2.9.8) we have:

$$\Theta^\iota = \mathrm{d}\theta^\iota + \omega \wedge \theta^\iota, \tag{2.9.13}$$

where the wedge product is taken with respect to the obvious bilinear pairing of $\mathfrak{gl}(E_0)$ and E_0.

The curvature tensor R of a connection ∇ on a vector bundle $\pi : E \to M$ can be identified with a smooth $\mathfrak{gl}(E)$-valued 2-form on M. We have the following:

LEMMA 2.9.16. *Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 endowed with a connection ∇. The curvature form Ω corresponding to the connection on the principal bundle of frames $\mathrm{FR}_{E_0}(E)$ is associated to the curvature tensor R; more explicitly:*

$$p \circ \Omega_p(\zeta_1, \zeta_2) \circ p^{-1} = R_x\big(\mathrm{d}\Pi_p(\zeta_1), \mathrm{d}\Pi_p(\zeta_2)\big) \in \mathrm{Lin}(E_x), \tag{2.9.14}$$

for all $x \in M$, $p \in \mathrm{FR}_{E_0}(E_x)$, $\zeta_1, \zeta_2 \in T_p\mathrm{FR}_{E_0}(E)$, where

$$\Pi : \mathrm{FR}_{E_0}(E) \longrightarrow M$$

denotes the projection.

PROOF. Let $s : U \to \mathrm{FR}_{E_0}(E)$ be a smooth local section and set $\bar\omega = s^*\omega$, $\overline{\Omega} = s^*\Omega$. Keeping in mind equality (2.9.5) and Remark 2.9.13, we see that the proof will be concluded once we show that:

$$s(x) \circ \big(\overline{\Omega}_x(v, w)\big) \circ s(x)^{-1} = R_x(v, w),$$

for all $x \in U$ and all $v, w \in T_xM$. Let $X, Y \in \mathbf{\Gamma}(TM|_U)$ and $\epsilon \in \mathbf{\Gamma}(E|_U)$ be fixed; denote by $\tilde\epsilon : U \to E_0$ the representation of ϵ with respect to s. We have to show that:

$$s(x)\big[\overline{\Omega}_x\big(X(x), Y(x)\big) \cdot \tilde\epsilon(x)\big] = R_x\big(X(x), Y(x)\big)\epsilon(x),$$

for all $x \in U$. We compute $R(X,Y)\epsilon$ using (2.5.4) as follows; the representation of $\nabla_Y\epsilon$ with respect to s is given by:

$$Y(\tilde{\epsilon}) + \bar{\omega}(Y)\cdot\tilde{\epsilon}.$$

Therefore, the representation of $\nabla_X\nabla_Y\epsilon$ with respect to s is equal to:

$$X\big(Y(\tilde{\epsilon})\big) + X\big(\bar{\omega}(Y)\big)\cdot\tilde{\epsilon} + \bar{\omega}(Y)\cdot X(\tilde{\epsilon}) + \bar{\omega}(X)\cdot Y(\tilde{\epsilon}) + \big(\bar{\omega}(X)\circ\bar{\omega}(Y)\big)\cdot\tilde{\epsilon}. \tag{2.9.15}$$

Similarly the representation of $\nabla_Y\nabla_X\epsilon$ with respect to s is equal to:

$$Y\big(X(\tilde{\epsilon})\big) + Y\big(\bar{\omega}(X)\big)\cdot\tilde{\epsilon} + \bar{\omega}(X)\cdot Y(\tilde{\epsilon}) + \bar{\omega}(Y)\cdot X(\tilde{\epsilon}) + \big(\bar{\omega}(Y)\circ\bar{\omega}(X)\big)\cdot\tilde{\epsilon}, \tag{2.9.16}$$

and the representation of $\nabla_{[X,Y]}\epsilon$ with respect to s is equal to:

$$[X,Y](\tilde{\epsilon}) + \bar{\omega}\big([X,Y]\big)\cdot\tilde{\epsilon}. \tag{2.9.17}$$

Hence, using (2.9.15), (2.9.16), (2.9.17) and Cartan's formula for exterior differentiation (A.3.3), we obtain that the representation of $R(X,Y)\epsilon$ with respect to s is equal to:

$$\mathrm{d}\bar{\omega}(X,Y)\cdot\tilde{\epsilon} + [\bar{\omega}(X),\bar{\omega}(Y)]\tilde{\epsilon} = \overline{\Omega}(X,Y)\cdot\tilde{\epsilon}.$$

The conclusion follows. □

PROPOSITION 2.9.17. *Let $\Pi : P \to M$ be a G-principal bundle endowed with a connection, E_0 be a real finite-dimensional vector space and $\rho : G \to \mathrm{GL}(E_0)$ be a smooth representation of G on E_0. Assume that the vector bundle $P\times_G E_0$ is endowed with connection defined in Example 2.5.9. If ℓ is a smooth $P\times_G E_0$-valued k-form on M and λ is the associated E_0-valued k-form on P then the covariant exterior differential $\mathrm{D}\lambda$ is associated to the covariant exterior differential $\mathrm{D}\ell$.*

PROOF. Let $s : U \to P$ be a smooth local section. Then, equality (2.9.5) holds. We have to prove that (see Remark 2.9.11):

$$[s(x), (s^*\mathrm{D}\lambda)_x(v_1,\ldots,v_{k+1})] = \mathrm{D}\ell_x(v_1,\ldots,v_{k+1}), \tag{2.9.18}$$

for all $x \in U$ and all $v_1,\ldots,v_{k+1} \in T_xM$. Define $\mathfrak{H} : P \to \mathrm{FR}_{E_0}(P\times_G E_0)$ as in (1.5.3) and set $s_1 = \mathfrak{H}\circ s$, so that $\hat{s} = \check{s}_1$ (see (1.5.5)). Let ω denote the $\mathfrak{g}$-valued connection form of the connection of P and let ω' denote the $\mathfrak{gl}(E_0)$-valued connection form of the connection of $\mathrm{FR}_{E_0}(P\times_G E_0)$. Since $\mathfrak{H}$ is connection preserving, we have:

$$\mathfrak{H}^*\omega' = \bar{\rho}\circ\omega,$$

where $\bar{\rho} = \mathrm{d}\rho(1) : \mathfrak{g} \to \mathfrak{gl}(E_0)$ (see (c) of Lemma 2.2.11). Setting $\bar{\omega} = s^*\omega$ then:

$$s_1^*\omega' = s^*\mathfrak{H}^*\omega' = s^*(\bar{\rho}\circ\omega) = \bar{\rho}\circ\bar{\omega}.$$

By Example 2.5.12, the vector bundle isomorphism:

$$\check{s}_1 : U\times E_0 \longrightarrow (P|_U)\times_G E_0$$

is connection preserving if the trivial vector bundle $U\times E_0$ is endowed with the connection $\mathrm{d}\mathbb{I} + s_1^*\omega' = \mathrm{d}\mathbb{I} + \bar{\rho}\circ\bar{\omega}$, where the $\mathfrak{gl}(E_0)$-valued 1-form $\bar{\rho}\circ\bar{\omega}$ on U is identified with the $C^\infty(U)$-bilinear map:

$$\boldsymbol{\Gamma}(TM|_U)\times\boldsymbol{\Gamma}(U\times E_0) \ni (X,\epsilon) \longmapsto \big(\bar{\rho}\circ\bar{\omega}(X)\big)(\epsilon) \in \boldsymbol{\Gamma}(U\times E_0).$$

Set $\tilde{\ell} = \check{s}_1^{-1} \circ \ell = \hat{s}^{-1} \circ \ell$; by (2.9.5), we have $\tilde{\ell} = s^*\lambda$. Denote by $\mathrm{D}\tilde{\ell}$ the exterior covariant derivative of the E_0-valued k-form $\tilde{\ell}$ associated to the connection $\mathrm{d}\mathbb{I} + \bar{\rho} \circ \bar{\omega}$ on the trivial bundle $U \times E_0$; since $\check{s}_1$ is connection preserving, by Proposition 2.7.20, we have:

$$\hat{s} \circ \mathrm{D}\tilde{\ell} = \check{s}_1 \circ \mathrm{D}\tilde{\ell} = \mathrm{D}\ell. \tag{2.9.19}$$

Moreover, by Corollary 2.7.21 and Example 2.7.19:

$$\mathrm{D}\tilde{\ell} = \mathrm{d}\tilde{\ell} + (\bar{\rho} \circ \bar{\omega}) \wedge \tilde{\ell},$$

where the wedge product is taken with respect to the obvious bilinear pairing of $\mathfrak{gl}(E_0)$ with E_0. If we consider the bilinear pairing of $\mathfrak{g}$ with E_0 given by:

$$\mathfrak{g} \times E_0 \ni (A, e) \longmapsto \bar{\rho}(A) \cdot e \in E_0$$

then $(\bar{\rho} \circ \bar{\omega}) \wedge \tilde{\ell} = \bar{\omega} \wedge \tilde{\ell}$, so that:

$$\mathrm{D}\tilde{\ell} = \mathrm{d}\tilde{\ell} + \bar{\omega} \wedge \tilde{\ell}.$$

Taking the pull-back by s on both sides of (2.9.8) and using that $s^*\lambda = \tilde{\ell}$ we obtain:

$$s^*\mathrm{D}\lambda = \mathrm{d}\tilde{\ell} + \bar{\omega} \wedge \tilde{\ell},$$

so that:

$$s^*\mathrm{D}\lambda = \mathrm{D}\tilde{\ell},$$

and, by (2.9.19):

$$\hat{s} \circ s^*\mathrm{D}\lambda = \mathrm{D}\ell,$$

proving (2.9.18). This concludes the proof. □

COROLLARY 2.9.18. *The ι-torsion form Θ^ι is associated to the ι-torsion tensor T^ι; more explicitly:*

$$p\big(\Theta^\iota_p(\zeta_1, \zeta_2)\big) = T_x\big(\mathrm{d}\Pi_p(\zeta_1), \mathrm{d}\Pi_p(\zeta_2)\big) \in E_x, \tag{2.9.20}$$

for all $x \in M$, $p \in \mathrm{FR}_{E_0}(E_x)$, $\zeta_1, \zeta_2 \in T_p\mathrm{FR}_{E_0}(E)$, where

$$\Pi : \mathrm{FR}_{E_0}(E) \longrightarrow M$$

denotes the projection.

PROOF. Follows immediately from Proposition 2.9.17 and from Example 2.7.22. □

2.10. Relating connections with principal subbundles

Let $\Pi : P \to M$ be a G-principal bundle endowed with a connection $\mathrm{Hor}(P)$. Let H be a Lie subgroup of G and let $Q \subset P$ be an H-principal subbundle of P. It may be the case that the distribution $\mathrm{Hor}(P)$ is tangent to the submanifold Q of P; in this case (and *only* in this case), the restriction of $\mathrm{Hor}(P)$ to Q is a connection on the H-principal bundle Q. If $\mathrm{Hor}(P)$ is tangent to Q, we say that the connection $\mathrm{Hor}(P)$ is *compatible* with the subbundle Q. Let us take a look at the general case.

Denote by ω the connection form of $\mathrm{Hor}(P)$. For each $x \in M$ and each $p \in P_x$, the map:

$$(\mathrm{d}\Pi_p, \omega_p) : T_pP \ni \zeta \xrightarrow{\cong} \big(\mathrm{d}\Pi_p(\zeta), \omega_p(\zeta)\big) \in T_xM \oplus \mathfrak{g} \tag{2.10.1}$$

is an isomorphism; namely we have a direct sum decomposition

$$T_pP = \mathrm{Hor}_p(P) \oplus \mathrm{Ver}_p(P),$$

the map $\mathrm{d}\Pi_p$ sends $\mathrm{Hor}_p(P)$ isomorphically onto T_xM and the map ω_p sends $\mathrm{Ver}_p(P)$ isomorphically onto $\mathfrak{g}$ (recall that the restriction of ω_p to $\mathrm{Ver}_p(P)$ is the inverse of the canonical isomorphism (1.3.3)). If the connection $\mathrm{Hor}(P)$ is compatible with subbundle Q then, for all $p \in Q_x$, the space T_pQ corresponds via (2.10.1) to the space $T_xM \oplus \mathfrak{h}$ (see Exercise 2.23). In the general case, we wish to define a tensor that measures how much $(\mathrm{d}\Pi_p, \omega_p)(T_pQ)$ deviates from $T_xM \oplus \mathfrak{h}$. We have the following:

LEMMA 2.10.1. *Let $\Pi : P \to M$ be a G-principal bundle endowed with a connection $\mathrm{Hor}(P)$, H be a Lie subgroup of G and $Q \subset P$ an H-principal subbundle of P. Given $x \in M$, $p \in Q_x$, then there exists a unique linear map $L : T_xM \to \mathfrak{g}/\mathfrak{h}$ such that the image of T_pQ under the isomorphism* (2.10.1) *is equal to:*

$$\big\{(v, X) \in T_xM \oplus \mathfrak{g} : L(v) = X + \mathfrak{h}\big\}.$$

Moreover, if $s : U \to Q$ is a smooth local section of Q with $s(x) = p$ and $\bar{\omega} = s^*\omega$ is the representation of the connection form ω with respect to s, then L is given by the composition of $\bar{\omega}_x : T_xM \to \mathfrak{g}$ with the canonical quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$.

PROOF. Let $S \subset T_xM \oplus \mathfrak{g}$ denote the image of T_pQ under the isomorphism (2.10.1). The existence and uniqueness of the desired map L is obtained by an elementary linear algebra argument, from the following two facts, that will be proven below:

(a) the restriction to S of the first projection $T_xM \oplus \mathfrak{g} \to T_xM$ is surjective;
(b) $S \cap (0 \oplus \mathfrak{g}) = 0 \oplus \mathfrak{h}$.

Assertion (a) follows from the fact that the restriction of $\mathrm{d}\Pi_p : T_pP \to T_xM$ to T_pQ is surjective and from the commutativity of the following diagram:

$$\begin{tikzcd} T_pP \arrow[rr, "(2.10.1)", "\cong"'] \arrow[dr, "\mathrm{d}\Pi_p"'] & & T_xM \oplus \mathfrak{g} \arrow[dl, "\text{first projection}"] \\ & T_xM & \end{tikzcd}$$

To prove (b), we observe first that $T_pQ \cap \mathrm{Ver}_p(P) = \mathrm{Ver}_p(Q)$; namely:

$$T_pQ \cap \mathrm{Ver}_p(P) = \mathrm{Ker}\big(\mathrm{d}\Pi_p|_{T_pQ}\big) = \mathrm{Ker}\big(\mathrm{d}(\Pi|_Q)_p\big) = \mathrm{Ver}_p(Q).$$

Since the isomorphism (2.10.1) carries $\mathrm{Ver}_p(P)$ to $0 \oplus \mathfrak{g}$, we have to show that (2.10.1) carries $\mathrm{Ver}_p(Q)$ to $0 \oplus \mathfrak{h}$. This follows by differentiating the commutative diagram below:

$$
\begin{tikzcd}
Q_x \arrow[r, "\text{inclusion}"] & P_x \\
H \arrow[u, "\beta_p", "\cong"'] \arrow[r, "\text{inclusion}"] & G \arrow[u, "\cong", "\beta_p"']
\end{tikzcd}
$$

Let now $s : U \to Q$ be a smooth local section of Q with $s(x) = p$ and set $\bar{\omega} = s^*\omega$. Clearly, the image of $\mathrm{d}s_x$ is contained in T_pQ. Given $v \in T_xM$ then $\mathrm{d}s_x(v)$ is in T_pQ and the image of $\mathrm{d}s_x(v)$ under (2.10.1) is equal to:

$$\big(v, \omega_p(\mathrm{d}s_x(v))\big) = \big(v, \bar{\omega}_x(v)\big).$$

Hence the graph of $\bar{\omega}_x : T_xM \to \mathfrak{g}$ is contained in S and the conclusion follows. □

REMARK 2.10.2. From Lemma 2.10.1 it follows in particular that, although the linear map $\bar{\omega}_x : T_xM \to \mathfrak{g}$ depends on the choice of the section s of Q with $s(x) = p$, the composite map $T_xM \xrightarrow{\bar{\omega}_x} \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ only depends on the choice of $p \in Q_x$.

Given $x \in M$, recall that we have identified the Lie group $\mathrm{Left}(Q_x)$ of left translations of the fiber Q_x with a Lie subgroup of the Lie group $\mathrm{Left}(P_x)$ of left translations of the fiber P_x; we have also identified the Lie algebra $\mathfrak{left}(Q_x)$ of $\mathrm{Left}(Q_x)$ with a Lie subalgebra of the Lie algebra $\mathfrak{left}(P_x)$ of $\mathrm{Left}(P_x)$. Given $p \in Q_x$ we have a Lie algebra isomorphism $\mathrm{Ad}_p : \mathfrak{g} \to \mathfrak{left}(P_x)$ that carries $\mathfrak{h}$ onto $\mathfrak{left}(Q_x)$ (recall (1.7.3)); therefore, we have an induced isomorphism $\overline{\mathrm{Ad}}_p : \mathfrak{g}/\mathfrak{h} \to \mathfrak{left}(P_x)/\mathfrak{left}(Q_x)$.

LEMMA 2.10.3. *Let $\Pi : P \to M$ be a G-principal bundle endowed with a connection whose connection form is ω and Q be a principal subbundle of P with structural group H. Let $s : U \to Q$ be a smooth local section of Q, $x \in U$ and set $p = s(x)$ and $\bar{\omega} = s^*\omega$. The map $\mathfrak{I}^Q_x : T_xM \to \mathfrak{left}(P_x)/\mathfrak{left}(Q_x)$ defined by the diagram:*

$$
\begin{tikzcd}
T_xM \arrow[r, "\bar{\omega}_x"] \arrow[rrr, bend right, "\mathfrak{I}^Q_x"'] & \mathfrak{g} \arrow[r, "\textit{quotient}"] & \mathfrak{g}/\mathfrak{h} \arrow[r, "\overline{\mathrm{Ad}}_p"] & \mathfrak{left}(P_x)/\mathfrak{left}(Q_x)
\end{tikzcd}
\tag{2.10.2}
$$

does not depend on the choice of the local section s.

PROOF. We observe first that the composition of $\bar{\omega}_x$ with the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ depends only on p, by Remark 2.10.2; in particular, $\mathfrak{I}^Q_x$ depends only on p. Let $p, p' \in Q_x$ be fixed. Write $p' = p \cdot h$, with $h \in H$. Denote by $\gamma_h : P \to P$ the diffeomorphism given by the action of h on P and consider the local section $s' = \gamma_h \circ s : U \to Q$ of Q; obviously, $s'(x) = p'$. Setting $\bar{\omega}' = s'^*\omega$ then it follows from (2.2.3) that $\bar{\omega}' = \mathrm{Ad}_{h^{-1}} \circ \bar{\omega}$. We have the following commutative diagram:

$$\begin{tikzcd}
 & \mathfrak{g} \arrow[r, "\text{quotient}"] \arrow[dd, "\mathrm{Ad}_{h^{-1}}"] & \mathfrak{g}/\mathfrak{h} \arrow[dd, "\overline{\mathrm{Ad}}_{h^{-1}}"] \arrow[dr, "\overline{\mathrm{Ad}}_p"] & \\
T_xM \arrow[ur, "\bar{\omega}_x"] \arrow[dr, "\bar{\omega}'_x"'] & & & \mathfrak{left}(P_x)/\mathfrak{left}(Q_x) \\
 & \mathfrak{g} \arrow[r, "\text{quotient}"'] & \mathfrak{g}/\mathfrak{h} \arrow[ur, "\overline{\mathrm{Ad}}_{p'}"'] &
\end{tikzcd}$$

where $\overline{\mathrm{Ad}}_{h^{-1}}$ is obtained from $\mathrm{Ad}_{h^{-1}}$ by passing to the quotient. The commutativity of the rightmost triangle on the diagram above follows from (1.7.1). This concludes the proof. $\square$

DEFINITION 2.10.4. The linear map $\mathfrak{I}_x^Q : T_xM \to \mathfrak{left}(P_x)/\mathfrak{left}(Q_x)$ defined by diagram (2.10.2) is called the *covariant derivative* of the subbundle Q at the point $x \in M$.

REMARK 2.10.5. It follows directly from Lemma 2.10.1 and from the definition of $\mathfrak{I}_x^Q$ that if $\Pi : P \to M$ is a G-principal bundle, Q is an H-principal subbundle of P and $p \in Q$ then the image of T_pQ under the isomorphism (2.10.1) is the subspace of $T_xM \oplus \mathfrak{g}$ given by:

$$\big\{(v, X) \in T_xM \oplus \mathfrak{g} : \big((\overline{\mathrm{Ad}}_p)^{-1} \circ \mathfrak{I}_x^Q\big)(v) = X + \mathfrak{h}\big\}.$$

Let P, P' be principal bundles over the same differentiable manifold M and let $\phi : P \to P'$ be an isomorphism of principal bundles. For each $x \in M$, the isomorphism of principal spaces $\phi_x : P_x \to P'_x$ induces a group isomorphism $\mathcal{I}_{\phi_x} : \mathrm{Left}(P_x) \to \mathrm{Left}(P'_x)$ (see Exercise 1.26); the commutativity of diagram (1.8) implies that $\mathcal{I}_{\phi_x}$ is in fact a Lie group isomorphism and therefore we may consider its differential at the identity, which we denote by:

$$\mathrm{Ad}_{\phi_x} = \mathrm{d}\mathcal{I}_{\phi_x}(1) : \mathfrak{left}(P_x) \longrightarrow \mathfrak{left}(P'_x).$$

Let $Q \subset P$ be a principal subbundle and set $Q' = \phi(Q)$. By the commutativity of diagram (1.9), Ad_{ϕ_x} carries $\mathfrak{left}(Q_x)$ onto $\mathfrak{left}(Q'_x)$ and therefore we get an induced map:

$$\overline{\mathrm{Ad}}_{\phi_x} : \mathfrak{left}(P_x)/\mathfrak{left}(Q_x) \longrightarrow \mathfrak{left}(P'_x)/\mathfrak{left}(Q'_x) \tag{2.10.3}$$

by passing to the quotient.

LEMMA 2.10.6. *Let P, P' be principal bundles over the same differentiable manifold M endowed with connections. Let $\phi : P \to P'$ be a connection preserving isomorphism of principal bundles. Let $Q \subset P$ be a principal subbundle and set $Q' = \phi(Q)$. Then, for all $x \in M$:*

$$\mathfrak{I}_x^{Q'} = \overline{\mathrm{Ad}}_{\phi_x} \circ \mathfrak{I}_x^Q,$$

where $\overline{\mathrm{Ad}}_{\phi_x}$ is defined in (2.10.3).

PROOF. Let G, G', H, H' denote the structural groups of P, P', Q and Q' respectively and let $\phi_0 : G \to G'$ denote the Lie group homomorphism subjacent to ϕ. Let $x \in M$ be fixed and choose a smooth local section $s : U \to Q$; set $p = s(x)$ and $p' = \phi(p)$. By differentiating diagram (1.8), we get another commutative diagram:

$$\begin{tikzcd} \mathfrak{left}(P) \arrow[r, "\mathrm{Ad}_\phi"] & \mathfrak{left}(P') \\ \mathfrak{g} \arrow[u, "\mathrm{Ad}_p"] \arrow[r, "\bar{\phi}_0"'] & \mathfrak{g}' \arrow[u, "\mathrm{Ad}_{p'}"'] \end{tikzcd}$$

By passing to the quotient, we obtain another commutative diagram:

(2.10.4)
$$\begin{tikzcd} \mathfrak{left}(P)/\mathfrak{left}(Q) \arrow[r, "\overline{\mathrm{Ad}}_\phi"] & \mathfrak{left}(P')/\mathfrak{left}(Q') \\ \mathfrak{g}/\mathfrak{h} \arrow[u, "\overline{\mathrm{Ad}}_p"] \arrow[r, "\tilde{\phi}_0"'] & \mathfrak{g}'/\mathfrak{h}' \arrow[u, "\overline{\mathrm{Ad}}_{p'}"'] \end{tikzcd}$$

where $\tilde{\phi}_0 : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}'/\mathfrak{h}'$ is obtained from $\bar{\phi}_0$ by passing to the quotient.

Let ω, ω' denote the connection forms of the connections of P and P' respectively; since the map ϕ is connection preserving, we have (see (c) of Lemma 2.2.11):

$$\phi^*\omega' = \bar{\phi}_0 \circ \omega,$$

where $\bar{\phi}_0 = \mathrm{d}\phi_0(1)$. We compute $\mathfrak{I}_x^Q$ using the smooth local section s of Q and $\mathfrak{I}_x^{Q'}$ using the smooth local section $\phi \circ s$ of Q'; set $\bar{\omega} = s^*\omega$ and $\bar{\omega}' = (\phi \circ s)^*\omega'$, so that:

$$\bar{\omega}'_x = \bar{\phi}_0 \circ \bar{\omega}_x. \tag{2.10.5}$$

The conclusion is now obtained from (2.10.4) and (2.10.5) observing that the following diagram:

$$\begin{tikzcd} \mathfrak{left}(P_x)/\mathfrak{left}(Q_x) \arrow[rr, "\overline{\mathrm{Ad}}_{\phi_x}"] & & \mathfrak{left}(P'_x)/\mathfrak{left}(Q'_x) \\ \mathfrak{g}/\mathfrak{h} \arrow[u, "\overline{\mathrm{Ad}}_p"'] \arrow[rr, "\tilde{\phi}_0"] & & \mathfrak{g}'/\mathfrak{h}' \arrow[u, "\overline{\mathrm{Ad}}_{p'}"] \\ \mathfrak{g} \arrow[u, "\text{quotient}"] \arrow[rr, "\bar{\phi}_0"] & & \mathfrak{g}' \arrow[u, "\text{quotient}"'] \\ & T_xM \arrow[ul, "\bar{\omega}_x"] \arrow[ur, "\bar{\omega}'_x"'] \arrow[uuul, dotted, bend left=60, "\mathfrak{I}_x^Q"] \arrow[uuur, dotted, bend right=60, "\mathfrak{I}_x^{Q'}"'] & \end{tikzcd}$$

commutes. $\square$

2.11. The inner torsion of a G-structure

Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 endowed with a connection ∇, let G be a Lie subgroup of $\mathrm{GL}(E_0)$ and $P \subset \mathrm{FR}_{E_0}(E)$ be a G-structure on E. The connection ∇ is associated to a unique connection $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ on the $\mathrm{GL}(E_0)$-principal bundle $\mathrm{FR}_{E_0}(E)$ (recall Proposition 2.5.4). We may therefore consider the covariant derivative $\mathfrak{I}_x^P : T_xM \to \mathfrak{gl}(E_x)/\mathfrak{g}_x$ of the G-principal subbundle P of $\mathrm{FR}_{E_0}(E)$ at a point $x \in M$ (recall the notation introduced in Section 1.8). We call $\mathfrak{I}_x^P$ the *inner torsion* of the G-structure P at the point x with respect to the connection ∇. The following lemma gives a simple way of computing $\mathfrak{I}_x^P$.

LEMMA 2.11.1. *Let $\pi : E \to M$ be a vector bundle with typical fiber E_0, let G be a Lie subgroup of $\mathrm{GL}(E_0)$ and let $P \subset \mathrm{FR}_{E_0}(E)$ be a G-structure on E; assume that a connection ∇ on E is given. If $s : U \to P$ is a smooth local E_0-frame of E compatible with P then the inner torsion $\mathfrak{I}_x^P : T_xM \to \mathfrak{gl}(E_x)/\mathfrak{g}_x$ of the G-structure P at the point x is given by the composition of the Christoffel tensor $\Gamma_x : T_xM \to \mathfrak{gl}(E_x)$ of the connection ∇ with respect to s and the quotient map $\mathfrak{gl}(E_x) \to \mathfrak{gl}(E_x)/\mathfrak{g}_x$.*

PROOF. Let ω denote the connection form of $\mathrm{Hor}\big(\mathrm{FR}_{E_0}(E)\big)$ and set $\bar\omega = s^*\omega$. From (2.5.5) and (1.7.2) we get $\Gamma_x = \mathrm{Ad}_p \circ \bar\omega_x$, where $p = s(x)$. The conclusion follows from the commutativity of the following diagram:

$$\begin{tikzcd}
T_xM \arrow[r, "\bar\omega_x"'] \arrow[rrr, bend left, "\mathfrak{I}_x^P"] \arrow[dr, "\Gamma_x"'] & \mathfrak{gl}(E_0) \arrow[r, "\text{quotient}"'] \arrow[d, "\mathrm{Ad}_p"] & \mathfrak{gl}(E_0)/\mathfrak{g} \arrow[r, "\overline{\mathrm{Ad}}_p"'] & \mathfrak{gl}(E_x)/\mathfrak{g}_x \\
& \mathfrak{gl}(E_x) \arrow[urr, "\text{quotient}"'] & &
\end{tikzcd}$$

□

EXAMPLE 2.11.2. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and let $s : M \to \mathrm{FR}_{E_0}(E)$ be a global smooth section. Then $P = s(M)$ is a G-structure on E with $G = \{\mathrm{Id}_{E_0}\}$. For each $x \in M$, we have $G_x = \{\mathrm{Id}_{E_x}\}$ and $\mathfrak{g}_x = \{0\}$. If ∇ is a connection in E then $\mathfrak{I}_x^P : T_xM \to \mathfrak{gl}(E_x)$ is equal to the Christoffel tensor $\Gamma_x : T_xM \to \mathfrak{gl}(E_x)$ corresponding to s.

EXAMPLE 2.11.3. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 endowed with a semi-Riemannian structure g of index r and let $\langle\cdot,\cdot\rangle_{E_0}$ be an indefinite inner product of index r on E_0. Then (recall Example 1.8.4) $P = \mathrm{FR}^{\mathrm{o}}_{E_0}(E)$ is a G-structure on E with $G = \mathrm{O}(E_0)$. If ∇ is a connection on E, let us compute the inner torsion of P with respect to ∇. Let $x \in M$ be fixed. The inner torsion $\mathfrak{I}_x^P$ is a linear map from T_xM to the quotient $\mathfrak{gl}(E_x)/\mathfrak{g}_x$. Clearly, G_x is the group of linear isometries of E_x (with respect to g_x) and $\mathfrak{g}_x$ is the Lie algebra of linear endomorphisms of E_x that are anti-symmetric (with respect to g_x). We identify $\mathfrak{gl}(E_x)/\mathfrak{g}_x$ with the space $\mathrm{sym}(E_x)$ of all linear endomorphisms of E_x that are symmetric (with respect to g_x) via the map:

$$\mathfrak{gl}(E_x)/\mathfrak{g}_x \ni T + \mathfrak{g}_x \longmapsto \tfrac{1}{2}(T + T^*) \in \mathrm{sym}(E_x), \tag{2.11.1}$$

where $T^*: E_x \to E_x$ denotes the *transpose* of T with respect to g_x, i.e., the unique linear endomorphism of E_x such that:

$$g_x\big(T(e), e'\big) = g_x\big(e, T^*(e')\big),$$

for all $e, e' \in E_x$. Thus, the inner torsion $\mathfrak{I}^P_x$ is identified with a linear map from T_xM to $\mathrm{sym}(E_x)$. Let $s: U \to P$ be a smooth local section with $x \in U$ and let $e, e' \in E_x$ be fixed; consider the local sections $\epsilon, \epsilon': U \to E$ defined by:

$$\epsilon(y) = \big(s(y) \circ s(x)^{-1}\big) \cdot e, \qquad \epsilon'(y) = \big(s(y) \circ s(x)^{-1}\big) \cdot e', \tag{2.11.2}$$

for all $y \in U$. Since the representations of ϵ and ϵ' with respect to s are constant, we have:

$$\nabla_v \epsilon = \mathrm{d}\mathbb{I}^s_v \epsilon + \Gamma_x(v) \cdot \epsilon(x) = \Gamma_x(v) \cdot \epsilon(x), \qquad \nabla_v \epsilon' = \mathrm{d}\mathbb{I}^s_v \epsilon' + \Gamma_x(v) \cdot \epsilon'(x) = \Gamma_x(v) \cdot \epsilon'(x), \tag{2.11.3}$$

for all $v \in T_xM$. Since s is a local section of $\mathrm{FR}^{\mathrm{o}}_{E_0}(E)$, it follows that:

$$g_y\big(\epsilon(y), \epsilon'(y)\big) = \langle s(x)^{-1} \cdot e, s(x)^{-1} \cdot e' \rangle_{E_0},$$

for all $y \in U$, so that the real-valued map $g(\epsilon, \epsilon')$ is constant. Thus:

$$\begin{aligned} 0 = v\big(g(\epsilon, \epsilon')\big) &= (\nabla_v g)(e, e') + g_x(\nabla_v \epsilon, e') + g_x(e, \nabla_v \epsilon') \\ &= (\nabla_v g)(e, e') + g_x\big(\Gamma_x(v) \cdot e, e'\big) + g_x\big(e, \Gamma_x(v) \cdot e'\big), \end{aligned}$$

for all $v \in T_xM$. Then:

$$g_x\big[\big(\Gamma_x(v) + \Gamma_x(v)^*\big) \cdot e, e'\big] = -(\nabla_v g)(e, e')$$

and (Lemma 2.11.1 and (2.11.1)):

$$g_x\big(\mathfrak{I}^P_x(v), \cdot\big) = \tfrac{1}{2} g_x\big[\big(\Gamma_x(v) + \Gamma_x(v)^*\big), \cdot\big] = -\tfrac{1}{2} \nabla_v g,$$

for all $x \in M$, $v \in T_xM$. Identifying $\nabla_v g : E_x \times E_x \to \mathbb{R}$ with a linear endomorphism of E_x, we obtain:

$$\mathfrak{I}^P_x(v) = -\tfrac{1}{2} \nabla_v g.$$

Thus, the inner torsion of P is essentially the covariant derivative of the semi-Riemannian structure g. In particular, $\mathfrak{I}^P = 0$ if and only if $\nabla g = 0$, i.e., ∇ is compatible with the semi-Riemannian structure g.

EXAMPLE 2.11.4. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and F be a vector subbundle of E. If F_0 is a subspace of E_0 such that $\dim(F_0) = \dim(F_x)$ for all $x \in M$ then the set $P = \mathrm{FR}_{E_0}(E; F_0, F)$ of all E_0-frames of E adapted to (F_0, F) is a G-structure on E with $G = \mathrm{GL}(E_0; F_0)$ (Example 1.8.5). Let ∇ be a connection on E and let us compute the inner torsion $\mathfrak{I}^P$. Let $x \in M$ be fixed. Clearly $G_x = \mathrm{GL}(E_x; F_x)$ and $\mathfrak{g}_x$ is the Lie algebra of linear endomorphisms $T : E_x \to E_x$ with $T(F_x) \subset F_x$. We identify the quotient $\mathfrak{gl}(E_x)/\mathfrak{g}_x$ with the space $\mathrm{Lin}(F_x, E_x/F_x)$ via the map:

$$\mathfrak{gl}(E_x)/\mathfrak{g}_x \ni T + \mathfrak{g}_x \longmapsto \mathfrak{q} \circ T|_{F_x} \in \mathrm{Lin}(F_x, E_x/F_x),$$

where $\mathfrak{q} : E_x \to E_x/F_x$ denotes the quotient map. Thus, the inner torsion $\mathfrak{I}^P_x$ is identified with a linear map from T_xM to $\mathrm{Lin}(F_x, E_x/F_x)$. Let $s : U \to P$ be a smooth local section with $x \in U$. Given $e \in F_x$, we define a local section $\epsilon : U \to E$ as in (2.11.2). Then the representation of ϵ with respect to s is constant and (2.11.3) holds, for all $v \in T_xM$. Moreover, since s takes values in $\mathrm{FR}_{E_0}(E; F_0, F)$, we have $\epsilon(U) \subset F$. Thus:

$$\nabla_v\epsilon + F_x = \alpha^F_x(v, e) \in E_x/F_x,$$

where α^F denotes the second fundamental form of the vector subbundle F (Exercise 2.20). Then:

$$\Gamma_x(v) \cdot e + F_x = \alpha^F_x(v, e)$$

and:

$$\mathfrak{I}^P_x(v) = \alpha^F_x(v, \cdot) \in \mathrm{Lin}(F_x, E_x/F_x),$$

for all $x \in M$, $v \in T_xM$. In particular, $\mathfrak{I}^P = 0$ if and only if $\alpha^F = 0$, i.e., if and only if the covariant derivative of any smooth section of F is a smooth section of F.

EXAMPLE 2.11.5. Let $\pi : E \to M$, F, E_0, F_0 be as in Example 2.11.4. Let g be a semi-Riemannian structure on E, $\langle\cdot,\cdot\rangle_{E_0}$ be an indefinite inner product on E_0 and assume that $\mathrm{FR}^{\mathrm{o}}_{E_0}(E_x; F_0, F_x) \neq \emptyset$, for all $x \in M$. Then $P = \mathrm{FR}^{\mathrm{o}}_{E_0}(E; F_0, F)$ is a G-structure on E with $G = \mathrm{O}(E_0; F_0)$ (Example 1.8.5). For simplicity, we assume that the restriction of g_x to $F_x \times F_x$ is nondegenerate, for all $x \in M$; thus, $E = F \oplus F^\perp$. Denote by $\mathfrak{q} : E \to F^\perp$ the projection. Let ∇ be a connection on E and let $x \in M$ be fixed. We compute $\mathfrak{I}^P_x$. We have $G_x = \mathrm{O}(E_x; F_x)$ and $\mathfrak{g}_x$ is the Lie algebra of linear endomorphisms $T : E_x \to E_x$ that are anti-symmetric (with respect to g_x) and satisfy $T(F_x) \subset F_x$. We have an isomorphism:

$$\begin{aligned}\mathfrak{gl}(E_x)/\mathfrak{g}_x &\longrightarrow \mathrm{sym}(E_x) \oplus \mathrm{Lin}(F_x, F_x^\perp)\\ T + \mathfrak{g}_x &\longmapsto \big(\tfrac{1}{2}(T + T^*), \tfrac{1}{2}\mathfrak{q}_x \circ (T - T^*)|_{F_x}\big),\end{aligned}$$

so that we identify $\mathfrak{I}^P_x$ with a linear map from T_xM to the space $\mathrm{sym}(E_x) \oplus \mathrm{Lin}(F_x, F_x^\perp)$. Consider the component:

$$\alpha \in \mathbf{\Gamma}\big(\mathrm{Lin}(TM, F; F^\perp)\big)$$

of ∇ with respect to the decomposition $E = F \oplus F^\perp$. Let $s : U \to P$ be a smooth local section with $x \in U$. As in Example 2.11.3, we have:

$$\tfrac{1}{2}\big(\Gamma_x(v) + \Gamma_x(v)^*\big) = -\tfrac{1}{2}\nabla_v g,$$

for all $v \in T_xM$. Moreover, arguing as in Example 2.11.4, we obtain:

$$\mathfrak{q}\big(\Gamma_x(v) \cdot e\big) = \alpha_x(v, e),$$

for all $v \in T_xM$, $e \in E_x$. Then:

$$\tfrac{1}{2}\big(\Gamma_x(v) - \Gamma_x(v)^*\big) = \Gamma_x(v) - \tfrac{1}{2}\big(\Gamma_x(v) + \Gamma_x(v)^*\big) = \Gamma_x(v) + \tfrac{1}{2}\nabla_v g, \tag{2.11.4}$$

and:

$$\mathfrak{I}^P_x(v) = \big(-\tfrac{1}{2}\nabla_v g, \alpha_x(v, \cdot) + \tfrac{1}{2}\mathfrak{q} \circ \nabla_v g|_{F_x}\big),$$

for all $x \in M$, $v \in T_xM$, where $\nabla_v g$ is identified with a linear endomorphism of E_x. In particular, $\mathfrak{I}^P = 0$ if and only if $\nabla g = 0$ and $\alpha = 0$, i.e., if and only if ∇ is compatible with g and the covariant derivative of any smooth section of F is a smooth section of F.

EXAMPLE 2.11.6. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 and $\epsilon \in \mathbf{\Gamma}(E)$ be a smooth section of E with $\epsilon(x) \neq 0$, for all $x \in M$. If $e_0 \in E_0$ is a nonzero vector then $P = \mathrm{FR}_{E_0}(E; e_0, \epsilon)$ is a G-structure on E with $G = \mathrm{GL}(E_0; e_0)$ (Example 1.8.6). Let ∇ be a connection on E and let us compute $\mathfrak{I}^P$. Let $x \in M$ be fixed. Then $G_x = \mathrm{GL}\big(E_x; \epsilon(x)\big)$ and $\mathfrak{g}_x$ is the Lie algebra of linear endomorphisms $T : E_x \to E_x$ such that $T\big(\epsilon(x)\big) = 0$. We identify the quotient $\mathfrak{gl}(E_x)/\mathfrak{g}_x$ with E_x via the map:

$$\mathfrak{gl}(E_x)/\mathfrak{g}_x \ni T + \mathfrak{g}_x \longmapsto T\big(\epsilon(x)\big) \in E_x.$$

Let $s : U \to P$ be a smooth local section with $x \in U$. Then $\mathfrak{I}^P_x$ is identified with a linear map from T_xM to E_x. Since s takes values in $\mathrm{FR}_{E_0}(E; e_0, \epsilon)$, we have:

$$\epsilon(y) = s(y) \cdot e_0,$$

so that the representation of ϵ with respect to s is constant and:

$$\nabla_v \epsilon = \Gamma_x(v) \cdot \epsilon(x), \tag{2.11.5}$$

for all $v \in T_xM$. Then:

$$\mathfrak{I}^P_x(v) = \nabla_v \epsilon,$$

for all $v \in T_xM$ and:

$$\mathfrak{I}^P_x = (\nabla \epsilon)(x),$$

for all $x \in M$. In particular, $\mathfrak{I}^P = 0$ if and only if the section ϵ is parallel. Assume now that g is a semi-Riemannian structure on E, $\langle \cdot, \cdot \rangle_{E_0}$ is an indefinite inner product on E_0 and that $\mathrm{FR}^{\mathrm{o}}_{E_0}\big(E_x; e_0, \epsilon(x)\big) \neq \emptyset$, for all $x \in M$. Then $P = \mathrm{FR}^{\mathrm{o}}_{E_0}(E; e_0, \epsilon)$ is a G-structure on E with $G = \mathrm{O}(E_0; e_0)$. Let us compute $\mathfrak{I}^P$. Let $x \in M$ be fixed. Then $G_x = \mathrm{O}\big(E_x; \epsilon(x)\big)$ and $\mathfrak{g}_x$ is the Lie algebra of anti-symmetric linear endomorphisms T of E_x such that $T\big(\epsilon(x)\big) = 0$. We have the following linear isomorphism:

$$\mathfrak{gl}(E_x)/\mathfrak{g}_x \ni T + \mathfrak{g}_x \longmapsto \big(\tfrac{1}{2}(T + T^*), \tfrac{1}{2}(T - T^*) \cdot \epsilon(x)\big) \in \mathrm{sym}(E_x) \oplus \epsilon(x)^\perp$$

where $\epsilon(x)^\perp$ denotes the kernel of $g_x\big(\epsilon(x), \cdot\big)$. Let $s : U \to P$ be a smooth local section with $x \in U$. As in Example 2.11.3, we have:

$$\tfrac{1}{2}\big(\Gamma_x(v) + \Gamma_x(v)^*\big) = -\tfrac{1}{2}\nabla_v g,$$

and, as in (2.11.4):

$$\tfrac{1}{2}\big(\Gamma_x(v) - \Gamma_x(v)^*\big) = \Gamma_x(v) + \tfrac{1}{2}\nabla_v g,$$

for all $v \in T_xM$. Moreover, (2.11.5) holds. Then:

$$\tfrac{1}{2}\big(\Gamma_x(v) - \Gamma_x(v)^*\big) \cdot \epsilon(x) = \nabla_v \epsilon + \tfrac{1}{2}(\nabla_v g)\big(\epsilon(x)\big).$$

Hence:

$$\mathfrak{I}^P_x(v) = \Big(-\tfrac{1}{2}\nabla_v g, \nabla_v \epsilon + \tfrac{1}{2}(\nabla_v g)\big(\epsilon(x)\big)\Big),$$

for all $x \in M$, $v \in T_xM$. In particular, $\mathfrak{I}^P = 0$ if and only if ∇ is compatible with g and ϵ is parallel.

