Short course
On the isometry group of Lorentz manifolds
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Global geometry: Riemannian vs. Lorentzian

Compact Riemannian manifolds:
- are complete and geodesically complete
- are geodesically connected
- have compact isometry group

Compact Lorentz manifolds:
- may be geodesically incomplete
- may fail to be geodesically connected
- have possibly non compact isometry group
Why is the set of Lorentzian isometries a (finite dimensional) Lie group?

Classify the Lie groups that act isometrically (and faithfully) on (compact) Lorentzian manifolds.

Discuss compactness and non compactness issues.

Can one obtain geometric information on the manifold when its isometry group is given?

Examples

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Isometry group of Lorentz manifolds
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- Examples
Let \((M, g)\) be a compact manifold, and let \(G\) be a connected Lie group acting by isometries locally faithfully on \(M\). Then either of the two possibilities occur:

- \(G\) is locally isomorphic to \(\text{SL}(2, \mathbb{R}) \times K\), with \(K\) compact;
- \(G\) is amenable (equiv., compact modulo radical).

If \((M, g)\) is a Lorentz manifold with finite volume, and let \(G\) be a Lie group acting locally faithfully and isometrically on \(M\). If the Zariski closure \(G_a\) of \(\text{Ad}(G)\) \(\subset \text{Aut}(g)\) has no co-compact normal algebraic subgroup, then:

- for almost every \(m \in M\), the stabilizer \(G_m \subset G\) is discrete;
- either \(g = \text{sl}(2, \mathbb{R}) \oplus \text{abelian}\), or there exists a one-dimensional ideal \(I_0 \subset g\) such that \(g/I_0\) is abelian.
R. J. Zimmer, Inventiones 1986:

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Some Bibliography. —2

If $(M, g)$ is a compact, simply connected real analytic Lorentz manifold, then $\text{Iso}(M, g)$ is compact.

Example of compact simply connected pseudo-Riemannian manifold of type $(7, 2)$ with non-compact isometry group.


If $G$ acts non properly on a compact Lorent manifold, and all $G$-stabilizers are discrete, then $G$ is locally isomorphic to $\text{SL}(2, \mathbb{R})$.

In general, if $G$ acts non properly, then $G$ is locally isomorphic to either $\text{SO}(1, n)$ or to $\text{SO}(2, n)$.

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Isometry group of Lorentz manifolds
G. D’Ambra, Inventiones 1988

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- *If* $G$ *acts non properly on a compact Lorent manifold, and all* $G$-*stabilizers are discrete, then* $G$ *is locally isomorphic to* $\text{SL}(2, \mathbb{R})$.*
- *In general, if* $G$ *acts non properly, then* $G$ *is locally isomorphic to either* $\text{SO}(1, n)$ *or to* $\text{SO}(2, n)$.
Let $G$ be a connected, simply connected Lie group. The following are equivalent:

1. $G$ is the universal cover of $\text{Iso}^0(M, g)$ for some compact Lorentz manifold $M$;
2. $G$ is isomorphic to $L \times K \times \mathbb{R}^d$, where $K$ is compact and semisimple, $d \geq 0$, and $L$ in the list:
   - $\tilde{\text{SL}}(2, \mathbb{R})$
   - Heisenberg group $\text{Heis}^{2n+1}$
   - An oscillator group.

A. Zeghib, 1998

Geometric description of compact Lorentz manifold with $\text{Iso}^0(M, g)$ non compact.

P. P., A. Zeghib, 2010

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Isometry group and covering spaces

\[ (\tilde{M}, g) \text{ semi-Riemannian manifold} \]

\[ \pi: \tilde{M} \to M \text{ covering}, \quad \tilde{g} = \pi^* (g) \]

\[ H = \{ \tilde{\psi} \in \text{Iso}(\tilde{M}, \tilde{g}) : \pi(\tilde{\psi}(x)) = \pi(\tilde{\psi}(y)) \text{ if } \pi(x) = \pi(y) \} \]

\[ H \ni \tilde{\psi} \Phi \mapsto \tilde{\psi} \in \text{Iso}(M, g) \quad (\psi \circ \pi = \pi \circ \tilde{\psi}) \]

\[ \Gamma = \text{Ker}(\Phi) \text{ group of covering automorphisms of } \pi \]

If \( \tilde{M} \) is the universal cover of \( M \) then:

\[ \Phi \text{ is onto} \quad H = \text{Nor}(\Gamma) \implies \text{Iso}(M, g) = \text{Nor}(\Gamma) / \Gamma. \]
(\(M, g\)) semi-Riemannian manifold
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\[ H \ni \tilde{\psi} \xrightarrow{\Phi} \psi \in \text{Iso}(M, g) \quad (\psi \circ \pi = \pi \circ \tilde{\psi}) \]
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If \(\tilde{M}\) is the *universal cover* of \(M\) then:

- \(\Phi\) is onto
- \(H = \text{Nor}(\Gamma) \implies \text{Iso}(M, g) = \text{Nor}(\Gamma)/\Gamma\).
\( M^n \) smooth manifold, \( L(M) \) frame bundle, \( \text{GL}(n) \)-principal bundle

- \( g \) semi-Riemannian metric on \( M \) of index \( k \)
- \( \text{Iso}(M, g) \) isometry group
- \( L_g(M) \) \( g \)-orthogonal frame bundle, \( \text{O}(n, k) \)-principal subbundle
- \( p \in L_g(M) \), orbit: \( O_p = \{ \psi \circ \psi : \psi \in \text{Iso}(M, g) \} \) is a closed subset of \( L_g(M) \)
- \( \text{Iso}(M, g) \ni \psi \mapsto \psi \circ p \in O_p \) is a homeomorphism
Two direct consequences

Proposition

- If \((M, g)\) is a compact Riemannian manifold, then \(\text{Iso}(M, g)\) is compact.

Corollary (J. L. Flores, M. A. Javaloyes, P. P.)
Compact stationary Lorentzian manifolds admit at least two non-trivial closed geodesics.
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- If \((M, g)\) is compact Lorentzian, \(K \in \text{Kill}(M, g), p \in M, g(K_p, K_p) < 0\), then the 1-parameter group of isometries generated by \(K\) is pre-compact.
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Compact stationary Lorentzian manifolds admit at least two non trivial closed geodesics.
Theorem 1 (Gleason 1952, Yamabe 1953)

If $G$ is a \textit{locally compact} topological group, then $G$ admits a (necessarily unique) Lie group structure compatible with the topology, if and only if $G$ doesn’t have \textit{small subgroups}.
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**Def.:** $G$ has *small subgroups* if every neighborhood of 1 contains a nontrivial subgroup of $G$. 
Riemannian isometries 1

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Theorem 2

If $G$ is a \textit{locally compact} topological group, $M$ is a smooth manifold that carries a continuous \textit{effective} action by diffeomorphisms $G \times M \to M$ of $G$, then $G$ admits a Lie group structure compatible with the topology, and the action is smooth.
Riemannian isometries 2

\((M, g)\) Riemannian manifold

The isometry group of a locally compact metric space is a locally compact topological group, endowed with the compact-open topology. It follows from Theorem 2 that \(\text{Iso}(M, g)\) is a Lie group with the compact-open topology.

This theory does not work for Lorentzian isometries/conformal diffeomorphisms

General question: For which groups \(G \subset \text{GL}(n, \mathbb{R})\) the set of automorphisms of a \(G\)-manifold is a Lie transformation group?
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### Definitions and examples

#### Notation

- $M$: differentiable (smooth) manifold of dimension $n < \infty$.
- $G$: Lie subgroup (not necessarily closed) of $GL(n)$.
- $L(M)$: the frame bundle of $M$.

**Obs.**: $G$ may be non-connected, but we require that it is second-countable.

---

**A $G$-structure** $B_G$ on $M$ is a reduction $\pi: B_G \to M$ of the structural group of $L(M)$ to $G$.  

**Characterization**

Given $p \in B_G$ and $g \in GL(n)$, we have $p \circ g \in B_G \iff g \in G$.  

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**Isometry group of Lorentz manifolds**
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When $G = \{1\}$, a $G$-structure is a choice of frame at each tangent space, $T_x M$; i.e., a parallelism on $M$. 
Definitions and examples

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$O(n, \nu)$-structures

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### \( O(n, \nu) \)-structures

When \( G = O(n, \nu) \), a \( G \)-structure is equivalent to a semi-Riemannian metric on \( M \) of index \( \nu \).

### \( Sp(n) \)-structures

When \( G = Sp(n) \), a \( G \)-structure is equivalent to an \textit{almost symplectic} structure on \( M \). This means that the symplectic form \( \Omega \) is not necessarily closed. One may write the integrability condition \( d\Omega = 0 \) in the language of \( G \)-structures, as we’ll see.
Example: $O(n, \nu)$-structures

To define a semi-Riemannian metric from a given $O(n, \nu)$-structure:

$(1)$

Conversely, to define a $O(n, \nu)$-structure from a given semi-Riemannian metric:

$(2)$
Example: $O(n, \nu)$-structures

To define a semi-Riemannian metric from a given $O(n, \nu)$-structure: given $u, v \in T_x M$, put

$$\langle u, v \rangle_x := \langle p^{-1}(u), p^{-1}(v) \rangle_{\mathbb{R}^n}, \quad p \in \pi^{-1}(x).$$  \hfill (1)
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$$\langle u, v \rangle_x := \langle p^{-1}(u), p^{-1}(v) \rangle_{\mathbb{R}^\nu}, \quad p \in \pi^{-1}(x). \quad (1)$$