EXAMPLE 2.11.7. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0, J be an almost complex structure on E and J_0 be a complex structure on E_0. The set $P = \mathrm{FR}^{\mathrm{c}}_{E_0}(E)$ is a G-structure on E with $G = \mathrm{GL}(E_0, J_0)$ (Example 1.8.7). Let ∇ be a connection on E and let us compute $\mathfrak{I}^P$. Let $x \in M$ be fixed. Then $G_x = \mathrm{GL}(E_x, J_x)$ and $\mathfrak{g}_x$ is the Lie algebra of linear endomorphisms $T : E_x \to E_x$ such that $T \circ J_x = J_x \circ T$. We have an isomorphism:

$$\mathfrak{gl}(E_x)/\mathfrak{g}_x \ni T + \mathfrak{g}_x \longmapsto [T, J_x] \in \overline{\mathrm{Lin}}(E_x, J_x),$$

where $[T, J_x] = T \circ J_x - J_x \circ T$ and $\overline{\mathrm{Lin}}(E_x, J_x)$ denotes the space of linear maps $T : E_x \to E_x$ such that $T \circ J_x + J_x \circ T = 0$. Let $s : U \to P$ be a smooth local section with $x \in U$ and let $e \in E_x$ be fixed. We define a local section $\epsilon : U \to E$ as in (2.11.2). Then $\epsilon(x) = e$ and the representation of ϵ with respect to s is constant; moreover, since s takes values in $\mathrm{FR}^{\mathrm{c}}_{E_0}(E)$, also the representation of $J(\epsilon)$ with respect to s is constant. Then:

$$\nabla_v \epsilon = \Gamma_x(v, e), \quad \nabla_v\big(J(\epsilon)\big) = \Gamma_x\big(v, J_x(e)\big),$$

and:

$$\nabla_v\big(J(\epsilon)\big) = (\nabla_v J)(e) + J_x(\nabla_v \epsilon),$$

for all $v \in T_xM$. We therefore obtain:

$$\Gamma_x(v) \circ J_x = \nabla_v J + J_x \circ \Gamma_x(v).$$

Hence:

$$\mathfrak{I}^P_x(v) = \nabla_v J,$$

for all $x \in M$ and all $v \in T_xM$. In particular, $\mathfrak{I}^P = 0$ if and only if J is parallel.

EXAMPLE 2.11.8. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0, J be an almost complex structure on E, g be a semi-Riemannian structure on E, J_0 be a complex structure on E_0 and $\langle \cdot, \cdot \rangle_{E_0}$ be an indefinite inner product on E_0. Assume that J_x is anti-symmetric with respect to g_x for all $x \in M$, that J_0 is anti-symmetric with respect to $\langle \cdot, \cdot \rangle_{E_0}$ and that g_x has the same index as $\langle \cdot, \cdot \rangle_{E_0}$, for all $x \in M$. Then the set $P = \mathrm{FR}^{\mathrm{u}}_{E_0}(E)$ is a G-structure on E with $G = \mathrm{U}(E_0)$ (Example 1.8.7). Let ∇ be a connection on E and let us compute $\mathfrak{I}^P$. Let $x \in M$ be fixed. We have $G_x = \mathrm{U}(E_x)$ and $\mathfrak{g}_x$ is the Lie algebra of linear maps $T : E_x \to E_x$ such that $T \circ J_x = J_x \circ T$ and such that T is anti-symmetric with respect to g_x. We have a linear isomorphism:

$$\begin{aligned} \mathfrak{gl}(E_x)/\mathfrak{g}_x &\longrightarrow \mathrm{sym}(E_x) \oplus \overline{\mathrm{Lin}}_{\mathrm{a}}(E_x, J_x) \\ T + \mathfrak{g}_x &\longmapsto \big(\tfrac{1}{2}(T + T^*), \tfrac{1}{2}[T - T^*, J_x]\big), \end{aligned}$$

where $\overline{\mathrm{Lin}}_{\mathrm{a}}(E_x, J_x)$ denotes the space of linear maps $T : E_x \to E_x$ that are anti-symmetric with respect to g_x and such that $T \circ J_x + J_x \circ T = 0$. Let $s : U \to P$ be a smooth local section with $x \in U$. As in Example 2.11.3, we have:

$$\tfrac{1}{2}\big(\Gamma_x(v) + \Gamma_x(v)^*\big) = -\tfrac{1}{2}\nabla_v g,$$

for all $v \in T_xM$. Moreover, as in Example 2.11.7, we have:

$$[\Gamma_x(v), J_x] = \nabla_v J,$$

for all $v \in T_xM$. Then:

$$\tfrac{1}{2}\big(\Gamma_x(v) - \Gamma_x(v)^*\big) = \Gamma_x(v) - \nabla_v g,$$

and:

$$\tfrac{1}{2}[\Gamma_x(v) - \Gamma_x(v)^*, J_x] = \nabla_v J - [\nabla_v g, J_x].$$

Hence:

$$\mathfrak{I}^P_x(v) = \big(-\tfrac{1}{2}\nabla_v g, \nabla_v J - [\nabla_v g, J_x]\big),$$

for all $x \in M$ and all $v \in T_xM$. In particular, $\mathfrak{I}^P = 0$ if and only if ∇ is compatible with g and J is parallel.

REMARK 2.11.9. Let M be an n-dimensional differentiable manifold endowed with a connection ∇, G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$ and $P \subset \mathrm{FR}(TM)$ be a G-structure on TM. We denote by $\mathrm{Hor}\big(\mathrm{FR}(TM)\big)$ the connection on $\mathrm{FR}(TM)$ associated to ∇ and by ω the corresponding connection form. Given $x \in M$, $p \in P_x$, we have a linear isomorphism:

$$(2.11.6)\qquad (\mathrm{d}\Pi_p, \omega_p) : T_p\mathrm{FR}(TM) \longrightarrow T_xM \oplus \mathfrak{gl}(\mathbb{R}^n)$$

as in (2.10.1). We have seen in Remark 2.10.5 that the image of T_pP under the isomorphism (2.11.6) is equal to:

$$\big\{(v, X) \in T_xM \oplus \mathfrak{gl}(\mathbb{R}^n) : \big((\overline{\mathrm{Ad}}_p)^{-1} \circ \mathfrak{I}^P_x\big)(v) = X + \mathfrak{g}\big\}.$$

Since p is an isomorphism from $\mathbb{R}^n$ to T_xM, by composing (2.11.6) with $p^{-1} \oplus \mathrm{Id}$ we obtain another linear isomorphism (recall (2.9.11)):

$$(2.11.7)\qquad (\theta_p, \omega_p) : T_p\mathrm{FR}(TM) \xrightarrow{\ \cong\ } \mathbb{R}^n \oplus \mathfrak{gl}(\mathbb{R}^n).$$

The image of T_pP under (2.11.7) is obviously equal to:

$$\big\{(u, X) \in \mathbb{R}^n \oplus \mathfrak{gl}(\mathbb{R}^n) : \big((\overline{\mathrm{Ad}}_p)^{-1} \circ \mathfrak{I}^P_x \circ p\big)(u) = X + \mathfrak{g}\big\}.$$

If $\mathfrak{I}^P = 0$, i.e., if ∇ is compatible with P and if $p : I \to \mathrm{FR}(E)$ is a smooth horizontal curve such that $p(t_0) \in P$ for some $t_0 \in I$ then $p(t) \in P$ for all $t \in I$. We now generalize this property to the case where $\mathfrak{I}^P$ is not necessarily zero.

PROPOSITION 2.11.10. *Let E be a vector bundle of rank k over a manifold M, ∇ be a connection on E, G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^k)$ and $P \subset \mathrm{FR}(E)$ be a G-structure on E. Let $p : I \to E$ be a smooth curve and set $\gamma = \Pi \circ p$, where $\Pi : \mathrm{FR}(E) \to M$ denotes the projection. Assume that $p(I) \cap P \neq \emptyset$. Then $p(I) \subset P$ if and only if:*

$$(2.11.8)\qquad \mathfrak{I}^P_{\gamma(t)}\big(\gamma'(t)\big) = (\nabla_1 p)(t) \circ p(t)^{-1} + \mathfrak{g}_{\gamma(t)},$$

for all $t \in I$.

PROOF. Since TP is invariant by the action of G in $T\big(\mathrm{FR}(E)\big)$, there exists a $\mathrm{GL}(\mathbb{R}^k)$-invariant smooth distribution $\mathcal{D}$ on the manifold $\mathrm{FR}(E)$ such that $\mathcal{D}_p = T_pP$, for all $p \in P$. Such distribution is integrable because $P \cdot g$ is an integral submanifold of $\mathcal{D}$, for all $g \in \mathrm{GL}(\mathbb{R}^k)$. For all $x \in M$ and all $p \in \mathrm{FR}(E_x)$, we define $\mathcal{L}_p : T_xM \to \mathfrak{gl}(\mathbb{R}^k)/\mathfrak{g}$ by setting $\mathcal{L}_p = \overline{\mathrm{Ad}}_p^{-1} \circ \mathfrak{I}_x^P$ and we define $\mathcal{V}_p$ by setting:

$$\mathcal{V}_p = \big\{(v, X) \in T_xM \oplus \mathfrak{gl}(\mathbb{R}^k) : \mathcal{L}_p(v) = X + \mathfrak{g}\big\}. \tag{2.11.9}$$

Clearly:

$$\mathcal{L}_{p\circ g} = \overline{\mathrm{Ad}}_{g^{-1}} \circ \mathcal{L}_p,$$

and therefore:

$$(\mathrm{Id} \oplus \mathrm{Ad}_{g^{-1}})(\mathcal{V}_p) = \mathcal{V}_{p\circ g}, \tag{2.11.10}$$

for all $p \in \mathrm{FR}(E)$ and all $g \in \mathrm{GL}(\mathbb{R}^k)$.

We claim that $(\mathrm{d}\Pi_p, \omega_p)(\mathcal{D}_p) = \mathcal{V}_p$, for all $p \in FR(E)$. Namely, by the definition of inner torsion, such equality holds for $p \in P$. The fact that the equality holds for any $p \in \mathrm{FR}(E)$ follows from (2.11.10) and from the fact that the diagram:

$$\begin{tikzcd}
T_{p\circ g}\mathrm{FR}(E) \arrow[r, "(\mathrm{d}\Pi_{p\circ g}, \omega_{p\circ g})", "\cong"'] & T_xM \oplus \mathfrak{gl}(\mathbb{R}^k) \\
T_p\mathrm{FR}(E) \arrow[u, "g"] \arrow[r, "\cong", "(\mathrm{d}\Pi_p, \omega_p)"'] & T_xM \oplus \mathfrak{gl}(\mathbb{R}^k) \arrow[u, "\mathrm{Id}\oplus\mathrm{Ad}_{g^{-1}}"']
\end{tikzcd}$$

commutes, for all $x \in M$, $p \in \mathrm{FR}(E_x)$ and all $g \in \mathrm{GL}(\mathbb{R}^k)$. Now:

$$(\mathrm{d}\Pi_{p(t)}, \omega_{p(t)})\big(p'(t)\big) = \big(\gamma'(t), p(t)^{-1} \circ (\nabla_1 p)(t)\big),$$

for all $t \in I$ and therefore p is tangent to $\mathcal{D}$ if and only if (2.11.8) holds. If $p(I) \subset P$ then, since P is an integral submanifold of $\mathcal{D}$, it follows that p is tangent to $\mathcal{D}$ and thus (2.11.8) holds. Conversely, assume that (2.11.8) holds and that $p(I) \cap P \neq \emptyset$. Since for all $g \in \mathrm{GL}(\mathbb{R}^k)$, $P \cdot g$ is an integral submanifold of $\mathcal{D}$, it follows that the set $p^{-1}(P \cdot g)$ is open in I; thus $p^{-1}(P)$ is both open and closed in I and the conclusion follows. $\square$

Exercises

The general concept of connection.

EXERCISE 2.1. Let V, W be vector spaces and let $T : V \to W$ be a linear map. Given a subspace Z of V, show that $V = Z \oplus \mathrm{Ker}(T)$ if and only if the map $T|_Z : Z \to T(V)$ is an isomorphism.

EXERCISE 2.2. Let V_1, V_2, V' be vector spaces and assume that we are given linear maps $T_1 : V_1 \to V'$, $T_2 : V_2 \to V'$, $L : V_1 \to V_2$ such that the diagram:

$$\begin{tikzcd} V_1 \arrow[rr, "L"] \arrow[dr, "T_1"'] & & V_2 \arrow[dl, "T_2"] \\ & V' & \end{tikzcd}$$

commutes and such that T_1 and T_2 have the same image (this is the case, for instance, if both T_1 and T_2 are surjective or if L is surjective). Let Z be a subspace of V_1 with $V_1 = Z \oplus \mathrm{Ker}(T_1)$. Show that the restriction of L to Z is injective and that $V_2 = L(Z) \oplus \mathrm{Ker}(T_2)$.

EXERCISE 2.3. Let W be a vector space and W_1, W_2, W_2' be subspaces of W such that $W = W_1 \oplus W_2$ and $W_1 \cap W_2' = \{0\}$. Show that $W_2 \subset W_2'$ if and only if $W_2 = W_2'$. Conclude that, under the hypotheses and notations of Exercise 2.2, if Z' is a subspace of V_2 with $V_2 = Z' \oplus \mathrm{Ker}(T_2)$ then $L(Z) \subset Z'$ if and only if $L(Z) = Z'$.

EXERCISE 2.4. Let V, W be vector spaces, $T : V \to W$ be a linear map and $V_0 \subset V$, $W_0 \subset W$ be subspaces such that $T|_{V_0} : V_0 \to W_0$ is an isomorphism. If H is a subspace of W with $W = H \oplus W_0$, show that:

$$V = T^{-1}(H) \oplus V_0.$$

EXERCISE 2.5. Let $\mathcal{E}$, M be differentiable manifolds, $\pi : \mathcal{E} \to M$ be a smooth submersion and $\epsilon : U \to E$ be a smooth local section of π. Show that for all $x \in U$, the image of $\mathrm{d}\epsilon(x)$ is a horizontal subspace of $T_{\epsilon(x)}\mathcal{E}$.

Connections on principal fiber bundles.

EXERCISE 2.6. Let $\Pi : P \to M$ be a G-principal bundle and let $M = \bigcup_{i \in I} U_i$ be an open cover of M. Assume that for every $i \in I$ it is given a connection $\mathrm{Hor}(P|_{U_i})$ on the principal bundle $P|_{U_i}$ and assume that for all $i, j \in I$ and all $x \in U_i \cap U_j$ we have $\mathrm{Hor}_x(P|_{U_i}) = \mathrm{Hor}_x(P|_{U_j})$. Show that there exists a unique connection $\mathrm{Hor}(P)$ on P such that $\mathrm{Hor}_x(P) = \mathrm{Hor}_x(P|_{U_i})$, for all $i \in I$ and all $x \in U_i$.

EXERCISE 2.7. Let $\Pi : P \to M$ be a G-principal bundle, V be a real finite-dimensional vector space and let $\rho : G \to \mathrm{GL}(V)$ be a smooth representation of G on V. Show that a V-valued differential form λ on P is ρ-pseudo G-invariant if and only if for every $x \in M$, there exists a point $p \in P_x$ such that:

$$(\gamma_g^* \lambda)_p = \rho(g)^{-1} \circ \lambda_p,$$

for all $g \in G$.

EXERCISE 2.8. Let P be a G-principal bundle endowed with a connection $\mathrm{Hor}(P)$ and denote by $\mathfrak{p}_{\mathrm{ver}} : TP \to \mathrm{Ver}(P)$, $\mathfrak{p}_{\mathrm{hor}} : TP \to \mathrm{Hor}(P)$ respectively the vertical and the horizontal projections determined by the horizontal distribution $\mathrm{Hor}(P)$. Given $g \in G$, $p \in P$, $\zeta \in T_pP$, show that:

$$\mathfrak{p}_{\mathrm{ver}}(\zeta \cdot g) = \mathfrak{p}_{\mathrm{ver}}(\zeta) \cdot g, \quad \mathfrak{p}_{\mathrm{hor}}(\zeta \cdot g) = \mathfrak{p}_{\mathrm{hor}}(\zeta) \cdot g.$$

EXERCISE 2.9. Let $\Pi : P \to M$ be a G-principal bundle and let ω be a Ad-pseudo G-invariant $\mathfrak{g}$-valued 1-form on P. Show that if for every $x \in M$ there exists $p \in P_x$ such that condition (2.2.5) holds then condition (2.2.5) holds for all $p \in P$.

EXERCISE 2.10. Let $\Pi : P \to M$ be a G-principal bundle, V be a real finite-dimensional vector space and $\rho : G \to \mathrm{GL}(V)$ be a smooth representation of G on V. Let λ^1, λ^2 be V-valued ρ-pseudo G-invariant k-forms on P and assume that:

$$\lambda^1_p(\zeta_1, \ldots, \zeta_k) = \lambda^2_p(\zeta_1, \ldots, \zeta_k),$$

for all $p \in P$, $\zeta_1, \ldots, \zeta_k \in T_pP$, provided that at least one of the vectors ζ_i is in $\mathrm{Ver}_p(P)$. Given a smooth local section $s : U \to P$ of P, show that if $s^*\lambda^1 = s^*\lambda^2$ then λ^1 and λ^2 are equal on $P|_U$.

EXERCISE 2.11. Let P, Q be principal bundles over the same differentiable manifold M, with structural groups G and H, respectively. Let $\phi : P \to Q$ be a morphism of principal bundles whose subjacent Lie group homomorphism is $\phi_0 : G \to H$. Denote by $\mathfrak{g}$, $\mathfrak{h}$ the Lie algebras of G and H respectively and by $\bar{\phi}_0 : \mathfrak{g} \to \mathfrak{h}$ the differential of ϕ_0 at the identity. For $p \in P$, $q \in Q$, denote by $\beta^P_p : G \to P$, $\beta^Q_q : H \to Q$ the maps given by action at p and by action at q, respectively; consider the linear isomorphisms:

$$\mathrm{d}\beta^P_p(1) : \mathfrak{g} \longrightarrow \mathrm{Ver}_p(P), \quad \mathrm{d}\beta^Q_q(1) : \mathfrak{h} \longrightarrow \mathrm{Ver}_q(Q).$$

Let ω be an $\mathfrak{h}$-valued 1-form on Q such that:

$$\omega_q|_{\mathrm{Ver}_q(Q)} = \big(\mathrm{d}\beta^Q_q(1)\big)^{-1},$$

for all $q \in Q$. Show that:

$$(\phi^*\omega)_p|_{\mathrm{Ver}_p(P)} = \bar{\phi}_0 \circ \big(\mathrm{d}\beta^P_p(1)\big)^{-1},$$

for all $p \in P$.

EXERCISE 2.12. Let P, Q be principal bundles over the same differentiable manifold M, with structural groups G and H, respectively. Let $\phi : P \to Q$ be a morphism of principal bundles and let $\phi_0 : G \to H$ denote its subjacent Lie group homomorphism. Let V be a real finite-dimensional vector space and let $\rho : H \to \mathrm{GL}(V)$ be a smooth representation of H on V. If λ is a ρ-pseudo H-invariant differential form on Q, show that $\phi^*\lambda$ is a $(\rho \circ \phi_0)$-pseudo G-invariant differential form on P.

Connections on vector bundles.

EXERCISE 2.13. Let V, W be vector spaces and $T : V \to W$ be a linear map. Given subspaces Z, Z' of V, show that $T(Z) = T(Z')$ if and only if $Z + \mathrm{Ker}(T) = Z' + \mathrm{Ker}(T)$.

EXERCISE 2.14. Let $\pi : E \to M$ be a vector bundle and ∇ be a connection on E. Given a smooth section $\epsilon \in \mathbf{\Gamma}(E)$ of E that vanishes on an open subset U of M, show that $\nabla_v\epsilon$ also vanishes, for all $v \in TM|_U$. Conclude that, if $\epsilon, \epsilon' \in \mathbf{\Gamma}(E)$

are equal on an open subset U of M then $\nabla_v \epsilon$ and $\nabla_v \epsilon'$ are also equal, for all $v \in TM|_U$.

EXERCISE 2.15. Let $\pi : E \to M$ be a vector bundle and ∇ be a connection on E.

- Given open subsets U, V of M with $V \subset U$, consider the connection ∇^U induced by ∇ on $E|_U$ and the connection $(\nabla^U)^V$ induced by ∇^U on $(E|_U)|_V = E|_V$. Show that $(\nabla^U)^V$ is the same as ∇^V.
- Let ∇' be another connection on E. If every point of M has an open neighborhood U in M such that $\nabla^U = \nabla'^U$, show that $\nabla = \nabla'$.

EXERCISE 2.16. Let ∇, ∇' be connections on a vector bundle $\pi : E \to M$. Assume that for all $x \in M$ and all $e \in E_x$ there exists a smooth local section $\epsilon : U \to E$ of E defined in an open neighborhood U of x in M such that $\epsilon(x) = e$ and:

$$\nabla_v \epsilon = \nabla'_v \epsilon,$$

for all $v \in T_x M$. Show that $\nabla = \nabla'$.

EXERCISE 2.17. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 endowed with a connection ∇ and let E_1 be a real vector space isomorphic to E_0. As we have seen in Exercise 1.61, $\pi : E \to M$ can be regarded also as a vector bundle with typical fiber E_1. Since the differential structure of E does not depend on the typical fiber, the space $\boldsymbol{\Gamma}(E)$ also doesn't depend on the typical fiber and hence ∇ is also a connection on the vector bundle $\pi : E \to M$ with typical fiber E_1. The connection ∇ is associated to connections on both principal bundles of frames $\mathrm{FR}_{E_0}(E)$ and $\mathrm{FR}_{E_1}(E)$. Show that:

- the horizontal distribution on E defined by ∇ does not depend on the typical fiber;
- for any linear isomorphism $\mathfrak{i} : E_1 \to E_0$, the isomorphism of principal bundles $\gamma_{\mathfrak{i}}$ defined in Exercise 1.61 is connection preserving.

Pull-back of connections on vector bundles.

EXERCISE 2.18. Assume that we are given a commutative diagram of sets and maps:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Given a subset S of B, show that $\phi\big(f^{-1}(S)\big) \subset g^{-1}\big(\psi(S)\big)$.

Functorial constructions with connections on vector bundles.

EXERCISE 2.19. Let $n \geq 1$ be fixed and let $\underline{\mathfrak{F}} : \underline{\mathfrak{Vec}}^n \to \underline{\mathfrak{Vec}}$ be a smooth functor. Let $E^1, \ldots, E^n$ be vector bundles over a differentiable manifold M with typical fibers $E_0^1, \ldots, E_0^n$, respectively. For each $i = 1, \ldots, n$, let $\overline{E}_0^i$ be a real vector space isomorphic to E_0^i. As we have seen in Exercise 1.61, the vector bundle E^i can also be regarded as a vector bundle with typical fiber $\overline{E}_0^i$; denote such vector bundle with changed typical fiber by $\overline{E}^i$. As we have seen in Exercise 1.68, the vector bundles $\underline{\mathfrak{F}}(E^1, \ldots, E^n)$ and $\underline{\mathfrak{F}}(\overline{E}^1, \ldots, \overline{E}^n)$ differ only by their typical fibers. For $i = 1, \ldots, n$, let ∇^i be a connection on E^i; then ∇^i is also a connection on $\overline{E}^i$ (recall Exercise 2.17). The reader should observe that the construction of the connection $\underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n)$ depends in principle not only on the connections ∇^i but also on the typical fibers of the vector bundles. Show that, in fact, the connection $\underline{\mathfrak{F}}(\nabla^1, \ldots, \nabla^n)$ does not depend on the typical fibers of the vector bundles involved.

The components of a linear connection.

EXERCISE 2.20. Let $\pi : E \to M$ be a vector bundle endowed with a connection ∇ and F be a vector subbundle of E. Denote by $\mathfrak{q} : E \to E/F$ the quotient map. Show that the map:

$$\mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(F) \ni (X, \epsilon) \longmapsto \mathfrak{q} \circ \nabla_X \epsilon \in \mathbf{\Gamma}(E/F)$$

is $C^\infty(M)$-bilinear. Conclude that there exists a smooth section α^F of $\mathrm{Lin}(TM, F; E/F)$ such that:

$$\nabla_v \epsilon + F_x = \alpha_x^F\big(v, \epsilon(x)\big) \in E_x/F_x,$$

for all $x \in M$, $v \in T_xM$. We call α^F the *second fundamental form* of the subbundle F.

EXERCISE 2.21. Let $\pi : E \to M$ be a vector bundle endowed with a semi-Riemannian structure g and a connection ∇ compatible with g. If R denotes the curvature tensor of ∇, show that for all $x \in M$, $v, w \in T_xM$, the linear operator $R_x(v, w) : E_x \ni e \mapsto R_x(v, w)e \in E_x$ is anti-symmetric with respect to g_x, i.e.:

$$g_x\big(R_x(v, w)e, e'\big) = -g_x\big(e, R_x(v, w)e'\big),$$

for all $e, e' \in E_x$.

EXERCISE 2.22. Let (M, g) be a semi-Riemannian manifold. Show that there exists a unique connection ∇ on M which is both symmetric and compatible with the semi-Riemannian metric g; such connection is defined by the equality:

$$g(\nabla_X Y, Z) = \tfrac{1}{2}\Big(X\big(g(Y, Z)\big) + Y\big(g(Z, X)\big) - Z\big(g(X, Y)\big) - g\big(X, [Y, Z]\big) + g\big(Y, [Z, X]\big) + g\big(Z, [X, Y]\big)\Big), \tag{2.11}$$

and is called the *Levi-Civita connection* of the semi-Riemannian manifold (M, g). Formula (2.11) is known as *Koszul formula*.

Relating connections with principal subbundles.

EXERCISE 2.23. Let $\Pi : P \to M$ be a G-principal bundle endowed with a connection $\mathrm{Hor}(P)$ and let Q be an H-principal subbundle of P; denote by ω the connection form of $\mathrm{Hor}(P)$. Show that the following conditions are equivalent:

- $\mathrm{Hor}_p(P) \subset T_pQ$, for all $p \in Q$;
- $T_pQ = \mathrm{Hor}_p(P) \oplus \mathrm{Ver}_p(Q)$, for all $p \in Q$;
- the 1-form $\omega|_Q$ takes values in $\mathfrak{h}$, i.e., $\omega_p(T_pQ) \subset \mathfrak{h}$, for all $p \in Q$;
- there exists a connection on the principal bundle Q such that the inclusion map $Q \to P$ is connection preserving;
- the isomorphism (2.10.1) carries T_pQ onto $T_xM \oplus \mathfrak{h}$, for all $x \in M$ and all $p \in Q_x$.

The inner torsion of a G-structure.

EXERCISE 2.24. Let $\pi : E \to M$ be a vector bundle with typical fiber E_0 endowed with a connection ∇, E_1 be a real vector space and $\mathfrak{i} : E_1 \to E_0$ be a linear isomorphism. Let G be a Lie subgroup of $\mathrm{GL}(E_0)$ and $P \subset \mathrm{FR}_{E_0}(E)$ be a G-structure on E. Then $\gamma_{\mathfrak{i}}(P) \subset \mathrm{FR}_{E_1}(E)$ is a $\mathcal{I}_{\mathfrak{i}}^{-1}(G)$-structure on E (see Exercises 1.61 and 1.47). Show that the inner torsion of P is equal to the inner torsion of $\gamma_{\mathfrak{i}}(P)$.

CHAPTER 3

Immersion theorems

3.1. Affine manifolds

DEFINITION 3.1.1. By a *connection* on a differentiable manifold M we mean a connection on its tangent bundle TM. An *affine manifold* is a pair (M, ∇), where M is a differentiable manifold and ∇ is a connection on M.

Affine geometry is the geometry of affine manifolds. This is a large class of manifolds containing in particular the class of *semi-Riemannian manifolds* (see Exercise 2.22).

DEFINITION 3.1.2. Let (M, ∇) be an affine manifold. A *geodesic* in M is a smooth curve $\gamma : I \to M$ such that $\gamma' : I \to TM$ is parallel.

Let $\mathcal{G} : TM \to TTM$ be the vector field on TM such that for all $v \in TM$, $\mathcal{G}(v) \in T_v TM$ is the unique horizontal vector such that $\mathrm{d}\pi_v\big(\mathcal{G}(v)\big) = v$. The vector field $\mathcal{G}$ is smooth and it is called the *geodesic vector field* of (M, ∇). Clearly, a curve $\tilde{\gamma} : I \to TM$ is an integral curve of $\mathcal{G}$ if and only if $\gamma = \Pi \circ \tilde{\gamma}$ is a geodesic and $\tilde{\gamma} = \gamma'$. If:

$$F^{\mathcal{G}} : \mathrm{Dom}(F^{\mathcal{G}}) \subset \mathbb{R} \times TM \longrightarrow TM$$

denotes the maximal flow of $\mathcal{G}$ then the map:

$$\exp : \big\{v \in TM : (1, v) \in \mathrm{Dom}(F^{\mathcal{G}})\big\} \ni v \longmapsto \pi\big(F^{\mathcal{G}}(1, v)\big) \in M$$

is called the *exponential map* of the affine manifold (M, ∇). Clearly the domain $\mathrm{Dom}(\exp)$ of $\exp$ is an open subset of TM and $\exp$ is a smooth map.

PROPOSITION 3.1.3. *Let (M, ∇) be an affine manifold. Then:*

(a) *for all $t_0 \in \mathbb{R}$ and all $v \in TM$, the curve:*

$$\gamma : \big\{t \in \mathbb{R} : (t - t_0)v \in \mathrm{Dom}(\exp)\big\} \ni t \longmapsto \exp\big((t - t_0)v\big) \in M$$

is a geodesic with $\gamma'(t_0) = v$;

(b) *if $\gamma : I \to M$ is a geodesic then for all $t_0 \in I$, $t \in I$, we have $\gamma(t) = \exp\big((t - t_0)v\big)$, where $v = \gamma'(t_0) \in TM$;*

(c) *given $x \in M$, if $\exp_x : \mathrm{Dom}(\exp) \cap T_x M \to M$ denotes the restriction of $\exp$ to $\mathrm{Dom}(\exp) \cap T_x M$ then $\exp_x(0) = x$ and $\mathrm{d}\exp_x(0)$ is the identity map of $T_x M$.*

PROOF. □

Observe that item (c) of Proposition 3.1.3 implies that for all $x \in M$, $\exp_x$ restricts to a smooth diffeomorphism of an open neighborhood of the origin in T_xM onto an open neighborhood of x in M.

DEFINITION 3.1.4. An affine manifold (M, ∇) is said to be *geodesically complete* if the domain of the exponential map is the whole tangent bundle TM, i.e., it for all $v \in TM$ there exists a geodesic $\gamma : \mathbb{R} \to M$ with $\gamma'(0) = v$.

DEFINITION 3.1.5. Let M', M be affine manifolds. An *affine map* from M' to M is a smooth map $f : M' \to M$ such that the morphism of vector bundles $\overleftarrow{\mathrm{d}f} : TM' \to f^*TM$ (recall Example 1.5.27) is connection preserving.

The composition of affine maps is an affine map. If a smooth diffeomorphism $f : M' \to M$ is an affine map then also $f^{-1} : M \to M'$ is an affine map (see Exercise 3.1).

3.2. Homogeneous affine manifolds

Let M be an affine manifold. The set $\mathrm{Aff}(M)$ of all affine smooth diffeomorphisms $f : M \to M$ is a subgroup of $\mathrm{Diff}(M)$. We have the following:

THEOREM 3.2.1. *Let M be an affine manifold with a finite number of connected components. Then the group $\mathrm{Aff}(M)$ admits a unique manifold structure such that $\mathrm{Aff}(M)$ is a Lie group and such that the topology of $\mathrm{Aff}(M)$ is the compact-open topology. Moreover, the canonical left action $\mathrm{Aff}(M) \times M \to M$ is smooth.*

PROOF. See [**8**]. $\square$

DEFINITION 3.2.2. An affine manifold is said to be *homogeneous* if the group $\mathrm{Aff}(M)$ acts transitively on M.

Let M be an homogeneous affine manifold having a finite number of connected components. If A is any Lie subgroup of $\mathrm{Aff}(M)$ that acts transitively on M (for instance, $A = \mathrm{Aff}(M)$) then, given $x_0 \in M$, we have a smooth diffeomorphism:

$$\bar{\beta}_{x_0} : A/A_{x_0} \ni gA_{x_0} \longmapsto g(x_0) \in M. \tag{3.2.1}$$

The manifold A/A_{x_0} can be endowed with a uniquely defined connection that makes (3.2.1) an affine diffeomorphism (see Exercise 3.2). Obviously such connection on A/A_{x_0} is A-*invariant*, i.e., for all $g \in A$, the smooth diffeomorphism:

$$\overline{L}_g : A/A_{x_0} \ni hA_{x_0} \longmapsto (gh)A_{x_0} \in A/A_{x_0}$$

is affine.

In the remainder of the section we the problem of determining the A-invariant connections on a manifold A/H, where A is a Lie group and H is a closed subgroup of A. Denote by $\mathfrak{a}$ and $\mathfrak{h}$, respectively, the Lie algebras of A and H. Let us fix an arbitrary subspace $\mathfrak{m}$ of $\mathfrak{a}$ with $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{m}$ and let us denote by $\mathfrak{p}_{\mathfrak{h}} : \mathfrak{a} \to \mathfrak{h}$, $\mathfrak{p}_{\mathfrak{m}} : \mathfrak{a} \to \mathfrak{m}$ the projections. Denote by $\mathfrak{q} : A \to A/H$ the quotient map and set $\bar{1} = \mathfrak{q}(1)$. The restriction of $\mathrm{d}\mathfrak{q}_1$ to $\mathfrak{m}$ is an isomorphism onto $T_{\bar{1}}(A/H)$; we will always identify $T_{\bar{1}}(A/H)$ with $\mathfrak{m}$ via such isomorphism. For $h \in H$, the linear isomorphism $\mathrm{Ad}_h : \mathfrak{a} \to \mathfrak{a}$ carries $\mathfrak{h}$ to $\mathfrak{h}$ and therefore we have an induced isomorphism $\overline{\mathrm{Ad}}_h : \mathfrak{m} \to \mathfrak{m}$ defined by:

$$\overline{\mathrm{Ad}}_h = \mathfrak{p}_{\mathfrak{h}} \circ \mathrm{Ad}_h|_{\mathfrak{m}}.$$

Notice that $\overline{\mathrm{Ad}} : H \to \mathrm{GL}(\mathfrak{m})$ is a smooth representation of H on $\mathfrak{m}$; we call $\overline{\mathrm{Ad}}$ the *isotropic representation* of H on $\mathfrak{m}$. The differential of $\overline{\mathrm{Ad}}$ at the identity will be denoted by $\overline{\mathrm{ad}} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m})$; we have:

$$\overline{\mathrm{ad}}_X(Y) = \mathfrak{p}_{\mathfrak{m}}\big([X, Y]\big),$$

for all $X \in \mathfrak{h}$, $Y \in \mathfrak{m}$. We have a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\mathcal{I}_h"] \arrow[d, "\mathfrak{q}"'] & A \arrow[d, "\mathfrak{q}"] \\ A/H \arrow[r, "\overline{L}_h"'] & A/H \end{tikzcd}$$

By differentiating such diagram we obtain:

$$\mathrm{d}\overline{L}_h(\bar{1}) = \overline{\mathrm{Ad}}_h, \tag{3.2.2}$$

for all $h \in H$.

Let $T(A/H)$ be the tangent bundle of A/H and consider the $\mathrm{GL}(\mathfrak{m})$-principal bundle[1] $\mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big)$. We have a smooth left action of A on $\mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big)$ defined by:

$$g \cdot p = \mathrm{d}\overline{L}_g(x) \circ p, \tag{3.2.3}$$

for all $x \in A/H$, $p \in \mathrm{FR}_{\mathfrak{m}}\big(T_x(A/H)\big)$ and all $g \in A$. We can therefore define a smooth left action of $A \times \mathrm{GL}(\mathfrak{m})$ on $\mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big)$ by setting:

$$(g, \tau) \cdot p = (g \cdot p) \circ \tau^{-1} = g \cdot (p \circ \tau^{-1}), \tag{3.2.4}$$

for all $p \in \mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big)$, $g \in A$, $\tau \in \mathrm{GL}(\mathfrak{m})$. Let:

$$\mathrm{Id}_{\mathfrak{m}} \in \mathrm{FR}_{\mathfrak{m}}\big(T_{\bar{1}}(A/H)\big)$$

denote the identity map of $\mathfrak{m}$ and $S \subset A \times \mathrm{GL}(\mathfrak{m})$ denote the isotropy group of $\mathrm{Id}_{\mathfrak{m}}$. Clearly, the action (3.2.4) is transitive and therefore we have a smooth diffeomorphism:

$$\begin{aligned} \Upsilon : \big(A \times \mathrm{GL}(\mathfrak{m})\big)/S \ni (g, \tau)S \longmapsto {} & (g, \tau) \cdot \bar{1} \\ & = \mathrm{d}\overline{L}_g(\bar{1}) \circ \tau^{-1} \in \mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big). \end{aligned} \tag{3.2.5}$$

---

[1] We may consider as the typical fiber of the tangent bundle $T(A/H)$ the space $\mathfrak{m}$ rather than $\mathbb{R}^n$ (see Exercise 1.61).

We have a commutative diagram:

$$
\begin{tikzcd}
\big(A\times\mathrm{GL}(\mathfrak{m})\big)/S \arrow[rr, "\Upsilon", "\cong"'] \arrow[dr, "\Pi'"'] & & \mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big) \arrow[dl, "\Pi"] \\
& A/H &
\end{tikzcd}
\tag{3.2.6}
$$

where Π' is defined by:

$$\Pi':\big(A\times\mathrm{GL}(\mathfrak{m})\big)/S\ni(g,\tau)S\longmapsto gH\in A/H. \tag{3.2.7}$$

Using (3.2.2), we get:

$$S=\mathrm{Gr}(\overline{\mathrm{Ad}})=\big\{(h,\overline{\mathrm{Ad}}_h):h\in H\big\}\subset H\times\mathrm{GL}(\mathfrak{m}).$$

The Lie algebra of S is therefore given by:

$$\mathfrak{s}=\mathrm{Gr}(\overline{\mathrm{ad}})=\big\{(X,\overline{\mathrm{ad}}_X):X\in\mathfrak{h}\big\}\subset\mathfrak{h}\oplus\mathfrak{gl}(\mathfrak{m}). \tag{3.2.8}$$

A connection on A/H is uniquely determined by a connection on the principal bundle $\mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big)$, which is determined by a smooth horizontal $\mathrm{GL}(\mathfrak{m})$-invariant distribution $\mathrm{Hor}\big[\mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big)\big]$; we denote by $\mathcal{D}$ the smooth distribution on the quotient $\big(A\times\mathrm{GL}(\mathfrak{m})\big)/S$ that corresponds to $\mathrm{Hor}\big[\mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big)\big]$ via the smooth diffeomorphism Υ. A connection on A/H is A-invariant if and only if the horizontal distribution on $\mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big)$ is A-invariant (see Exercise 3.3). Hence, the A-invariant connections of A/H are in one to one correspondence with the $\big(A\times\mathrm{GL}(\mathfrak{m})\big)$-invariant (necessarily smooth) distributions $\mathcal{D}$ on $\big(A\times\mathrm{GL}(\mathfrak{m})\big)/S$ that are horizontal with respect to Π'.

From equality (3.2.8) it follows that $\mathfrak{m}\oplus\mathfrak{gl}(\mathfrak{m})$ is a complement of $\mathfrak{s}$ in the space $\mathfrak{a}\oplus\mathfrak{gl}(\mathfrak{m})$. Therefore, the differential at the point $(\bar{1},\mathrm{Id}_{\mathfrak{m}})S$ of the quotient map:

$$A\times\mathrm{GL}(\mathfrak{m})\ni(g,\tau)\longmapsto(g,\tau)S\in\big(A\times\mathrm{GL}(\mathfrak{m})\big)/S$$

restricts to an isomorphism from $\mathfrak{m}\oplus\mathfrak{gl}(\mathfrak{m})$ to the tangent space of $\big(A\times\mathrm{GL}(\mathfrak{m})\big)/S$ at $(\bar{1},\mathrm{Id}_{\mathfrak{m}})S$; we will therefore identify this tangent space with $\mathfrak{m}\oplus\mathfrak{gl}(\mathfrak{m})$ via such isomorphism. Consider the isotropic representation

$$\overline{\overline{\mathrm{Ad}}}:S\longrightarrow\mathrm{GL}\big(\mathfrak{m}\oplus\mathfrak{gl}(\mathfrak{m})\big) \tag{3.2.9}$$

of S on $\mathfrak{m}\oplus\mathfrak{gl}(\mathfrak{m})$. A $\big(A\times\mathrm{GL}(\mathfrak{m})\big)$-invariant distribution $\mathcal{D}$ on $\big(A\times\mathrm{GL}(\mathfrak{m})\big)/S$ is uniquely determined by a subspace $\mathfrak{d}=\mathcal{D}_{(\bar{1},\mathrm{Id}_{\mathfrak{m}})S}$ of $\mathfrak{m}\oplus\mathfrak{gl}(\mathfrak{m})$ that is invariant under the isotropic representation (3.2.9) (see Exercise 3.5). It is easily computed that the isotropic representation (3.2.9) is given by:

$$\overline{\overline{\mathrm{Ad}}}_s:(X,\kappa)\longmapsto\big(\overline{\mathrm{Ad}}_h(X),\overline{\mathrm{Ad}}_h\circ\kappa\circ\overline{\mathrm{Ad}}_h^{-1}-\overline{\mathrm{ad}}_{\mathfrak{p}_{\mathfrak{h}}\mathrm{Ad}_h(X)}\big), \tag{3.2.10}$$

for all $X\in\mathfrak{m}$, $\kappa\in\mathfrak{gl}(\mathfrak{m})$, $h\in H$, where $s=(h,\overline{\mathrm{Ad}}_h)\in S$. The differential of $\overline{\overline{\mathrm{ad}}}:\mathfrak{s}\to\mathfrak{gl}\big(\mathfrak{m}\oplus\mathfrak{gl}(\mathfrak{m})\big)$ of (3.2.9) is given by:

$$\overline{\overline{\mathrm{ad}}}_\sigma:(X,\kappa)\longmapsto\big(\overline{\mathrm{ad}}_Y(X),[\overline{\mathrm{ad}}_Y,\kappa]-\overline{\mathrm{ad}}_{\mathfrak{p}_{\mathfrak{h}}[Y,X]}\big), \tag{3.2.11}$$

for all $X \in \mathfrak{m}$, $\kappa \in \mathfrak{gl}(\mathfrak{m})$, $Y \in \mathfrak{h}$, where $\sigma = (Y, \overline{\mathrm{ad}}_Y) \in \mathfrak{s}$. The differential of Π' at $(\bar{1}, \mathrm{Id}_{\mathfrak{m}})S$ is given by (see (3.2.7)):

$$\mathrm{d}\Pi'\big((\bar{1}, \mathrm{Id}_{\mathfrak{m}})S\big) : \mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{m}) \ni (X, \kappa) \longmapsto X \in \mathfrak{m}. \tag{3.2.12}$$

Thus, $\mathcal{D}$ is horizontal with respect to Π' if and only if $\mathfrak{d}$ is a complement of $\mathfrak{gl}(\mathfrak{m})$ in $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{m})$. We have the following:

PROPOSITION 3.2.3. *The A-invariant connections on A/H are in one to one correspondence with the linear maps $\lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ satisfying the condition:*

$$\overline{\mathrm{Ad}}_h \circ \lambda(X) \circ \overline{\mathrm{Ad}}_h^{-1} - \overline{\mathrm{ad}}_{\mathfrak{p}_{\mathfrak{h}} \mathrm{Ad}_h(X)} = \lambda\big(\overline{\mathrm{Ad}}_h(X)\big), \tag{3.2.13}$$

for all $h \in H$ and all $X \in \mathfrak{m}$. Condition (3.2.13) *implies:*

$$[\overline{\mathrm{ad}}_Y, \lambda(X)] - \overline{\mathrm{ad}}_{\mathfrak{p}_{\mathfrak{h}}[Y,X]} = \lambda\big(\overline{\mathrm{ad}}_Y(X)\big), \tag{3.2.14}$$

for all $X \in \mathfrak{m}$ and all $Y \in \mathfrak{h}$. If H is connected then condition (3.2.13) *is equivalent to condition* (3.2.14).

PROOF. A subspace $\mathfrak{d}$ of $\mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{m})$ is a complement of $\mathfrak{gl}(\mathfrak{m})$ if and only if it is the graph of a linear map $\lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$. Using (3.2.10), it is easily seen that $\mathrm{Gr}(\lambda)$ is invariant under the isotropic representation (3.2.9) if and only if condition (3.2.13) holds. The rest of the statement follows from (3.2.11) and from the result of Exercise 3.6. □

We will now compute the curvature and the torsion of an A-invariant connection ∇ on A/H in terms of the corresponding linear map λ. To this aim, recall that $\mathfrak{q} : A \to A/H$ is an H-principal bundle (see Example 1.3.4) and consider the morphism of principal bundles:

$$\phi : A \ni g \longmapsto \mathrm{d}\overline{L}_g(\bar{1}) \in \mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big) \tag{3.2.15}$$

whose subjacent Lie group homomorphism is the isotropic representation $\overline{\mathrm{Ad}} : H \to \mathrm{GL}(\mathfrak{m})$. Denote by ω and θ respectively the connection form and the canonical form on $\mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big)$. By the result of Exercise 3.4 (with $f = \overline{L}_g$), ω and θ are invariant by the action of A on $\mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big)$. It then follows that the differential forms $\phi^*\omega$ and $\phi^*\theta$ on A are left invariant. Clearly:

$$\begin{aligned}(\phi^*\Omega)_1(X,Y) &= \mathrm{d}(\phi^*\omega)_1(X,Y) + \tfrac{1}{2}\big((\phi^*\omega)_1 \wedge (\phi^*\omega)_1\big)(X,Y)\\ &= -(\phi^*\omega)_1\big([X,Y]\big) + [(\phi^*\omega)_1(X), (\phi^*\omega)_1(Y)],\end{aligned}$$

$$\begin{aligned}(\phi^*\Theta)_1(X,Y) &= \mathrm{d}(\phi^*\theta)_1(X,Y) + \big((\phi^*\omega)_1 \wedge (\phi^*\theta)_1\big)(X,Y)\\ &= -(\phi^*\theta)_1\big([X,Y]\big) + (\phi^*\omega)_1(X) \cdot (\phi^*\theta)_1(Y)\\ &\quad - (\phi^*\omega)_1(Y) \cdot (\phi^*\theta)_1(X),\end{aligned} \tag{3.2.16}$$

for all $X, Y \in \mathfrak{a}$. Our strategy is to compute $\phi^*\omega$, $\phi^*\theta$ and then use (3.2.16) to compute $\phi^*\Omega$ and $\phi^*\Theta$. From $\phi^*\Omega$ and $\phi^*\Theta$ the curvature and torsion tensor of the connection ∇ are easily computed. We have the following:

LEMMA 3.2.4. *The left invariant forms $\phi^*\omega$ and $\phi^*\theta$ on A are given by:*

$$(\phi^*\omega)_1(X) = \lambda\big(\mathfrak{p}_{\mathfrak{m}}(X)\big) + \overline{\mathrm{ad}}_{\mathfrak{p}_{\mathfrak{h}}(X)}, \tag{3.2.17}$$

$$(\phi^*\theta)_1(X) = \mathfrak{p}_{\mathfrak{m}}(X), \tag{3.2.18}$$

for all $X \in \mathfrak{a}$.

PROOF. We start by computing $\Upsilon^*\omega$ and $\Upsilon^*\theta$. Consider the diffeomorphism:

$$\beta_{\mathrm{Id}_{\mathfrak{m}}} : \mathrm{GL}(\mathfrak{m}) \ni \tau \longmapsto \tau \in \mathrm{FR}_{\mathfrak{m}}\big(T_{\bar{1}}(A/H)\big)$$

given by action at $\mathrm{Id}_{\mathfrak{m}}$. The differential of $\beta_{\mathrm{Id}_{\mathfrak{m}}}$ at $\mathrm{Id}_{\mathfrak{m}} \in \mathrm{GL}(\mathfrak{m})$ is an isomorphism:

$$\mathrm{d}\beta_{\mathrm{Id}_{\mathfrak{m}}}(\mathrm{Id}_{\mathfrak{m}}) : \mathfrak{gl}(\mathfrak{m}) \longrightarrow \mathrm{Ver}_{\mathrm{Id}_{\mathfrak{m}}}\big[\mathrm{FR}_{\mathfrak{m}}\big(T_{\bar{1}}(A/H)\big)\big]. \tag{3.2.19}$$

The restriction of ω to $\mathrm{Ver}_{\mathrm{Id}_{\mathfrak{m}}}\big[\mathrm{FR}_{\mathfrak{m}}\big(T_{\bar{1}}(A/H)\big)\big]$ is the inverse of (3.2.19). The isomorphism:

$$\mathrm{d}\Upsilon\big((1, \mathrm{Id}_{\mathfrak{m}})S\big) : \mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{m}) \to T_{\mathrm{Id}_{\mathfrak{m}}}\mathrm{FR}_{\mathfrak{m}}\big(T_{\bar{1}}(A/H)\big) \tag{3.2.20}$$

carries $\mathfrak{gl}(\mathfrak{m})$ (which is the kernel of (3.2.12)) to $\mathrm{Ver}_{\mathrm{Id}_{\mathfrak{m}}}\big[\mathrm{FR}_{\mathfrak{m}}\big(T_{\bar{1}}(A/H)\big)\big]$ (see (3.2.6)). The restriction of $\Upsilon^*\omega$ to $\mathfrak{gl}(\mathfrak{m})$ is equal to the composition of the restriction of (3.2.20) to $\mathfrak{gl}(\mathfrak{m})$ with the inverse of (3.2.19). Such composition is the differential at $(1, \mathrm{Id}_{\mathfrak{m}})S$ of the map:

$$\begin{aligned}\beta_{\mathrm{Id}_{\mathfrak{m}}}^{-1} \circ \Upsilon : \Pi'^{-1}(\bar{1}) = \big(H \times \mathrm{GL}(\mathfrak{m})\big)/S &\longrightarrow \mathrm{GL}(\mathfrak{m}) \\ (h, \tau)S &\longmapsto \overline{\mathrm{Ad}}_h \circ \tau^{-1}.\end{aligned}$$

This is computed easily as:

$$\Upsilon^*\omega_{(1,\mathrm{Id}_{\mathfrak{m}})S} : \mathfrak{gl}(\mathfrak{m}) \ni \kappa \longmapsto -\kappa \in \mathfrak{gl}(\mathfrak{m}).$$

The map (3.2.20) carries $\mathrm{Gr}(\lambda)$ to $\mathrm{Hor}_{\mathrm{Id}_{\mathfrak{m}}}\big[\mathrm{FR}_{\mathfrak{m}}\big(T_{\bar{1}}(A/H)\big)\big]$ and therefore $\Upsilon^*\omega$ vanishes on $\mathrm{Gr}(\lambda)$. This yields:

$$\Upsilon^*\omega_{(1,\mathrm{Id}_{\mathfrak{m}})S} : \mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{m}) \ni (X, \kappa) \longmapsto \lambda(X) - \kappa \in \mathfrak{gl}(\mathfrak{m}). \tag{3.2.21}$$

As to $\Upsilon^*\theta$, we have:

$$\Upsilon^*\theta_{(1,\mathrm{Id}_{\mathfrak{m}})S} = \mathrm{d}\Pi_{\mathrm{Id}_{\mathfrak{m}}} \circ \mathrm{d}\Upsilon_{(1,\mathrm{Id}_{\mathfrak{m}})S} = \mathrm{d}\Pi'_{\mathrm{Id}_{\mathfrak{m}}},$$

i.e.:

$$\Upsilon^*\theta_{(1,\mathrm{Id}_{\mathfrak{m}})S} : \mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{m}) \ni (X, \kappa) \longmapsto X \in \mathfrak{m}. \tag{3.2.22}$$

Let us now compute $\phi^*\omega$ and $\phi^*\theta$. We have:

$$\phi^*\omega = (\Upsilon^{-1} \circ \phi)^*\Upsilon^*\omega, \quad \phi^*\theta = (\Upsilon^{-1} \circ \phi)^*\Upsilon^*\theta, \tag{3.2.23}$$

where:

$$\Upsilon^{-1} \circ \phi : A \ni g \longmapsto (g, \mathrm{Id}_{\mathfrak{m}})S \in \big(A \times \mathrm{GL}(\mathfrak{m})\big)/S.$$

The differential of $\Upsilon^{-1} \circ \phi$ at $1 \in A$ is given by:

$$\mathrm{d}(\Upsilon^{-1} \circ \phi)_1 : \mathfrak{a} \ni X \longmapsto \big(\mathfrak{p}_{\mathfrak{m}}(X), -\overline{\mathrm{ad}}_{\mathfrak{p}_{\mathfrak{h}}(X)}\big) \in \mathfrak{m} \oplus \mathfrak{gl}(\mathfrak{m}). \tag{3.2.24}$$

The conclusion follows from (3.2.21), (3.2.22), (3.2.23) and (3.2.24). □

COROLLARY 3.2.5. *The left invariant forms $\phi^*\Omega$ and $\phi^*\Theta$ on A are given by:*

$$(\phi^*\Omega)_1(X,Y) = -(\lambda\circ\mathfrak{p}_\mathfrak{m})\big([X,Y]\big) - \overline{\mathrm{ad}}_{\mathfrak{p}_\mathfrak{h}[X,Y]} + \big[\lambda\big(\mathfrak{p}_\mathfrak{m}(X)\big) + \overline{\mathrm{ad}}_{\mathfrak{p}_\mathfrak{h}(X)}, \lambda\big(\mathfrak{p}_\mathfrak{m}(Y)\big) + \overline{\mathrm{ad}}_{\mathfrak{p}_\mathfrak{h}(Y)}\big], \tag{3.2.25}$$

$$\begin{aligned}(\phi^*\Theta)_1(X,Y) = {} & -\mathfrak{p}_\mathfrak{m}\big([X,Y]\big) + \lambda\big(\mathfrak{p}_\mathfrak{m}(X)\big)\cdot\mathfrak{p}_\mathfrak{m}(Y) \\ & - \lambda\big(\mathfrak{p}_\mathfrak{m}(Y)\big)\cdot\mathfrak{p}_\mathfrak{m}(X) \\ & + \overline{\mathrm{ad}}_{\mathfrak{p}_\mathfrak{h}(X)}\big(\mathfrak{p}_\mathfrak{m}(Y)\big) \\ & - \overline{\mathrm{ad}}_{\mathfrak{p}_\mathfrak{h}(Y)}\big(\mathfrak{p}_\mathfrak{m}(X)\big),\end{aligned} \tag{3.2.26}$$

for all $X, Y \in \mathfrak{a}$.

PROOF. Follows from Lemma 3.2.4 and from (3.2.16). $\square$

THEOREM 3.2.6. *Let A be a Lie group and H be a closed subgroup of A. Denote by $\mathfrak{a}$, $\mathfrak{h}$ respectively the Lie algebras of A and H; let $\mathfrak{m}$ be an arbitrary subspace of $\mathfrak{a}$ with $\mathfrak{a} = \mathfrak{h}\oplus\mathfrak{m}$ and denote by $\mathfrak{p}_\mathfrak{h} : \mathfrak{a}\to\mathfrak{h}$, $\mathfrak{p}_\mathfrak{m} : \mathfrak{a}\to\mathfrak{m}$ the projections. Let $\lambda : \mathfrak{m}\to\mathfrak{gl}(\mathfrak{m})$ be a linear map satisfying condition* (3.2.13) *and let the manifold A/H be endowed with the A-invariant connection ∇ whose horizontal space $\mathrm{Hor}_{\mathrm{Id}_\mathfrak{m}}\big[\mathrm{FR}_\mathfrak{m}\big(T(A/H)\big)\big]$ is the image under $\mathrm{d}\Upsilon$ (see* (3.2.5)*) of the graph of λ. Then, the curvature and torsion tensors of ∇ at the point $\bar{1} = 1\cdot H\in A/H$ are given by:*

$$\begin{aligned} R_{\bar 1} : \mathfrak{m}\times\mathfrak{m}\ni(X,Y) &\longmapsto [\lambda(X),\lambda(Y)] - \overline{\mathrm{ad}}_{\mathfrak{p}_\mathfrak{h}[X,Y]} \\ &\qquad - (\lambda\circ\mathfrak{p}_\mathfrak{m})\big([X,Y]\big)\in\mathfrak{gl}(\mathfrak{m}), \\ T_{\bar 1} : \mathfrak{m}\times\mathfrak{m}\ni(X,Y) &\longmapsto -\mathfrak{p}_\mathfrak{m}\big([X,Y]\big) + \lambda(X)\cdot Y - \lambda(Y)\cdot X\in\mathfrak{m},\end{aligned}$$

where we identify $\mathfrak{m}$ with $T_{\bar 1}(A/H)$ by the differential of the quotient map $A\to A/H$.

PROOF. By Lemma 2.9.16 and Corollary 2.9.18, we have:

$$R_{\bar 1}(X,Y) = \Omega(\zeta_1,\zeta_2),\quad T(X,Y) = \Theta(\zeta_1,\zeta_2),$$

where $\zeta_1,\zeta_2\in T_{\mathrm{Id}_\mathfrak{m}}\mathrm{FR}_\mathfrak{m}\big(T_{\bar 1}(A/H)\big)$ are chosen with:

$$\mathrm{d}\Pi(\zeta_1) = X,\quad \mathrm{d}\Pi(\zeta_2) = Y. \tag{3.2.27}$$

If ϕ is defined by (3.2.15) then $\zeta_1 = \mathrm{d}\phi_1(X)$, $\zeta_2 = \mathrm{d}\phi_1(Y)$ satisfy (3.2.27); thus:

$$R_{\bar 1}(X,Y) = (\phi^*\Omega)_1(X,Y),\quad T_{\bar 1}(X,Y) = (\phi^*\Theta)_1(X,Y),$$

for all $X, Y\in\mathfrak{m}$. The conclusion follows from Corollary 3.2.5. $\square$

3.3. Homogeneous affine manifolds with G-structure

DEFINITION 3.3.1. Let M be an n-dimensional differentiable manifold, G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$ and assume that M is endowed with a connection ∇ and a G-structure $P\subset\mathrm{FR}(TM)$. The triple (M,∇,P) is said to be a *homogeneous affine manifold with G-structure* if for every $x, y\in M$ and every $p\in P_x$, $q\in P_y$, there exists a smooth affine G-structure preserving diffeomorphism $f : M \to M$ such that $f(x) = y$ and $\mathrm{d}f(x) \circ p = q$.

Given an affine manifold M endowed with a G-structure P we denote by $\mathrm{Aff}_G(M)$ the subgroup of $\mathrm{Aff}(M)$ consisting of G-structure preserving affine diffeomorphisms of M. We have the following:

PROPOSITION 3.3.2. *Let (M, ∇) be a connected affine manifold with G-structure P and assume that (M, ∇, P) is homogeneous. Then $\mathrm{Aff}_G(M)$ is a Lie subgroup of $\mathrm{Aff}(M)$.*

PROOF. Since M is connected, the action of $\mathrm{Aff}(M)$ on $\mathrm{FR}(TM)$ is free; given $p \in P$, the orbit $\mathrm{Aff}(M)p \subset \mathrm{FR}(TM)$ is an almost embedded submanifold of $\mathrm{FR}(TM)$, since it is an integral submanifold of a smooth involutive distribution on $\mathrm{FR}(TM)$. The assumption that (M, ∇, P) is homogeneous implies that $P = \mathrm{Aff}_G(M)p$. Since P is an immersed submanifold of $\mathrm{FR}(TM)$ contained in the almost embedded submanifold $\mathrm{Aff}(M)p$ then P is also an immersed submanifold of $\mathrm{Aff}(M)p$. The smooth diffeomorphism $\beta_p : \mathrm{Aff}(M) \to \mathrm{Aff}(M)p$ carries $\mathrm{Aff}_G(M)$ to P and thus $\mathrm{Aff}_G(M)$ is an immersed submanifold of $\mathrm{Aff}(M)$. The conclusion follows. □

Let (M, ∇) be a connected affine manifold with G-structure P and assume that (M, ∇, P) is homogeneous. Set $A = \mathrm{Aff}_G(M)$, so that A is a Lie group and the left action of A on M is smooth and transitive. Given a point $x_0 \in M$, then we have a smooth diffeomorphism $\bar\beta_{x_0}$ from A/A_{x_0} to M (see (3.2.1)) and A/A_{x_0} is endowed with a unique connection that makes $\bar\beta_{x_0}$ an affine diffeomorphism; such connection is A-invariant. Moreover, A/A_{x_0} is endowed with a unique G-structure that makes $\bar\beta_{x_0}$ G-structure preserving (see Exercise 1.74); such G-structure is given by:

$$\big\{\mathrm{d}(\bar\beta_{x_0}^{-1}) \circ p : p \in P\big\}. \tag{3.3.1}$$

Since (M, ∇, P) is homogeneous, if we fix $p_0 \in P_{x_0}$, then:

$$P = \big\{\mathrm{d}\gamma_g \circ p_0 : g \in A\big\}. \tag{3.3.2}$$

Since $\bar\beta_{x_0}^{-1} \circ \gamma_g = \overline{L}_g \circ \bar\beta_{x_0}^{-1}$, it follows from (3.3.2) that the G-structure (3.3.1) on A/A_{x_0} is equal to:

$$\big\{\mathrm{d}\overline{L}_g(\bar 1) \circ \mathrm{d}(\bar\beta_{x_0}^{-1}) \circ p_0 : g \in A\big\}. \tag{3.3.3}$$

Setting $i_0 = \mathrm{d}(\bar\beta_{x_0}^{-1}) \circ p_0 : \mathbb{R}^n \to \mathfrak{a}/\mathfrak{a}_{x_0}$ then (3.3.3) is just the orbit of $i_0 \in \mathrm{FR}\big(T_{\bar 1}(A/A_{x_0})\big)$ under the action of A.

We now consider the following setup:

- a Lie group A with Lie algebra $\mathfrak{a}$;
- a closed Lie subgroup H of A with Lie algebra $\mathfrak{h}$;
- a complement $\mathfrak{m}$ of $\mathfrak{h}$ on $\mathfrak{a}$ (as usual, we identify $T_{\bar 1}(A/H)$ with $\mathfrak{m}$);
- an A-invariant connection ∇ on A/H corresponding to a linear map $\lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ as in Proposition 3.2.3;
- a linear isomorphism $\mathfrak{i} : \mathbb{R}^n \to \mathfrak{m}$.

Consider the isotropic representation $\overline{\mathrm{Ad}} : H \to \mathrm{GL}(\mathfrak{m})$, the group isomorphism $\mathcal{I}_{\mathfrak{i}} : \mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(\mathfrak{m})$ defined by $\mathcal{I}_{\mathfrak{i}}(\tau) = \mathfrak{i} \circ \tau \circ \mathfrak{i}^{-1}$, for all $\tau \in \mathrm{GL}(\mathfrak{m})$ and set:

$$G^{\mathfrak{i}} = \mathcal{I}_{\mathfrak{i}}^{-1}\big(\overline{\mathrm{Ad}}(H)\big) \subset \mathrm{GL}(\mathbb{R}^n).$$

Consider the smooth left action of A on $\mathrm{FR}\big(T(A/H)\big)$ defined as in (3.2.3) and let $P^{\mathfrak{i}} \subset \mathrm{FR}\big(T(A/H)\big)$ be the A-orbit of $\mathfrak{i}$. Then $P^{\mathfrak{i}}$ is a $G^{\mathfrak{i}}$-structure on A/H (see Exercise 1.73). The group $G^{\mathfrak{i}}_{\bar{1}}$ of all $G^{\mathfrak{i}}$-structure preserving endomorphisms of $T_{\bar{1}}(A/H) = \mathfrak{m}$ is just $\overline{\mathrm{Ad}}(H)$; its Lie algebra $\mathfrak{g}^{\mathfrak{i}}_{\bar{1}}$ is thus equal to $\overline{\mathrm{ad}}(\mathfrak{h})$.

We will now determine the inner torsion $\mathfrak{I}^{P^{\mathfrak{i}}}_{\bar{1}} : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})/\overline{\mathrm{ad}}(\mathfrak{h})$ of the $G^{\mathfrak{i}}$-structure $P^{\mathfrak{i}}$ on A/H. By the result of Exercise 2.24, we may as well compute the inner torsion of the G-structure:

$$P = \gamma_{\mathfrak{i}}^{-1}(P^{\mathfrak{i}}) = \big\{\mathrm{d}\overline{L}_g : g \in A\big\} \subset \mathrm{FR}_{\mathfrak{m}}\big(T(A/H)\big), \tag{3.3.4}$$

where $G = \mathcal{I}_{\mathfrak{i}}(G^{\mathfrak{i}}) = \overline{\mathrm{Ad}}(H)$. Notice that P is just the image of the morphism of principal spaces ϕ defined in (3.2.15).

PROPOSITION 3.3.3. *Let ∇ be an A-invariant connection on A/H corresponding to a linear map λ, as in Proposition 3.2.3. The inner torsion $\mathfrak{I}^{P^{\mathfrak{i}}}_{\bar{1}} : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})/\overline{\mathrm{ad}}(\mathfrak{h})$ of the $\overline{\mathrm{Ad}}(H)$-structure* (3.3.4) *on A/H is equal to the composition of $\lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ with the quotient map $\mathfrak{gl}(\mathfrak{m}) \to \mathfrak{gl}(\mathfrak{m})/\overline{\mathrm{ad}}(\mathfrak{h})$.*

PROOF. Let $s : U \to A$ be a smooth local section of the quotient map $A \to A/H$ with $\bar{1} \in U$, $s(\bar{1}) = 1$ and $\mathrm{d}s_{\bar{1}}(\mathfrak{m}) = \mathfrak{m}$; notice that $\mathrm{d}s_{\bar{1}}$ is just the inclusion map of $\mathfrak{m}$ in $\mathfrak{a}$. Notice that $\phi \circ s$ is a smooth local section of $P \to A/H$. Set $\bar{\omega} = (\phi \circ s)^*\omega$, where ω denotes the connection form of ∇. By diagram (2.10.2), in order to conclude the proof, it suffices to show that $\bar{\omega}_{\bar{1}}$ is equal to λ. We have $\bar{\omega} = s^*(\phi^*\omega)$ and therefore:

$$\bar{\omega}_{\bar{1}} = (\phi^*\omega)_1 \circ \mathrm{d}s_{\bar{1}}.$$

The conclusion follows directly from (3.2.17). □

3.4. Affine immersions in homogeneous spaces

Let M be an n-dimensional differentiable manifold, G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$ and assume that M is endowed with a connection ∇ and a G-structure $P \subset \mathrm{FR}(TM)$. For each $x \in M$ we denote by G_x the Lie subgroup of $\mathrm{GL}(T_xM)$ consisting of G-structure preserving endomorphisms of T_xM, by $\mathfrak{g}_x \subset \mathfrak{gl}(T_xM)$ the Lie algebra of G_x and by $\mathfrak{I}^P_x : T_xM \to \mathfrak{gl}(T_xM)/\mathfrak{g}_x$ the inner torsion of the G-structure P (recall Section 1.8). The triple (M, ∇, P) will be called an *affine manifold with G-structure*. Given points $x, y \in M$ and a G-structure preserving map $\sigma : T_xM \to T_yM$ then the Lie group isomorphism $\mathcal{I}_\sigma : \mathrm{GL}(T_xM) \to \mathrm{GL}(T_yM)$ defined by:

$$\mathcal{I}_\sigma : \mathrm{GL}(T_xM) \ni T \longmapsto \sigma \circ T \circ \sigma^{-1} \in \mathrm{GL}(T_yM)$$

carries G_x onto G_y. Its differential at the identity $\mathrm{Ad}_\sigma : \mathfrak{gl}(T_xM) \to \mathfrak{gl}(T_yM)$ carries $\mathfrak{g}_x$ onto $\mathfrak{g}_y$ and therefore it induces a linear isomorphism

$$\overline{\mathrm{Ad}}_\sigma : \mathfrak{gl}(T_xM)/\mathfrak{g}_x \longrightarrow \mathfrak{gl}(T_yM)/\mathfrak{g}_y.$$

DEFINITION 3.4.1. Let V, V' be real vector spaces and $\sigma : V \to V'$ be a linear isomorphism. Given a multilinear map $B' \in \mathrm{Lin}_k(V', V')$ then the *pull-back* of B' by σ is the multilinear map $\sigma^* B \in \mathrm{Lin}_k(V, V)$ defined by:

$$(\sigma^* B)(v_1, \ldots, v_k) = \sigma^{-1}\big[B\big(\sigma(v_1), \ldots, \sigma(v_k)\big)\big],$$

for all $v_1, \ldots, v_k \in V$. Given multilinear maps $B \in \mathrm{Lin}_k(V, V)$, $B' \in \mathrm{Lin}_k(V', V')$ and a (not necessarily invertible) linear map $\sigma : V \to V'$ then B is said to be *σ-related* with B' if:

$$B'\big(\sigma(v_1), \ldots, \sigma(v_k)\big) = \sigma\big(B(v_1, \ldots, v_k)\big), \tag{3.4.1}$$

for all $v_1, \ldots, v_k \in V$. More generally, if $V_1, \ldots, V_k$ are subspaces of V and if $B \in \mathrm{Lin}(V_1, \ldots, V_k; V)$, $B' \in \mathrm{Lin}_k(V', V')$ are multilinear maps then B is said to be *σ-related* with B' if (3.4.1) holds for all $v_1 \in V_1, \ldots, v_k \in V_k$.

Clearly, if $\sigma : V \to V'$ is a linear isomorphism and if $B' \in \mathrm{Lin}_k(V', V')$ then $\sigma^* B'$ is the only multilinear map B in $\mathrm{Lin}_k(V, V)$ that is σ-related with B'.

DEFINITION 3.4.2. Let M be an n-dimensional differentiable manifold, $\overline{M}$ be an $\bar{n}$-dimensional differentiable manifold and let $\pi : E \to M$ be a vector bundle over M with typical fiber $\mathbb{R}^k$, where $\bar{n} = n + k$. Set $\widehat{E} = TM \oplus E$, so that $\widehat{E}$ is a vector bundle over M with typical fiber $\mathbb{R}^{\bar{n}}$. Let $\widehat{\nabla}$ and $\overline{\nabla}$ be connections on $\widehat{E}$ and on $T\overline{M}$ respectively. By an *affine immersion* of $(M, E, \widehat{\nabla})$ into the affine manifold $(\overline{M}, \overline{\nabla})$ we mean a pair (f, L), where $f : M \to \overline{M}$ is a smooth map, $L : \widehat{E} \to f^* T\overline{M}$ is a connection preserving vector bundle isomorphism and:

$$L_x|_{T_xM} = \mathrm{d}f_x, \tag{3.4.2}$$

for all $x \in M$, where $f^* T\overline{M}$ is endowed with the connection $f^*\overline{\nabla}$. By a *local affine immersion* of $(M, E, \widehat{\nabla})$ into $(\overline{M}, \overline{\nabla})$ we mean an affine immersion (f, L) of $(U, E|_U, \widehat{\nabla})$ into $(\overline{M}, \overline{\nabla})$, where U is an open subset of M; we call U the *domain* of the local affine immersion (f, L).

Observe that if (f, L) is a (local) affine immersion, condition (3.4.2) implies that f is an immersion.

There exists in the literature a notion of affine immersion between affine manifolds (see [**11**, Definition 1.1, Chapter II]). Using our terminology, such notion of affine immersion is:

DEFINITION 3.4.3. Given affine manifolds (M, ∇), $(\overline{M}, \overline{\nabla})$ then a smooth map $f : M \to \overline{M}$ is said to be an *affine immersion* of (M, ∇) into $(\overline{M}, \overline{\nabla})$ if there exists a vector bundle $\pi : E \to M$, a connection $\widehat{\nabla}$ on $\widehat{E} = TM \oplus E$ and a vector bundle isomorphism $L : \widehat{E} \to f^* T\overline{M}$ such that (f, L) is an affine immersion of

$(M, E, \widehat{\nabla})$ into $(\overline{M}, \overline{\nabla})$ and such that ∇ is a component of $\widehat{\nabla}$ with respect to the decomposition $\widehat{E} = TM \oplus E$, i.e.:

$$\nabla_X Y = \mathrm{pr}_1(\widehat{\nabla}_X Y),$$

for all $X, Y \in \mathbf{\Gamma}(TM)$, where $\mathrm{pr}_1 : \widehat{E} \to TM$ denotes the first projection.

LEMMA 3.4.4. *Fix objects M, $\overline{M}$, $\pi : E \to M$, $\widehat{E}$, $\widehat{\nabla}$ and $\overline{\nabla}$ as in Definition 3.4.2. Let $s : U \to \mathrm{FR}(\widehat{E})$ be a smooth local frame of $\widehat{E}$, $f : U \to \overline{M}$ be a map and $L : \widehat{E}|_U \to f^*T\overline{M}$ be a bijective fiberwise linear map. Define $F : U \to \mathrm{FR}(T\overline{M})$ by setting:*

$$F(x) = L_x \circ s(x) \in \mathrm{FR}(T_{f(x)}\overline{M}), \tag{3.4.3}$$

for all $x \in U$. Denote by $\omega^{\overline{M}}$ the connection form on $\mathrm{FR}(T\overline{M})$ corresponding to the connection $\mathrm{Hor}\big(\mathrm{FR}(T\overline{M})\big)$ associated to $\overline{\nabla}$ and by ω^M the connection form on $\mathrm{FR}(\widehat{E})$ corresponding to the connection $\mathrm{Hor}\big(\mathrm{FR}(\widehat{E})\big)$ associated to $\widehat{\nabla}$. Denote also by $\theta^{\overline{M}}$ the canonical form of $\mathrm{FR}(T\overline{M})$ and by θ^M the ι-canonical form of $\mathrm{FR}(\widehat{E})$, where $\iota : TM \to \widehat{E}$ denotes the inclusion map. Then (f, L) is a local affine immersion with domain U if and only if the map F is smooth and:

$$F^*\theta^{\overline{M}} = s^*\theta^M, \tag{3.4.4}$$

$$F^*\omega^{\overline{M}} = s^*\omega^M. \tag{3.4.5}$$

PROOF. Denote by $L_* : \mathrm{FR}(\widehat{E}) \to \mathrm{FR}(f^*T\overline{M}) = f^*\mathrm{FR}(T\overline{M})$ the map induced by L and by $\bar{f} : f^*\mathrm{FR}(T\overline{M}) \to \mathrm{FR}(T\overline{M})$ the canonical map of the pull-back $f^*\mathrm{FR}(T\overline{M})$. Clearly:

$$F = \bar{f} \circ L_* \circ s. \tag{3.4.6}$$

We claim that F is smooth if and only if both f and L are smooth. Namely, if both f and L are smooth then equality (3.4.6) implies that F is smooth. Conversely, if F is smooth then f is also smooth, since $f = \overline{\Pi} \circ F$, where $\overline{\Pi} : \mathrm{FR}(T\overline{M}) \to \overline{M}$ denotes the projection. Moreover, F is a local section of $\mathrm{FR}(T\overline{M})$ along f and:

$$L_* \circ s = \overleftarrow{F},$$

so that $L_* \circ s$ is smooth by Corollary 1.3.19. Since s is an atlas of local sections for the principal bundle $\mathrm{FR}(\widehat{E})|_U$, it follows from the result of Exercise 1.45 that $L_* : \mathrm{FR}(\widehat{E})|_U \to \mathrm{FR}(f^*T\overline{M})$ is a (smooth) isomorphism of principal bundles whose subjacent Lie group homomorphism is the identity map of $\mathrm{GL}(\mathbb{R}^n)$. Hence L is smooth by Lemma 1.5.18.

Now, assuming that F, f and L are smooth, we prove that L is connection preserving if and only if (3.4.5) holds. Recall from (c) of Lemma 2.5.10 that L is connection preserving if and only if $L_* : \mathrm{FR}(\widehat{E}) \to \mathrm{FR}(f^*T\overline{M})$ is connection preserving. By definition, the connection form of the pull-back $\mathrm{FR}(f^*T\overline{M}) =$

$f^*\mathrm{FR}(T\overline{M})$ is equal to $\bar{f}^*\omega^{\overline{M}}$; thus, by part (d) of Lemma 2.2.11, L_* is connection preserving if and only if:

$$(L_* \circ s)^*(\bar{f}^*\omega^{\overline{M}}) = s^*\omega^M. \tag{3.4.7}$$

But (3.4.7) is obviously the same as (3.4.5), by (3.4.6).

Finally, let us prove that $L_x|_{T_xM} = \mathrm{d}f_x$ for all $x \in U$ if and only if (3.4.4) holds. Using (2.9.12), we see that (3.4.4) holds if and only if:

$$F(x)^{-1} \circ \mathrm{d}\overline{\Pi}_{F(x)} \circ \mathrm{d}F_x = s(x)^{-1} \circ \iota_x, \tag{3.4.8}$$

for all $x \in U$. Since $\overline{\Pi} \circ F = f$, we see that (3.4.8) holds if and only if:

$$F(x)^{-1} \circ \mathrm{d}f_x = s(x)^{-1}|_{T_xM}, \tag{3.4.9}$$

for all $x \in U$. Finally, since $F(x) = L_x \circ s(x)$, it is clear that (3.4.9) holds if and only if $L_x|_{T_xM} = \mathrm{d}f_x$. This concludes the proof. □

COROLLARY 3.4.5 (uniqueness of affine immersions with initial data). *Let M, $\overline{M}$, $\pi : E \to M$, $\widehat{E}$, $\widehat{\nabla}$ and $\overline{\nabla}$ be as in Definition 3.4.2; assume that M is connected. If (f^1, L^1), (f^2, L^2) are both affine immersions of $(M, E, \widehat{\nabla})$ into $(\overline{M}, \overline{\nabla})$ and if there exists $x_0 \in M$ with:*

$$f^1(x_0) = f^2(x_0), \quad L^1_{x_0} = L^2_{x_0},$$

then $(f^1, L^1) = (f^2, L^2)$.

PROOF. Denote by $\bar{f}^i : (f^i)^*T\overline{M} \to T\overline{M}$ the canonical map of the pull-back $(f^i)^*T\overline{M}$, $i = 1, 2$. Clearly $(f^1, L^1) = (f^2, L^2)$ if and only if the maps:

$$\bar{f}^1 \circ L^1 : M \longrightarrow T\overline{M}, \quad \bar{f}^2 \circ L^2 : M \longrightarrow T\overline{M} \tag{3.4.10}$$

are equal. The set of points of M where the maps (3.4.10) coincide is obviously closed and, by our hypotheses, nonempty. Let us check that such set is also open. Let $x \in M$ be a point at which the maps (3.4.10) coincide. Let $s : U \to \mathrm{FR}(\widehat{E})$ be a smooth local frame of $\widehat{E}$ where U is a connected open neighborhood of x in M. For $i = 1, 2$, define $F^i : U \to \mathrm{FR}(T\overline{M})$ by setting $F^i(y) = L^i_y \circ s(y)$, for all $y \in U$. Then $F^1(x) = F^2(x)$ and Lemma 3.4.4 implies that F^i is a smooth map satisfying:

$$(F^i)^*(\theta^{\overline{M}}, \omega^{\overline{M}}) = (s^*\theta^M, s^*\omega^M),$$

for $i = 1, 2$. Since for each $p \in \mathrm{FR}(T\overline{M})$, the linear map:

$$(\theta^{\overline{M}}_p, \omega^{\overline{M}}_p) : T_p\mathrm{FR}(T\overline{M}) \longrightarrow \mathbb{R}^{\bar{n}} \oplus \mathfrak{gl}(\mathbb{R}^{\bar{n}})$$

is an isomorphism (recall (2.11.7)) then Lemma A.4.9 implies that $F^1 = F^2$. Hence the maps (3.4.10) coincide in U and we are done. □

DEFINITION 3.4.6. An affine manifold with G-structure (M, ∇, P) is said to be *infinitesimally homogeneous* if for all $x, y \in M$ and all G-structure preserving map $\sigma : T_xM \to T_yM$, the following conditions hold:

- $\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^P_x = \mathfrak{I}^P_y \circ \sigma$;
- T_x is σ-related with T_y;

- R_x is σ-related with R_y.

The condition of infinitesimal homogeneity means that curvature, torsion and inner torsion are constant with respect to frames that are in the G-structure. This statement is made more precise in the following:

LEMMA 3.4.7. *Let (M, ∇, P) be an n-dimensional affine manifold with G-structure, where G is a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$. Then (M, ∇, P) is infinitesimally homogeneous if and only if there exists multilinear maps $R_0 \in \mathrm{Lin}_3(\mathbb{R}^n, \mathbb{R}^n)$, $T_0 \in \mathrm{Lin}_2(\mathbb{R}^n, \mathbb{R}^n)$ and a linear map $\mathfrak{I}_0 : \mathbb{R}^n \to \mathfrak{gl}(\mathbb{R}^n)/\mathfrak{g}$ such that:*

$$\begin{gathered} p^* R_x = R_0, \quad p^* T_x = T_0, \\ \overline{\mathrm{Ad}}_p \circ \mathfrak{I}_0 = \mathfrak{I}^P_x \circ p, \end{gathered} \tag{3.4.11}$$

for all $x \in M$ and all $p \in P_x$.

PROOF. Assume the existence of R_0, T_0, $\mathfrak{I}_0$ such that (3.4.11) holds for all $x \in M$ and all $p \in P_x$. Let $x, y \in M$ and a G-structure preserving map $\sigma : T_xM \to T_yM$ be fixed. Choose any $p \in P_x$ and set $q = \sigma \circ p$, so that $q \in P_y$. Then:

$$p^* R_x = R_0 = q^* R_y = p^* \sigma^* R_x,$$

and then $R_x = \sigma^* R_y$, i.e., R_x is σ-related with R_y. Similarly, T_x is σ-related with T_y. Moreover $\overline{\mathrm{Ad}}_p \circ \mathfrak{I}_0 = \mathfrak{I}^P_x \circ p$, $\overline{\mathrm{Ad}}_q \circ \mathfrak{I}_0 = \mathfrak{I}^P_y \circ q$ and therefore:

$$\mathfrak{I}^P_y \circ \sigma \circ p = \mathfrak{I}^P_y \circ q = \overline{\mathrm{Ad}}_q \circ \mathfrak{I}_0 = \overline{\mathrm{Ad}}_\sigma \circ \overline{\mathrm{Ad}}_p \circ \mathfrak{I}_0 = \overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^P_x \circ p,$$

proving $\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^P_x = \mathfrak{I}^P_y \circ \sigma$. Conversely, assume that (M, ∇, P) is infinitesimally homogeneous. Choose any $x \in M$ and any $p \in P_x$ and set:

$$R_0 = p^* R_x, \quad T_0 = p^* T_x, \quad \mathfrak{I}_0 = (\overline{\mathrm{Ad}}_p)^{-1} \circ \mathfrak{I}^P_x \circ p.$$

Given any $y \in M$, $q \in P_y$ then $\sigma = q \circ p^{-1} : T_xM \to T_yM$ is a G-structure preserving map and therefore $\sigma^* R_y = R_x$, $\sigma^* T_y = T_x$ and $\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^P_x = \mathfrak{I}^P_y \circ \sigma$. Then:

$$q^* R_y = p^* \sigma^* R_y = p^* R_x = R_0, \quad q^* T_y = p^* \sigma^* T_y = p^* T_x = T_0;$$

moreover:

$$\overline{\mathrm{Ad}}_q \circ \mathfrak{I}_0 = \overline{\mathrm{Ad}}_q \circ (\overline{\mathrm{Ad}}_p)^{-1} \circ \mathfrak{I}^P_x \circ p = \overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^P_x \circ p = \mathfrak{I}^P_y \circ \sigma \circ p = \mathfrak{I}^P_y \circ q.$$

This concludes the proof. □

Roughly speaking, an affine manifold with G-structure (M, ∇, P) is infinitesimally homogeneous if one can describe the inner torsion $\mathfrak{I}^P$, the torsion tensor T and the curvature tensor R by formulas that involve only the G-structure. A better understanding of this statement can be obtained by considering the following examples.

EXAMPLE 3.4.8. Let (M, g) be an n-dimensional semi-Riemannian manifold with $\mathrm{n}_-(g) = r$ having *constant sectional curvature* $c \in \mathbb{R}$. This means that:

$$g_x\big(R_x(v, w)v, w\big) = c\big(g_x(v, w)^2 - g_x(v, v) g_x(w, w)\big),$$

for all $x \in M$ and all $v, w \in T_xM$, where R denotes the curvature tensor of the Levi-Civita connection ∇ of (M, g). It is well-known (see Exercise 3.8) that if (M, g) has constant sectional curvature c then the curvature tensor R is given by:

$$R_x(v, w)u = c\big(g_x(w, u)v - g_x(v, u)w\big), \tag{3.4.12}$$

for all $x \in M$ and all $v, w, u \in T_xM$.

If $P = \mathrm{FR}^{\mathrm{o}}(TM)$ is the $\mathrm{O}_r(\mathbb{R}^n)$-structure on M consisting of all orthonormal frames then the triple (M, ∇, P) is infinitesimally homogeneous. Namely, $\mathfrak{I}^P = 0$, $T = 0$ and formula (3.4.12) says that the curvature tensor R is constant on frames that belong to the G-structure (the curvature tensor R can be described using only the G-structure P, that can be identified with the metric tensor g). In this situation, the multilinear maps R_0, T_0, $\mathfrak{I}_0$ of Lemma 3.4.7 are given by $T_0 = 0$, $\mathfrak{I}_0 = 0$ and:

$$R_0 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (v, w, u) \longmapsto \langle w, u\rangle v - \langle v, u\rangle w \in \mathbb{R}^n,$$

where $\langle\cdot,\cdot\rangle$ denotes the Minkowski bilinear form of index r in $\mathbb{R}^n$.

EXAMPLE 3.4.9. Let A be an n-dimensional Lie group and ∇ be a left invariant connection on A, i.e., the left translations of A are affine maps. Denote by $\mathfrak{a}$ the Lie algebra of A. The connection ∇ is determined by a linear map $\Gamma : \mathfrak{a} \to \mathrm{Lin}(\mathfrak{a})$ and it is given by:

$$\nabla_v X = g\big(\mathrm{d}\widetilde{X}_g(v) + \Gamma(g^{-1}v) \cdot \widetilde{X}(g)\big), \tag{3.4.13}$$

for all $g \in A$, $v \in T_gA$ and all $X \in \boldsymbol{\Gamma}(TA)$, where $\widetilde{X}(g) = g^{-1}X(g)$. The curvature tensor of ∇ at $1 \in A$ is easily computed as:

$$R_1(X, Y) = [\Gamma(X), \Gamma(Y)] - \Gamma\big([X, Y]\big),$$

for all $X, Y \in \mathfrak{a}$. Choose any linear isomorphism $p_0 : \mathbb{R}^n \to \mathfrak{a}$. Consider the global smooth section $s : A \to \mathrm{FR}(TA)$ defined by:

$$s(g) = \mathrm{d}L_g(1) \circ p_0 \in \mathrm{FR}(T_gA), \tag{3.4.14}$$

for all $g \in A$, where $L_g : A \to A$ denotes left translation by g. Then $P = s(A)$ is a G-structure on A with $G = \{\mathrm{Id}_{\mathbb{R}^n}\}$. Since the left translations of A are affine G-structure preserving diffeomorphisms, it follows that (A, ∇, P) is a homogeneous (and infinitesimally homogeneous) affine manifold with G-structure. The Christoffel tensor of ∇ with respect to s is given by:

$$T_gA \ni v \longmapsto \mathrm{d}L_g(1) \circ \Gamma(g^{-1}v) \circ \mathrm{d}L_g(1)^{-1} \in \mathrm{Lin}(T_gA)$$

for all $g \in A$. The inner torsion $\mathfrak{I}^P$ coincides with the Christoffel tensor (Example 2.11.2).

EXAMPLE 3.4.10. Let (M_1, g^1), (M_2, g^2) be semi-Riemannian manifolds with $\dim(M_i) = n_i$, $\mathrm{n}_-(g^i) = r_i$, $i = 1, 2$. Assume that (M_i, g^i) has constant sectional curvature $c_i \in \mathbb{R}$, $i = 1, 2$. Consider the product $M = M_1 \times M_2$ endowed with the metric g obtained by taking the orthogonal sum of g^1 and g^2, i.e.:

$$g_{(x_1,x_2)}\big((v_1, v_2), (w_1, w_2)\big) = g^1_{x_1}(v_1, w_1) + g^2_{x_2}(v_2, w_2),$$

for all $x_1 \in M_1$, $x_2 \in M_2$, $v_1, w_1 \in T_{x_1}M_1$ and all $v_2, w_2 \in T_{x_2}M_2$. The curvature tensor R of the Levi-Civita connection ∇ of (M, g) is given by (recall (3.4.12)):

$$\begin{aligned}(3.4.15)\quad R_{(x_1,x_2)}\big((v_1, v_2), (w_1, w_2)\big)(u_1, u_2) &\\ = c_1\big(g^1_{x_1}(w_1, u_1)v_1 - g^1_{x_1}(v_1, u_1)w_1\big) &\\ + c_2\big(g^2_{x_2}(w_2, u_2)v_2 - g^2_{x_2}(v_2, u_2)w_2\big),&\end{aligned}$$

for all $x_1 \in M_1$, $x_2 \in M_2$, $u_1, v_1, w_1 \in T_{x_1}M_1$ and all $u_2, v_2, w_2 \in T_{x_2}M_2$. Set:

$$P = \mathrm{FR}^{\mathrm{o}}\big(TM; \mathbb{R}^{n_1} \oplus \{0\}^{n_2}, \mathrm{pr}_1^*(TM_1)\big),$$

where $\mathrm{pr}_1 : M \to M_1$ denotes the first projection and $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ is endowed with the orthogonal sum of the Minkowski bilinear forms of indexes r_1 and r_2. More explicitly, for all $(x_1, x_2) \in M$, $P_{(x_1,x_2)}$ is the set of all linear isometries $p : \mathbb{R}^{n_1+n_2} \to T_{(x_1,x_2)}M$ such that $p\big(\mathbb{R}^{n_1} \oplus \{0\}^{n_2}\big) = T_{x_1}M_1 \oplus \{0\}$ and (automatically) $p\big(\{0\}^{n_1} \oplus \mathbb{R}^{n_2}\big) = \{0\} \oplus T_{x_2}M_2$. Then P is a G-structure on M with:

$$G = \mathrm{O}\big(\mathbb{R}^{n_1+n_2}; \mathbb{R}^{n_1} \oplus \{0\}^{n_2}\big) \cong \mathrm{O}_{r_1}(\mathbb{R}^{n_1}) \times \mathrm{O}_{r_2}(\mathbb{R}^{n_2}).$$

We claim that (M, ∇, P) is infinitesimally homogeneous. Since ∇ is compatible with g and the covariant derivative of sections of $\mathrm{pr}_1^*(TM_1)$ are sections of $\mathrm{pr}_1^*(TM_1)$, it follows from Example 2.11.5 that the inner torsion $\mathfrak{I}^P$ is zero. Moreover, the torsion of ∇ is zero and formula (3.4.15) implies that R is constant on frames that belong to the G-structure P.

EXAMPLE 3.4.11. Let (M, g) be a semi-Riemannian manifold and let J be an almost complex structure on M such that J_x is anti-symmetric with respect to g_x, for all $x \in M$. Assume that J is parallel with respect to the Levi-Civita connection ∇. Then (M, g, J) is called a *semi-Kähler* manifold; when g is a Riemannian metric, we call (M, g, J) a *Kähler* manifold. We say that (M, g, J) has *constant holomorphic curvature* $c \in \mathbb{R}$ if:

$$g_x\big[R_x\big(v, J(v)\big)v, Jv\big] = -c\,g_x(v, v)^2,$$

for all $x \in M$ and all $v \in T_xM$. It is well-known (see Exercise 3.9) that if (M, g, J) has constant holomorphic curvature c then the curvature tensor R is given by:

$$\begin{aligned}(3.4.16)\quad R_x(v, w)u = -\tfrac{c}{4}\big[g_x(v, u)w - g_x(w, u)v - g_x\big(v, J_x(u)\big)J_x(w) &\\ + g_x\big(w, J_x(u)\big)J_x(v) - 2g_x\big(v, J_x(w)\big)J_x(u)\big],&\end{aligned}$$

for all $x \in M$ and all $v, w, u \in T_xM$. If (M, g, J) is a semi-Kähler manifold with constant holomorphic curvature and if $P = \mathrm{FR}^{\mathrm{u}}(TM)$ then (M, ∇, P) is infinitesimally homogeneous. Namely, the inner torsion $\mathfrak{I}^P$ is zero (Example 2.11.8), the torsion is zero and formula (3.4.16) shows that R is constant in frames that belong to P.

EXAMPLE 3.4.12. Let (M, g) be an n-dimensional semi-Riemannian manifold where g has index r and let $\xi \in \mathbf{\Gamma}(TM)$ be a smooth vector field on M with $g_x\big(\xi(x), \xi(x)\big) = 1$, for all $x \in M$. Let us endow $\mathbb{R}^n$ with the Minkowski bilinear form $\langle \cdot, \cdot \rangle$ of index r; denote by $e_1, \ldots, e_n$ the canonical basis of $\mathbb{R}^n$. Assume that there exists a trilinear map $R_0 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and a linear map $L_0 : \mathbb{R}^n \to \mathbb{R}^n$ such that for every $x \in M$ and every linear isometry $p : \mathbb{R}^n \to T_xM$ with $p(e_1) = \xi(x)$, the following conditions holds:

(a) R_0 is p-related with R_x;
(b) $p \circ L_0 = (\nabla \xi)_x \circ p$.

Set $P = \mathrm{FR}^{\mathrm{o}}(TM; e_1, \xi)$, so that P is a G-structure on M with $G = \mathrm{O}(\mathbb{R}^n; e_1)$ (Example 2.11.6). Then (M, ∇, P) is infinitesimally homogeneous. Namely, this follows from Lemma 3.4.7, keeping in mind that, since ∇ is compatible with g, the inner torsion $\mathfrak{I}^P$ can be identified with $\nabla \xi$ (Example 2.11.6). It will also be interesting to consider the case where M is oriented and (a) and (b) above hold only for orientation preserving linear isometries $p : \mathbb{R}^n \to T_xM$ with $p(e_1) = \xi(x)$. In this case, one considers the open subset of P consisting of orientation preserving frames, which is a principal bundle with structural group:

$$\big\{T \in \mathrm{O}(\mathbb{R}^n; e_1) : \det(T) = 1\big\}.$$

Interesting examples of Riemannian manifolds satisfying the conditions above are the homogeneous 3-dimensional Riemannian manifolds with an isometry group of dimension 4 (see, for instance, [**7**]).

DEFINITION 3.4.13. Fix objects M, $\overline{M}$, $\pi : E \to M$, $\widehat{E}$, $\widehat{\nabla}$ and $\overline{\nabla}$ as in Definition 3.4.2. Let G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^{\bar{n}})$ and assume that $\widehat{E}$ and $T\overline{M}$ are endowed with G-structures $\widehat{P}$ and $\overline{P}$, respectively. A (local) affine immersion (f, L) of $(M, E, \widehat{\nabla})$ into $(\overline{M}, \overline{\nabla})$ is said to be *G-structure preserving* if L is a G-structure preserving isomorphism of vector bundles, where $f^*T\overline{M}$ is endowed with the G-structure $f^*\overline{P}$ (recall Example 1.8.3).

THEOREM 3.4.14. *Fix objects M, $\overline{M}$, $\pi : E \to M$, $\widehat{E}$, $\widehat{\nabla}$, $\overline{\nabla}$, G, $\widehat{P}$ and $\overline{P}$ as in Definition 3.4.13. Denote by $\widehat{T}$, $\widehat{R}$, $\overline{T}$, $\overline{R}$, respectively the ι-torsion of $\widehat{\nabla}$, the curvature of $\widehat{\nabla}$, the torsion of $\overline{\nabla}$ and the curvature of $\overline{\nabla}$, where $\iota : TM \to \widehat{E}$ denotes the inclusion map. Assume that $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous and that for all $x \in M$, $y \in \overline{M}$ and every G-structure preserving map $\sigma : \widehat{E}_x \to T_y\overline{M}$, the following conditions hold:*

(a) $\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^{\widehat{P}}_x = \mathfrak{I}^{\overline{P}}_y \circ \sigma|_{T_xM}$;
(b) *$\widehat{T}_x : T_xM \times T_xM \to \widehat{E}_x$ is σ-related with $\overline{T}_y : T_y\overline{M} \times T_y\overline{M} \to T_y\overline{M}$;*
(c) *$\widehat{R}_x : T_xM \times T_xM \times \widehat{E}_x \to \widehat{E}_x$ is σ-related with $\overline{R}_y : T_y\overline{M} \times T_y\overline{M} \times T_y\overline{M} \to T_y\overline{M}$.*

Then, for all $x_0 \in M$, all $y_0 \in \overline{M}$ and for every G-structure preserving map $\sigma_0 : \widehat{E}_{x_0} \to T_{y_0}\overline{M}$ there exists a G-structure preserving local affine immersion (f, L) of $(M, E, \widehat{\nabla})$ into $(\overline{M}, \overline{\nabla})$ whose domain is an open neighborhood U of x_0 in M and such that $f(x_0) = y_0$, $L_{x_0} = \sigma_0$.

PROOF. Denote by $\omega^{\overline{M}}$ the connection form on $\mathrm{FR}(T\overline{M})$ corresponding to the connection $\mathrm{Hor}\big(\mathrm{FR}(T\overline{M})\big)$ associated to $\overline{\nabla}$ and by ω^M the connection form on $\mathrm{FR}(\widehat{E})$ corresponding to the connection $\mathrm{Hor}\big(\mathrm{FR}(\widehat{E})\big)$ associated to $\widehat{\nabla}$. Denote also by $\theta^{\overline{M}}$ the canonical form of $\mathrm{FR}(T\overline{M})$ and by θ^M the ι-canonical form of $\mathrm{FR}(\widehat{E})$, where $\iota : TM \to \widehat{E}$ denotes the inclusion map. Let $s : V \to \widehat{P}$ be a smooth local section with $x_0 \in V$. Denote by $\lambda^{\overline{P}}$ the 1-form on $\overline{P}$ obtained by restricting the $\mathbb{R}^{\bar{n}} \oplus \mathfrak{gl}(\mathbb{R}^{\bar{n}})$-valued 1-form $(\theta^{\overline{M}}, \omega^{\overline{M}})$ and by λ^V the $\mathbb{R}^{\bar{n}} \oplus \mathfrak{gl}(R^{\bar{n}})$-valued 1-form on V defined by:

$$\lambda^V = s^*(\theta^M, \omega^M) = (s^*\theta^M, s^*\omega^M).$$

Since $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous, by Lemma 3.4.7, there exists a linear map $\mathfrak{I}_0 : \mathbb{R}^{\bar{n}} \to \mathfrak{gl}(\mathbb{R}^{\bar{n}})/\mathfrak{g}$ such that:

$$\overline{\mathrm{Ad}}_{\bar{p}} \circ \mathfrak{I}_0 = \mathfrak{I}^{\overline{P}}_y \circ \bar{p}, \tag{3.4.17}$$

for all $y \in \overline{M}$ and all $\bar{p} \in \overline{P}_y$. Let us show that for all $x \in M$ and all $p \in P_x$ we have:

$$(\overline{\mathrm{Ad}}_p)^{-1} \circ \mathfrak{I}^{\widehat{P}}_x = \mathfrak{I}_0 \circ p^{-1}|_{T_xM}. \tag{3.4.18}$$

Namely, choose any $y \in \overline{M}$, $\bar{p} \in \overline{P}_y$ and set $\sigma = \bar{p} \circ p^{-1}$, so that $\sigma : \widehat{E}_x \to T_y\overline{M}$ is G-structure preserving (notice that $\bar{p} = \sigma \circ p$ and use Remark 1.1.14). Then:

$$\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^{\widehat{P}}_x = \mathfrak{I}^{\overline{P}}_y \circ \sigma|_{T_xM} = \mathfrak{I}^{\overline{P}}_y \circ \bar{p} \circ p^{-1}|_{T_xM} \overset{(3.4.17)}{=} \overline{\mathrm{Ad}}_{\bar{p}} \circ \mathfrak{I}_0 \circ p^{-1}|_{T_xM},$$

and:

$$\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^{\widehat{P}}_x = \overline{\mathrm{Ad}}_{\bar{p}} \circ (\overline{\mathrm{Ad}}_p)^{-1} \circ \mathfrak{I}^{\widehat{P}}_x,$$

so that:

$$\overline{\mathrm{Ad}}_{\bar{p}} \circ (\overline{\mathrm{Ad}}_p)^{-1} \circ \mathfrak{I}^{\widehat{P}}_x = \overline{\mathrm{Ad}}_{\bar{p}} \circ \mathfrak{I}_0 \circ p^{-1}|_{T_xM},$$

proving (3.4.18).

We divide the rest of the proof into steps.

Step 1. *The thesis of the theorem follows once it is shown the existence of a smooth map $F : U \to \overline{P}$ defined in an open neighborhood U of x_0 in V such that $F^*\lambda^{\overline{P}} = \lambda^V|_U$ and $F(x_0) = \sigma_0 \circ s(x_0)$.*

Assume that we are given a smooth map $F : U \to \overline{P}$ defined in an open neighborhood U of x_0 in V such that $F^*\lambda^{\overline{P}} = \lambda^V|_U$ and $F(x_0) = \sigma_0 \circ s(x_0)$. Set $f = \overline{\Pi} \circ F : U \to \overline{M}$, where $\overline{\Pi}$ denotes the projection of the principal bundle $\overline{P}$. We define a fiberwise linear map $L : \widehat{E}|_U \to f^*T\overline{M}$ by setting:

$$L_x = F(x) \circ s(x)^{-1} : \widehat{E}_x \longrightarrow T_{f(x)}\overline{M} = (f^*T\overline{M})_x,$$

for all $x \in U$; thus (3.4.3) holds. Clearly $f(x_0) = y_0$ and $L_{x_0} = \sigma_0$. Since F is smooth and:

$$F^*(\theta^{\overline{M}}, \omega^{\overline{M}}) = \lambda^V|_U = \big((s|_U)^*\theta^M, (s|_U)^*\omega^M\big),$$

Lemma 3.4.4 implies that the pair (f, L) is a local affine immersion of $(M, E, \widehat{\nabla})$ into $(\overline{M}, \overline{\nabla})$ with domain U. Since, for all $x \in U$, $s(x)$ is in $\widehat{P}_x$ and $F(x)$ is in $\overline{P}_{f(x)}$, equality (3.4.3) implies that L is G-structure preserving (see Remark 1.1.14).

Step 2. *For all $p \in \overline{P}$, the linear map $\lambda_p^{\overline{P}}$ maps $T_p\overline{P}$ isomorphically onto the space:*

$$\big\{(u, X) \in \mathbb{R}^{\bar{n}} \oplus \mathfrak{gl}(\mathbb{R}^{\bar{n}}) : \mathfrak{I}_0(u) = X + \mathfrak{g}\big\}. \tag{3.4.19}$$

Follows directly from Remark 2.11.9 and from equality (3.4.17).

Step 3. *The 1-form λ^V takes values in the space* (3.4.19).

Let $x \in V$ and $v \in T_xM$ be fixed. We have:

$$\lambda_x^V(v) = \big((s^*\theta^M)_x(v), (s^*\omega^M)_x(v)\big) \overset{(2.9.12)}{=} \big(s(x)^{-1} \cdot v, (s^*\omega^M)_x(v)\big).$$

We have to check that:

$$\mathfrak{I}_0\big(s(x)^{-1} \cdot v\big) = (s^*\omega^M)_x(v) + \mathfrak{g}.$$

By the definition of $\mathfrak{I}_x^{\widehat{P}}$, we have:

$$(\overline{\mathrm{Ad}}_{s(x)})^{-1}\big(\mathfrak{I}_x^{\widehat{P}}(v)\big) = (s^*\omega^M)_x(v) + \mathfrak{g}.$$

But formula (3.4.18) with $p = s(x)$ gives:

$$(\overline{\mathrm{Ad}}_{s(x)})^{-1}\big(\mathfrak{I}_x^{\widehat{P}}(v)\big) = \mathfrak{I}_0\big(s(x)^{-1} \cdot v\big).$$

Step 4. *There exists a smooth map $F : U \to \overline{P}$ as in step 1.*

We apply Proposition A.4.7. Observe that, since σ_0 is G-structure preserving and $s(x_0) \in \widehat{P}$, we have $\sigma_0 \circ s(x_0) \in \overline{P}$; thus, once the hypotheses of Proposition A.4.7 have been checked, its thesis will give us a smooth map $F : U \to \overline{P}$ defined in an open neighborhood U of x_0 in V with $F(x_0) = \sigma_0 \circ s(x_0)$ and $F^*\lambda^{\overline{P}} = \lambda^V|_U$. Let $x \in V$, $y \in \overline{M}$, $\bar{p} \in \overline{P}_y$ be fixed. By step 3, the linear map λ_x^V maps T_xM to (3.4.19) and by step 2 the linear map $\lambda_{\bar{p}}^{\overline{P}}$ maps $T_{\bar{p}}\overline{P}$ isomorphically onto (3.4.19); therefore, we get a linear map:

$$\tau = (\lambda_{\bar{p}}^{\overline{P}})^{-1} \circ \lambda_x^V : T_xM \longrightarrow T_{\bar{p}}\overline{P}.$$

In order to apply Proposition A.4.7, we need to check that:

$$\tau^*\mathrm{d}\lambda_{\bar{p}}^{\overline{P}} = \mathrm{d}\lambda_x^V. \tag{3.4.20}$$

Obviously (3.4.20) is the same as:

$$\tau^*\mathrm{d}\theta_{\bar{p}}^{\overline{M}} = (s^*\mathrm{d}\theta^M)_x, \quad \tau^*\mathrm{d}\omega_{\bar{p}}^{\overline{M}} = (s^*\mathrm{d}\omega^M)_x. \tag{3.4.21}$$

Clearly:

$$\tau^*\theta_{\bar{p}}^{\overline{M}} = (s^*\theta^M)_x, \quad \tau^*\omega_{\bar{p}}^{\overline{M}} = (s^*\omega^M)_x,$$

so that (3.4.21) is equivalent to:

$$\tag{3.4.22} \begin{aligned} \tau^*(\mathrm{d}\theta^{\overline{M}} + \omega^{\overline{M}} \wedge \theta^{\overline{M}})_{\bar p} &= \big(s^*(\mathrm{d}\theta^{M} + \omega^{M} \wedge \theta^{M})\big)_x,\\ \tau^*(\mathrm{d}\omega^{\overline{M}} + \tfrac12\,\omega^{\overline{M}} \wedge \omega^{\overline{M}})_{\bar p} &= \big(s^*(\mathrm{d}\omega^{M} + \tfrac12\,\omega^{M} \wedge \omega^{M})\big)_x. \end{aligned}$$

But, by (2.9.2) and (2.9.13), (3.4.22) is the same as:

$$\tag{3.4.23} \tau^*\Theta^{\overline{M}}_{\bar p} = (s^*\Theta^M)_x, \quad \tau^*\Omega^{\overline{M}}_{\bar p} = (s^*\Omega^M)_x,$$

where $\Theta^{\overline{M}}$ denotes the torsion form of $\mathrm{FR}(TM)$, $\Omega^{\overline{M}}$ denotes the curvature form of the connection of $\mathrm{FR}(TM)$, Θ^M denotes the ι-torsion form of $\mathrm{FR}(\widehat{E})$ and Ω^M denotes the curvature form of the connection of $\mathrm{FR}(\widehat{E})$. Equalities (3.4.23) hold if and only if:

$$\tag{3.4.24} \begin{aligned} \Theta^{\overline{M}}_{\bar p}\big(\tau(v),\tau(w)\big) &= \Theta^M_{s(x)}\big(\mathrm{d}s_x(v),\mathrm{d}s_x(w)\big),\\ \Omega^{\overline{M}}_{\bar p}\big(\tau(v),\tau(w)\big) &= \Omega^M_{s(x)}\big(\mathrm{d}s_x(v),\mathrm{d}s_x(w)\big), \end{aligned}$$

for all $v,w \in T_xM$. Denote by $\widehat{\Pi} : \mathrm{FR}(\widehat{E}) \to M$ the projection; using (2.9.20) and (2.9.14), keeping in mind that $\mathrm{d}\widehat{\Pi}_{s(x)} \circ \mathrm{d}s_x$ is the identity of T_xM, we obtain that (3.4.24) is equivalent to:

$$\tag{3.4.25} \begin{aligned} \bar p^{-1}\Big(\overline{T}_y\big(\mathrm{d}\overline{\Pi}_{\bar p}[\tau(v)],\mathrm{d}\overline{\Pi}_{\bar p}[\tau(w)]\big)\Big) &= s(x)^{-1}\big(\widehat{T}_x(v,w)\big),\\ \bar p^{-1}\circ \overline{R}_y\big(\mathrm{d}\overline{\Pi}_{\bar p}[\tau(v)],\mathrm{d}\overline{\Pi}_{\bar p}[\tau(w)]\big)\circ \bar p &= s(x)^{-1}\circ \widehat{R}_x(v,w)\circ s(x). \end{aligned}$$

Let us compute $\mathrm{d}\overline{\Pi}_{\bar p}\circ\tau : T_xM \to T_y\overline{M}$. Given $u \in \mathbb{R}^{\bar n}$, $X \in \mathfrak{gl}(\mathbb{R}^{\bar n})$ with (u,X) in (3.4.19) then $(\lambda^{\overline{P}}_{\bar p})^{-1}(u,X) = \zeta$, where $\zeta \in T_{\bar p}\overline{P}$ satisfies:

$$\theta^{\overline{M}}_{\bar p}(\zeta) = \bar p^{-1}\big(\mathrm{d}\overline{\Pi}_{\bar p}(\zeta)\big) = u;$$

thus:

$$\big(\mathrm{d}\overline{\Pi}_{\bar p}\circ(\lambda^{\overline{P}}_{\bar p})^{-1}\big)(u,X) = \bar p(u).$$

Given $v \in T_xM$ then, using (2.9.12), we see that the first component of $\lambda^V_x(v)$ is $s(x)^{-1}\cdot v$; therefore:

$$(\mathrm{d}\overline{\Pi}_{\bar p}\circ\tau)(v) = \big(\mathrm{d}\overline{\Pi}_{\bar p}\circ(\lambda^{\overline{P}}_{\bar p})^{-1}\circ\lambda^V_x\big)(v) = \big(\bar p\circ s(x)^{-1}\big)(v).$$

Setting $\sigma = \bar p\circ s(x)^{-1} : \widehat{E}_x \to T_y\overline{M}$ then (3.4.25) is equivalent to:

$$\begin{aligned} \bar p^{-1}\Big(\overline{T}_y\big(\sigma(v),\sigma(w)\big)\Big) &= s(x)^{-1}\big(\widehat{T}_x(v,w)\big),\\ \bar p^{-1}\circ\overline{R}_y\big(\sigma(v),\sigma(w)\big)\circ\bar p &= s(x)^{-1}\circ\widehat{R}_x(v,w)\circ s(x), \end{aligned}$$

which is the same as:

$$\tag{3.4.26} \begin{aligned} \overline{T}_y\big(\sigma(v),\sigma(w)\big) &= \sigma\big(\widehat{T}_x(v,w)\big),\\ \overline{R}_y\big(\sigma(v),\sigma(w)\big) &= \sigma\circ\widehat{R}_x(v,w)\circ\sigma^{-1}. \end{aligned}$$

Finally, since σ is G-structure preserving, our hypotheses say that $\sigma^*\overline{T}_y = \widehat{T}_x$ and $\sigma^*\overline{R}_y = \widehat{R}_x$, i.e., (3.4.26) holds. This concludes the proof. □

3.4.1. The global affine immersions theorem.

THEOREM 3.4.15. *Under the assumptions of Theorem 3.4.14, if M is simply-connected and $(\overline{M},\overline{\nabla})$ is geodesically complete then, for all x_0 in M, all $y_0 \in \overline{M}$ and for all G-structure preserving map $\sigma_0 : \widehat{E}_{x_0} \to T_{y_0}\overline{M}$ there exists a G-structure preserving affine immersion (f,L) of $(M,E,\widehat{\nabla})$ into $(\overline{M},\overline{\nabla})$ such that $f(x_0)=y_0$, $L_{x_0}=\sigma_0$. Moreover if M is connected then, by Corollary 3.4.5, such affine immersion (f,L) is unique.*

LEMMA 3.4.16. *Let $(M,\widehat{\nabla})$, $(\overline{M},\overline{\nabla})$ be n-dimensional affine manifolds, G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$, $\widehat{P}\subset \mathrm{FR}(TM)$ be a G-structure on M, $\overline{P}\subset \mathrm{FR}(T\overline{M})$ be a G-structure on $\overline{M}$ and $s : V \to \widehat{P}$ be a smooth local section of $\widehat{P}$. Denote by θ^M, ω^M, $\theta^{\overline{M}}$, $\omega^{\overline{M}}$ respectively the canonical form of $\mathrm{FR}(TM)$, the connection form of $\mathrm{FR}(TM)$, the canonical form of $\mathrm{FR}(T\overline{M})$ and the connection form of $\mathrm{FR}(T\overline{M})$. Set:*

$$\lambda^V = (s^*\theta^M, s^*\omega^M)$$

and denote by $\lambda^{\overline{P}}$ the restriction to $\overline{P}$ of $(\theta^{\overline{M}},\omega^{\overline{M}})$. Let $\gamma : I \to V$, $\mu : I \to \overline{M}$ be geodesics and $\tilde{\mu} : I \to \overline{P}$ be a parallel lifting of μ. Assume that $s\circ\gamma$ is a parallel lifting of γ and that:

$$s\big(\gamma(t_0)\big)^{-1}\cdot\gamma'(t_0) = \tilde{\mu}(t_0)^{-1}\cdot\mu'(t_0), \tag{3.4.27}$$

for some $t_0\in I$. Then:

$$\lambda^{\overline{P}}_{\tilde{\mu}(t)}\big(\tilde{\mu}'(t)\big) = \lambda^V_{\gamma(t)}\big(\gamma'(t)\big), \tag{3.4.28}$$

for all $t\in I$.

PROOF. Since $s\circ\gamma$ and $\tilde{\mu}$ are both parallel, we have:

$$(s^*\omega^M)_{\gamma(t)}\big(\gamma'(t)\big) = \omega^M_{(s\circ\gamma)(t)}\big((s\circ\gamma)'(t)\big) = 0, \quad \omega^{\overline{M}}_{\tilde{\mu}(t)}\big(\tilde{\mu}'(t)\big) = 0,$$

for all $t\in I$, so that (3.4.28) is equivalent to:

$$(s^*\theta^M)_{\gamma(t)}\big(\gamma'(t)\big) = \theta^{\overline{M}}_{\tilde{\mu}(t)}\big(\tilde{\mu}'(t)\big),$$

for all $t\in I$. By (2.9.12), we have:

$$(s^*\theta^M)_{\gamma(t)}\big(\gamma'(t)\big) = s\big(\gamma(t)\big)^{-1}\cdot\gamma'(t),$$

for all $t\in I$; moreover:

$$\theta^{\overline{M}}_{\tilde{\mu}(t)}\big(\tilde{\mu}'(t)\big) = \tilde{\mu}(t)^{-1}\cdot\mu'(t),$$

for all $t\in I$. Since γ and μ are geodesics, the curves $\gamma' : I\to TM$ and $\mu' : I\to T\overline{M}$ are parallel; since $s\circ\gamma : I\to \mathrm{FR}(TM)$ and $\tilde{\mu} : I\to\mathrm{FR}(T\overline{M})$ are also parallel, the maps:

$$I\ni t\longmapsto s\big(\gamma(t)\big)^{-1}\cdot\gamma'(t)\in\mathbb{R}^n, \quad I\ni t\longmapsto \tilde{\mu}(t)^{-1}\cdot\mu'(t)\in\mathbb{R}^n$$

are constant and therefore (3.4.27) implies that:

$$s\big(\gamma(t)\big)^{-1}\cdot\gamma'(t) = \tilde{\mu}(t)^{-1}\cdot\mu'(t),$$

for all $t \in I$. The conclusion follows. $\square$

LEMMA 3.4.17. *Let $(M, \widehat{\nabla})$, $(\overline{M}, \overline{\nabla})$ be n-dimensional affine manifolds, G be a Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$, $\widehat{P} \subset \mathrm{FR}(TM)$ be a G-structure on M, $\overline{P} \subset \mathrm{FR}(T\overline{M})$ be a G-structure on $\overline{M}$; assume that $(\overline{M}, \overline{\nabla})$ is geodesically complete. Denote by $\widehat{T}$, $\widehat{R}$, $\overline{T}$, $\overline{R}$, respectively the torsion of $\widehat{\nabla}$, the curvature of $\widehat{\nabla}$, the torsion of $\overline{\nabla}$ and the curvature of $\overline{\nabla}$. Assume that for all $x \in M$, $y \in \overline{M}$ and every G-structure preserving map $\sigma : T_xM \to T_y\overline{M}$, the following conditions hold:*

(a) $\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^{\widehat{P}}_x = \mathfrak{I}^{\overline{P}}_y \circ \sigma$;
(b) $\widehat{T}_x : T_xM \times T_xM \to T_xM$ *is σ-related with* $\overline{T}_y : T_y\overline{M} \times T_y\overline{M} \to T_y\overline{M}$;
(c) $\widehat{R}_x : T_xM \times T_xM \times T_xM \to T_xM$ *is σ-related with* $\overline{R}_y : T_y\overline{M} \times T_y\overline{M} \times T_y\overline{M} \to T_y\overline{M}$.

Let $x_1 \in M$ be fixed and let V_0 be an open subset of $T_{x_1}M$ that is star-shaped at the origin and such that $\exp_{x_1}$ maps V_0 diffeomorphically onto an open subset V of M. Then, for all $x_0 \in V$, all $y_0 \in \overline{M}$ and for every G-structure preserving map $\sigma_0 : T_{x_0}M \to T_{y_0}\overline{M}$ there exists a G-structure preserving affine map $f : V \to \overline{M}$ such that $f(x_0) = y_0$, $\mathrm{d}f_{x_0} = \sigma_0$.

REMARK 3.4.18. Observe that, if M is nonempty, conditions (a), (b) and (c) in the statement of Lemma 3.4.17 imply that $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous. A similar observation *does not* holds in the case of Theorem 3.4.14, because the relations that appear in conditions (a), (b) and (c) in the statement of Theorem 3.4.14 involve restrictions of the tensors.

PROOF. By Lemma 2.2.30, there exists a smooth local section $s : V \to \widehat{P}$ such that for all $v \in T_{x_1}M$, the curve $t \mapsto s\big(\exp_{x_1}(tv)\big) \in \widehat{P}$ is a parallel lifting of the geodesic $t \mapsto \exp_{x_1}(tv)$. Define θ^M, ω^M, $\theta^{\overline{M}}$, $\omega^{\overline{M}}$, λ^V and $\lambda^{\overline{P}}$ as in the statement of Lemma 3.4.16. Our strategy is to employ Proposition A.4.10 to obtain a smooth map $F : V \to \overline{P}$ such that $F(x_0) = \sigma_0 \circ s(x_0)$ and $F^*\lambda^{\overline{P}} = \lambda^V$. Once this map F is obtained, we set $f = \overline{\Pi} \circ F$, where $\overline{\Pi} : \overline{P} \to \overline{M}$ denotes the projection; then, arguing as in step 1 of the proof of Theorem 3.4.14, it will follow that f is a G-structure preserving affine map such that $f(x_0) = y_0$, $\mathrm{d}f_{x_0} = \sigma_0$. Let us check the validity of the hypotheses of Proposition A.4.10. Hypothesis (a) is obtained as in the proof of steps 2 and 3 of Theorem 3.4.14 and hypothesis (b) is obtained as in the proof of step 4 of Theorem 3.4.14. Hypothesis (c) (i.e., the simply-connectedness of V) follows from the fact that V is homeomorphic to a star-shaped open subset of $T_{x_1}M$. To prove that hypothesis (d) holds, we consider the set $\mathcal{C}$ of all geodesics $\gamma : [0,1] \to V$ such that $s \circ \gamma$ is a parallel lifting of γ. The fact that $\mathcal{C}$ is rich follows by considering the map:

$$H : [0,1] \times V \ni (t,x) \longmapsto \exp_{x_1}\big(t \exp_{x_1}^{-1}(x)\big) \in V.$$

Finally, given $\gamma \in \mathcal{C}$ and $\bar{p} \in \overline{P}$, we have to show that there exists a smooth curve $\tilde{\mu} : [0,1] \to \overline{P}$ such that $\tilde{\mu}(0) = \bar{p}$ and such that (3.4.28) holds, for all $t \in [0,1]$. Since $(\overline{M}, \overline{\nabla})$ is geodesically complete, there exists a geodesic $\mu : [0,1] \to \overline{M}$

with $\mu(0) = \overline{\Pi}(\bar{p})$ and:

$$\mu'(0) = \big[\bar{p} \circ s\big(\gamma(0)\big)^{-1}\big] \cdot \gamma'(0).$$

Let $\tilde{\mu} : [0,1] \to \overline{P}$ be a parallel lifting of μ with $\tilde{\mu}(0) = \bar{p}$ (Proposition 2.2.28). By Lemma 3.4.16, (3.4.28) holds, for all $t \in [0,1]$. This concludes the proof. □

We can now prove a global version of Theorem 3.4.14 in codimension zero.

PROPOSITION 3.4.19. *Under the conditions of Lemma 3.4.17, if M is simply-connected then for all $x_0 \in M$, all $y_0 \in \overline{M}$ and for every G-structure preserving map $\sigma_0 : T_{x_0}M \to T_{y_0}\overline{M}$ there exists a G-structure preserving affine map $f : M \to \overline{M}$ such that $f(x_0) = y_0$ and $\mathrm{d}f_{x_0} = \sigma_0$. If M is connected then such f is unique, by Corollary 3.4.5.*

PROOF. We may assume without loss of generality that M is connected. Our plan is to use the globalization theory explained in Section B.4. Let us define a pre-sheaf on M as follows: for every open subset U of M, $\mathfrak{P}(U)$ is the set of all G-structure preserving affine maps $f : U \to \overline{M}$ and given open subsets $U, V \subset M$ with $V \subset U$, the map $\mathfrak{P}_{U,V} : \mathfrak{P}(U) \to \mathfrak{P}(V)$ is given by $f \mapsto f|_V$. The fact that the pre-sheaf $\mathfrak{P}$ has the localization property is trivial. The fact that $\mathfrak{P}$ has the uniqueness property follows from Corollary 3.4.5. Moreover, given $x_1 \in M$, if V_0 is an open subset of $T_{x_1}M$, star-shaped at the origin, such that $\exp_{x_1}$ maps V_0 diffeomorphically onto an open subset V of M then it follows easily from Lemma 3.4.17 that V has the extension property with respect to $\mathfrak{P}$. Thus, $\mathfrak{P}$ has the extension property. We are therefore under the hypotheses of Corollary B.4.22. Now, let $\bar{f} : V \to \overline{M}$ be a G-structure preserving affine map defined on a connected open neighborhood V of x_0 with $\bar{f}(x_0) = y_0$ and $\mathrm{d}\bar{f}_{x_0} = \sigma_0$ (the existence of $\bar{f}$ can be obtained either from Lemma 3.4.17 or from Theorem 3.4.14). By Corollary B.4.22, there exists $f \in \mathfrak{P}(X)$ such that $f|_V = \bar{f}$. This concludes the proof. □

REMARK 3.4.20. Under the conditions of Proposition 3.4.19, if in addition (M, ∇) is geodesically complete, $\overline{M}$ is simply-connected and both M and $\overline{M}$ are connected then the map f given by the thesis of the proposition is a smooth diffeomorphism. Namely, one can interchange the roles of M and $\overline{M}$ to obtain a smooth inverse for the map f.

PROPOSITION 3.4.21. *Let (M, ∇) be an affine manifold endowed with a G-structure P. If M is connected and simply-connected, (M, ∇) is geodesically complete and (M, ∇, P) is infinitesimally homogeneous then (M, ∇, P) is a homogeneous affine manifold with G-structure.*

PROOF. Take $(\overline{M}, \overline{\nabla}, \overline{P}) = (M, \nabla, P)$ in Proposition 3.4.19 and use Remark 3.4.20. □

PROOF OF THEOREM 3.4.15. We can assume without loss of generality that M is connected. We will first prove the theorem under the additional assumption that $\overline{M}$ is simply-connected so that, by Proposition 3.4.21, $(\overline{M}, \overline{\nabla}, \overline{P})$ is a homogeneous affine manifold with G-structure. We will use the globalization theory explained in Section B.4. Let us define a pre-sheaf on M as follows: for every open subset U of M, $\mathfrak{P}(U)$ is the set of all G-structure preserving local affine immersions (f, L) of $(M, E, \widehat{\nabla})$ into $(\overline{M}, \overline{\nabla})$ with domain U; given open subsets $U, V \subset M$ with $V \subset U$, the map $\mathfrak{P}_{U,V} : \mathfrak{P}(U) \to \mathfrak{P}(V)$ is given by $(f, L) \mapsto (f|_V, L|_{\widehat{E}|_V})$. The fact that the pre-sheaf $\mathfrak{P}$ has the localization property is trivial. The fact that $\mathfrak{P}$ has the uniqueness property follows from Corollary 3.4.5. Let us now show that every open subset U of M such that $\mathfrak{P}(U)$ is nonempty has the extension property with respect to $\mathfrak{P}$; since, by Theorem 3.4.14, the set of such open sets U cover M, it will follow that the pre-sheaf $\mathfrak{P}$ has the extension property. Let then U be an open subset of M such that $\mathfrak{P}(U)$ is nonempty and let $(\hat{f}, \hat{L})$ in $\mathfrak{P}(U)$ be fixed. Given a nonempty connected open subset V of U and an affine immersion (f, L) in $\mathfrak{P}(V)$, we show that (f, L) admits an extension to U. Choose any $x_0 \in V$; the linear map:

$$L_{x_0} \circ \hat{L}_{x_0}^{-1} : T_{\hat{f}(x_0)}\overline{M} \longrightarrow T_{f(x_0)}M \tag{3.4.29}$$

is G-structure preserving. Thus, by the homogeneity of $(\overline{M}, \overline{\nabla}, \overline{P})$, there exists a affine G-structure preserving diffeomorphism $g : \overline{M} \to \overline{M}$ such that $g\big(\hat{f}(x_0)\big) = f(x_0)$ and $\mathrm{d}g_{x_0}$ is equal to (3.4.29). Then:

$$(\bar{f}, \bar{L}) = \big(g \circ \hat{f}, (\hat{f}^* \overleftarrow{\mathrm{d}g}) \circ \hat{L}\big)$$

is in $\mathfrak{P}(U)$ and $\bar{f}(x_0) = f(x_0)$, $\bar{L}_{x_0} = L_{x_0}$. Since V is connected, by Corollary 3.4.5, the restriction of $(\bar{f}, \bar{L})$ to V is equal to (f, L). This concludes the proof that $\mathfrak{P}$ has the extension property. We are therefore under the hypotheses of Corollary B.4.22 which allows us to extend a G-structure preserving local affine immersion given by Theorem 3.4.14 to the desired G-structure preserving affine immersion of $(M, E, \widehat{\nabla})$ into $(\overline{M}, \overline{\nabla})$. The general case in which $\overline{M}$ is not simply-connected can be obtained by considering the universal covering of $\overline{M}$. □

3.5. Isometric immersions into homogeneous semi-Riemannian manifolds

DEFINITION 3.5.1. Suppose we are given the following data:

- an $\bar{n}$-dimensional semi-Riemannian manifold $(\overline{M}, \bar{g})$, where the semi-Riemannian metric $\bar{g}$ has index $\bar{r}$;
- an n-dimensional semi-Riemannian manifold (M, g), where the semi-Riemannian metric g has index r;
- a vector bundle $\pi : E \to M$ with typical fiber $\mathbb{R}^k$ endowed with a semi-Riemannian structure g^E of index s, where $\bar{n} = n + k$ and $\bar{r} = r + s$;
- a connection ∇^E on E compatible with g^E;
- a smooth section α^0 of $\mathrm{Lin}_2^{\mathrm{s}}(TM, E)$.

By a *local solution of the semi-Riemannian isometric immersion problem* corresponding to the data above we mean a pair (f, S), where $f : U \to \overline{M}$ is an isometric immersion defined in an open subset U of M and $S : E|_U \to f^{\perp}$ is an isomorphism of vector bundles such that:

- $\bar{g}_{f(x)}\big(S_x(e), S_x(e')\big) = g^E_x(e, e')$, for all $x \in U$ and all $e, e' \in E_x$;
- S is connection preserving if E is endowed with ∇^E and $f^\perp$ is endowed with the normal connection $\nabla^\perp$;
- S carries α^0 to the second fundamental form α of the isometric immersion f, i.e., $S_x \circ \alpha^0_x = \alpha_x$, for all $x \in U$.

We call U the *domain* of the local solution (f, S). By a *solution of the semi-Riemannian isometric immersion problem* we mean a local solution (f, S) whose domain is M.

Consider the vector bundle $\widehat{E} = TM \oplus E$ endowed with the semi-Riemannian structure $\hat{g}$ whose restrictions to TM and E are g and g^E respectively and such that TM and E are orthogonal. Let G be a Lie subgroup of $\mathrm{O}_{\bar{r}}(\mathbb{R}^{\bar{n}})$, $\widehat{P}$ be a G-structure on $\widehat{E}$ and $\overline{P}$ be a G-structure on $\overline{M}$ such that $\widehat{P} \subset \mathrm{FR}^{\mathrm{o}}(\widehat{E})$ and $\overline{P} \subset \mathrm{FR}^{\mathrm{o}}(T\overline{M})$. A local solution (f, S) of the semi-Riemannian isometric immersion problem with domain $U \subset M$ is said to be *G-structure preserving* if for all $x \in U$, the linear isomorphism:

$$\mathrm{d}f_x \oplus S_x : \widehat{E}_x = T_xM \oplus E_x \longrightarrow \mathrm{d}f_x(T_xM) \oplus f^\perp_x = T_{f(x)}\overline{M}$$

is G-structure preserving.

THEOREM 3.5.2. *Suppose we are given data as in Definition 3.5.1; denote by ∇ the Levi-Civita connection of (M, g) and by $\overline{\nabla}$ the Levi-Civita connection of $(\overline{M}, \bar{g})$. Consider the vector bundle $\widehat{E} = TM \oplus E$ endowed with the semi-Riemannian structure $\hat{g}$ whose restrictions to TM and E are g and g^E respectively and such that TM and E are orthogonal. Let $\widehat{\nabla}$ be the connection on $\widehat{E}$ that is compatible with $\hat{g}$ and whose components are ∇, ∇^E and α^0 (see Subsection 2.8.1). Let G be a Lie subgroup of $\mathrm{O}_{\bar{r}}(\mathbb{R}^{\bar{n}})$, $\widehat{P}$ be a G-structure on $\widehat{E}$ and $\overline{P}$ be a G-structure on $\overline{M}$ such that $\widehat{P} \subset \mathrm{FR}^{\mathrm{o}}(\widehat{E})$ and $\overline{P} \subset \mathrm{FR}^{\mathrm{o}}(T\overline{M})$. Assume that $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous and that for all $x \in M$, $y \in \overline{M}$ and every G-structure preserving map $\sigma : \widehat{E}_x \to T_y\overline{M}$, the following conditions hold:*

(a) *σ relates the inner torsion of $\widehat{P}$ with the inner torsion of $\overline{P}$, i.e.:*

$$\overline{\mathrm{Ad}}_\sigma \circ \mathfrak{I}^{\widehat{P}}_x = \mathfrak{I}^{\overline{P}}_y \circ \sigma;$$

(b) *the Gauss equation holds:*

$$\begin{aligned}\bar{g}_y\big[\overline{R}_y\big(\sigma(v), \sigma(w)\big)\sigma(u), \sigma(z)\big] &= g_x\big(R_x(v, w)u, z\big)\\ &\quad - g^E_x\big(\alpha^0_x(w, u), \alpha^0_x(v, z)\big) + g^E_x\big(\alpha^0_x(v, u), \alpha^0_x(w, z)\big),\end{aligned}$$

for all $u, v, w, z \in T_xM$;

(c) *the Codazzi equation holds:*

$$\begin{aligned}\bar{g}_y\big[\overline{R}_y\big(\sigma(v), \sigma(w)\big)\sigma(u), \sigma(e)\big] &= g^E_x\big((\nabla^\otimes\alpha^0)_x(v, w, u), e\big)\\ &\quad - g^E_x\big((\nabla^\otimes\alpha^0)_x(w, v, u), e\big),\end{aligned}$$

for all $u, v, w \in T_xM$ and all $e \in E_x$, where $\nabla^\otimes$ denotes the connection induced by ∇ and ∇^E on $\mathrm{Lin}_2(TM, E)$;

(d) *the Ricci equation holds:*

$$\bar{g}_y\big[\overline{R}_y\big(\sigma(v),\sigma(w)\big)\sigma(e),\sigma(e')\big] = g^E_x\big(R^E_x(v,w)e,e'\big) \\ + g_x\big(\alpha^0_x(v)^*\cdot e,\alpha^0_x(w)^*\cdot e'\big) - g_x\big(\alpha^0_x(w)^*\cdot e,\alpha^0_x(v)^*\cdot e'\big),$$

for all $v,w\in T_xM$ and all $e,e'\in E_x$, where R^E denotes the curvature tensor of ∇^E.

Then, for all $x_0\in M$, all $y_0\in\overline{M}$ and for every G-structure preserving map $\sigma_0 : \widehat{E}_{x_0}\to T_{y_0}\overline{M}$ there exists a G-structure preserving local solution (f,S) of the semi-Riemannian isometric immersion problem whose domain is an open neighborhood U of x_0 such that $f(x_0)=y_0$,

$$\sigma_0 = \mathrm{d}f_{x_0}\oplus S_{x_0} : \widehat{E}_{x_0} = T_{x_0}M\oplus E_{x_0} \longrightarrow \mathrm{d}f_{x_0}(T_{x_0}M)\oplus f^\perp_{x_0} = T_{y_0}\overline{M}. \tag{3.5.1}$$

If M is connected and simply-connected and if $(\overline{M},\overline{\nabla})$ is geodesically complete then one can find a unique G-structure preserving global solution (f,S) of the semi-Riemannian isometric immersion problem satisfying the initial condition above.

PROOF. This is an application of Theorems 3.4.14 and 3.4.15. First, notice that if (f,L) is a G-structure preserving local affine immersion of $(M,E,\widehat{\nabla})$ into $(\overline{M},\overline{\nabla})$ then, setting $S = L|_E : E\to f^\perp$, the pair (f,S) is a G-structure preserving local solution of the semi-Riemannian isometric immersion problem; conversely, if (f,S) is a G-structure preserving local solution of the semi-Riemannian isometric immersion problem then, setting $L = \overleftarrow{\mathrm{d}f}\oplus S$, the pair (f,L) is a G-structure preserving local affine immersion of $(M,E,\widehat{\nabla})$ into $(\overline{M},\overline{\nabla})$. Now observe that:

- hypothesis (a) of Theorem 3.4.14 is the same as hypothesis (a) of this theorem;
- hypothesis (b) of Theorem 3.4.14 follows from the symmetry of the Levi-Civita connection ∇ and from the symmetry of α^0 (see Example 2.8.3);
- hypothesis (c) of Theorem 3.4.14 follows from the Gauss, Codazzi and Ricci equations (Proposition 2.8.1).

This concludes the proof. □

Let us see some explicit examples of applications of Theorem 3.5.2, by looking closer at its hypotheses in particular situations.

EXAMPLE 3.5.3. Assume that $(\overline{M},\bar{g})$ has constant sectional curvature $c\in\mathbb{R}$ (recall Example 3.4.8). Set $G = \mathrm{O}_{\bar{r}}(\mathbb{R}^{\bar{n}})$, $\widehat{P} = \mathrm{FR}^o(\widehat{E})$ and $\overline{P} = \mathrm{FR}^o(T\overline{M})$. Then $(\overline{M},\overline{\nabla},\overline{P})$ is infinitesimally homogeneous. We have $\mathfrak{I}^{\widehat{P}} = 0$ and $\mathfrak{I}^{\overline{P}} = 0$ because the connections $\widehat{\nabla}$ and $\overline{\nabla}$ are compatible with the semi-Riemannian structures $\hat{g}$ and $\bar{g}$, respectively (Example 2.11.3). Thus, hypothesis (a) of Theorem 3.5.2 is automatically satisfied. By (3.4.12), the lefthand side of the Gauss equation becomes:

$$c\big(g_x(w,u)g_x(v,z) - g_x(v,u)g_x(w,z)\big)$$

and the lefthand sides of the Codazzi and Ricci equations vanish. Thus, in this case, Theorem 3.5.2 gives us the classical fundamental theorem of isometric immersions (see for instance [**5**]). More explicitly, for all $x_0 \in M$, all $y_0 \in \overline{M}$ and for every linear isometry $\sigma_0 : \widehat{E}_{x_0} \to T_{y_0}\overline{M}$ there exists a local solution (f, S) of the semi-Riemannian isometric immersion problem whose domain is an open neighborhood U of x_0 such that $f(x_0) = y_0$ and (3.5.1) holds.

EXAMPLE 3.5.4. Let (M_1, g^1), (M_2, g^2) be semi-Riemannian manifolds with $\dim(M_i) = n_i$, $\mathrm{n}_-(g^i) = r_i$, $i = 1, 2$. Assume that (M_i, g^i) has constant sectional curvature $c_i \in \mathbb{R}$, $i = 1, 2$. Consider the product $\overline{M} = M_1 \times M_2$ endowed with the metric $\bar{g}$ obtained by taking the orthogonal sum of g^1 and g^2 (as in Example 3.4.10). Set $\bar{n} = n_1 + n_2$, $\bar{r} = r_1 + r_2$,

$$\overline{P} = \mathrm{FR}^{\mathrm{o}}\big(T\overline{M}; \mathbb{R}^{n_1} \oplus \{0\}^{n_2}, \mathrm{pr}_1^*(TM_1)\big),$$

where $\mathrm{pr}_1 : \overline{M} \to M_1$ denotes the first projection and $\mathbb{R}^{\bar{n}} = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ is endowed with the orthogonal sum of the Minkowski bilinear forms of indexes r_1 and r_2. Then P is a G-structure on $\overline{M}$ with:

$$G = \mathrm{O}\big(\mathbb{R}^{\bar{n}}; \mathbb{R}^{n_1} \oplus \{0\}^{n_2}\big) \cong \mathrm{O}_{r_1}(\mathbb{R}^{n_1}) \times \mathrm{O}_{r_2}(\mathbb{R}^{n_2})$$

and $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous. Let F be a vector subbundle of $\widehat{E}$ whose fibers are n_1-dimensional and set:

$$\widehat{P} = \mathrm{FR}^{\mathrm{o}}(\widehat{E}; \mathbb{R}^{n_1} \oplus \{0\}^{n_2}, F).$$

Let us assume that $\widehat{P}_x$ is nonempty[2] for all $x \in M$, so that $\widehat{P}$ is a G-structure on $\widehat{E}$ (Example 1.8.5). Since $\mathfrak{I}^{\overline{P}} = 0$, hypothesis (a) of Theorem 3.5.2 means that $\mathfrak{I}^{\widehat{P}} = 0$, i.e., the covariant derivative $\widehat{\nabla}$ of sections of F are sections of F (Example 2.11.5). Denote by $\pi^F : \widehat{E} \to F$, $\pi^{F^\perp} : \widehat{E} \to F^\perp$ the projections corresponding to the direct sum decomposition $\widehat{E} = F \oplus F^\perp$. The lefthand side of the Gauss equation becomes (recall (3.4.15)):

$$\begin{aligned} c_1\big[\hat{g}_x\big(\pi^F(w), \pi^F(u)\big)\hat{g}_x\big(\pi^F(v), \pi^F(z)\big) \\ - \hat{g}_x\big(\pi^F(v), \pi^F(u)\big)\hat{g}_x\big(\pi^F(w), \pi^F(z)\big)\big] \\ + c_2\big[\hat{g}_x\big(\pi^{F^\perp}(w), \pi^{F^\perp}(u)\big)\hat{g}_x\big(\pi^{F^\perp}(v), \pi^{F^\perp}(z)\big) \\ - \hat{g}_x\big(\pi^{F^\perp}(v), \pi^{F^\perp}(u)\big)\hat{g}_x\big(\pi^{F^\perp}(w), \pi^{F^\perp}(z)\big)\big]. \end{aligned}$$

The lefthand side of the Codazzi and Ricci equations are zero. The thesis of Theorem 3.5.2 becomes: for all $x_0 \in M$, all $y^0 = (y_1^0, y_2^0) \in \overline{M}$ and for every linear isometry $\sigma_0 : \widehat{E}_{x_0} \to T_{y^0}\overline{M}$ with $\sigma_0(F_{x_0}) = T_{y_1^0}M_1 \oplus \{0\}$ there exists a local solution (f, S) of the semi-Riemannian isometric immersion problem whose domain is an open neighborhood U of x_0 such that $f(x_0) = y^0$, satisfying (3.5.1) and such that:

$$(\mathrm{d}f_x \oplus S_x)(F_x) = T_{(\mathrm{pr}_1 \circ f)(x)}M_1 \oplus \{0\},$$

[2]This is equivalent to a compatibility condition between indexes of suitable restrictions of $\hat{g}$. It holds automatically, for instance, in the Riemannian case.

for all $x \in U$.

EXAMPLE 3.5.5. Assume that $\overline{M} = A$ is a Lie group and that $\bar{g}$ is a left invariant semi-Riemannian metric on $\overline{M}$. Then the connection $\overline{\nabla}$ is also left invariant and it is given by (3.4.13), where $\Gamma : \mathfrak{a} \to \mathrm{Lin}(\mathfrak{a})$ is equal to (see (2.11)):

$$\Gamma(X) \cdot Y = [X, Y] - (\mathrm{ad}_Y)^*(X) - (\mathrm{ad}_X)^*(Y), \tag{3.5.2}$$

for all $X, Y \in \mathfrak{a}$, where $\mathrm{ad}_X(Y) = [X, Y]$. Choose a linear isometry $p_0 : \mathbb{R}^{\bar{n}} \to \mathfrak{a}$ and consider the smooth global section $s : \overline{M} \to \mathrm{FR}^{\mathrm{o}}(T\overline{M})$ defined by (3.4.14). Then $\overline{P} = s(\overline{M})$ is a G-structure on $\overline{M}$ with $G = \{\mathrm{Id}_{\mathbb{R}^{\bar{n}}}\}$ and $(\overline{M}, \overline{\nabla}, \overline{P})$ is (infinitesimally) homogeneous. Let $\hat{s} : M \to \mathrm{FR}^{\mathrm{o}}(\widehat{E})$ be a global smooth frame of $\widehat{E}$, so that $\widehat{P} = \hat{s}(M)$ is a G-structure on $\widehat{E}$. Hypothesis (a) of Theorem 3.5.2 means that for all $x \in M$ the diagram:

$$\begin{tikzcd}
T_xM \arrow[r, "\widehat{\Gamma}_x"] \arrow[d, "\sigma|_{T_xM}"'] & \mathrm{Lin}(\widehat{E}_x) \arrow[d, "\mathrm{Ad}_\sigma"] \\
\mathfrak{a} \arrow[r, "\Gamma"'] & \mathrm{Lin}(\mathfrak{a})
\end{tikzcd}$$

commutes, where $\sigma = p_0 \circ \hat{s}(x)^{-1} : \widehat{E}_x \to \mathfrak{a}$, $\widehat{\Gamma}$ denotes the Christoffel tensor of $\widehat{\nabla}$ with respect to $\hat{s}$ and Γ is given by (3.5.2). In the lefthand side of the Gauss, Codazzi and Ricci equations, one should replace $\bar{g}_y$ by $\bar{g}_1$, $\overline{R}_y$ by $\overline{R}_1$ and σ should be understood as $p_0 \circ \hat{s}(x)^{-1}$; $\overline{R}_1$ is given by:

$$\overline{R}_1(X, Y) = [\Gamma(X), \Gamma(Y)] - \Gamma\big([X, Y]\big),$$

for all $X, Y \in \mathfrak{a}$. The thesis of Theorem 3.5.2 becomes: for all $x_0 \in M$ and all $y_0 \in \overline{M}$ there exists a local solution (f, S) of the semi-Riemannian isometric immersion problem whose domain is an open neighborhood U of x_0 such that $f(x_0) = y_0$ and such that:

$$(\mathrm{d}f_x \oplus S_x) \circ \hat{s}(x) = s\big(f(x)\big),$$

for all $x \in U$.

EXAMPLE 3.5.6. Assume that $\overline{M}$ is endowed with an almost complex structure $\overline{J}$ such that $(\overline{M}, \bar{g}, \overline{J})$ is a semi-Kähler manifold with constant holomorphic curvature $c \in \mathbb{R}$ (recall Example 3.4.11). Set $\overline{P} = \mathrm{FR}^{\mathrm{u}}(T\overline{M})$ and $G = \mathrm{U}_{\bar{r}}(\mathbb{R}^{\bar{n}})$, so that $\overline{P}$ is a G-structure on $\overline{M}$ and $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous. Let J be an almost complex structure on M and J^E be an almost complex structure on E; we define an almost complex structure $\widehat{J}$ on $\widehat{E}$ by setting $\widehat{J}_x(v, e) = \big(J_x(v), J^E_x(e)\big)$, for all $x \in M$, $v \in T_xM$, $e \in E_x$. Assume that, for all $x \in M$, J_x and J^E_x are anti-symmetric with respect to g_x and g^E_x, respectively, so that $\widehat{J}_x$ is anti-symmetric with respect to $\hat{g}$. Set $\widehat{P} = \mathrm{FR}^{\mathrm{u}}(\widehat{E})$, so that $\widehat{P}$ is a G-structure on $\widehat{E}$. We have $\mathfrak{I}^{\overline{P}} = 0$ because the connection $\overline{\nabla}$ is compatible with the semi-Riemannian structure $\bar{g}$ and $\overline{J}$ is parallel (Example 2.11.8). Thus, hypothesis (a) of Theorem 3.5.2 means that $\widehat{J}$ is parallel with respect to $\widehat{\nabla}$. An easy computation shows that $\widehat{J}$ is parallel with respect to $\widehat{\nabla}$ if and only if the following conditions hold:

- J is parallel with respect to ∇, i.e., (M, g, J) is Kähler;
- J^E is parallel with respect to ∇^E;
- α^0 is $\mathbb{C}$-bilinear, i.e., $\alpha^0_x(J_x\cdot, \cdot) = \alpha^0_x(\cdot, J_x\cdot) = J^E_x \circ \alpha^0$, for all $x \in M$.

By (3.4.16), the lefthand side of the Gauss equation becomes:

$$-\tfrac{c}{4}\big[g_x(v,u)g_x(w,z) - g_x(w,u)g_x(v,z) - g_x\big(v, J_x(u)\big)g_x\big(J_x(w), z\big) \\ + g_x\big(w, J_x(u)\big)g_x\big(J_x(v), z\big) - 2g_x\big(v, J_x(w)\big)g_x\big(J_x(u), z\big)\big],$$

and the lefthand sides of the Codazzi and Ricci equations vanish. Thus, in this case, Theorem 3.5.2 gives us a fundamental theorem for isometric immersions of Kähler manifolds. More explicitly, for all $x_0 \in M$, all $y_0 \in \overline{M}$ and for every $\mathbb{C}$-linear isometry $\sigma_0 : \widehat{E}_{x_0} \to T_{y_0}\overline{M}$ (i.e., $\overline{J}_{y_0} \circ \sigma_0 = \sigma_0 \circ \widehat{J}_{x_0}$) there exists a local solution (f, S) of the semi-Riemannian isometric immersion problem whose domain is an open neighborhood U of x_0 such that $f(x_0) = y_0$, such that (3.5.1) holds and $\mathrm{d}f_x \oplus S_x : \widehat{E}_x \to T_{f(x)}\overline{M}$ is $\mathbb{C}$-linear, for all $x \in U$.

EXAMPLE 3.5.7. Assume that $(\overline{M}, \bar{g})$ is endowed with a smooth vector field ξ such that $\bar{g}(\xi, \xi) \equiv 1$ and such that the conditions described in Example 3.4.12 hold. Set $P = \mathrm{FR}^{\mathrm{o}}(TM; e_1, \xi)$, so that P is a G-structure on M with $G = \mathrm{O}(\mathbb{R}^{\bar{n}}; e_1)$. Then $(\overline{M}, \overline{\nabla}, \overline{P})$ is infinitesimally homogeneous. Let $\epsilon : M \to \widehat{E}$ be a smooth global section of $\widehat{E}$ (ϵ is determined by a smooth vector field on M and by a smooth global section of E) and assume that $\hat{g}(\epsilon, \epsilon) \equiv 1$. Set $\widehat{P} = \mathrm{FR}^{\mathrm{o}}(\widehat{E}; e_1, \epsilon)$, so that $\widehat{P}$ is a G-structure on $\widehat{E}$. Hypothesis (a) in Theorem 3.5.2 means that for every $x \in M$ and every $p \in \widehat{P}_x$, we have:

$$p \circ L_0|_{p^{-1}(T_xM)} = (\widehat{\nabla}\epsilon)_x \circ p|_{p^{-1}(T_xM)}.$$

The lefthand side of the Gauss equation becomes:

$$\langle R_0\big(p^{-1}(v), p^{-1}(w)\big)p^{-1}(u), p^{-1}(z)\rangle,$$

where $p \in \widehat{P}_x$ is chosen arbitrarily. Similar considerations hold for the Codazzi and Ricci equations. The thesis of Theorem 3.5.2 becomes: for all $x_0 \in M$, all $y_0 \in \overline{M}$ and for every linear isometry $\sigma_0 : \widehat{E}_{x_0} \to T_{y_0}\overline{M}$ with $\sigma_0\big(\epsilon(x_0)\big) = \xi(y_0)$ there exists a local solution (f, S) of the semi-Riemannian isometric immersion problem whose domain is an open neighborhood U of x_0 such that $f(x_0) = y_0$, satisfying (3.5.1) and such that:

$$(\mathrm{d}f_x \oplus S_x)\big(\epsilon(x)\big) = \xi\big(f(x)\big),$$

for all $x \in U$. Considering the case where $(\overline{M}, \bar{g})$ is a homogeneous 3-dimensional Riemannian manifold with an isometry group of dimension 4, we obtain the results concerning the existence of isometric immersions that appear in [**7**].

Exercises

Affine manifolds.

EXERCISE 3.1. Let M', M be affine manifolds and $f : M' \to M$ be a smooth diffeomorphism. Given a vector field X on M', we denote by f_*X the vector field on M defined by:

$$f_*X = \mathrm{d}f \circ X \circ f^{-1}.$$

Show that if f is affine if and only if:

$$f_*(\nabla_X Y) = \nabla_{f_*X}(f_*Y),$$

for all $X, Y \in \mathbf{\Gamma}(TM)$. Conclude that if f is affine then also f^{-1} is affine.

EXERCISE 3.2. Let M', M be differentiable manifolds and $f : M' \to M$ be a smooth diffeomorphism. Given a connection ∇' on M', show that there exists a unique connection ∇ on M such that f is affine.

EXERCISE 3.3. Let M', M be affine manifolds and $f : M' \to M$ be a smooth local diffeomorphism. Consider the smooth map $(\mathrm{d}f)_*$ defined in (1.8.1). Show that f is affine if and only if:

$$\mathrm{d}\big((\mathrm{d}f)_*\big)_p\big[\mathrm{Hor}_p\big(\mathrm{FR}(TM')\big)\big] = \mathrm{Hor}_q\big(\mathrm{FR}(TM)\big),$$

for all $p \in \mathrm{FR}(TM')$, where $q = (\mathrm{d}f)_*(p)$.

EXERCISE 3.4. Let M', M be affine manifolds and $f : M' \to M$ be a smooth local diffeomorphism. Consider the smooth map $(\mathrm{d}f)_*$ defined in (1.8.1). Denote by ω, ω' respectively the connection forms on $\mathrm{FR}(TM)$ and on $\mathrm{FR}(TM')$. Denote also by θ, θ' respectively the canonical forms of $\mathrm{FR}(TM)$ and of $\mathrm{FR}(TM')$. Show that:

- the pull-back of θ by $(\mathrm{d}f)_*$ is equal to θ';
- f is affine if and only if the pull-back of ω by $(\mathrm{d}f)_*$ is equal to ω'.

Homogeneous affine manifolds.

EXERCISE 3.5. Let G be a Lie group and H be a closed subgroup of G; denote by $\mathfrak{g}$, $\mathfrak{h}$ respectively the Lie algebras of G and H. Let $\mathfrak{m}$ be a complement of $\mathfrak{h}$ on $\mathfrak{g}$. We identify the tangent space of G/H at $\bar{1} = 1H$ via the differential of the quotient map $G \to G/H$ and we consider the isotropic representation $\overline{\mathrm{Ad}} : H \to \mathrm{GL}(\mathfrak{m})$ of H on $\mathfrak{m}$. Show that if $\mathcal{D}$ is a (necessarily smooth) G-invariant distribution on G/H then $\mathcal{D}_{\bar{1}} \subset \mathfrak{m}$ is an $\overline{\mathrm{Ad}}$-invariant subspace of $\mathfrak{m}$. Conversely, if $\mathfrak{d}$ is an $\overline{\mathrm{Ad}}$-invariant subspace of $\mathfrak{m}$ then there exists a unique G-invariant distribution $\mathcal{D}$ on G/H with $\mathcal{D}_{\bar{1}} = \mathfrak{d}$.

EXERCISE 3.6. Let G be a Lie group, V be a real finite-dimensional vector space and $\rho : G \to \mathrm{GL}(V)$ a smooth representation of G on V. Denote by $\bar{\rho} : \mathfrak{g} \to \mathfrak{gl}(V)$ the differential of ρ at the identity. Given a subspace W of V, show that if $\rho(g)(W) = W$, for all $g \in G$ then $\bar{\rho}(X)(W) \subset W$, for all $X \in \mathfrak{g}$. Conversely, if G is connected, show that if $\bar{\rho}(X)(W) \subset W$, for all $X \in \mathfrak{g}$ then $\rho(g)(W) = W$, for all $g \in G$.

Affine immersions in homogeneous spaces.

EXERCISE 3.7. Let V, V', V'' be real vector spaces and let:

$$\sigma : V \longrightarrow V', \quad \sigma' : V' \longrightarrow V''$$

be linear maps. Given multilinear maps:

$$B \in \mathrm{Lin}_k(V, V), \quad B' \in \mathrm{Lin}_k(V', V'), \quad B'' \in \mathrm{Lin}_k(V'', V''),$$

show that:

- if B is σ-related with B' and B' is σ'-related with B'' then B is $(\sigma' \circ \sigma)$-related with B'';
- if σ is an isomorphism and B is σ-related with B' then B' is σ^{-1}-related with B.

EXERCISE 3.8. Let V be a real-finite dimensional vector space, $g : V \times V \to \mathbb{R}$ be a nondegenerate symmetric bilinear form on V and $R : V \times V \times V \to V$ be a trilinear map such that:

$$R(v, w)u = -R(w, v)u, \quad g\big(R(v, w)u, z\big) = -g\big(R(v, w)z, u\big),$$
$$R(v, w)u + R(u, v)w + R(w, u)v = 0,$$

for all $v, w, u, z \in V$. Given $c \in \mathbb{R}$, show that the following conditions are equivalent:

- $g\big(R(v, w)v, w\big) = c\big(g(v, w)^2 - g(v, v)g(w, w)\big)$, for all $v, w \in V$;
- $R(v, w)u = c\big(g(w, u)v - g(v, u)w\big)$, for all $v, w, u \in V$.

EXERCISE 3.9. Let V be a real-finite dimensional vector space, $g : V \times V \to \mathbb{R}$ be a nondegenerate symmetric bilinear form on V, J be a g-anti-symmetric complex structure on V and $R : V \times V \times V \to V$ be a trilinear map such that:

$$R(v, w)u = -R(w, v)u, \quad g\big(R(v, w)u, z\big) = -g\big(R(v, w)z, u\big),$$
$$R(v, w)u + R(u, v)w + R(w, u)v = 0,$$
$$R(v, w)J(u) = J\big(R(v, w)u\big),$$

for all $v, w, u, z \in V$. Given $c \in \mathbb{R}$, show that the following conditions are equivalent:

- $g\big[R\big(v, J(v)\big)v, J(v)\big] = -cg(v, v)^2$, for all $v \in V$;
- $R(v, w)u = -\frac{c}{4}\big[g(v, u)w - g(w, u)v - g\big(v, J(u)\big)J(w) + g\big(w, J(u)\big)J(v) - 2g\big(v, J(w)\big)J(u)\big]$, for all $v, w, u \in V$.

APPENDIX A

Vector fields and differential forms

A.1. Differentiable manifolds

Basic knowledge of the theory of differentiable manifolds (standard references for the subject are [**1, 3, 9, 12**]) is a prerequisite for reading this book. Many authors define differentiable manifold by starting with a topological space and then introducing a differentiable atlas. In many situations (for instance, Sections 1.3 and 1.4), one does not have a natural topology to star with and thus it is easier to define differentiable manifolds by starting only with a set and then, later, introducing a topology that is induced by the atlas. We adopt this point of view and, for the reader's convenience, we present here a complete definition of differentiable manifold.

Let M be a set. By an *n-dimensional local chart* (or, more simply, a *local chart*) on M we mean a bijective map $\varphi : U \to \widetilde{U}$ where U is an arbitrary subset of M and $\widetilde{U}$ is an open subset of $\mathbb{R}^n$. Given n-dimensional local charts $\varphi : U \to \widetilde{U}$ and $\psi : V \to \widetilde{V}$ on M then the *transition map* from φ to ψ is the bijective map:

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V).$$

We say that φ and ψ are *compatible* if $\varphi(U \cap V)$ and $\psi(U \cap V)$ are both open in $\mathbb{R}^n$ and if the transition map $\psi \circ \varphi^{-1}$ is a smooth diffeomorphism (by "smooth" we will always mean "of class C^∞"). Notice that the local charts φ and ψ are compatible when their domains are disjoint. By an *n-dimensional atlas* (or, more simply, an *atlas*) on M we mean a set $\mathcal{A}$ of n-dimensional local charts on M such that:

- the union of the domains of the local charts $\varphi \in \mathcal{A}$ is M;
- given $\varphi, \psi \in \mathcal{A}$ then φ and ψ are compatible.

If $\mathcal{A}$ is an atlas on M and if two local charts φ, ψ on M are compatible with every local chart that belongs to $\mathcal{A}$ then φ and ψ are compatible with each other; thus, every atlas $\mathcal{A}$ on M is contained in a unique maximal atlas. Such maximal atlas consists of all local charts on M that are compatible with every local chart that belongs to $\mathcal{A}$. A maximal atlas on a set M is also called a *differential structure* on M.

If $\mathcal{A}$ is an atlas on a set M then there exists a unique topology τ on M such that for every local chart $\varphi : U \to \widetilde{U}$ that belongs to $\mathcal{A}$ the set U is open in M and the map φ is a homeomorphism; the topology τ consists of all subsets A of M such that $\varphi(U \cap A)$ is open in $\mathbb{R}^n$, for every local chart $\varphi : U \to \widetilde{U}$ that belongs to

$\mathcal{A}$. We call τ the topology *induced* by the atlas $\mathcal{A}$. If $\mathcal{A}$, $\mathcal{A}'$ are atlases on M and $\mathcal{A} \subset \mathcal{A}'$ then clearly $\mathcal{A}$ and $\mathcal{A}'$ induce the same topology on M.

DEFINITION A.1.1. An *n-dimensional differentiable manifold* (or simply a *differentiable manifold*) is a set M endowed with a maximal n-dimensional atlas $\mathcal{A}$ such that the topology induced by $\mathcal{A}$ on M is Hausdorff and second countable (i.e., admits a countable basis of open subsets).

We adopt the following convention: if M is a differentiable manifold with maximal atlas $\mathcal{A}$ then by a "local chart" of M we mean a local chart of M that belongs to $\mathcal{A}$.

A.1.1. Submanifolds. By "submanifold" we will always mean "embedded submanifold", unless otherwise stated. In some occasions, we will also talk about immersed submanifolds and almost embedded submanifolds. We list the definitions and some basic results (without proof) below.

DEFINITION A.1.2. Let M be an n-dimensional differentiable manifold. A subset N of M is said to be a *smooth submanifold* if there exists an integer k, $0 \le k \le n$, such that for all $x \in N$, there exists a local chart $\varphi : U \to \widetilde{U}$ such that $\varphi(U \cap N) = \widetilde{U} \cap \big(\mathbb{R}^k \oplus \{0\}^{n-k}\big)$ and $x \in U$. Such a local chart is said to be a *submanifold chart* for N.

If N is a smooth submanifold and we consider the restriction to N of all submanifold charts, we obtain an atlas for N that makes it into a k-dimensional differentiable manifold; moreover, the inclusion map of N in M is a smooth embedding, i.e., it is a smooth immersion and a homeomorphism onto its image[1]. It is well-known that if N, M are differentiable manifolds and $f : N \to M$ is a smooth embedding then $f(N)$ is a smooth submanifold of M and the map $f : N \to f(N)$ is a smooth diffeomorphism when $f(N)$ is endowed with the atlas obtained by restriction of the submanifold charts. Thus, smooth submanifolds are also called *embedded submanifolds*.

DEFINITION A.1.3. Let M be a differentiable manifold. By an *immersed submanifold* of M we mean a differentiable manifold N that is contained in M (as a set) and such that the inclusion map of N in M is a smooth immersion.

The following result is a well-known consequence of the local form of immersions:

PROPOSITION A.1.4. *Let P, M be differentiable manifolds and N be an immersed submanifold of M. If $f : P \to M$ is a smooth map with $f(P) \subset N$ and $f_0 : P \to N$ is obtained from f by restriction of counter-domain then f_0 is smooth if and only if it is continuous. In particular, if N is embedded in M then f is smooth if and only if f_0 is smooth.* $\square$

[1] The condition that the inclusion map of N in M is a homeomorphism onto its image means that the topology on N induced by the atlas coincides with the topology that N inherits from M.

The annoying fact about immersed submanifolds is that reduction of counter-domain does not necessarily maintain smoothness. However, the class of embedded submanifolds is not general enough for us to work with maximal integral submanifolds of involutive distributions and with Lie subgroups. Thus, we have the following intermediate situation.

DEFINITION A.1.5. Let M be a differentiable manifold and N be an immersed submanifold of M. We say that N is *almost embedded* if every point $x \in N$ has an open neighborhood U in M such that the connected component containing x with respect to the topology inherited from M of $U \cap N$ is an embedded submanifold of M.

We have the following:

PROPOSITION A.1.6. *Let M be a differentiable manifold, N be an almost embedded submanifold of M and X be a locally connected topological space. If $f : X \to M$ is a continuous map with $f(X) \subset N$ and if $f_0 : X \to N$ is obtained from f by reduction of counter-domain then f_0 is also continuous. In particular, by Proposition A.1.4, if X is a differentiable manifold and f is smooth then also f_0 is smooth.* $\square$

It turns out that integral submanifolds of involutive distributions are almost embedded (see Remark A.4.4). In particular, Lie subgroups of Lie groups are almost embedded, since they are integral submanifolds of distributions obtained by left translation of a Lie algebra.

A.1.2. Lie groups. Some basic knowledge of Lie groups is also assumed from the reader. A standard reference is [**12**]. We just give here the basic terminology.

A *Lie group* is a group G endowed with a differentiable structure $\mathcal{A}$ such that the group operations:

$$G \times G \ni (x, y) \longmapsto xy \in G, \quad G \ni x \longmapsto x^{-1} \in G$$

are smooth and G is a differentiable manifold, i.e., the topology induced from $\mathcal{A}$ is Hausdorff and second countable[2]. By a *Lie subgroup* of G we mean an immersed submanifold H of G that is also a subgroup of G and such that H is a Lie group endowed with the multiplication induced from G (actually, it is proven in [**12**] that if H is both an immersed submanifold of G and a subgroup of G then H is indeed a Lie subgroup of G). A Lie subgroup H of G is an embedded submanifold of G if and only if it is a closed subset of G; moreover, any closed subgroup H of G is automatically a Lie subgroup.

If G is a Lie group then for all $g \in G$, the left translation $L_g : G \to G$ and the right translation $R_g : G \to G$ are smooth diffeomorphisms. Informally speaking, any object that lives on G is said to be left (resp., right) invariant if it is preserved by all left (resp., right) translations. For instance, a vector field X on G is left

[2]Actually, Hausdorff is automatic because every T1 topological group is automatically Hausdorff. Moreover, second countability is equivalent to the assumption that G has a countable number of connected components; namely, this follows from the observation that if G is connected and U is a neighborhood of the identity $1 \in G$ then $G = \bigcup_{n=1}^{\infty} U^n$.

(resp., right) invariant if X is L_g-related (resp., R_g-related) to itself, for all $g \in G$ (see Definition A.2.1). Left (resp., right) invariant vector fields are automatically smooth. If $\mathfrak{g}$ denotes the tangent space to G at the unit element $1 \in G$ then the map $X \mapsto X(1)$ gives an isomorphism from the space of left invariant vector fields onto $\mathfrak{g}$. We endow $\mathfrak{g}$ with the Lie bracket operations by such isomorphism from the usual Lie bracket of vector fields (Proposition A.2.2 implies that the Lie bracket of left invariant vector fields is left invariant). The space $\mathfrak{g}$ endowed with such bracket operation is called the *Lie algebra* of $\mathfrak{g}$.

We have a natural left (resp., right) action of G on its tangent bundle obtained by differentiating the left (resp., right) action of G on itself by left (resp., right) translations; more explicitly, $g \in G$ acts on $v \in TG$ and gives $\mathrm{d}L_g(v)$ (resp., $\mathrm{d}R_g(v)$). We then set:

$$gv = \mathrm{d}L_g(v), \quad vg = \mathrm{d}R_g(v),$$

for all $g \in G$ and all $v \in TG$. Given $X \in \mathfrak{g}$ then the unique left invariant (resp., right invariant) vector field on G whose value at 1 is X is denoted by X^L (resp., X^R) and it is given by $X^L(g) = gX$ (resp., $X^R(g) = Xg$), for all $g \in G$.

Given Lie groups G, H, then by a *Lie group homomorphism* from G to H we mean a group homomorphism $f : G \to H$ which is smooth (actually, a continuous group homomorphism between Lie groups is automatically smooth). If f is a Lie group homomorphism then its differential $\bar{f} = \mathrm{d}f(1)$ at the identity gives us a Lie algebra homomorphism[3] from the Lie algebra $\mathfrak{g}$ of G to the Lie algebra $\mathfrak{h}$ of H. In particular, if H is a Lie subgroup of G then the inclusion map $i : H \to G$ is a Lie group homomorphism and the differential of i at the identity allows us to identify $\mathfrak{h}$ with a Lie subalgebra of $\mathfrak{g}$. For every $g \in G$, the inner automorphism $\mathcal{I}_g : G \to G$ is a Lie group isomorphism and its differential at the identity, denoted by $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra isomorphism. The map:

$$\mathrm{Ad} : G \ni g \longmapsto \mathrm{Ad}_g \in \mathrm{GL}(\mathfrak{g})$$

is known as the *adjoint representation* of G in $\mathfrak{g}$. Its differential at the identity is a linear map:

$$\mathrm{ad} : \mathfrak{g} \ni X \longmapsto \mathrm{ad}_X \in \mathfrak{gl}(\mathfrak{g})$$

called the *adjoint representation* of $\mathfrak{g}$ on itself. We have:

$$\mathrm{ad}_X(Y) = [X, Y],$$

for all $X, Y \in \mathfrak{g}$.

Given $X \in \mathfrak{g}$ then there exists exactly one Lie group homomorphism $\gamma : \mathbb{R} \to G$ with $\gamma'(0) = X$. This is called the *one-parameter subgroup* of G generated by X. The smooth curve γ is an integral curve of both X^L and X^R. The map $\exp : \mathfrak{g} \ni X \to \gamma(1) \in G$ is called the *exponential map* of G. The exponential map of a Lie group is smooth and for all $X \in \mathfrak{g}$, the corresponding one-parameter subgroup is given by $t \mapsto \exp(tX)$.

A distribution $\mathcal{D} \subset TG$ in a Lie group G is said to be left (resp., right) invariant if $\mathrm{d}L_g(\mathcal{D}) = \mathcal{D}$ (resp., $\mathrm{d}R_g(\mathcal{D}) = \mathcal{D}$), for all $g \in G$. A left or right invariant

[3] A Lie algebra homomorphism is a linear map that preserves Lie brackets.

distribution $\mathcal{D}$ on G is completely determined by the subspace $\mathcal{D}_1$ of $\mathfrak{g}$; moreover, left or right invariant distributions on a Lie group are automatically smooth. A left or right invariant distribution $\mathcal{D}$ on G is involutive (Definition A.4.2) if and only if $\mathcal{D}_1$ is a Lie subalgebra of $\mathfrak{g}$. If H is a Lie subgroup of G then H is an integral submanifold (Definition A.4.1) of the involutive left invariant distribution $\mathcal{D}$ on G with $\mathcal{D}_1 = \mathfrak{h}$. In particular (see Remark A.4.4), Lie subgroups are always almost embedded submanifolds.

A.2. Vector fields and flows

We continue our summary of the basic theory of differentiable manifolds. Again, more details can be found in [**1, 3, 9, 12**].

By a *vector field* on a differentiable manifold M we mean a section X of the tangent bundle TM, i.e., a map $X : M \to TM$ with $X(x) \in T_xM$, for all $x \in M$. In the terminology of Subsection 1.5.1, the space of all smooth vector fields on M is denoted by $\mathbf{\Gamma}(TM)$.

Given a vector field X in M and a smooth map $f : M \to \mathbb{R}$, we denote by $X(f) : M \to \mathbb{R}$ the map defined by $X(f)(x) = \mathrm{d}f(x) \cdot X(x)$, for all $x \in M$. We use such notation also if f takes values in a fixed real finite-dimensional vector space.

Given smooth vector fields $X, Y \in \mathbf{\Gamma}(TM)$ then there exists a unique vector field Z on M such that $Z(f) = X\big(Y(f)\big) - Y\big(X(f)\big)$. The vector field Z is smooth and it is called the *Lie bracket* of X and Y. We write $Z = [X, Y]$.

DEFINITION A.2.1. Let M, N be differentiable manifolds and $f : M \to N$ be a smooth map. We say that two vector fields $X \in \mathbf{\Gamma}(TM)$, $Y \in \mathbf{\Gamma}(TN)$ are *f-related* if:

$$Y\big(f(x)\big) = \mathrm{d}f_x\big(X(x)\big),$$

for all $x \in M$.

We recall the following:

PROPOSITION A.2.2. *Let M, N be differentiable manifolds, $f : M \to N$ be a smooth map and $X, X' \in \mathbf{\Gamma}(TM)$, $Y, Y' \in \mathbf{\Gamma}(TN)$ be vector fields. Assume that Y is f-related with X and that Y' is f-related with X'. Then $[Y, Y']$ is f-related with $[X, X']$.*

PROOF. See [**12**]. □

Let G be a Lie group and N be a differentiable manifold; assume that we are given a (left or right) smooth action of G on N.

DEFINITION A.2.3. Given a vector X in the Lie algebra $\mathfrak{g}$ of G, we denote by X^N the *induced vector field* on the differentiable manifold N defined by:

$$X^N(n) = \mathrm{d}\beta_n(1) \cdot X \in T_nN,$$

for all $n \in N$, where $\beta_n : G \to N$ is the map given by action on the element n.

It can be shown that the induced vector field X^N is smooth.

The following result was used in the proof of Lemma 2.3.3.

LEMMA A.2.4. *Let G be a Lie group and N be a differentiable manifold; assume that we are given a (left or right) smooth action of G on N. Assume that the action of G is effective on a subset A of N. Then, given $X \in \mathfrak{g}$, if $X^N(n) = 0$ for all $n \in A$ then $X = 0$.*

PROOF. If $X^N(n) = 0$ then $\exp(tX)$ is in the isotropy group G_n, for all $t \in \mathbb{R}$. Thus, if $X^N(n) = 0$ for all $n \in A$ then $\exp(tX) \in \bigcap_{n\in A} G_n = \{1\}$, for all $t \in \mathbb{R}$. Hence $X = 0$. $\square$

DEFINITION A.2.5. Let $\pi : \mathcal{E} \to M$ be a smooth submersion and let $\mathrm{Hor}(\mathcal{E})$ be a generalized connection on $\mathcal{E}$ with respect to π. Given a vector field X on M then the *horizontal lift* of X is the unique vector field X^{hor} on $\mathcal{E}$ such that $X^{\mathrm{hor}}(e) \in \mathrm{Hor}_e(\mathcal{E})$ and:

$$\mathrm{d}\pi_e\big(X^{\mathrm{hor}}(e)\big) = X\big(\pi(e)\big),$$

for all $e \in \mathcal{E}$.

It can be shown that if X is smooth then X^{hor} is also smooth.

DEFINITION A.2.6. Let M be a differentiable manifold and $X \in \mathbf{\Gamma}(TM)$ be a smooth vector field on M. By an *integral curve* of X we mean a smooth map $\gamma : I \to M$ defined in an interval $I \subset \mathbb{R}$ with:

$$\gamma'(t) = X\big(\gamma(t)\big),$$

for all $t \in I$. Given $x_0 \in M$ then a *maximal integral curve of X through x_0* is an integral curve $\gamma : I \to M$ of X such that:

- $0 \in I$ and $\gamma(0) = x_0$;
- if $\mu : J \to M$ is an integral curve of X with $0 \in J$ and $\mu(0) = x_0$ then $J \subset I$ and $\mu = \gamma|_J$.

THEOREM A.2.7 (maximal flow of a vector field). *Let M be a differentiable manifold and $X \in \mathbf{\Gamma}(TM)$ be a smooth vector field on M. Then:*

- *for each $x_0 \in M$, there exists a unique maximal integral curve of X through x_0 and its domain is an open interval;*
- *if $\gamma_{x_0} : I_{x_0} \to M$ denotes the maximal integral curve of X through x_0 then the set:*

$$\bigcup_{x_0\in M} \big(\{x_0\} \times I_{x_0}\big) = \big\{(x_0, t) : x_0 \in M,\ t \in I_{x_0}\big\}$$

is open in $M \times \mathbb{R}$ and the maximal flow *of X defined by:*

$$F^X : \bigcup_{x_0\in M} \big(\{x_0\} \times I_{x_0}\big) \ni (x_0, t) \longmapsto \gamma_{x_0}(t) \in M$$

is a smooth map.

PROOF. See [**1**]. $\square$

COROLLARY A.2.8. *Let M be a differentiable manifold and $X \in \mathbf{\Gamma}(TM)$ be a smooth vector field on M. If $\gamma_1 : I \to M$, $\gamma_2 : I \to M$ are integral curves of X with $\gamma_1(t_0) = \gamma_2(t_0)$ for some $t_0 \in I$ then $\gamma_1 = \gamma_2$.*

PROOF. Observe that both $t \mapsto \gamma_1(t+t_0)$ and $t \mapsto \gamma_2(t+t_0)$ are restrictions of the maximal integral curve of X through $x_0 = \gamma_1(t_0)$. □

DEFINITION A.2.9. Let M be a differentiable manifold. By a *time dependent vector field* on M we mean a map $X : A \to TM$ such that $X(t,x) \in T_xM$, for all $(t,x) \in A$, where A is an open subset of $\mathbb{R} \times M$.

In other words, a time dependent vector field on M, is a local section of TM along the projection $\mathbb{R} \times M \ni (t,x) \mapsto x \in M$. We have a version of Definition A.2.6 for time-dependent vector fields:

DEFINITION A.2.10. Let M be a differentiable manifold and $X : A \subset \mathbb{R} \times M \to TM$ be a smooth time-dependent vector field on M. By an *integral curve* of X we mean a smooth map $\gamma : I \to M$ defined in an interval $I \subset \mathbb{R}$ with $\big(t, \gamma(t)\big) \in A$ and:

$$\gamma'(t) = X\big(t, \gamma(t)\big),$$

for all $t \in I$. Given $(t_0, x_0) \in A$ then a *maximal integral curve of X through* (t_0, x_0) is an integral curve $\gamma : I \to M$ of X such that:

- $t_0 \in I$ and $\gamma(t_0) = x_0$;
- if $\mu : J \to M$ is an integral curve of X with $t_0 \in J$ and $\mu(t_0) = x_0$ then $J \subset I$ and $\mu = \gamma|_J$.

Theorem A.2.7 generalizes to time-dependent vector fields, as follows:

THEOREM A.2.11 (maximal flow of a time-dependent vector field). *Let M be a differentiable manifold and $X : A \subset \mathbb{R}\times M \to TM$ be a smooth time-dependent vector field on M. Then:*

- *for each $(t_0, x_0) \in A$, there exists a unique maximal integral curve of X through (t_0, x_0) and its domain is an open interval;*
- *if $\gamma_{(t_0,x_0)} : I_{(t_0,x_0)} \to M$ denotes the maximal integral curve of X through (t_0, x_0) then the set:*

$$\bigcup_{(t_0,x_0)\in A} \big(\{(t_0,x_0)\} \times I_{(t_0,x_0)}\big) = \big\{(t_0,x_0,t) : (t_0,x_0) \in A,\ t \in I_{(t_0,x_0)}\big\}$$

is open in $\mathbb{R} \times M \times \mathbb{R}$ and the maximal flow *of X defined by:*

$$F^X : \bigcup_{(t_0,x_0)\in A} \big(\{(t_0,x_0)\} \times I_{(t_0,x_0)}\big) \ni (t_0,x_0,t) \longmapsto \gamma_{(t_0,x_0)}(t) \in M$$

is a smooth map.

PROOF. Follows by applying Theorem A.2.7 to the smooth vector field:

$$A \ni (t,x) \longmapsto \big(1, X(t,x)\big) \in \mathbb{R} \oplus T_xM \cong T_{(1,x)}A$$

on the open set A. □

COROLLARY A.2.12. *Let M be a differentiable manifold and $X : A \subset \mathbb{R} \times M \to TM$ be a smooth time-dependent vector field on M. If $\gamma_1 : I \to M$, $\gamma_2 : I \to M$ are integral curves of X with $\gamma_1(t_0) = \gamma_2(t_0)$ for some $t_0 \in I$ then $\gamma_1 = \gamma_2$.*

PROOF. Observe that both γ_1 and γ_2 are restrictions of the maximal integral curve of X through (t_0, x_0), where $x_0 = \gamma_1(t_0) = \gamma_2(t_0)$. $\square$

DEFINITION A.2.13. Let M be a differentiable manifold, $I \subset \mathbb{R}$ be an open interval and $X : I \times M \to TM$ be a time-dependent vector field. We say that X is *spatially homogeneous* if for every $x, y \in M$ there exists a smooth diffeomorphism $f : M \to M$ such that $f(x) = y$ and:

$$X\big(t, f(z)\big) = \mathrm{d}f_z\big(X(t, z)\big), \tag{A.2.1}$$

for all $t \in I$, $z \in M$.

Observe that if a smooth diffeomorphism $f : M \to M$ satisfying (A.2.1) for all $t \in I$, $z \in M$ and if $\gamma : J \to M$ is an integral curve of X then $f \circ \gamma$ is also an integral curve of X.

LEMMA A.2.14. *Let M be a differentiable manifold, $I \subset \mathbb{R}$ be an open interval and $X : I \times M \to TM$ be a smooth time-dependent spatially homogeneous vector field. Then the domain of the maximal flow of X is $I \times M \times I$, i.e., for every $(t_0, x_0) \in I \times M$, the domain of the maximal integral curve of X through (t_0, x_0) is I.*

PROOF. Let $\gamma : J \to M$ be the maximal integral curve through (t_0, x_0); then $t_0 \in J$ and J is an open interval contained in I. Assume by contradiction that J is properly contained in I; then, for instance, $b = \sup J$ is in I. The triple (b, x_0, b) is obviously in the domain A of the maximal flow of X (recall Theorem A.2.11). Since A is open in $\mathbb{R} \times M \times \mathbb{R}$, there exists $\varepsilon > 0$ such that $]b-\varepsilon, b+\varepsilon[\times \{x_0\} \times]b-\varepsilon, b+\varepsilon[$ is contained in A; we can also take $\varepsilon > 0$ sufficiently small so that $b - \frac{\varepsilon}{2} \in J$. By the definition of A, there exists an integral curve μ of X through $\big(b-\frac{\varepsilon}{2}, x_0\big)$ defined on the interval $]b-\varepsilon, b+\varepsilon[$. Since X is spatially homogeneous, there exists a smooth diffeomorphism $f : M \to M$ such that (A.2.1) holds for all $t \in I$, $z \in M$ and such that $f(x_0) = \gamma\big(b - \frac{\varepsilon}{2}\big)$. Then $f \circ \mu$ is an integral curve of X with $(f \circ \mu)\big(b - \frac{\varepsilon}{2}\big) = \gamma\big(b - \frac{\varepsilon}{2}\big)$ and, therefore, by Corollary A.2.12, γ and $f \circ \mu$ are equal on the intersection of their domains. We can therefore use $f \circ \mu$ to extend γ to an integral curve of X defined in $J \cup]b-\varepsilon, b+\varepsilon[$, which contradicts the maximality of γ. This concludes the proof. $\square$

COROLLARY A.2.15. *Let G be a Lie group, $I \subset \mathbb{R}$ be an interval and $X : I \to \mathfrak{g}$ be a smooth curve in the Lie algebra $\mathfrak{g}$ of G. Then for every $t_0 \in I$ and every $g_0 \in G$ there exists a smooth curve $g : I \to G$ such that $g(t_0) = g_0$ and:*

$$g'(t) = X(t)g(t), \tag{A.2.2}$$

for all $t \in I$.

PROOF. We can assume without loss of generality that I is open; otherwise, take an arbitrary smooth extension of X to an open interval. Clearly, g satisfies (A.2.2) if and only if g is an integral curve of the time-dependent vector field given by:

$$I \times G \ni (t, h) \longmapsto X(t)h \in T_hG.$$

Such vector field is smooth and spatially homogeneous; namely, the smooth diffeomorphism f that is required by Definition A.2.13 can be taken to be a right translation. The conclusion follows from Lemma A.2.14. $\square$

A.3. Differential forms

DEFINITION A.3.1. Let $V_1, \dots, V_k, V_1', \dots, V_l', W, W'$ be real finite-dimensional vector spaces and let $B : V_1 \times \cdots \times V_k \to W$, $B' : V_1' \times \cdots \times V_l' \to W'$ be multilinear maps. Suppose that a bilinear map:

$$W \times W' \ni (w, w') \longmapsto w \cdot w' \in W'' \tag{A.3.1}$$

is fixed, where W'' is a real finite-dimensional vector space. We define the *tensor product* of B by B' (with respect to the *bilinear pairing* (A.3.1)) to be the multilinear map $(B \otimes B') : V_1 \times \cdots \times V_k \times V_1' \times \cdots \times V_l' \to W''$ given by:

$$(B \otimes B')(v_1, \dots, v_k, v_1', \dots, v_l') = B(v_1, \dots, v_k) \cdot B'(v_1', \dots, v_l'),$$

for all $v_1 \in V_1, \dots, v_k \in V_k, v_1' \in V_1', \dots, v_l' \in V_l'$.

Denote by S_k the symmetric group on k elements (see Example 1.1.2). Given $\sigma \in S^k$ and a multilinear map $B \in \mathrm{Lin}_k(V, W)$, we denote by $\sigma \cdot B$ the multilinear map defined by:

$$(\sigma \cdot B)(v_0, \dots, v_{k-1}) = B(v_{\sigma(0)}, \dots, v_{\sigma(k-1)}),$$

for all $v_0, \dots, v_{k-1} \in V$. The *alternator* of B is defined by:

$$\mathrm{Alt}(B) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)(\sigma \cdot B).$$

Clearly, $\mathrm{Alt}(B)$ is always alternating; moreover, if B is alternating then $\mathrm{Alt}(B) = (k!)B$.

REMARK A.3.2. Clearly, we have:

$$\mathrm{Alt}(\sigma \cdot B) = \mathrm{sgn}(\sigma)\mathrm{Alt}(B).$$

Thus, if there exists an odd permutation σ with $\sigma \cdot B = B$ then $\mathrm{Alt}(B) = 0$; in particular, if B is symmetric with respect to some pair of variables then $\mathrm{Alt}(B) = 0$.

Given $B \in \mathrm{Lin}_k^{\mathrm{a}}(V, W)$, $B' \in \mathrm{Lin}_l^{\mathrm{a}}(V, W')$, then the tensor product $B \otimes B'$ is not in general antisymmetric. We give the following:

DEFINITION A.3.3. Let V, W, W', W'' be real finite-dimensional vector spaces, $B \in \mathrm{Lin}_k^{\mathrm{a}}(V, W)$, $B' \in \mathrm{Lin}_l^{\mathrm{a}}(V, W')$ and assume that we are given a bilinear pairing (A.3.1). The *exterior product* (or *wedge product*) of B by B' (with respect to (A.3.1)) is defined by:

$$B \wedge B' = \frac{1}{k!\, l!} \mathrm{Alt}(B \otimes B') \in \mathrm{Lin}_{k+l}^{\mathrm{a}}(V, W'').$$

The reader should recall from Example 1.6.7 the notion of (vector valued) differential forms.

We now recall the definition of exterior differential of a smooth differential form. We start with the case of open subsets of $\mathbb{R}^n$. A differential k-form on an open subset U of $\mathbb{R}^n$ can be identified with a map ω from U to the vector space $\mathrm{Lin}_k^{\mathrm{a}}(\mathbb{R}^n, \mathbb{R})$. If ω is a smooth differential k-form on U then we can consider the standard differential of ω at a point $x \in U$, denoted by $\mathbb{d}\omega_x$, is a linear map from $\mathbb{R}^n$ to $\mathrm{Lin}_k^{\mathrm{a}}(\mathbb{R}^n, \mathbb{R})$ that can be identified with the $(k+1)$-linear map:

$$\mathbb{R}^n \times \cdots \times \mathbb{R}^n \ni (v_0, \ldots, v_k) \longmapsto \mathbb{d}\omega_x(v_0) \cdot (v_1, \ldots, v_k) \in \mathbb{R}.$$

Thus $\mathbb{d}\omega : x \mapsto \mathbb{d}\omega_x$ is a smooth map from U to $\mathrm{Lin}_{k+1}(\mathbb{R}^n, \mathbb{R})$. We set:

$$\mathrm{d}\omega_x = \frac{1}{k!}\mathrm{Alt}(\mathbb{d}\omega_x),$$

for all $x \in U$, so that $\mathrm{d}\omega$ is a smooth $(k+1)$-form on U. The following results are standard:

(a) if ω, λ are respectively a k-form and an l-form on U then $\mathrm{d}(\omega^\lambda) = (\mathrm{d}\omega)^\lambda + (-1)^k \omega \wedge \mathrm{d}\lambda$;
(b) $\mathrm{d}\mathrm{d}\omega = 0$;
(c) if V is an open subset of $\mathbb{R}^m$, U is an open subset of $\mathbb{R}^n$, $f : V \to U$ is a smooth map and ω is a smooth differential k-form on U then $\mathrm{d}(f^*\omega) = f^*\mathrm{d}\omega$.

Property (c) allows us to define the exterior differential of a smooth k-form ω on a differentiable manifold M so that if $\varphi : U \to \widetilde{U} \subset \mathbb{R}^n$ is a local chart of M then $\mathrm{d}\omega|_U = \varphi^*\mathrm{d}\big((\varphi^{-1})^*\omega\big)$. With such definition, properties (a), (b) and (c) also hold for the exterior differential of forms on manifolds.

A direct formula for computing the exterior differential of a k-form on a manifold is given by the following:

LEMMA A.3.4 (Cartan's formula for exterior differential). *Let λ be a (possibly vector valued) smooth k-form on a differentiable manifold M. Given vector fields $X_0, \ldots, X_k \in \mathbf{\Gamma}(TM)$, then:*

$$\text{(A.3.2)}\quad \mathrm{d}\lambda(X_0, X_1, \ldots, X_k) = \sum_{i=0}^{k}(-1)^i X_i\big(\lambda(X_0, \ldots, \widehat{X_i}, \ldots, X_k)\big) + \sum_{i<j}(-1)^{i+j}\,\lambda\big([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k\big),$$

where the hat indicates that the term is omitted in the list.

PROOF. See [**12**]. □

Cartan's formula for exterior differentiation becomes:

$$\text{(A.3.3)}\qquad \mathrm{d}\lambda(X, Y) = X\big(\lambda(Y)\big) - Y\big(\lambda(X)\big) - \lambda\big([X, Y]\big).$$

for 1-forms.

A.4. The Frobenius theorem

DEFINITION A.4.1. Let M be a differentiable manifold and $\mathcal{D} \subset TM$ be a smooth distribution on M. By an *integral submanifold* of $\mathcal{D}$ we mean an immersed submanifold $S \subset M$ such that $T_xS = \mathcal{D}_x$, for all $x \in S$. We say that $\mathcal{D}$ is *integrable* if for every $x \in M$ there exists an integral submanifold S of $\mathcal{D}$ with $x \in S$.

Observe that if $\mathcal{D}$ is integral then for every $x \in M$ there exists an *embedded* integral submanifold S of $\mathcal{D}$ with $x \in S$; namely, if S is an immersed integral submanifold of $\mathcal{D}$ with $x \in S$ then a sufficiently small open neighborhood of x in S is an embedded integral submanifold of $\mathcal{D}$.

DEFINITION A.4.2. A smooth distribution $\mathcal{D} \subset TM$ is said to be *involutive* if for all $X, Y \in \mathbf{\Gamma}(\mathcal{D})$ the Lie bracket $[X, Y]$ is in $\mathbf{\Gamma}(\mathcal{D})$.

THEOREM A.4.3 (Frobenius). *Let M be a differentiable manifold. A smooth distribution $\mathcal{D} \subset TM$ on M is involutive if and only if it is integrable.*

PROOF. See [**12**]. □

REMARK A.4.4. If $\mathcal{D}$ is a smooth involutive distribution in a differentiable manifold M then every integral submanifold $S \subset M$ of $\mathcal{D}$ is almost embedded in M.

Let M be a differentiable manifold and $\mathcal{D} \subset TM$ be a smooth distribution on M. By a *maximal integral submanifold* of $\mathcal{D}$ we mean a connected immersed integral submanifold $S \subset M$ of $\mathcal{D}$ which is not properly contained in any connected immersed integral submanifold of $\mathcal{D}$.

THEOREM A.4.5 (Global Frobenius). *Let M be a differentiable manifold and $\mathcal{D} \subset TM$ be a smooth involutive distribution on M. Then for every $x \in M$ there exists a unique maximal integral submanifold $S \subset M$ of $\mathcal{D}$ with $x \in S$.*

PROOF. See [**12**]. □

REMARK A.4.6. If $\mathcal{D}$ is a smooth involutive distribution in a differentiable manifold M and if $S \subset M$ is a maximal integral submanifold of $\mathcal{D}$ then the following condition holds: if $\gamma : [a, b] \to M$ is a smooth curve such that $\gamma'(t) \in \mathcal{D}_{\gamma(t)}$, for every $t \in [a, b]$ and if $\gamma(t_0) \in S$ for some $t_0 \in [a, b]$ then $\gamma(t) \in S$, for all $t \in [a, b]$.

Frobenius theorem can be seen as a result concerning the existence of solutions of a certain class of first order partial differential equations called *total PDEs*; informally speaking, a total PDE is an equation (on the unknown f) of the form $\mathrm{d}f_x = F\big(x, f(x)\big)$. The relation between solutions of total PDEs and integral submanifolds of distributions is that the graph of a solution of a total PDE is an integral submanifold of an appropriate distribution. We now study a particular case of this situation in the language of differential forms.

Let M, N be differentiable manifolds, V be a real finite-dimensional vector space and λ^M, λ^N, be V-valued smooth 1-forms on M and on N respectively;

assume that $\lambda^N_y : T_yN \to V$ is an isomorphism, for all $y \in N$. We are interested in finding smooth maps $f : U \to N$ defined on an open subset U of M with:

$$f^*\lambda^N = \lambda^M|_U. \tag{A.4.1}$$

Notice that (A.4.1) is equivalent to:

$$\mathrm{d}f(x) = \tau_{xf(x)}, \quad x \in U,$$

where, for $x \in M$, $y \in N$, $\tau_{xy} : T_xM \to T_yN$ denotes the linear map defined by:

$$\tau_{xy} = (\lambda^N_y)^{-1} \circ \lambda^M_x. \tag{A.4.2}$$

Consider the smooth distribution $\mathcal{D}$ on $M \times N$ defined by:

$$\mathcal{D}_{(x,y)} = \mathrm{Gr}(\tau_{xy}) \subset T_xM \oplus T_yN \cong T_{(x,y)}(M \times N), \tag{A.4.3}$$

for all $x \in M$, $y \in N$. Clearly, a smooth map $f : U \to N$ defined on an open subset U of M satisfies (A.4.1) if and only if the graph of f is an integral submanifold of $\mathcal{D}$. Hence, the existence of a map f satisfying (A.4.1) can be obtained as an application of the Frobenius theorem.

We have the following:

PROPOSITION A.4.7. *Let M, N be differentiable manifolds, V be a real finite-dimensional vector space and λ^M, λ^N, be V-valued smooth 1-forms on M and on N respectively; assume that $\lambda^N_y : T_yN \to V$ is an isomorphism, for all $y \in N$. The following conditions are equivalent:*

(a) *for all $x \in M$, $y \in N$ there exists a smooth map $f : U \to N$ defined in an open neighborhood U of x in M with $f(x) = y$ such that* (A.4.1) *holds;*
(b) *for all $x \in M$, $y \in N$, $\tau^*_{xy}\mathrm{d}\lambda^N_y = \mathrm{d}\lambda^M_x$, where $\tau_{xy} : T_xM \to T_yN$ is the linear map defined in* (A.4.2).

PROOF. The fact that (a) implies (b) follows by taking the exterior differential in both sides of (A.4.1). Now assume (b). Consider the smooth V-valued 1-form θ on $M \times N$ defined by:

$$\theta = \mathrm{pr}_2^*\lambda^N - \mathrm{pr}_1^*\lambda^M,$$

where pr_1, pr_2 denote the projections of the cartesian product $M \times N$. Clearly, for all $(x, y) \in M \times N$, $\theta_{(x,y)} : T_xM \oplus T_yN \to V$ is surjective and its kernel equals (A.4.3). By Lemma A.4.8, $\mathcal{D}$ is involutive if and only if $\mathrm{d}\theta_{(x,y)}$ vanishes on $\mathcal{D}_{(x,y)} \times \mathcal{D}_{(x,y)}$, for all $(x, y) \in M \times N$; but clearly such condition is equivalent to (b). To conclude the proof, let $x \in M$, $y \in N$ be fixed and let $S \subset M \times N$ be an integral submanifold of $\mathcal{D}$ with $(x, y) \in S$. Since the first projection $T_xM \oplus T_yN \to T_xM$ carries $\mathcal{D}_{(x,y)}$ isomorphically onto T_xM then, by taking S to be sufficiently small, we may assume that the map $\mathrm{pr}_1|_S : S \to M$ is a smooth diffeomorphism onto an open neighborhood U of x in M. The map $f : U \to N$ defined by:

$$f = \mathrm{pr}_2 \circ (\mathrm{pr}_1|_S)^{-1} \tag{A.4.4}$$

is therefore smooth and its graph equals S. Hence f satisfies (A.4.1). $\square$

LEMMA A.4.8. *Let Q be a differentiable manifold, V be a real finite-dimensional vector space and θ be a smooth V-valued 1-form on Q. Assume that $\theta_x : T_xQ \to V$ is surjective, for all $x \in Q$. Then the smooth distribution $\mathcal{D} = \mathrm{Ker}(\theta)$ is involutive if and only if $\mathrm{d}\theta_x$ vanishes on $\mathcal{D}_x \times \mathcal{D}_x$, for all $x \in Q$.*

PROOF. Follows easily from Cartan's formula for exterior differentiation (see (A.3.3)). □

We now wish to prove a global version of Proposition A.4.7. To this aim, we use the technique of "solving the total PDE (A.4.1) along curves on M".

If $f : U \subset M \to N$ is a smooth function satisfying (A.4.1) and if $\gamma : I \to U$ is an arbitrary smooth curve then the smooth curve $\mu = f \circ \gamma : I \to N$ satisfies:

$$\lambda^N_{\mu(t)}\big(\mu'(t)\big) = \lambda^M_{\gamma(t)}\big(\gamma'(t)\big), \tag{A.4.5}$$

for all $t \in I$. Clearly (A.4.5) is equivalent to:

$$\mu'(t) = \tau_{\gamma(t)\mu(t)}\big(\gamma'(t)\big). \tag{A.4.6}$$

Notice that $\mu : I \to N$ satisfies (A.4.6) for all $t \in I$ if and only if μ is an integral curve of the smooth time-dependent vector field on N defined by[4]:

$$I \times N \ni (t, y) \longmapsto \tau_{\gamma(t)y}\big(\gamma'(t)\big) \in TN. \tag{A.4.7}$$

We can now prove a uniqueness result for solutions of (A.4.1).

LEMMA A.4.9. *Let M, N be differentiable manifolds, V be a real finite-dimensional vector space and λ^M, λ^N, be V-valued smooth 1-forms on M and on N respectively; assume that M is connected and that $\lambda^N_y : T_yN \to V$ is an isomorphism, for all $y \in N$. If $f_1 : M \to N$, $f_2 : M \to N$ are smooth maps with $f_1^*\lambda^N = \lambda^M$, $f_2^*\lambda^N = \lambda^M$ and if $f_1(x_0) = f_2(x_0)$ for some $x_0 \in M$ then $f_1 = f_2$.*

PROOF. If $\gamma : I \to M$ is a smooth curve such that $f_1\big(\gamma(t_0)\big) = f_2\big(\gamma(t_0)\big)$ for some $t_0 \in I$ then $f_1 \circ \gamma = f_2 \circ \gamma$; namely, both $f_1 \circ \gamma$ and $f_2 \circ \gamma$ are integral curves of the smooth time-dependent vector field (A.4.7) (see Corollary A.2.12). The conclusion follows from the observation that, since M is connected, every two points of M can be joined by a piecewise smooth curve. □

Finally, we have the following global result:

PROPOSITION A.4.10. *Let M, N be differentiable manifolds, V be a real finite-dimensional vector space and λ^M, λ^N, be V-valued smooth 1-forms on M and on N respectively. Assume that:*

(a) *$\lambda^N_y : T_yN \to V$ is an isomorphism, for all $y \in N$;*
(b) *condition (b) in the statement of Proposition A.4.7 holds;*
(c) *M is simply-connected;*

[4]If the interval I is not open, we can always consider a smooth extension of γ to an open interval I' containing I so that the time-dependent vector field (A.4.7) is defined on the open subset $I' \times N$ of $\mathbb{R} \times N$.

(d) *there exists a rich family* $\mathcal{C}$ *of smooth curves* $\gamma : [0,1] \to M$ *such that for every* γ *in* $\mathcal{C}$ *and every* $y \in N$ *there exists a smooth curve* $\mu : [0,1] \to N$ *such that* $\mu(0) = y$ *and* (A.4.5) *holds, for all* $t \in [0,1]$.

Then, for all $x_0 \in M$, $y_0 \in N$, *there exists a smooth map* $f : M \to N$ *with* $f(x_0) = y_0$ *such that* $f^*\lambda^N = \lambda^M$.

PROOF. We may assume without loss of generality that M is connected. Consider the smooth distribution $\mathcal{D}$ on $M \times N$ defined by (A.4.3). As in the proof of Proposition A.4.7, we see that $\mathcal{D}$ is involutive. Let $x_0 \in M$, $y_0 \in N$ be fixed and let $S \subset M \times N$ be a maximal integral submanifold of $\mathcal{D}$ containing (x_0, y_0). Since for all $(x,y) \in S$, the first projection $T_xM \oplus T_yN \to T_xM$ maps $T_{(x,y)}S = \mathcal{D}_{(x,y)}$ isomorphically onto T_xM, we have that the map $\mathrm{pr}_1|_S : S \to M$ is a smooth local diffeomorphism, where $\mathrm{pr}_1 : M \times N \to M$ denotes the first projection. We will now use Corollary B.3.13 to establish that $\mathrm{pr}_1|_S : S \to M$ is a covering map. To this aim, we have to check that every $\gamma : [0,1] \to M$ in $\mathcal{C}$ admits liftings with arbitrary initial conditions with respect to $\mathrm{pr}_1|_S$. Let $\gamma : [0,1] \to M$ in $\mathcal{C}$ be fixed and let $(x,y) \in S$ be such that $\mathrm{pr}_1(x,y) = x = \gamma(0)$. By our hypotheses, there exists a smooth curve $\mu : [0,1] \to N$ such that $\mu(0) = y$ and (A.4.5) holds, for all $t \in [0,1]$. Clearly $\big(\gamma'(t), \mu'(t)\big) \in \mathcal{D}_{(\gamma(t),\mu(t))}$, for all $t \in [0,1]$ so that, by Remark A.4.6, the image of the curve $(\gamma, \mu) : [0,1] \to M \times N$ is contained in S. Since S is almost embedded in $M \times N$ (Remark A.4.4), the curve $(\gamma,\mu) : [0,1] \to S$ is continuous and it is therefore a lifting of γ with initial condition (x,y). This concludes the proof that $\mathrm{pr}_1|_S : S \to M$ is a covering map. Since M is connected and simply-connected, Corollary B.3.19 implies that $\mathrm{pr}_1|_S : S \to M$ is a homeomorphism and hence a smooth diffeomorphism. The conclusion is now obtained by defining $f : M \to N$ as in (A.4.4). $\square$

A.5. Horizontal liftings of curves

DEFINITION A.5.1. Let $\mathcal{E}$, M be differentiable manifolds, $\pi : \mathcal{E} \to M$ be a smooth submersion and $\mathrm{Hor}(\mathcal{E})$ be a generalized connection on $\mathcal{E}$ with respect to π (recall Definition 2.1.1). A smooth curve $\tilde\gamma : I \to \mathcal{E}$ is said to be *horizontal* with respect to $\mathrm{Hor}(\mathcal{E})$ if $\tilde\gamma'(t) \in \mathrm{Hor}_{\tilde\gamma(t)}(\mathcal{E})$, for all $t \in I$. If $\gamma : I \to M$ is a smooth curve then a *horizontal lifting* of γ with respect to π and $\mathrm{Hor}(\mathcal{E})$ is a smooth curve $\tilde\gamma : I \to \mathcal{E}$ which is horizontal with respect to $\mathrm{Hor}(\mathcal{E})$ and satisfies $\pi \circ \tilde\gamma = \gamma$.

LEMMA A.5.2. *Let* $\pi : \mathcal{E} \to M$ *be a smooth submersion and* $\mathrm{Hor}(\mathcal{E})$ *be a generalized connection on* $\mathcal{E}$ *with respect to* π. *If* $\tilde\gamma_1 : I \to \mathcal{E}$, $\tilde\gamma_2 : I \to \mathcal{E}$ *are both horizontal liftings of a smooth curve* $\gamma : I \to M$ *and if* $\tilde\gamma_1(t_0) = \tilde\gamma_2(t_0)$ *for some* $t_0 \in I$ *then* $\tilde\gamma_1 = \tilde\gamma_2$.

PROOF. If the interval I is not open, we start by considering a smooth extension of γ to an open interval I' containing I; such extension will still be denoted by γ. Consider the pull-back $\pi_1 : \gamma^*\mathcal{E} \to I'$ endowed with the generalized connection $\mathrm{Hor}(\gamma^*\mathcal{E})$ obtained from $\mathrm{Hor}(\mathcal{E})$ by pull-back and denote by $\bar\gamma : \gamma^*\mathcal{E} \to \mathcal{E}$ the canonical map. Let X be the vector field on I' such that $X(t) = 1 \in T_tI'$ for all

$t \in I'$ and let $X^{\rm hor}$ be the vector field on $\gamma^*\mathcal{E}$ obtained from X by horizontal lift. Using the property of pull-backs described in diagram (1.17), we obtain smooth curves $(\tilde\gamma_1)^{\leftarrow} : I \to \gamma^*\mathcal{E}$, $(\tilde\gamma_2)^{\leftarrow} : I \to \gamma^*\mathcal{E}$, with $\pi_1 \circ (\tilde\gamma_i)^{\leftarrow} : I \to I'$ the inclusion map of I in I' and with $\bar\gamma \circ (\tilde\gamma_i)^{\leftarrow} = \gamma_i$, $i = 1, 2$. Since $\tilde\gamma_i$ is parallel, also $(\tilde\gamma_i)^{\leftarrow}$ is parallel (Lemma 2.1.12); thus, for all $t \in I'$, we have:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\tilde\gamma_i)^{\leftarrow}(t) \in \mathrm{Hor}(\gamma^*\mathcal{E}), \quad \mathrm{d}\pi_1\big(\tfrac{\mathrm{d}}{\mathrm{d}t}(\tilde\gamma_i)^{\leftarrow}(t)\big) = 1 = X(t),$$

so that:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\tilde\gamma_i)^{\leftarrow}(t) = X^{\rm hor}\big((\tilde\gamma_i)^{\leftarrow}(t)\big),$$

and $(\tilde\gamma_i)^{\leftarrow}$ is an integral curve of $X^{\rm hor}$, $i = 1, 2$. Since:

$$(\tilde\gamma_1)^{\leftarrow}(t_0) = \big(t_0, \tilde\gamma_1(t_0)\big) = \big(t_0, \tilde\gamma_2(t_0)\big) = (\tilde\gamma_2)^{\leftarrow}(t_0),$$

it follows from Corollary A.2.8 that $(\tilde\gamma_1)^{\leftarrow} = (\tilde\gamma_2)^{\leftarrow}$. Hence $\tilde\gamma_1 = \tilde\gamma_2$. □

In the terminology introduced in Definition 2.1.7, a smooth curve $\tilde\gamma : I \to \mathcal{E}$ is horizontal if and only if it is parallel. If $\tilde\gamma$ is a horizontal lift of γ with $\tilde\gamma(t_0) = e$ we say also that $\tilde\gamma$ is the *parallel transport* of e along γ.

LEMMA A.5.3. *Let $\pi : \mathcal{E} \to M$ be a smooth submersion and $\mathrm{Hor}(\mathcal{E})$ be a generalized connection on $\mathcal{E}$ with respect to π. If $\gamma : I \to M$ is a smooth curve then for all $t_0 \in I$ and all $e \in \pi^{-1}\big(\gamma(t_0)\big)$ there exists an open connected neighborhood J of t_0 in I and a horizontal lift $\tilde\gamma : J \to \mathcal{E}$ of $\gamma|_J$ with $\tilde\gamma(t_0) = e$.*

PROOF. By considering an extension of γ, we can assume that I is open. Define $\pi_1 : \gamma^*\mathcal{E} \to I$, $\bar\gamma : \gamma^*\mathcal{E} \to \mathcal{E}$, X and $X^{\rm hor}$ as in the proof of Lemma A.5.2. Let $\eta : J \to \gamma^*\mathcal{E}$ be an integral curve of $X^{\rm hor}$ with $\eta(t_0) = (t_0, e) \in \gamma^*\mathcal{E}$, where J is an open interval containing t_0. Since $X^{\rm hor}$ is π_1-related with X, we have that $\pi_1 \circ \eta$ is an integral curve of X; thus $(\pi_1 \circ \eta)(t) = t$, for all $t \in J$. It follows that $J \subset I$ and that $\tilde\gamma = \bar\gamma \circ \eta$ is a lifting of $\gamma|_J$; moreover, since η is parallel, also $\tilde\gamma$ is parallel (Lemma 2.1.12). Hence $\tilde\gamma$ is an horizontal lifting of $\gamma|_J$ with $\tilde\gamma(t_0) = e$. □

COROLLARY A.5.4. *Let $\pi : \mathcal{E} \to M$ be a smooth submersion and $\mathrm{Hor}(\mathcal{E})$ be a generalized connection on $\mathcal{E}$ with respect to π. Let $\gamma : I \to M$ be a smooth curve, t_0 be an interior point of I and set $I_1 = I \cap \,]-\infty, t_0]$, $I_2 = I \cap [t_0, +\infty[$. If $\tilde\gamma : I \to \mathcal{E}$ is a map such that $\tilde\gamma|_{I_1}$ is a horizontal lifting of $\gamma|_{I_1}$ and $\tilde\gamma|_{I_2}$ is a horizontal lifting of $\gamma|_{I_2}$ then $\tilde\gamma$ (is smooth and) it is a horizontal lifting of γ.*

PROOF. By Lemma A.5.3, there exists an open subinterval J of I containing t_0 and a horizontal lifting $\hat\gamma : J \to \mathcal{E}$ of $\gamma|_J$ such that $\hat\gamma(t_0) = \tilde\gamma(t_0)$. By Lemma A.5.2, we have:

$$\hat\gamma|_{J\cap I_1} = \tilde\gamma|_{J\cap I_1}, \quad \hat\gamma|_{J\cap I_2} = \tilde\gamma|_{J\cap I_2},$$

so that $\hat\gamma|_J = \tilde\gamma|_J$. The conclusion follows. □

DEFINITION A.5.5. Let $\pi : \mathcal{E} \to M$ be a smooth submersion endowed with a generalized connection $\mathrm{Hor}(\mathcal{E})$. An open subset U of M is said to have the *horizontal lifting property for paths* if for every smooth curve $\gamma : I \to U$, every $t_0 \in I$ and every $e \in \pi^{-1}\big(\gamma(t_0)\big)$ there exists a horizontal lifting $\tilde\gamma : I \to \mathcal{E}$ of γ with $\tilde\gamma(t_0) = e$.

LEMMA A.5.6. *Let $\pi : \mathcal{E} \to M$ be a smooth submersion endowed with a generalized connection $\mathrm{Hor}(\mathcal{E})$. If M can be covered by open sets having the horizontal lifting property for paths then M itself has the horizontal lifting property for paths.*

PROOF. Let $\gamma : [a, b] \to M$ be a smooth curve and let $e \in \pi^{-1}\big(\gamma(a)\big)$. By the result of Exercise A.5, it will suffice to prove that γ admits a horizontal lifting starting at e. The family of all sets of the form $\gamma^{-1}(U)$, where U runs over the open subsets of M having the horizontal lifting property for paths, is an open cover of the compact space $[a, b]$; let $\delta > 0$ be a Lebesgue number of such open cover, i.e., every subset of $[a, b]$ having diameter less than δ is contained in some $\gamma^{-1}(U)$. Consider a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$ with $t_{i+1} - t_i < \delta$, for $i = 0, 1, \ldots, k-1$. Since for each i, $\gamma\big([t_i, t_{i+1}]\big)$ is contained in an open subset of M having the horizontal lifting property for paths, we can find horizontal liftings $\tilde{\gamma}_i : [t_i, t_{i+1}] \to \mathcal{E}$ of $\gamma|_{[t_i, t_{i+1}]}$, $i = 0, 1, \ldots, k-1$, with $\tilde{\gamma}_0(t_0) = e$ and $\tilde{\gamma}_{i+1}(t_{i+1}) = \tilde{\gamma}_i(t_{i+1})$, for $i = 0, 1, \ldots, k-1$. Let $\tilde{\gamma} : [a, b] \to \mathcal{E}$ be the map such that $\tilde{\gamma}|_{[t_i, t_{i+1}]} = \tilde{\gamma}_i$, for all i. It follows from Corollary A.5.4 that $\tilde{\gamma}$ is a horizontal lifting of γ. □

DEFINITION A.5.7. Let Λ and M be differentiable manifolds. By a *smooth Λ-parametric family of curves* we mean a smooth map $H : A \to M$ where A is an open subset of $\Lambda \times \mathbb{R}$ such that for all $\lambda \in \Lambda$, the set:

$$A_\lambda = \big\{t \in \mathbb{R} : (\lambda, t) \in A\big\}$$

is an interval (possibly empty).

EXAMPLE A.5.8. If X is a smooth vector field on a differentiable manifold M then, by Theorem A.2.7, the maximal flow of X is a smooth M-parametric family of curves on M containing $M \times \{0\}$.

PROPOSITION A.5.9. *Let $\pi : \mathcal{E} \to M$ be a smooth submersion endowed with a generalized connection $\mathrm{Hor}(\mathcal{E})$, let Λ be a differentiable manifold and let $H : A \to M$ be a smooth Λ-parametric family of curves in M with $\Lambda \times \{0\} \subset A$. If $\widetilde{H} : A \to \mathcal{E}$ is a map such that:*

- *for all $\lambda \in \Lambda$, the curve $A_\lambda \ni t \mapsto \widetilde{H}(\lambda, t)$ is a horizontal lifting of the curve $A_\lambda \ni t \mapsto H(\lambda, t)$,*
- *the map $\Lambda \ni \lambda \mapsto \widetilde{H}(\lambda, 0) \in \mathcal{E}$ is smooth,*

then $\widetilde{H}$ is smooth.

PROOF. Consider the pull-back $\pi_1 : H^*\mathcal{E} \to A$ endowed with the generalized connection $\mathrm{Hor}(H^*\mathcal{E})$ obtained from $\mathrm{Hor}(\mathcal{E})$ by pull-back and consider the canonical map $\overline{H} : H^*\mathcal{E} \to \mathcal{E}$. Since $\widetilde{H}$ is a section of $\mathcal{E}$ along H, there exists a unique section $(\widetilde{H})^{\leftarrow} : A \to H^*\mathcal{E}$ of $\pi_1 : H^*\mathcal{E} \to A$ with $\overline{H} \circ (\widetilde{H})^{\leftarrow} = \widetilde{H}$. The property of pull-backs described in diagram (1.17) implies that the map:

$$\Lambda \ni \lambda \longmapsto (\widetilde{H})^{\leftarrow}(\lambda, 0) \in H^*\mathcal{E}$$

is smooth. To conclude the proof, it suffices to show that the map $(\widetilde{H})^{\leftarrow}$ is smooth. Consider the smooth vector field X on A defined by:

$$X(\lambda, t) = (0, 1) \in T_\lambda\Lambda \oplus T_t\mathbb{R},$$

for all $(\lambda, t) \in A$; denote by X^{hor} the vector field on $H^*\mathcal{E}$ obtained from X by horizontal lift. Given $\lambda \in \Lambda$ then, since the curve $t \mapsto \widetilde{H}(\lambda, t)$ is parallel, also the curve $t \mapsto (\widetilde{H})^{\leftarrow}(\lambda, t)$ is parallel (Lemma 2.1.12); thus, for all $(\lambda, t) \in A$, we have:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\widetilde{H})^{\leftarrow}(\lambda, t) \in \mathrm{Hor}(H^*\mathcal{E}), \quad \mathrm{d}\pi_1\big(\tfrac{\mathrm{d}}{\mathrm{d}t}(\widetilde{H})^{\leftarrow}(\lambda, t)\big) = (0, 1) = X(\lambda, t),$$

so that:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\widetilde{H})^{\leftarrow}(\lambda, t) = X^{\mathrm{hor}}\big((\widetilde{H})^{\leftarrow}(\lambda, t)\big),$$

for all $(\lambda, t) \in A$. We have proven that $t \mapsto (\widetilde{H})^{\leftarrow}(\lambda, t)$ is an integral curve of X^{hor}, for all $\lambda \in \Lambda$. Hence, if F denotes the maximal flow of X^{hor} (see Theorem A.2.7), then:

$$(\widetilde{H})^{\leftarrow}(\lambda, t) = F\big((\widetilde{H})^{\leftarrow}(\lambda, 0), t\big),$$

for all $(\lambda, t) \in A$. This concludes the proof. □

Exercises

Differentiable manifolds.

EXERCISE A.1. Let M be a set and let $(N_i)_{i\in I}$ be a family of sets N_i, each of them endowed with a differential structure. For each $i \in I$ let $\varphi_i : U_i \to N_i$ be a bijective map defined in a subset U_i of M. Assume that $M = \bigcup_{i\in I} U_i$ and that the maps φ_i are pairwise *compatible* in the sense that for all $i, j \in I$ the set $\varphi_i(U_i \cap U_j)$ is open in N_i, the set $\varphi_j(U_i \cap U_j)$ is open in N_j and the *transition map*:

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

is smooth. Show that there exists a unique differential structure on the set M such that for every $i \in I$ the set U_i is open in M and the map $\varphi_i : U_i \to N_i$ is a smooth diffeomorphism.

EXERCISE A.2. Let M, N be differentiable manifolds and let $f : M \to N$ be a map. Show that f is a smooth embedding if and only if for every $x \in M$ there exists an open neighborhood U of $f(x)$ in N such that $f^{-1}(U)$ is open in M and $f|_{f^{-1}(U)} : f^{-1}(U) \to N$ is a smooth embedding.

Vector fields on manifolds.

EXERCISE A.3. Let G be a Lie group, N be a differentiable manifold and assume that we are given a smooth left (resp., right) action of G on N. Given $X \in \mathfrak{g}$, we denote by X^L, X^R respectively the left-invariant and the right-invariant vector fields on G whose value at $1 \in G$ is X. Given a vector field Y on N, show that the following conditions are equivalent:

- $Y = X^N$;

- for all $n \in N$, Y is β_n-related with X^R (resp., with X^L).

EXERCISE A.4. Let G be a Lie group, N be a differentiable manifold and assume that we are given a smooth left (resp., right) action of G on N. Given $X, Y \in \mathfrak{g}$, show that $[X^N, Y^N] = -[X, Y]^N$ (resp., $[X^N, Y^N] = [X, Y]^N$).

Horizontal liftings of curves.

EXERCISE A.5. Let $\pi : \mathcal{E} \to M$ be a smooth submersion endowed with a generalized connection $\mathrm{Hor}(\mathcal{E})$. Let U be an open subset of M having the following property: for every smooth curve $\gamma : [a, b] \to U$ and every $e \in \pi^{-1}\big(\gamma(a)\big)$, there exists a horizontal lifting $\tilde{\gamma} : [a, b] \to \mathcal{E}$ of γ with $\tilde{\gamma}(a) = e$. Show that U has the horizontal lifting property for paths.

APPENDIX B

Topological tools

B.1. Compact-Open Topology

Let X, Y be topological spaces. Denote by $\mathfrak{C}(X,Y)$ the set of continuous maps $f : X \to Y$. Given a compact subset $K \subset X$ and an open subset $U \subset Y$ we set:

$$\mathcal{V}(K;U) = \big\{ f \in \mathfrak{C}(X,Y) : f(K) \subset U \big\}.$$

The smallest topology on $\mathfrak{C}(X,Y)$ containing the sets $\mathcal{V}(K;U)$ (with $K \subset X$ compact and $U \subset Y$ open) is called the *compact-open topology* on $\mathfrak{C}(X,Y)$. The collection of all finite intersections:

$$\bigcap_{i=1}^{r} \mathcal{V}(K_i;U_i),$$

with $K_1,\ldots,K_r \subset X$ compact, $U_1,\ldots,U_r \subset Y$ open and r a positive integer, form a basis for the compact-open topology on $\mathfrak{C}(X,Y)$; indeed, observe that the sets $\mathcal{V}(K;U)$ cover $\mathfrak{C}(X,Y)$ because $\mathcal{V}(K;U) = \mathfrak{C}(X,Y)$ if K and U are both empty.

The main properties of the compact-open topology are described by the following two lemmas.

LEMMA B.1.1. *Let Λ, X, Y be topological spaces and let $f : \Lambda \times X \to Y$ be a continuous map. Then, if $\mathfrak{C}(X,Y)$ is endowed with the compact-open topology, the map:*

$$\tilde{f} : \Lambda \longrightarrow \mathfrak{C}(X,Y),$$

defined by $\tilde{f}(\lambda)(x) = f(\lambda,x)$, for all $\lambda \in \Lambda$, $x \in X$, is continuous.

PROOF. It is sufficient to prove that the set $\tilde{f}^{-1}\big(\mathcal{V}(K;U)\big)$ is open in Λ for every compact set $K \subset X$ and every open set $U \subset Y$. Let $\lambda \in \tilde{f}^{-1}\big(\mathcal{V}(K;U)\big)$ be fixed. The set $f^{-1}(U)$ is open in the product $\Lambda \times X$ and it contains $\{\lambda\} \times K$; since K is compact, $f^{-1}(U)$ also contains $V \times K$ for some neighborhood V of λ in Λ. Hence $V \subset \tilde{f}^{-1}\big(\mathcal{V}(K;U)\big)$ and we are done. $\square$

LEMMA B.1.2. *Let Λ, X, Y be topological spaces and let $\tilde{f} : \Lambda \to \mathfrak{C}(X,Y)$ be a continuous map, where the space $\mathfrak{C}(X,Y)$ is endowed with the compact-open topology. Assume that X is locally compact*[1]*. Then the map $f : \Lambda \times X \to Y$ defined by $f(\lambda,x) = \tilde{f}(\lambda)(x)$, for all $\lambda \in \Lambda$, $x \in X$ is continuous.*

[1]This means that any neighborhood of an arbitrary point $x \in X$ contains a compact neighborhood of x.

PROOF. Let $\lambda \in \Lambda$, $x \in X$ be fixed and let U be an open neighborhood of $f(\lambda, x)$ in Y. Since the map $\tilde{f}(\lambda) : X \to Y$ is continuous, the set $\tilde{f}(\lambda)^{-1}(U)$ is an open neighborhood of x in X. Let K be a compact neighborhood of x contained in $\tilde{f}(\lambda)^{-1}(U)$. Then $\tilde{f}(\lambda)$ is in $\mathcal{V}(K; U)$ and therefore we can find a neighborhood V of λ in Λ with $\tilde{f}(V) \subset \mathcal{V}(K; U)$. Hence $V \times K$ is a neighborhood of (λ, x) in $\Lambda \times X$ and $f(V \times K) \subset U$. $\square$

We now focus on the space $\mathfrak{C}\big([a, b], X\big)$ of continuous curves $\gamma : [a, b] \to X$ on a fixed topological space X. By a *partition* of the interval $[a, b]$ we mean a finite subset P of $[a, b]$ containing a and b; we write $P = \{t_0, \ldots, t_r\}$ meaning that $a = t_0 < t_1 < \cdots < t_r = b$. Given a partition $P = \{t_0, \ldots, t_r\}$ of $[a, b]$ and a sequence $U_1, U_2, \ldots, U_r$ of open subsets of X, we write:

$$\begin{aligned} &\mathfrak{V}(P; U_1, \ldots, U_r) \\ &\qquad = \big\{\gamma \in \mathfrak{C}\big([a, b], X\big) : \gamma\big([t_{i-1}, t_i]\big) \subset U_i,\ i = 1, \ldots, r\big\}. \end{aligned} \tag{B.1.1}$$

Obviously $\mathfrak{V}(P; U_1, \ldots, U_r)$ is an open subset of $\mathfrak{C}\big([a, b], X\big)$ with respect to the compact-open topology. Moreover, we have the following:

LEMMA B.1.3. *Let X be a topological space and $\mathcal{B}$ be a basis of open subsets for X. The sets $\mathfrak{V}(P; U_1, \ldots, U_r)$, where P runs over the partitions of $[a, b]$ and $U_1, \ldots, U_r$ run over $\mathcal{B}$, form a basis of open subsets for the compact-open topology on $\mathfrak{C}\big([a, b], X\big)$.*

PROOF. Let $\mathcal{Z}$ be an open subset of $\mathfrak{C}\big([a, b], X\big)$ with respect to the compact-open topology and let $\gamma \in \mathcal{Z}$ be fixed. We'll find a partition $P = \{t_0, \ldots, t_r\}$ of $[a, b]$ and basic open sets $U_1, \ldots, U_r \in \mathcal{B}$ such that:

$$\gamma \in \mathfrak{V}(P; U_1, \ldots, U_r) \subset \mathcal{Z}. \tag{B.1.2}$$

By the definition of the compact-open topology, we can find compact subsets $K_1, \ldots, K_s \subset [a, b]$ and open subsets $V_1, \ldots, V_s \subset X$ such that:

$$\gamma \in \bigcap_{i=1}^{s} \mathcal{V}(K_i; V_i) \subset \mathcal{Z}.$$

Let $u \in [a, b]$ be fixed. The set:

$$\bigcap_{\substack{i=1,\ldots,s \\ u \in K_i}} V_i$$

is an open neighborhood of $\gamma(u)$ in X and therefore it contains a basic open set $B_u \in \mathcal{B}$ such that $\gamma(u) \in B_u$. Set:

$$I_u = \gamma^{-1}(B_u) \cap \bigcap_{\substack{i=1,\ldots,s \\ u \notin K_i}} \big([a, b] \setminus K_i\big). \tag{B.1.3}$$

Then $u \in I_u$ and I_u is open in $[a, b]$. Let $\delta > 0$ be a Lebesgue number for the open cover $\bigcup_{u \in [a,b]} I_u$ of the compact metric space $[a, b]$; this means that every subset of $[a, b]$ having diameter less than δ is contained in some I_u. Let $P = \{t_0, \ldots, t_r\}$

be a partition of $[a,b]$ with $t_j - t_{j-1} < \delta$, for $j = 1, \dots, r$. For each $j = 1, \dots, r$ we can find $u_j \in [a,b]$ with $[t_{j-1}, t_j] \subset I_{u_j}$; set $U_j = B_{u_j}$. We claim that (B.1.2) holds.

Since for $j = 1, \dots, r$, $[t_{j-1}, t_j] \subset I_{u_j}$ and $\gamma(I_{u_j}) \subset B_{u_j} = U_j$, we have $\gamma \in \mathfrak{V}(P; U_1, \dots, U_r)$. To complete the proof, choose $\mu \in \mathfrak{V}(P; U_1, \dots, U_r)$ and let us prove that $\mu \in \bigcap_{i=1}^{s} \mathcal{V}(K_i; V_i)$. Let $i = 1, \dots, s$ and $t \in K_i$ be fixed. We have $t \in [t_{j-1}, t_j]$, for some $j = 1, \dots, r$. We claim that $u_j \in K_i$; namely, otherwise I_{u_j} would be contained in $[a,b] \setminus K_i$ (recall (B.1.3)), but t is in $I_{u_j} \cap K_i$. But $u_j \in K_i$ implies $U_j = B_{u_j} \subset V_i$. Finally, since $\mu \in \mathfrak{V}(P; U_1, \dots, U_r)$, we have $\mu(t) \in U_j \subset V_i$. This proves that $\mu(K_i) \subset V_i$ for $i = 1, \dots, s$ and completes the prove of the lemma. □

B.2. Liftings

DEFINITION B.2.1. Let X, $\widetilde{X}$, Y be topological spaces and $\pi : \widetilde{X} \to X$, $f : Y \to X$ be continuous maps. By a *lifting* of f with respect to π we mean a continuous map $\tilde{f} : Y \to \widetilde{X}$ such that $\pi \circ \tilde{f} = f$.

LEMMA B.2.2. *Let X, $\widetilde{X}$, Y be topological spaces, $f : Y \to X$ be a continuous map, $\pi : \widetilde{X} \to X$ be a continuous locally injective map*[2] *and let $\tilde{f}_1 : Y \to \widetilde{X}$, $\tilde{f}_2 : Y \to \widetilde{X}$ be liftings of f with respect to π. The set:*

$$\big\{ y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y) \big\} \tag{B.2.1}$$

is open in Y.

PROOF. Let $y \in Y$ be fixed with $\tilde{f}_1(y) = \tilde{f}_2(y)$. If A is an open neighborhood of $\tilde{f}_1(y)$ in $\widetilde{X}$ such that $\pi|_A$ in injective then $\tilde{f}_1^{-1}(A) \cap \tilde{f}_2^{-1}(A)$ is an open neighborhood of y in Y contained in (B.2.1). □

COROLLARY B.2.3. *Let X, $\widetilde{X}$, Y be topological spaces, with $\widetilde{X}$ Hausdorff and Y connected. Let $f : Y \to X$ be a continuous map, $\pi : \widetilde{X} \to X$ be a continuous locally injective map and let $\tilde{f}_1 : Y \to \widetilde{X}$, $\tilde{f}_2 : Y \to \widetilde{X}$ be liftings of f with respect to π. If $\tilde{f}_1$ and $\tilde{f}_2$ agree on some point of Y then $\tilde{f}_1 = \tilde{f}_2$.*

PROOF. Since $\widetilde{X}$ is Hausdorff, the set (B.2.1) is closed in Y and it also open by Lemma B.2.2. Moreover, it is nonempty, by our hypotheses. The conclusion follows from the connectedness of Y. □

Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a *local homeomorphism*, i.e., for every $\tilde{x} \in \widetilde{X}$ there exists an open neighborhoood A of $\tilde{x}$ in $\widetilde{X}$ such that $\pi(A)$ is open in X and the restriction $\pi|_A : A \to \pi(A)$ is a homeomorphism. Corollary B.2.3 implies that, if $\widetilde{X}$ is Hausdorff and Y is connected then a continuous map $f : Y \to X$ admits *at most* one lifting $\tilde{f} : Y \to \widetilde{X}$ satisfying a prescribed condition of the form $f(y_0) = \tilde{x}_0$.

In what follows we will be mostly concerned with liftings of continuous curves $\gamma : [a,b] \to X$.

[2]This means that every point of $\widetilde{X}$ has a neighborhood in which π is injective.

LEMMA B.2.4. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism. Denote by $\mathcal{L}$ the set of pairs $(\tilde{x}_0, \gamma) \in \widetilde{X} \times \mathfrak{C}\big([a,b], X\big)$ such that there exists a unique*[3] *lifting $\tilde{\gamma} : [a,b] \to \widetilde{X}$ of γ with respect to π satisfying the initial condition $\tilde{\gamma}(a) = x_0$. Endow the sets $\mathfrak{C}\big([a,b], X\big)$ and $\mathfrak{C}\big([a,b], \widetilde{X}\big)$ with the compact-open topology. The map:*

$$L : \mathcal{L} \longrightarrow \mathfrak{C}\big([a,b], \widetilde{X}\big)$$

defined by $L(\tilde{x}_0, \gamma) = \tilde{\gamma}$, where $\tilde{\gamma} : [a,b] \to \widetilde{X}$ is the unique lifting of γ such that $\tilde{\gamma}(a) = \tilde{x}_0$, is continuous.

PROOF. Let $\mathcal{B}$ denote the collection of all open subsets A of $\widetilde{X}$ such that $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism. Since π is a local homeomorphism, the set $\mathcal{B}$ is a basis of open subsets of $\widetilde{X}$. Let $(\tilde{x}_0, \gamma) \in \mathcal{L}$ be fixed and set $\tilde{\gamma} = L(\tilde{x}_0, \gamma)$. Let $P = \{t_0, \ldots, t_r\}$ be a partition of $[a,b]$ and let $A_1, \ldots, A_r \in \mathcal{B}$ be such that (recall (B.1.1)):

$$\tilde{\gamma} \in \mathfrak{V}(P; A_1, \ldots, A_r).$$

By Lemma B.1.3, in order to complete the proof, it suffices to find a neighborhood of $(\tilde{x}_0, \gamma)$ in $\mathcal{L}$ that is mapped by L into $\mathfrak{V}(P; A_1, \ldots, A_r)$. Let $\mathcal{Z}$ denote the set of pairs $(\tilde{y}_0, \mu)$ in $\mathcal{L}$ such that:

- $\tilde{y}_0 \in A_1$;
- $\mu\big([t_{i-1}, t_i]\big) \subset \pi(A_i)$, for $i = 1, \ldots, r$;
- $\mu(t_i) \in \pi(A_i \cap A_{i+1})$, for $i = 1, \ldots, r-1$.

By definition of the compact-open topology in $\mathfrak{C}\big([a,b], X\big)$, it is immediate that $\mathcal{Z}$ is open in $\mathcal{L}$. Moreover, $(\tilde{x}_0, \gamma)$ is in $\mathcal{Z}$. We will show that $L(\mathcal{Z}) \subset \mathfrak{V}(P; A_1, \ldots, A_r)$. Let $(\tilde{y}_0, \mu) \in \mathcal{Z}$ be fixed. For $i = 1, \ldots, r$, we consider the continuous curve $\tilde{\mu}_i : [t_{i-1}, t_i] \to A_i \subset \widetilde{X}$ defined by:

$$\tilde{\mu}_i = (\pi|_{A_i})^{-1} \circ \mu|_{[t_{i-1}, t_i]}.$$

We claim that $\tilde{\mu}_i(t_i) = \tilde{\mu}_{i+1}(t_i)$, for $i = 1, \ldots, r-1$. Namely, since $\mu(t_i)$ is in $\pi(A_i \cap A_{i+1})$, there exists $p \in A_i \cap A_{i+1}$ with $\mu(t_i) = \pi(p)$. Since $\pi|_{A_i}$ is injective, $\tilde{\mu}_i(t_i)$ and p are in A_i and $\pi\big(\tilde{\mu}_i(t_i)\big) = \mu(t_i) = \pi(p)$, it follows that $\tilde{\mu}_i(t_i) = p$. Similarly, since $\pi|_{A_{i+1}}$ is injective, $\tilde{\mu}_{i+1}(t_i)$ and p are in A_{i+1} and $\pi\big(\tilde{\mu}_{i+1}(t_i)\big) = \mu(t_i) = \pi(p)$, it follows that $\tilde{\mu}_{i+1}(t_i) = p$. This proves the claim.

Since $\tilde{\mu}_i(t_i) = \tilde{\mu}_{i+1}(t_i)$, for $i = 1, \ldots, r-1$, we can consider the curve $\tilde{\mu} : [a,b] \to \widetilde{X}$ such that $\tilde{\mu}|_{[t_{i-1}, t_i]} = \tilde{\mu}_i$, for $i = 1, \ldots, r$. The curve $\tilde{\mu}$ is a lifting of μ. Moreover, since $\pi|_{A_1}$ is injective, $\tilde{y}_0$ and $\tilde{\mu}(a)$ are in A_1 and:

$$\pi(\tilde{y}_0) = \mu(a) = \pi\big(\tilde{\mu}(a)\big),$$

it follows that $\tilde{\mu}(a) = \tilde{y}_0$. Therefore $L(\tilde{y}_0, \mu) = \tilde{\mu}$. The proof is completed by observing that $\tilde{\mu} \in \mathfrak{V}(P; A_1, \ldots, A_r)$. $\square$

---

[3]Notice that, by Corollary B.2.3, if $\widetilde{X}$ is Hausdorff, the uniqueness of $\tilde{\gamma}$ is automatic.

COROLLARY B.2.5. *Let $\widetilde{X}$, X, Y be topological spaces, $\pi : \widetilde{X} \to X$ be a local homeomorphism and $f : Y \times [a,b] \to X$, $\tilde{f}_0 : Y \to \widetilde{X}$ be continuous maps such that for every $y \in Y$, the curve $\gamma_y : [a,b] \ni t \mapsto f(y,t) \in X$ has a unique lifting $\tilde{\gamma}_y : [a,b] \to \widetilde{X}$ satisfying the initial condition $\tilde{\gamma}_y(a) = \tilde{f}_0(y)$. Then f has a unique lifting $\tilde{f} : Y \times [a,b] \to \widetilde{X}$ such that:*

$$\tilde{f}(y,a) = \tilde{f}_0(y),$$

for all $y \in Y$.

PROOF. By Lemma B.1.1, the map:

$$F : Y \ni y \longmapsto \gamma_y \in \mathfrak{C}\big([a,b], X\big)$$

is continuous. By our hypotheses, the continuous map:

$$(\tilde{f}_0, F) : Y \longrightarrow \widetilde{X} \times \mathfrak{C}\big([a,b], \widetilde{X}\big)$$

takes values in $\mathcal{L}$. It is clear that there exists a unique map $\tilde{f} : Y \times [a,b] \to \widetilde{X}$ such that $\pi \circ \tilde{f} = f$ and $\tilde{f}(y,a) = \tilde{f}_0(y)$, for all $y \in Y$; such map is given by:

$$\tilde{f}(y,t) = L\big(\tilde{f}_0(y), F(y)\big)(t),$$

for all $y \in Y$, $t \in [a,b]$. It follows from Lemmas B.2.4 and B.1.2 that $\tilde{f}$ is indeed continuous. □

DEFINITION B.2.6. Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a continuous map. We say that π has the *unique lifting property for paths* if for any continuous map $\gamma : [a,b] \to X$ and any $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ there exists a unique lifting $\tilde{\gamma} : [a,b] \to \widetilde{X}$ of γ with $\tilde{\gamma}(a) = \tilde{x}_0$.

DEFINITION B.2.7. By a *loop* in a topological space X we mean a continuous map $\gamma : [a,b] \to X$ with $\gamma(a) = \gamma(b)$. We say that the loop γ is *contractible* in X if there exists a continuous map $H : [0,1] \times [a,b] \to X$ such that:

- $H(0,t) = \gamma(t)$, for all $t \in [a,b]$;
- $H(s,a) = H(s,b)$, for all $s \in [0,1]$;
- the map $[a,b] \ni t \mapsto H(1,t) \in X$ is constant.

We say that X is *simply-connected* if every loop in X is contractible in X. We say that X is *semi-locally simply-connected* if every point of X has a neighborhood V such that any loop in V is contractible in X.

LEMMA B.2.8. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism. Assume that π has the unique lifting property for paths. Let A be an arc-connected subset of $\widetilde{X}$ such that every loop in $\pi(A)$ is contractible in X. Then $\pi|_A$ is injective.*

PROOF. Assume that $\tilde{x}_1, \tilde{x}_2 \in A$ and that $\pi(\tilde{x}_1) = \pi(\tilde{x}_2)$. Since A is arc-connected, there exists a continuous map $\tilde{\gamma} : [a,b] \to A$ with $\tilde{\gamma}(a) = \tilde{x}_1$ and $\tilde{\gamma}(b) = \tilde{x}_2$. Then $\gamma = \pi \circ \tilde{\gamma}$ is a loop in $\pi(A)$; therefore γ is contractible in X, i.e., there exists a continuous map $H : [0,1] \times [a,b] \to X$ as in Definition B.2.7.

Since π has the unique lifting property for paths, Corollary B.2.5 gives us a lifting $\widetilde{H} : [0,1] \times [a,b] \to \widetilde{X}$ of H such that $\widetilde{H}(0,t) = \tilde{\gamma}(t)$, for all $t \in [a,b]$.

Since the map $[a,b] \ni t \mapsto H(1,t) \in X$ is constant, the unique lifting property for paths implies that its lifting $[a,b] \ni t \mapsto \widetilde{H}(1,t) \in \widetilde{X}$ is also constant. In particular, $\widetilde{H}(1,a) = \widetilde{H}(1,b)$; therefore, the paths:

$$[0,1] \ni s \longmapsto \widetilde{H}(1-s,a) \in \widetilde{X}, \quad [0,1] \ni s \longmapsto \widetilde{H}(1-s,b) \in \widetilde{X},$$

are liftings of the same path in X and they agree on $s = 0$. Again, by the unique lifting property for paths, it follows that $\widetilde{H}(1-s,a) = \widetilde{H}(1-s,b)$, for all $s \in [0,1]$. In particular:

$$\tilde{x}_1 = \tilde{\gamma}(a) = \widetilde{H}(0,a) = \widetilde{H}(0,b) = \tilde{\gamma}(b) = \tilde{x}_2.$$

This concludes the proof. $\square$

COROLLARY B.2.9. *Under the hypotheses of Lemma B.2.8, if in addition the set A is open in $\widetilde{X}$ then $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism.*

PROOF. Simply observe that, being a local homeomorphism, π is an open mapping; moreover, if A is open in $\widetilde{X}$ and the restriction of π to A is injective then $\pi|_A : A \to \pi(A)$ is a continuous, bijective open mapping. $\square$

B.3. Covering Maps

DEFINITION B.3.1. Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be an arbitrary map. An open subset $U \subset X$ is called a *fundamental open subset* of X if $\pi^{-1}(U)$ equals a disjoint union $\bigcup_{i \in I} U_i$ of open subsets U_i of $\widetilde{X}$ such that $\pi|_{U_i} : U_i \to U$ is a homeomorphism for all $i \in I$. We say that π is a *covering map* if X can be covered by fundamental open subsets.

Obviously every covering map is a local homeomorphism.

DEFINITION B.3.2. We say that a topological space X is *locally arc-connected* (resp., *locally connected*) if every point $x \in X$ has a fundamental system of arc-connected (resp., connected) neighborhoods, i.e., if every neighborhood of x contains a (not necessarily open) arc-connected (resp., connected) neighborhood of x.

Obviously if X is locally arc-connected (resp., locally connected) then every open subset of X is also locally arc-connected (resp., locally connected), when endowed with the topology induced from X.

REMARK B.3.3. If X is locally arc-connected (resp., locally connected) then every point of X has a fundamental system of *open* arc-connected (resp., connected) neighborhoods, i.e., for every $x \in X$ and every neighborhood V of x, there exists an arc-connected (resp., a connected) open set C with $x \in C \subset V$. Namely, take C to be the arc-connected component (resp., connected component) of the interior of V containing x and use the fact that the arc-connected components (resp., connected components) of an open subset are open (see Exercise B.1).

LEMMA B.3.4. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism. Assume that π has the unique lifting property for paths and that $\widetilde{X}$ is locally arc-connected. Let U be an arc-connected open subset of X such that every loop in U is contractible in X. Then U is a fundamental open subset of X.*

PROOF. Let $(U_i)_{i\in I}$ denote the arc-connected components of $\pi^{-1}(U)$. Since $\pi^{-1}(U)$ is open in $\widetilde{X}$, each U_i is open in $\widetilde{X}$ (see Exercise B.1). Obviously:

$$\pi^{-1}(U) = \bigcup_{i\in I} U_i$$

and such union is disjoint. Let $i \in I$ be fixed and let us show that $\pi|_{U_i}$ is a homeomorphism onto U. Obviously $\pi(U_i) \subset U$. We claim that $\pi(U_i) = U$. Given $x \in U$, choose an arbitrary point $\tilde{x}_0 \in U_i$ and let $\gamma : [a,b] \to U$ be a continuous map with $\gamma(a) = \pi(\tilde{x}_0)$ and $\gamma(b) = x$. By the unique lifting property for paths, we can find a lifting $\tilde{\gamma} : [a,b] \to \widetilde{X}$ of γ such that $\tilde{\gamma}(a) = \tilde{x}_0$. Since $\tilde{\gamma}$ is a continuous curve in $\pi^{-1}(U)$ starting at a point of U_i and since U_i is an arc-connected component of $\pi^{-1}(U)$, it follows that $\tilde{\gamma}$ takes values in U_i. In particular $\tilde{\gamma}(b) \in U_i$ and $\pi\big(\tilde{\gamma}(b)\big) = \gamma(b) = x$. Finally, Corollary B.2.9 implies that $\pi|_{U_i}$ is a homeomorphism onto $\pi(U_i) = U$. $\square$

COROLLARY B.3.5. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism. Assume that π has the unique lifting property for paths and that X is locally arc-connected and semi-locally simply-connected. Then π is a covering map.*

PROOF. Observe that, since π is a local homeomorphism and X is locally arc-connected, it follows that also $\widetilde{X}$ is locally arc-connected. The conclusion follows from Lemma B.3.4 (recall also Remark B.3.3). $\square$

Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism; assume that $\widetilde{X}$ is Hausdorff, so that Corollary B.2.3 guarantees the uniqueness of liftings (with prescribed initial conditions) of curves in X. Now let $\gamma : [a,b] \to X$ be a continuous curve and let $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ be such that γ *does not* admit a lifting $\tilde{\gamma} : [a,b] \to \widetilde{X}$ with $\tilde{\gamma}(a) = \tilde{x}_0$. Consider the set:

$$\big\{t \in \,]a,b] : \gamma|_{[a,t]} \text{ admits a lifting } \tilde{\gamma} : [a,t] \to \widetilde{X} \text{ with } \tilde{\gamma}(a) = \tilde{x}_0\big\}. \tag{B.3.1}$$

The set (B.3.1) is not empty; namely, if A is an open neighborhood of $\tilde{x}_0$ such that $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism then there exists $\varepsilon > 0$ with $\gamma\big([a,a+\varepsilon]\big) \subset \pi(A)$ and therefore $\tilde{\gamma} = (\pi|_A)^{-1} \circ \gamma|_{[a,a+\varepsilon]}$ is a lifting of $\gamma|_{[a,a+\varepsilon]}$ with $\tilde{\gamma}(a) = \tilde{x}_0$.

Obviously if t is in (B.3.1) and t' is in $]a,t]$ then also t' is in (B.3.1). Therefore (B.3.1) is an interval whose left endpoint is a. Let $t_0 \in \,]a,b]$ be the supremum of (B.3.1). Then $]a,t_0[$ is contained in (B.3.1). For each $t \in \,]a,t_0[$, let $\tilde{\gamma}_t : [a,t] \to \widetilde{X}$ be a lifting of $\gamma|_{[a,t]}$ with $\tilde{\gamma}_t(a) = \tilde{x}_0$. Given $t, t' \in \,]a,t_0[$, with $t' < t$ then $\tilde{\gamma}_{t'}$ and $\tilde{\gamma}_t|_{[a,t']}$ are both liftings of the same curve having the same initial condition;

therefore $\tilde{\gamma}_{t'} = \tilde{\gamma}_t|_{[a,t']}$. Therefore, there exists a unique curve $\tilde{\gamma} : [a, t_0[\to \widetilde{X}$ such that:

$$\tilde{\gamma}|_{[a,t]} = \tilde{\gamma}_t,$$

for all $t \in]a, t_0[$. The curve $\tilde{\gamma}$ is continuous, since its restriction to $]a, t]$ is continuous for all $t \in]a, t_0[$. Moreover, $\tilde{\gamma}$ is a lifting of $\gamma|_{[a,t_0[}$ satisfying the initial condition $\tilde{\gamma}(a) = \tilde{x}_0$. We call $\tilde{\gamma}$ the *maximal partial lifting* of γ starting at $\tilde{x}_0$.

We have the following:

LEMMA B.3.6. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism; assume that $\widetilde{X}$ is Hausdorff. Let $\gamma : [a, b] \to X$ be a continuous curve and let $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ be such that γ does not admit a lifting starting at $\tilde{x}_0$. Let $\tilde{\gamma} : [a, t_0[\to \widetilde{X}$ be the maximal partial lifting of γ starting at $\tilde{x}_0$, where $t_0 \in]a, b]$. Then $\gamma|_{[a,t_0]}$ does not admit a lifting starting at $\tilde{x}_0$ (i.e., t_0 is not in* (B.3.1)*).*

PROOF. If $t_0 = b$ then $\gamma|_{[a,t_0]} = \gamma$ and, by our hypotheses, γ does not admit a lifting starting at $\tilde{x}_0$. Assume that $t_0 < b$ and assume by contradiction that $\gamma|_{[a,t_0]}$ admits a lifting $\tilde{\gamma} : [a, t_0] \to \widetilde{X}$ with $\tilde{\gamma}(a) = \tilde{x}_0$. Let A be an open neighborhood of $\tilde{\gamma}(t_0)$ in $\widetilde{X}$ such that $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism. Then $\gamma\big([t_0, t_0 + \varepsilon]\big)$ is contained in $\pi(A)$ for some $\varepsilon > 0$. Consider the curve $\tilde{\mu} : [t_0, t_0 + \varepsilon] \to A$ defined by $\tilde{\mu} = (\pi|_A)^{-1} \circ \gamma|_{[t_0,t_0+\varepsilon]}$. Then $\tilde{\mu}$ is a lifting of $\gamma|_{[t_0,t_0+\varepsilon]}$ starting at $\tilde{\gamma}(t_0)$. Therefore the concatenation of $\tilde{\gamma}$ with $\tilde{\mu}$ is a lifting of $\gamma|_{[a,t_0+\varepsilon]}$ starting at $\tilde{x}_0$. Thus $t_0 + \varepsilon$ is in (B.3.1), contradicting the fact that t_0 is the supremum of (B.3.1). $\square$

Recall that a point p in a topological space Y is called a *limit value* of a map $f : [a, b[\to Y$ at the point b if for any neighborhood V of p and any $\varepsilon > 0$ there exists $t \in]b - \varepsilon, b[$ with $f(t) \in V$. We have the following:

LEMMA B.3.7. *Let $\widetilde{X}$, X be Hausdorff topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism. Let $\gamma : [a, b] \to X$ be a continuous curve and let $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ be such that γ does not admit a lifting starting at $\tilde{x}_0$. Denote by $\tilde{\gamma} : [a, t_0[\to \widetilde{X}$ the maximal partial lifting of γ starting at $\tilde{x}_0$, where $t_0 \in]a, b]$. Then the map $\tilde{\gamma}$ has no limit values at the point t_0.*

PROOF. Assume by contradiction that $p \in \widetilde{X}$ is a limit value of $\tilde{\gamma}$ at the point t_0. We claim that $\pi(p) = \gamma(t_0)$. Otherwise, we could find disjoint open sets $U_1, U_2 \subset X$ with $\pi(p) \in U_1$ and $\gamma(t_0) \in U_2$; then $\gamma\big(]t_0 - \varepsilon, t_0]\big) \subset U_2$ for some $\varepsilon > 0$ and there exists $t \in]t_0 - \varepsilon, t_0[$ with $\tilde{\gamma}(t) \in \pi^{-1}(U_1)$. This implies $\gamma(t) = \pi\big(\tilde{\gamma}(t)\big) \in U_1$, contradicting $U_1 \cap U_2 = \emptyset$. The claim is proved.

Let now A be an open neighborhood of p in $\widetilde{X}$ such that $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism. Since $\gamma(t_0) = \pi(p)$ is in $\pi(A)$, we can find $\varepsilon > 0$ with $\gamma\big(]t_0 - \varepsilon, t_0]\big) \subset \pi(A)$. Now there exists $t \in]t_0 - \varepsilon, t_0[$ with $\tilde{\gamma}(t) \in A$. define $\tilde{\mu} : [t, t_0] \to A$ by setting:

$$\tilde{\mu} = (\pi|_A)^{-1} \circ \gamma|_{[t,t_0]}.$$

Then $\tilde{\mu}$ is a lifting of $\gamma|_{[t,t_0]}$ starting at $\tilde{\gamma}(t)$; the concatenation of $\tilde{\gamma}|_{[a,t]}$ with $\tilde{\mu}$ is therefore a lifting of $\gamma|_{[a,t_0]}$ starting at $\tilde{x}_0$. This contradicts Lemma B.3.6. $\square$

COROLLARY B.3.8. *Under the assumptions of Lemma B.3.7, we have:*

(a) *if $(t_n)_{n\geq 1}$ is a sequence in $[a, t_0[$ converging to t_0 then the sequence $\big(\tilde{\gamma}(t_n)\big)_{n\geq 1}$ has no converging subsequence in $\widetilde{X}$;*

(b) *if K is a compact subset of $\widetilde{X}$ then there exists $\varepsilon > 0$ such that:*

$$\tilde{\gamma}\big(\,]t_0 - \varepsilon, t_0[\,\big) \cap K = \emptyset.$$

PROOF. If $\big(\tilde{\gamma}(t_n)\big)_{n\geq 1}$ had a converging subsequence to a point $p \in \widetilde{X}$ then p would be a limit value of $\tilde{\gamma}$ at the point t_0. Thus (a) is proven. Let us prove (b). For each point $p \in K$, since p is not a limit value of $\tilde{\gamma}$ at the point t_0, we can find an open neighborhood U_p of p in $\widetilde{X}$ and a positive number $\varepsilon_p > 0$ such that $\tilde{\gamma}\big(\,]t_0 - \varepsilon_p, t_0[\,\big)$ is disjoint from U_p. The open cover $\bigcup_{p\in K} U_p$ of K has a finite subcover $\bigcup_{i=1}^{r} U_{p_i}$. Let $\varepsilon = \min_{i=1}^{r} \varepsilon_{p_i} > 0$. Then $\tilde{\gamma}\big(\,]t_0 - \varepsilon, t_0[\,\big)$ is disjoint from K. $\square$

DEFINITION B.3.9. Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a continuous map. We will say that a continuous curve $\gamma : [a, b] \to X$ *admits liftings with arbitrary initial conditions* with respect to π if for every $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ there exists a lifting $\tilde{\gamma} : [a, b] \to \widetilde{X}$ of γ satisfying the initial condition $\tilde{\gamma}(a) = \tilde{x}_0$.

In order to check that a local homeomorphism $\pi : \widetilde{X} \to X$ has the unique lifting property for paths, it suffices to show that a “sufficiently rich” set of curves in X admits lifting with arbitrary initial conditions. This is made more precise in the following:

DEFINITION B.3.10. Let X be a topological spaces and let $\mathcal{C}$ be a subset of $\mathfrak{C}\big([0, 1], X\big)$. We say that $\mathcal{C}$ is *rich* in X if for every point $p \in X$ there exists a neighborhood U of p in X, a point $p_0 \in X$ and a continuous map $H : [0, 1] \times U \to X$ such that the following conditions hold:

- $H(0, x) = p_0$ and $H(1, x) = x$, for all $x \in U$;
- for any $x \in U$, the curves:

$$[0, 1] \ni t \longmapsto H(t, x) \in X, \quad [0, 1] \ni t \longmapsto H(1 - t, x) \in X, \tag{B.3.2}$$

are in $\mathcal{C}$.

EXAMPLE B.3.11. If X is a differentiable manifold then the set of all smooth curves $\gamma : [0, 1] \to X$ is rich. Moreover, if X is a semi-Riemannian manifold then the set of all geodesics $\gamma : [0, 1] \to X$ is rich.

LEMMA B.3.12. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism; assume that $\widetilde{X}$ is Hausdorff. If there exists a rich set $\mathcal{C}$ of continuous curves $\gamma : [0, 1] \to X$ such that every $\gamma \in \mathcal{C}$ admits liftings with arbitrary initial conditions with respect to π then π has the unique lifting property for paths.*

PROOF. Let $\gamma : [a,b] \to X$ be a continuous curve and let $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$ be fixed. Assume by contradiction that γ does not admit a lifting starting at $\tilde{x}_0$. Let $\tilde{\gamma} : [a, t_0[\ \to \widetilde{X}$ be the maximal partial lifting of γ starting at $\tilde{x}_0$, where $t_0 \in \ ]a,b]$. Set $p = \gamma(t_0)$ and let U, p_0 and H be as in Definition B.3.10. Let $\varepsilon > 0$ be such that $\gamma\big([t_0 - \varepsilon, t_0]\big) \subset U$ and let $\tilde{\mu} : [0,1] \to \widetilde{X}$ be a lifting of the curve $[0,1] \ni t \mapsto H\big(1-t, \gamma(t_0 - \varepsilon)\big) \in X$ such that $\tilde{\mu}(0) = \tilde{\gamma}(t_0 - \varepsilon)$. Then $\tilde{p}_0 = \tilde{\mu}(1)$ is a point in $\widetilde{X}$ such that $\pi(\tilde{p}_0) = p_0$. Since for every $x \in U$ the curve $[0,1] \ni t \mapsto H(t,x) \in X$ admits a lifting starting at $\tilde{p}_0$, Corollary B.2.5 gives us a lifting $\widetilde{H} : [0,1] \times U \to \widetilde{X}$ of H such that $\widetilde{H}(0,x) = \tilde{p}_0$, for all $x \in U$. The curves $[0,1] \ni t \mapsto \tilde{\mu}(1-t) \in \widetilde{X}$ and $[0,1] \ni t \mapsto \widetilde{H}\big(t, \gamma(t_0 - \varepsilon)\big) \in \widetilde{X}$ are liftings of the same curve in X and they both start at the point $\tilde{p}_0$; therefore they are equal. In particular:

$$\tilde{\mu}(0) = \tilde{\gamma}(t_0 - \varepsilon) = \widetilde{H}\big(1, \gamma(t_0 - \varepsilon)\big).$$

Therefore $[t_0 - \varepsilon, t_0] \ni t \mapsto \widetilde{H}\big(1, \gamma(t)\big) \in \widetilde{X}$ is a lifting of $\gamma|_{[t_0 - \varepsilon, t_0]}$ starting at $\tilde{\gamma}(t_0 - \varepsilon)$; setting $\tilde{\gamma}(t_0) = \widetilde{H}\big(1, \gamma(t_0)\big)$ we thus obtain a lifting of $\gamma|_{[a,t_0]}$ starting at $\tilde{x}_0$. This contradicts Lemma B.3.6. □

COROLLARY B.3.13. *Under the conditions of Lemma B.3.12, if in addition, X is locally arc-connected and semi-locally simply-connected then $\pi : \widetilde{X} \to X$ is a covering map.*

PROOF. Follows readily from Lemma B.3.12 and Corollary B.3.5. □

In next lemma we show that uniqueness of liftings works for covering maps $\pi : \widetilde{X} \to X$ even if the space $\widetilde{X}$ is not Hausdorff (compare with Corollary B.2.3).

LEMMA B.3.14. *Let X, $\widetilde{X}$, Y be topological spaces, with Y connected. Let $f : Y \to X$ be a continuous map, $\pi : \widetilde{X} \to X$ be a covering map and let $\tilde{f}_1 : Y \to \widetilde{X}$, $\tilde{f}_2 : Y \to \widetilde{X}$ be liftings of f with respect to π. If $\tilde{f}_1$ and $\tilde{f}_2$ agree on some point of Y then $\tilde{f}_1 = \tilde{f}_2$.*

PROOF. Consider the set defined in (B.2.1). Since π is locally injective, (B.2.1) is open, by Lemma B.2.2; moreover, (B.2.1) is nonempty, by our hypotheses. We complete the proof by showing that (B.2.1) is closed (without using that $\widetilde{X}$ is Hausdorff). Let $y \in Y$ be a point not in (B.2.1), i.e., $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. We have $\pi\big(\tilde{f}_1(y)\big) = \pi\big(\tilde{f}_2(y)\big) = f(y)$; let $U \subset X$ be a fundamental open set containing $f(y)$. Then $\pi^{-1}(U) = \bigcup_{i \in I} U_i$, where $(U_i)_{i \in I}$ is a family of disjoint open subsets of $\widetilde{X}$ and π maps U_i homeomorphically onto U, for all $i \in I$. We have $\tilde{f}_1(y) \in U_i$ and $\tilde{f}_2(y) \in U_j$, for some $i, j \in I$. Since $\pi|_{U_i}$ is injective, it must be $i \neq j$. Set $V = \tilde{f}_1^{-1}(U_i) \cap \tilde{f}_2^{-1}(U_j)$. Then V is an open neighborhood of y in Y. Moreover, $\tilde{f}_1(V) \subset U_i$, $\tilde{f}_2(V) \subset U_j$ and $U_i \cap U_j = \emptyset$; therefore V is disjoint from (B.2.1). This completes the proof. □

LEMMA B.3.15. *If $\pi : \widetilde{X} \to X$ is a covering map then π has the unique lifting property for paths.*

PROOF. Let $\gamma : [a,b] \to X$ be a continuous map and fix a point $\tilde{x}_0 \in \pi^{-1}\big(\gamma(a)\big)$. We will show that γ has a lifting $\tilde{\gamma} : [a,b] \to \widetilde{X}$ with $\tilde{\gamma}(a) = \tilde{x}_0$; by Lemma B.3.14, such lifting is unique.

Let us start with the case where the image of γ is contained in a fundamental open subset U of X. Write $\pi^{-1}(U) = \bigcup_{i\in I} U_i$, where $(U_i)_{i\in I}$ is a family of disjoint open subsets of $\widetilde{X}$ and π maps U_i homeomorphically onto U for all $i \in I$. Since $\tilde{x}_0 \in \pi^{-1}(U)$, we have $\tilde{x}_0 \in U_i$, for some $i \in I$. Then $\tilde{\gamma} = (\pi|_{U_i})^{-1} \circ \gamma$ is a lifting of γ with $\tilde{\gamma}(a) = \tilde{x}_0$.

Let us now go to the general case. Since the fundamental open subsets of X form an open cover of X, its inverse images by γ form an open cover of the compact metric space $[a,b]$; let $\delta > 0$ be a Lebesgue number for this open cover, i.e., every subset of $[a,b]$ having diameter less than δ is contained in the inverse image by γ of some fundamental open subset of X. Let $P = \{t_0,\ldots,t_r\}$ be a partition of $[a,b]$ with $t_i - t_{i-1} < \delta$, $i = 1,\ldots,r$. Then $\gamma\big([t_{i-1},t_i]\big)$ is contained in a fundamental open subset of X; by the first part of the proof, the curve $\gamma|_{[t_{i-1},t_i]}$ admits liftings with arbitrary initial conditions, for all $i = 1,\ldots,r$. We construct a lifting $\tilde{\gamma}_i$ of $\gamma|_{[t_{i-1},t_i]}$ by induction on i as follows. Let $\tilde{\gamma}_1$ be a lifting of $\gamma|_{[t_0,t_1]}$ with $\tilde{\gamma}_1(a) = \tilde{x}_0$. Assuming that $\tilde{\gamma}_i$ is constructed for some $i < r$, we consider the lifting $\tilde{\gamma}_{i+1}$ of $\gamma|_{[t_i,t_{i+1}]}$ with $\tilde{\gamma}_{i+1}(t_i) = \tilde{\gamma}_i(t_i)$.

Since the continuous curves $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r$ satisfy:

$$\tilde{\gamma}_i(t_i) = \tilde{\gamma}_{i+1}(t_i),$$

for all $i = 1,\ldots,r-1$, we can define a continuous curve $\tilde{\gamma} : [a,b] \to \widetilde{X}$ by setting $\tilde{\gamma}|_{[t_{i-1},t_i]} = \tilde{\gamma}_i$, for $i = 1,\ldots,r$. Then $\tilde{\gamma}$ is a lifting of γ and $\tilde{\gamma}(a) = \tilde{x}_0$. This concludes the proof. □

COROLLARY B.3.16. *Assume that $\pi : \widetilde{X} \to X$ is a covering map and that $\widetilde{X}$ is locally arc-connected. If U is an arc-connected open subset of X such that every loop in U is contractible in X (in particular, if U is simply-connected) then U is a fundamental open subset of X.*

PROOF. Follows from Lemmas B.3.15 and B.3.4. □

LEMMA B.3.17. *If $\pi : \widetilde{X} \to X$ is a covering map then the image of π is closed in X.*

PROOF. Let $x \in X$ be a point outside the image of π. Let U be a fundamental open subset of X containing x. Then $\pi^{-1}(U) = \bigcup_{i\in I} U_i$, where $(U_i)_{i\in I}$ is a family of disjoint open subsets of $\widetilde{X}$ and π maps U_i homeomorphically onto U for all $i \in I$. We claim that $I = \emptyset$; namely, otherwise there would exist some $i \in I$ and $U = \pi(U_i)$ would be contained in the image of π. Since $I = \emptyset$, it follows that $\pi^{-1}(U) = \emptyset$, i.e., U is disjoint from the image of π. □

COROLLARY B.3.18. *If $\pi : \widetilde{X} \to X$ is a covering map, $\widetilde{X}$ is nonempty and X is connected then π is surjective.*

PROOF. The image of π is nonempty (because $\widetilde{X}$ is nonempty), open in X (because π is a local homeomorphism) and, by Lemma B.3.17, closed in X. □

COROLLARY B.3.19. *Assume that $\pi : \widetilde{X} \to X$ is a covering map, $\widetilde{X}$ is nonempty and arc-connected and X is connected and simply-connected. Then π is a homeomorphism.*

PROOF. By Corollary B.3.18, π is surjective and by Lemma B.3.15, π has the unique lifting property for paths. It follows from Corollary B.2.9 (with $A = \widetilde{X}$) that π is a homeomorphism. □

DEFINITION B.3.20. Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a continuous map. By a *local section* of π we mean a map $s : U \to \widetilde{X}$ defined on an open subset U of X such that $\pi \circ s$ equals the inclusion map of U in X.

Notice that a continuous local section $s : U \to \widetilde{X}$ of π is the same as a lifting with respect to π of the inclusion map $U \to X$ of an open subset U of X.

LEMMA B.3.21. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a locally injective continuous map. If $s : U \to \widetilde{X}$, $s' : U' \to \widetilde{X}$ are continuous local sections of π such that $s(x) = s'(x)$ for some $x \in U \cap U'$ then there exists an open neighborhood V of x contained in $U \cap U'$ such that $s|_V = s'|_V$.*

PROOF. Since $s|_{U\cap U'}$ and $s'|_{U\cap U'}$ are both liftings of the inclusion map of $U \cap U'$ into X, it follows from Lemma B.2.2 that the set of points of $U \cap U'$ where s coincides with s' is open. □

LEMMA B.3.22. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a locally injective continuous map; assume that $\widetilde{X}$ is Hausdorff. Let $s : U \to \widetilde{X}$, $s' : U \to \widetilde{X}$ be continuous local sections of π with U connected. If $s(x) = s'(x)$ for some $x \in U$ then $s = s'$.*

PROOF. It follows from Corollary B.2.3, observing that s and s' are both liftings of the inclusion map of U into X. □

LEMMA B.3.23. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism. If $s : U \to \widetilde{X}$ is a continuous local section of π then $s(U)$ is open in $\widetilde{X}$ and $s : U \to s(U)$ is a homeomorphism.*

PROOF. The map $s : U \to s(U)$ is continuous, bijective and its inverse, which is equal to $\pi|_{s(U)} : s(U) \to U$, is also continuous; thus $s : U \to s(U)$ is a homeomorphism. To complete the proof we show that $s(U)$ is open in $\widetilde{X}$. Given $x \in U$, we will find a neighborhood of $s(x)$ in $\widetilde{X}$ contained in $s(U)$. Let $A \subset \widetilde{X}$ be an open subset such that $s(x) \in A$, $\pi(A)$ is open in X and $\pi|_A : A \to \pi(A)$ is a homeomorphism. Then $s' = (\pi|_A)^{-1} : \pi(A) \to \widetilde{X}$ is a continuous local section of π and $s'(x) = s(x)$. By Lemma B.3.21, there exists an open subset V of X with $x \in V$, $V \subset U \cap \pi(A)$ and $s|_V = s'|_V$. Since s' is a homeomorphism onto an open subset of $\widetilde{X}$, it follows that $s'(V)$ is open in $\widetilde{X}$; moreover:

$$s(x) \in s'(V) = s(V) \subset s(U).$$

Hence $s'(V)$ is a neighborhood of $s(x)$ contained in $s(U)$. □

DEFINITION B.3.24. Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a continuous map. An open subset U of X is said to be *quasi-fundamental* with respect to π if for every $x \in U$ and every $\tilde{x} \in \pi^{-1}(x)$ there exists a continuous local section $s : U \to \widetilde{X}$ of π such that $s(x) = \tilde{x}$.

REMARK B.3.25. Clearly, if U is a fundamental open subset of X with respect to a continuous map $\pi : \widetilde{X} \to X$ then U is also quasi-fundamental. Namely, write $\pi^{-1}(U) = \bigcup_{i \in I} U_i$, where $(U_i)_{i \in I}$ is a family of pairwise disjoint open subsets of $\widetilde{X}$ such that π maps U_i homeomorphically onto U for all $i \in I$. Given $x \in U$ and $\tilde{x} \in \pi^{-1}(x)$ then $\tilde{x} \in U_i$ for some $i \in I$. Let $s = (\pi|_{U_i})^{-1} : U \to \widetilde{X}$. Clearly s is a continuous local section of π and $s(x) = \tilde{x}$.

LEMMA B.3.26. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism; assume that $\widetilde{X}$ is Hausdorff. If U is a quasi-fundamental connected open subset of X with respect to π then U is a fundamental open subset of X with respect to π.*

PROOF. Let $\mathcal{S}$ be the set of all continuous local sections of π defined in U. We claim that:

$$\pi^{-1}(U) = \bigcup_{s \in \mathcal{S}} s(U).$$

Indeed, if $s \in \mathcal{S}$ then obviously $s(U) \subset \pi^{-1}(U)$; moreover, given $\tilde{x} \in \pi^{-1}(U)$ then $x = \pi(\tilde{x}) \in U$ and, since U is quasi-fundamental, there exists $s \in \mathcal{S}$ with $s(x) = \tilde{x}$. Thus $\tilde{x} \in s(U)$. This proves the claim. Now observe that, by Lemma B.3.23, $s(U)$ is open in $\widetilde{X}$ for all $s \in \mathcal{S}$; moreover, $\pi|_{s(U)} : s(U) \to U$ is a homeomorphism, being the inverse of $s : U \to s(U)$. To complete the proof, we show that the union $\bigcup_{s \in \mathcal{S}} s(U)$ is disjoint. Pick $s, s' \in \mathcal{S}$ with $s(U) \cap s'(U) \neq \emptyset$. Then there exists $x, y \in U$ with $s(x) = s'(y)$. Observe that:

$$x = \pi\big(s(x)\big) = \pi\big(s'(y)\big) = y,$$

and thus $s(x) = s'(x)$. Since U is connected and $\widetilde{X}$ is Hausdorff, using Lemma B.3.22 we get that $s = s'$. $\square$

COROLLARY B.3.27. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism; assume that $\widetilde{X}$ is Hausdorff and that X is locally connected. If X can be covered by quasi-fundamental open sets then π is a covering map.*

PROOF. Given $x \in X$, there exists a quasi-fundamental open subset U of X containing x. Since X is locally connected, U contains an open connected neighborhood U' of x (see Remark B.3.3). Obviously also U' is quasi-fundamental. Thus U' is a fundamental open subset of X, by Lemma B.3.26. $\square$

LEMMA B.3.28. *Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be a local homeomorphism. If Y is a subset of X then the map:*

$$\pi' = \pi|_{\pi^{-1}(Y)} : \pi^{-1}(Y) \longrightarrow Y$$

is a local homeomorphism; moreover, if $U \subset X$ is a fundamental open subset with respect to π then $U \cap Y$ is a fundamental open subset of Y with respect to π'.

PROOF. Since π is a local homeomorphism, given $\tilde{x} \in \pi^{-1}(Y)$ we can find an open subset A of $\widetilde{X}$ with $\pi(A)$ open in X and $\pi|_A : A \to \pi(A)$ a homeomorphism. Now $A \cap \pi^{-1}(Y)$ is an open subset of $\pi^{-1}(Y)$ containing $\tilde{x}$ and $\pi\big(A \cap \pi^{-1}(Y)\big) = \pi(A) \cap Y$ is open in Y; moreover, π maps $A \cap \pi^{-1}(Y)$ homeomorphically onto $\pi(A) \cap Y$. Thus π' is a local homeomorphism. Now let us prove that $U \cap Y$ is fundamental for π'. Write $\pi^{-1}(U) = \bigcup_{i \in I} U_i$, where $(U_i)_{i \in I}$ is a family of disjoint open subsets of $\widetilde{X}$ and π maps U_i homeomorphically onto U, for all $i \in I$. We have:

$$\pi'^{-1}(U \cap Y) = \pi^{-1}(U) \cap \pi^{-1}(Y) = \bigcup_{i \in I} \big(U_i \cap \pi^{-1}(Y)\big),$$

and $\big(U_i \cap \pi^{-1}(Y)\big)_{i \in I}$ is a family of disjoint open subsets of $\pi^{-1}(Y)$. Moreover, π' maps $U_i \cap \pi^{-1}(Y)$ homeomorphically onto $U \cap Y$, for all $i \in I$. □

COROLLARY B.3.29. *If $\pi : \widetilde{X} \to X$ is a covering map and Y is a subset of X then $\pi|_{\pi^{-1}(Y)} : \pi^{-1}(Y) \to Y$ is also a covering map.* □

LEMMA B.3.30. *If $\pi : \widetilde{X} \to X$ is a covering map, X is locally arc-connected and $\widetilde{Y}$ is an arc-connected component of $\widetilde{X}$ then $\pi|_{\widetilde{Y}} : \widetilde{Y} \to X$ is also a covering map.*

PROOF. Let U be a fundamental arc-connected open subset of X (relatively to π). We will show that U is also fundamental relatively to $\pi|_{\widetilde{Y}}$. Write $\pi^{-1}(U) = \bigcup_{i \in I} U_i$, where $(U_i)_{i \in I}$ is a family of disjoint open subsets of $\widetilde{X}$ and π maps U_i homeomorphically onto U, for every $i \in I$. Since U_i is homeomorphic to U, we have that U_i is arc-connected for every $i \in I$; since $\widetilde{Y}$ is an arc-connected component of $\widetilde{X}$, we have either $U_i \subset \widetilde{Y}$ or $U_i \cap \widetilde{Y} = \emptyset$, for all $i \in I$. Set:

$$I' = \big\{ i \in I : U_i \subset \widetilde{Y} \big\}.$$

Then $(\pi|_{\widetilde{Y}})^{-1}(U) = \pi^{-1}(U) \cap \widetilde{Y} = \bigcup_{i \in I'} U_i$. This proves that U is fundamental for $\pi|_{\widetilde{Y}}$. Since π is a covering map and X is locally arc-connected, the result of Exercise B.2 implies that the fundamental arc-connected open subsets of X form a covering of X. This concludes the proof. □

COROLLARY B.3.31. *Assume that $\pi : \widetilde{X} \to X$ is a covering map. Let Y be a connected, locally arc-connected and simply-connected subset of X and let $\widetilde{Y}$ be an arc-connected component of $\pi^{-1}(Y)$. Then $\pi|_{\widetilde{Y}} : \widetilde{Y} \to Y$ is a homeomorphism.*

PROOF. By Corollary B.3.29, $\pi|_{\pi^{-1}(Y)} : \pi^{-1}(Y) \to Y$ is a covering map. Since Y is locally arc-connected and $\widetilde{Y}$ is an arc-connected component of $\pi^{-1}(Y)$, Lemma B.3.30 implies that $\pi|_{\widetilde{Y}} : \widetilde{Y} \to Y$ is also a covering map. The conclusion follows from Corollary B.3.19. □

COROLLARY B.3.32. *Assume that $\pi : \widetilde{X} \to X$ is a covering map and that X is simply-connected and locally arc-connected. Assume also that the image of π intersects every connected component of X. Then π admits a continuous* global section*, i.e., a continuous local section $s : X \to \widetilde{X}$ whose domain is X.*

PROOF. Write $X = \bigcup_{i \in I} X_i$, where each X_i is a connected component of X. Since X is locally arc-connected (and, in particular, locally connected), each X_i is open in X; thus each X_i is also locally arc-connected. The fact that X is simply-connected implies that each X_i is also simply-connected. Let $\widetilde{X}_i$ be an arc-connected component of $\pi^{-1}(X_i)$; observe that, since the image of π intersects X_i, the set $\pi^{-1}(X_i)$ is nonempty and thus such an arc-connected component does exist. It follows from Corollary B.3.31 that π maps $\widetilde{X}_i$ homeomorphically onto X_i. Let $s_i : X_i \to \widetilde{X}_i$ be the inverse of the homeomorphism $\pi|_{\widetilde{X}_i} : \widetilde{X}_i \to X_i$. Then each s_i is a section of π. The desired global section $s : X \to \widetilde{X}$ is obtained by setting $s|_{X_i} = s_i$, for every $i \in I$. □

B.4. Sheaves and Pre-Sheaves

DEFINITION B.4.1. Let X be a topological space. A *pre-sheaf* on X is a map $\mathfrak{P}$ that assigns to each open subset $U \subset X$ a set $\mathfrak{P}(U)$ and to each pair of open subsets $U, V \subset X$ with $V \subset U$ a map $\mathfrak{P}_{U,V} : \mathfrak{P}(U) \to \mathfrak{P}(V)$ such that the following properties hold:

- for every open subset $U \subset X$ the map $\mathfrak{P}_{U,U}$ is the identity map of the set $\mathfrak{P}(U)$;
- given open sets, $U, V, W \subset X$ with $W \subset V \subset U$ then:

$$\mathfrak{P}_{V,W} \circ \mathfrak{P}_{U,V} = \mathfrak{P}_{U,W}.$$

REMARK B.4.2. A pre-sheaf on X is simply a contravariant functor from the category of open subsets of X to the category of sets and maps. The morphisms in the category of open subsets of X are defined as follows; if $U, V \subset X$ are open then the set of morphisms from V to U has a single element if $V \subset U$ and it is empty otherwise.

DEFINITION B.4.3. Given a topological space X, a *sheaf* over X is a pair $(\mathcal{S}, \pi)$, where $\mathcal{S}$ is a topological space and $\pi : \mathcal{S} \to X$ is a local homeomorphism.

EXAMPLE B.4.4. If $(\mathcal{S}, \pi)$ is a sheaf over a topological space X then the following pre-sheaf $\mathfrak{P}$ is naturally associated to $(\mathcal{S}, \pi)$: for every open subset $U \subset X$ let $\mathfrak{P}(U)$ be the set of continuous local sections of π whose domain is U. Given open subsets $U, V \subset X$ with $V \subset U$ then the map $\mathfrak{P}_{U,V}$ is defined by:

$$\mathfrak{P}_{U,V}(s) = s|_V,$$

for all $s \in \mathfrak{P}(U)$.

Let $\mathfrak{P}$ be a pre-sheaf over a topological space X. Given a point $x \in X$, consider the disjoint union of all sets $\mathfrak{P}(U)$, where U is an open neighborhood of x in X. We define an equivalence relation $\sim$ on such disjoint union as follows;

given $f_1 \in \mathfrak{P}(U_1)$, $f_2 \in \mathfrak{P}(U_2)$, where U_1, U_2 are open neighborhoods of x in X then $f_1 \sim f_2$ if and only if there exists an open neighborhood V of x contained in $U_1 \cap U_2$ such that $\mathfrak{P}_{U_1,V}(f_1) = \mathfrak{P}_{U_2,V}(f_2)$. If U is an open neighborhood of x in X and $f \in \mathfrak{P}(U)$ then the equivalence class of f corresponding to the equivalence relation $\sim$ will be denoted by $[f]_x$ and will be called the *germ* of f at the point x. We set:

$$\mathcal{S}_x = \big\{[f]_x : f \in \mathfrak{P}(U), \text{ for some open neighborhood } U \text{ of } x \text{ in } X\big\}.$$

REMARK B.4.5. The set $\mathcal{S}_x$ is simply the direct limit of the net $U \mapsto \mathfrak{P}(U)$, where U runs over the set of open neighborhoods of x ordered by reverse inclusion.

Let $\mathcal{S}$ denote the disjoint union of all $\mathcal{S}_x$, with $x \in X$. Let $\pi : \mathcal{S} \to X$ denote the map that carries $\mathcal{S}_x$ to the point x. Our goal now is to define a topology on $\mathcal{S}$. Given an open subset $U \subset X$ and an element $f \in \mathfrak{P}(U)$ we set:

$$\mathcal{V}(f) = \big\{[f]_x : x \in U\big\} \subset \mathcal{S}.$$

Observe that if V is an open subset of U then:

$$\mathcal{V}\big(\mathfrak{P}_{U,V}(f)\big) = \big\{[f]_x : x \in V\big\};$$

namely, we have $\big[\mathfrak{P}_{U,V}(f)\big]_x = [f]_x$, for all $x \in V$.

We claim that the set:

$$\big\{\mathcal{V}(f) : f \in \mathfrak{P}(U), U \text{ an open subset of } X\big\} \tag{B.4.1}$$

is a basis for a topology on $\mathcal{S}$. First, it is obvious that (B.4.1) is a covering of $\mathcal{S}$. Second, we have to prove the following property; given open subsets $U_1, U_2 \subset X$, $f_1 \in \mathfrak{P}(U_1)$, $f_2 \in \mathfrak{P}(U_2)$ and $\mathfrak{g} \in \mathcal{V}(f_1) \cap \mathcal{V}(f_2)$, there exists an element of (B.4.1) containing $\mathfrak{g}$ and contained in $\mathcal{V}(f_1) \cap \mathcal{V}(f_2)$. Let us find such element of (B.4.1). Since $\mathfrak{g} \in \mathcal{V}(f_1) \cap \mathcal{V}(f_2)$ we have $\mathfrak{g} = [f_1]_x = [f_2]_x$, for some $x \in U_1 \cap U_2$. Since $[f_1]_x = [f_2]_x$, there must exist an open neighborhood V of x contained in $U_1 \cap U_2$ such that $\mathfrak{P}_{U_1,V}(f_1) = \mathfrak{P}_{U_2,V}(f_2)$. Now it is easy to see that $\mathcal{V}\big(\mathfrak{P}_{U_1,V}(f_1)\big)$ is an element of (B.4.1) containing $\mathfrak{g}$ and contained in $\mathcal{V}(f_1) \cap \mathcal{V}(f_2)$.

In what follows we consider the set $\mathcal{S}$ endowed with the topology having (B.4.1) as a basis. Our goal is to show that $(\mathcal{S}, \pi)$ is a sheaf over X. We start with the following:

LEMMA B.4.6. *Let $U \subset X$ be an open subset. Given $x \in U$ and $f \in \mathfrak{P}(U)$ then the set:*

$$\big\{\mathcal{V}\big(\mathfrak{P}_{U,V}(f)\big) : V \textit{ an open neighborhood of } x \textit{ contained in } U\big\} \tag{B.4.2}$$

is a fundamental system of open neighborhoods of $[f]_x$ in $\mathcal{S}$ (i.e., every neighborhood of $[f]_x$ in $\mathcal{S}$ contains an element of (B.4.2)*).*

PROOF. Let $\mathcal{W}$ be a neighborhood of $[f]_x$ in $\mathcal{S}$; since (B.4.1) is a basis of open subsets for $\mathcal{S}$, we can find an open subset $U_1 \subset X$ and $f_1 \in \mathfrak{P}(U_1)$ with $[f]_x \in \mathcal{V}(f_1) \subset \mathcal{W}$. Since $[f]_x \in \mathcal{V}(f_1)$, it must be $x \in U_1$ and $[f]_x = [f_1]_x$; thus there exists an open neighborhood V of x contained in $U \cap U_1$ such that $\mathfrak{P}_{U,V}(f) = \mathfrak{P}_{U_1,V}(f_1)$. Then $\mathcal{V}\big(\mathfrak{P}_{U_1,V}(f_1)\big)$ belongs to (B.4.2) and is contained in $\mathcal{W}$. $\square$

Given an open subset $U \subset X$ and an element $f \in \mathfrak{P}(U)$ we define a map $\hat{f} : U \to \mathcal{S}$ by setting:

$$\hat{f}(x) = [f]_x,$$

for all $x \in U$.

LEMMA B.4.7. *If $U \subset X$ is an open subset and $f \in \mathfrak{P}(U)$ then the map $\hat{f}$ maps U homeomorphically onto $\mathcal{V}(f)$.*

PROOF. It is clear hat $\hat{f} : U \to \mathcal{V}(f)$ is a bijection. Moreover, if V is open in U (and hence in X), we have $\hat{f}(V) = \mathcal{V}\big(\mathfrak{P}_{U,V}(f)\big)$; thus $\hat{f}$ is an open mapping. To complete the proof, we show that $\hat{f}$ is continuous. Let $x \in U$ be fixed and let $\mathcal{V}\big(\mathfrak{P}_{U,V}(f)\big)$ be an element of the fundamental system of neighborhoods (B.4.2) of $\hat{f}(x) = [f]_x$; by V we denote an open neighborhood of x contained in U. Then $\hat{f}(V) = \mathcal{V}\big(\mathfrak{P}_{U,V}(f)\big)$; this proves the continuity of $\hat{f}$ and completes the proof of the lemma. $\square$

COROLLARY B.4.8. *The map $\pi : \mathcal{S} \to X$ is a local homeomorphism. Thus $(\mathcal{S}, \pi)$ is a sheaf over X.*

PROOF. If $U \subset X$ is an open subset and $f \in \mathfrak{P}(U)$ then π maps the open set $\mathcal{V}(f)$ homeomorphically onto the open subset U of X; namely, the map:

$$\pi|_{\mathcal{V}(f)} : \mathcal{V}(f) \longrightarrow U$$

is the inverse of the map $\hat{f} : U \to \mathcal{V}(f)$. The conclusion follows by observing that the sets $\mathcal{V}(f)$ cover $\mathcal{S}$. $\square$

We call $(\mathcal{S}, \pi)$ the *sheaf of germs* associated to the pre-sheaf $\mathfrak{P}$. Observe that if U is an open subset of X and $f \in \mathfrak{P}(U)$ then $\hat{f}$ is a section of the sheaf of germs defined in U.

DEFINITION B.4.9. We say that the pre-sheaf $\mathfrak{P}$ has the *localization property* if, given a family $(U_i)_{i \in I}$ of open subsets of X and setting $U = \bigcup_{i \in I} U_i$ then the map:

$$\mathfrak{P}(U) \ni f \longmapsto \big(\mathfrak{P}_{U,U_i}(f)\big)_{i \in I} \in \prod_{i \in I} \mathfrak{P}(U_i) \tag{B.4.3}$$

is injective and its image consists of the families $(f_i)_{i \in I}$ in $\prod_{i \in I} \mathfrak{P}(U_i)$ such that $\mathfrak{P}_{U_i, U_i \cap U_j}(f_i) = \mathfrak{P}_{U_j, U_i \cap U_j}(f_j)$, for all $i, j \in I$.

REMARK B.4.10. Observe that if $\mathfrak{P}$ has the localization property then the set $\mathfrak{P}(\emptyset)$ has exactly one element. Namely, consider the empty family $(U_i)_{i \in I}$, i.e., I is the empty set. Then $U = \bigcup_{i \in I} U_i$ is the empty set and the image of the map (B.4.3) has exactly one element (the empty family $(f_i)_{i \in I}$). Thus $\mathfrak{P}(\emptyset)$ has exactly one element as well.

DEFINITION B.4.11. Given pre-sheafs $\mathfrak{P}$ and $\mathfrak{P}'$ over a topological space X then an *isomorphism* from $\mathfrak{P}$ to $\mathfrak{P}'$ is a map λ that associates to each open subset

$U \subset X$ a bijection $\lambda_U : \mathfrak{P}(U) \to \mathfrak{P}'(U)$ such that, given open subsets $U, V \subset X$ with $V \subset U$ then the diagram:

$$\begin{CD} \mathfrak{P}(U) @>{\lambda_U}>> \mathfrak{P}'(U) \\ @V{\mathfrak{P}_{U,V}}VV @VV{\mathfrak{P}'_{U,V}}V \\ \mathfrak{P}(V) @>>{\lambda_V}> \mathfrak{P}'(V) \end{CD} \tag{B.4.4}$$

commutes.

LEMMA B.4.12. *If the pre-sheaf $\mathfrak{P}$ has the localization property then, for every open subset $U \subset X$, the map $f \mapsto \hat{f}$ gives a bijection between the set $\mathfrak{P}(U)$ and the set of continuous local sections of the sheaf of germs defined in U. More precisely, such bijections give an isomorphism between the pre-sheaf $\mathfrak{P}$ and the pre-sheaf naturally associated to the sheaf of germs $(\mathcal{S}, \pi)$ (recall Example B.4.4).*

PROOF. We start by observing that, once we prove that the maps $f \mapsto \hat{f}$ are bijections, it will follow easily that they give an isomorphism of pre-sheaves (i.e., diagram (B.4.4) commutes). Namely, given open subsets $U, V \subset X$ with $V \subset U$ and given $f \in \mathfrak{P}(U)$, the commutativity of diagram (B.4.4) is equivalent to:

$$\hat{g} = \hat{f}|_V,$$

where $g = \mathfrak{P}_{U,V}(f)$.

Let $U \subset X$ be an open subset. Let us prove that the map $\mathfrak{P}(U) \ni f \mapsto \hat{f}$ is injective. Let $f_1, f_2 \in \mathfrak{P}(U)$ be fixed and assume that $\widehat{f_1} = \widehat{f_2}$. For every $x \in U$ we have $[f_1]_x = [f_2]_x$ and thus there exists an open neighborhood U_x of x contained in U such that $\mathfrak{P}_{U,U_x}(f_1) = \mathfrak{P}_{U,U_x}(f_2)$. Now $U = \bigcup_{x \in U} U_x$ and thus the localization property implies that $f_1 = f_2$. This proves the injectivity of $f \mapsto \hat{f}$.

Now let $s : U \to \mathcal{S}$ be a continuous local section of π and let us find $f \in \mathfrak{P}(U)$ with $s = \hat{f}$. For every $x \in U$, $s(x)$ is an element of $\mathcal{S}_x$; thus there exists an open neighborhood U_x of x and an element $f_x \in \mathfrak{P}(U_x)$ such that $s(x) = [f_x]_x$. Since s and $\widehat{f_x}$ are both continuous local sections of π and since $s(x) = \widehat{f_x}(x)$, there exists an open neighborhood V_x of x contained in $U_x \cap U$ such that $s|_{V_x} = \widehat{f_x}|_{V_x}$ (recall Lemma B.3.21). Set $g_x = \mathfrak{P}_{U_x,V_x}(f_x)$, for all $x \in U$; we claim that there exists $f \in \mathfrak{P}(U)$ with $\mathfrak{P}_{U,V_x}(f) = g_x$, for all $x \in U$. Since $\bigcup_{x \in U} V_x$ is an open cover of U, by the localization property, in order to prove the claim it suffices to show that for every $x, y \in U$ we have:

$$\mathfrak{P}_{V_x, V_x \cap V_y}(g_x) = \mathfrak{P}_{V_y, V_x \cap V_y}(g_y).$$

Let $x, y \in U$ be fixed and set $h_1 = \mathfrak{P}_{V_x, V_x \cap V_y}(g_x)$, $h_2 = \mathfrak{P}_{V_y, V_x \cap V_y}(g_y)$. We have:

$$\widehat{h_1} = \widehat{g_x}|_{V_x \cap V_y} = \widehat{f_x}|_{V_x \cap V_y} = s|_{V_x \cap V_y} = \widehat{f_y}|_{V_x \cap V_y} = \widehat{g_y}|_{V_x \cap V_y} = \widehat{h_2}.$$

By the first part of the proof, we get $h_1 = h_2$. This proves the claim, i.e., there exists $f \in \mathfrak{P}(U)$ with $\mathfrak{P}_{U,V_x}(f) = g_x$, for all $x \in U$. This implies:

$$[f]_x = [g_x]_x = [f_x]_x = s(x),$$

for all $x \in U$. Hence $\hat{f} = s$. □

REMARK B.4.13. It is easily seen that the pre-sheaf naturally associated to a sheaf (recall Example B.4.4) always satisfies the localization property. Thus the localization property is indeed an essential hypothesis in Lemma B.4.12.

DEFINITION B.4.14. We say that the pre-sheaf $\mathfrak{P}$ has the *uniqueness property* if for every connected open subset $U \subset X$ and every nonempty open subset $V \subset U$ the map $\mathfrak{P}_{U,V}$ is injective.

LEMMA B.4.15. *If the pre-sheaf $\mathfrak{P}$ has the uniqueness property and if X is locally connected and Hausdorff then the space $\mathcal{S}$ is Hausdorff.*

PROOF. Let $U_1, U_2 \subset X$ be open sets, $f_1 \in \mathfrak{P}(U_1)$, $f_2 \in \mathfrak{P}(U_2)$, $x \in U_1$, $y \in U_2$ be fixed with $[f_1]_x \neq [f_2]_y$. We have to find disjoint open neighborhoods of $[f_1]_x$ and $[f_2]_y$ in $\mathcal{S}$. If $x \neq y$, we can find disjoint open subsets $V_1, V_2 \subset X$ with $x \in V_1$ and $y \in V_2$. Then $\pi^{-1}(V_1)$ and $\pi^{-1}(V_2)$ are disjoint open neighborhoods of $[f_1]_x$ and $[f_2]_y$, respectively. Assume now that $x = y$. Let U be a connected open neighborhood of x contained in $U_1 \cap U_2$. Then $\mathcal{V}\big(\mathfrak{P}_{U_1,U}(f_1)\big)$ is an open neighborhood of $[f_1]_x$ and $\mathcal{V}\big(\mathfrak{P}_{U_2,U}(f_2)\big)$ is an open neighborhood of $[f_2]_x$. We claim that $\mathcal{V}\big(\mathfrak{P}_{U_1,U}(f_1)\big)$ and $\mathcal{V}\big(\mathfrak{P}_{U_2,U}(f_2)\big)$ are disjoint. Otherwise, there would exist $z \in U$ with $[f_1]_z = [f_2]_z$ and thus there would exist an open neighborhood V of z contained in U such that $\mathfrak{P}_{U_1,V}(f_1) = \mathfrak{P}_{U_2,V}(f_2)$. This implies:

$$(\mathfrak{P}_{U,V} \circ \mathfrak{P}_{U_1,U})(f_1) = (\mathfrak{P}_{U,V} \circ \mathfrak{P}_{U_2,U})(f_2);$$

by the uniqueness property, $\mathfrak{P}_{U,V}$ is injective and so

$$\mathfrak{P}_{U_1,U}(f_1) = \mathfrak{P}_{U_2,U}(f_2).$$

In particular, $[f_1]_x = [f_2]_x$, contradicting our hypothesis. □

DEFINITION B.4.16. We say that an open subset $U \subset X$ has the *extension property with respect to the pre-sheaf* $\mathfrak{P}$ if for every connected nonempty open subset V of U the map $\mathfrak{P}_{U,V}$ is surjective. We say that the pre-sheaf $\mathfrak{P}$ has the *extension property* if X can be covered by open sets having the extension property with respect to $\mathfrak{P}$.

LEMMA B.4.17. *Assume that X is locally connected. If U is an open subset of X having the extension property with respect to the pre-sheaf $\mathfrak{P}$ then U is quasi-fundamental with respect to $\pi : \mathcal{S} \to X$.*

PROOF. Let $x \in U$ and $\tilde{x} \in \mathcal{S}$ be fixed, with $\pi(\tilde{x}) = x$. We have to find a section $s : U \to \mathcal{S}$ of π with $s(x) = \tilde{x}$. Since $\tilde{x} \in \mathcal{S}_x$, there exists an open neighborhood W of x and $f \in \mathfrak{P}(W)$ with $\tilde{x} = [f]_x$. Let V be a connected open neighborhood of x contained in $U \cap W$. Since U has the extension property with respect to $\mathfrak{P}$, we can find $g \in \mathfrak{P}(U)$ with $\mathfrak{P}_{U,V}(g) = \mathfrak{P}_{W,V}(f)$. Hence $s = \hat{g}$ is a section of π defined in U and $s(x) = [g]_x = [f]_x = \tilde{x}$. □

COROLLARY B.4.18. *Assume that X is Hausdorff and locally connected and that the pre-sheaf $\mathfrak{P}$ has the uniqueness property. If U is a connected open subset of X having the extension property with respect to the pre-sheaf $\mathfrak{P}$ then U is a fundamental open subset of X with respect to the map π.*

PROOF. By Lemma B.4.17, the set U is quasi-fundamental and by Lemma B.4.15 the space $\mathcal{S}$ is Hausdorff. The conclusion follows from Lemma B.3.26. □

COROLLARY B.4.19. *Assume that X is Hausdorff and locally connected and that the pre-sheaf $\mathfrak{P}$ has the uniqueness property and the extension property. Then the map $\pi : \mathcal{S} \to X$ is a covering map.*

PROOF. By Lemma B.4.15, $\mathcal{S}$ is Hausdorff. The conclusion follows from Corollary B.3.27. □

The following is a converse of Lemma B.4.17.

LEMMA B.4.20. *Assume that the pre-sheaf $\mathfrak{P}$ has the localization property and the uniqueness property. If an open subset $U \subset X$ is quasi-fundamental with respect to $\pi : \mathcal{S} \to X$ then U has the extension property with respect to the pre-sheaf $\mathfrak{P}$.*

PROOF. Let V be a connected nonempty open subset of U. Let $f \in \mathfrak{P}(V)$ be fixed. We have to find an element $g \in \mathfrak{P}(U)$ with $\mathfrak{P}_{U,V}(g) = f$. Choose an arbitrary point $x \in V$. The germ $[f]_x$ is an element of $\mathcal{S}$ with $\pi\big([f]_x\big) = x$. Since $x \in U$ and U is quasi-fundamental, it follows that there exists a continuous local section $s : U \to \mathcal{S}$ of π with $s(x) = [f]_x$. Since $\mathfrak{P}$ has the localization property, Lemma B.4.12 gives us an element $g \in \mathfrak{P}(U)$ with $s = \hat{g}$. Then $[g]_x = s(x) = [f]_x$ and therefore there exists an open neighborhood W of x contained in V such that $\mathfrak{P}_{U,W}(g) = \mathfrak{P}_{V,W}(f)$; thus:

$$\mathfrak{P}_{V,W}\big(\mathfrak{P}_{U,V}(g)\big) = \mathfrak{P}_{V,W}(f).$$

Since $\mathfrak{P}$ has the uniqueness property and W is a nonempty open subset of the connected open set V, we have $\mathfrak{P}_{U,V}(g) = f$. This concludes the proof. □

Finally, we prove our main results.

LEMMA B.4.21. *Assume that X is Hausdorff, locally arc-connected and that the pre-sheaf $\mathfrak{P}$ has the localization property, the uniqueness property and the extension property. If U is an arc-connected open subset of X such that every loop in U is contractible in X (in particular, if U is simply-connected) then U has the extension property with respect to $\mathfrak{P}$.*

PROOF. By Corollary B.4.19, the map $\pi : \mathcal{S} \to X$ is a covering map. Observe that, since X is locally arc-connected and $\pi : \mathcal{S} \to X$ is a local homeomorphism then $\mathcal{S}$ is also locally arc-connected; thus, by Corollary B.3.16, U is a fundamental open subset of X. By Remark B.3.25, U is quasi-fundamental and hence Lemma B.4.20 implies that U has the extension property. □

COROLLARY B.4.22. *Assume that X is Hausdorff, locally arc-connected, arc-connected, simply-connected and that the pre-sheaf $\mathfrak{P}$ has the localization property, the uniqueness property and the extension property. Then for every connected nonempty open subset $V \subset X$ and every $f \in \mathfrak{P}(V)$ there exists $g \in \mathfrak{P}(X)$ with $\mathfrak{P}_{X,V}(g) = f$.*

PROOF. It follows from Lemma B.4.21 that X itself is an open subset of X having the extension property. Thus, since V is open, connected and nonempty, it follows that the map $\mathfrak{P}_{X,V} : \mathfrak{P}(X) \to \mathfrak{P}(V)$ is surjective. $\square$

LEMMA B.4.23. *Assume that X is Hausdorff, locally arc-connected and simply-connected and that the pre-sheaf $\mathfrak{P}$ has the localization property, the uniqueness property and the extension property. Assume also that every connected component of X contains a nonempty open set U such that $\mathfrak{P}(U)$ is nonempty. Then the set $\mathfrak{P}(X)$ is nonempty.*

PROOF. By Corollary B.4.19, the map $\pi : \mathcal{S} \to X$ is a covering map. Since every connected component of X contains a nonempty set U such that $\mathfrak{P}(U)$ is nonempty, it follows that the image of π intersects every connected component of X. It follows from Corollary B.3.32 that π admits a continuous global section $s : X \to \mathcal{S}$. By Lemma B.4.12, there exists $f \in \mathfrak{P}(X)$ with $s = \hat{f}$. Hence $\mathfrak{P}(X)$ is nonempty. $\square$

EXAMPLE B.4.24. Let X be simply-connected differentiable manifold and let θ be a smooth closed 1-form on X. Let us prove that θ is exact. For every open subset $U \subset X$ let $\mathfrak{P}(U)$ be the set of smooth maps $f : U \to \mathbb{R}$ with $\mathrm{d}f = \theta|_U$. If $U, V \subset X$ are open subsets with $V \subset U$, define:

$$\mathfrak{P}_{U,V}(f) = f|_V,$$

for all $f \in \mathfrak{P}(U)$. It is immediate that $\mathfrak{P}$ is a pre-sheaf over X satisfying the localization property. If U is a connected open subset of X and if $f_1, f_2 \in \mathfrak{P}(U)$ are equal at one point of U then $f_1 = f_2$; this implies that $\mathfrak{P}$ satisfies the uniqueness property. Assuming the well-known fact that every smooth closed 1-form on an open ball in Euclidean space is exact, we conclude that for every open subset U of X that is diffeomorphic to an open ball in Euclidean space the set $\mathfrak{P}(U)$ is nonempty; in particular, every connected component of X contains a nonempty open subset U such that $\mathfrak{P}(U)$ is nonempty. Finally, let us prove that $\mathfrak{P}$ has the extension property. To this aim, we prove that if U is an open subset of X that is diffeomorphic to an open ball in Euclidean space then U has the extension property with respect to $\mathfrak{P}$. Namely, let V be a connected nonempty open subset of U and let $f \in \mathfrak{P}(V)$ be fixed. Since U is diffeomorphic to an open ball in Euclidean space, there exists a smooth map $f_1 : U \to \mathbb{R}$ with $\mathrm{d}f_1 = \theta|_U$. Since V is connected, $f_1|_V - f$ is constant and equal to some $c \in \mathbb{R}$. Hence $f_1 - c \in \mathfrak{P}(U)$ and $(f_1 - c)|_V = f$. This concludes the proof of the extension property. Now Lemma B.4.23 implies that $\mathfrak{P}(X)$ is nonempty, i.e., there exists a smooth map $f : X \to \mathbb{R}$ with $\mathrm{d}f = \theta$. Hence θ is exact.

Exercises

EXERCISE B.1. Let X be a locally arc-connected (resp., locally connected) topological space and let U be an open subset of X. Show that the arc-connected components (resp., connected components) of U are open in X.

EXERCISE B.2. Let $\widetilde{X}$, X be topological spaces and $\pi : \widetilde{X} \to X$ be an arbitrary map. If $U \subset X$ is a fundamental open subset with respect to π, show that every open subset V of U is also fundamental.

Bibliography

[1] M. Berger and B. Gostiaux, *Differential geometry: manifolds, curves, and surfaces*, Graduate Texts in Mathematics, 115. Springer–Verlag, New York, 1988.

[2] L. Biliotti, F. Mercuri, D. V. Tausk, *A note on tensor fields in Hilbert spaces*, Anais da Academia Brasileira de Ciências, **74**, n$^{\underline{o}}$ 2 (2002), pgs. 207–210.

[3] W. M. Boothby, *An introduction to differentiable manifolds and Riemannian geometry*, Second edition. Pure and Applied Mathematics, 120. Academic Press, Inc., Orlando, FL, 1986.

[4] M. P. do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992.

[5] M. Dajczer, *Submanifolds and Isometric Immersions*, Mathematics Lecture Series, Publish or Perish, 1990.

[6] B. Daniel, *Isometric immersions into $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$ and applications to minimal surfaces*, preprint (`perso-math.univ-mlv.fr/users/daniel.benoit/codazzi.pdf`).

[7] B. Daniel, *Isometric Immersions into 3-Dimensional Homogeneous Manifolds*, 2005, to appear in Commentarii Mathematici Helvetici.

[8] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol I*, Interscience Publishers, New York-London, 1963.

[9] S. Lang, *Introduction to differentiable manifolds*. Second edition. Universitext. Springer-Verlag, New York, 2002.

[10] B. O'Neill, *Semi-Riemannian geometry*, Academic Press, 1983.

[11] K. Nomizu and T. Sasaki, *Affine differential geometry*, Cambridge University Press, 1994.

[12] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Glenview, Scott, 1971.

[13] J. A. Wolf, *Spaces of constant curvature*. New York, Mc-Graw-Hill, 1967.

List of Symbols

Symbols

Index

I

K

L