Conversely, to define a $O(n, \nu)$-structure from a given semi-Riemannian metric:

given $x \in M$, put

$$\pi^{-1}(x) := \{ p \in Iso(\mathbb{R}^n; T_x M) : p \text{ is a linear isometry} \}. \quad (2)$$
Isomorphisms

Let $B^1_G, B^2_G$ be $G$-structures on $M_1, M_2$, respectively.
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Let $B^1_G, B^2_G$ be $G$-structures on $M_1, M_2$, respectively.

Each diffeomorphism $f : M_1 \rightarrow M_2$ induces a map $f_\ast$ such that the following diagram commutes.

\[
\begin{array}{ccc}
L(M_1) & \xrightarrow{f_\ast} & L(M_2) \\
\pi^1 \downarrow & & \downarrow \pi^2 \\
M_1 & \xrightarrow{f} & M_2 \\
\end{array}
\]
Isomorphisms

Let $B^1_G$, $B^2_G$ be $G$-structures on $M_1$, $M_2$, respectively.

Each diffeomorphism $f : M_1 \to M_2$ induces a map $f_*$ such that the following diagram commutes.

We say that $f$ is an isomorphism of $B^1_G$ onto $B^2_G$ if $f_*(B^1_G) = B^2_G$.

In this case, $f$ is an automorphism when $M_1 = M_2$ and $B^1_G = B^2_G$. 

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Isometry group of Lorentz manifolds
Let $\pi : B_G \to M$ be a $G$-structure. There’s a canonical vector-valued 1-form defined on $B_G$:

$$\theta_p : T_p(B_G) \to \mathbb{R}^n$$

$$X \mapsto p^{-1} \circ d\pi_p(X) \quad (3)$$

Differentiating $\theta_p$, we get

$$d\theta_p : T_p(B_G) \wedge T_p(B_G) \to \mathbb{R}^n \quad (4)$$

Choose $H$ such that $H \oplus V_p = T_p(B_G)$. Then, we can define $c_H : \mathbb{R}^n \wedge \mathbb{R}^n \to \mathbb{R}^n$

$$u \wedge v \mapsto d\theta_p(X \wedge Y) \quad (5)$$

where $X, Y \in H$, $\theta_p(X) = u$, $\theta_p(Y) = v$. 

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First-order structure function

Preliminaries

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Isometry group of Lorentz manifolds
Dependence on horizontal complements

Let $H_1, H_2$ be such that $H_1 \oplus V_p \simeq H_2 \oplus V_p \simeq T_p(B_G)$. 

Define $S_{H_2, H_1} : \mathbb{R}^n \rightarrow V_p \simeq \mathbb{R}^n$ such that $X_2 - X_1$, (6)

Then, the following equation holds

$c_{H_2}(u \wedge v) - c_{H_1}(u \wedge v) = A(S_{H_2, H_1})(u \wedge v)$, (7)

where $A : \text{hom}(\mathbb{R}^n; \mathbb{R}) \rightarrow \text{hom}(\mathbb{R}^n \wedge \mathbb{R}^n; \mathbb{R})$ is the antisymmetrization operator.
Dependence on horizontal complements

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Define

$$S_{H_2,H_1} : \mathbb{R}^n \to \mathcal{V}_p \simeq g$$

$$u \mapsto X_2 - X_1,$$

where $X_i \in H_i, \ i = 1, 2$, and $\theta_p(X_2) = \theta_p(X_1) = u$. 

(6)
Dependence on horizontal complements

Let $H_1, H_2$ be such that $H_1 \oplus \mathcal{V}_p \simeq H_2 \oplus \mathcal{V}_p \simeq T_p(B_G)$.

Define

$$S_{H_2, H_1} : \mathbb{R}^n \to \mathcal{V}_p \simeq g$$

$$u \mapsto X_2 - X_1,$$  \hspace{1cm} (6)

where $X_i \in H_i$, $i = 1, 2$, and $\theta_p(X_2) = \theta_p(X_1) = u$.

Then, the following equation holds

$$c_{H_2}(u \wedge v) - c_{H_1}(u \wedge v) = \mathcal{A}(S_{H_2, H_1})(u \wedge v),$$  \hspace{1cm} (7)

where

$$\mathcal{A} : \text{hom}(\mathbb{R}^n; g) \to \text{hom}(\mathbb{R}^n \wedge \mathbb{R}^n; \mathbb{R}^n)$$

is the antisymmetrization operator.
The first-order structure function of $B_G$ is

$$c : B_G \rightarrow \frac{\text{hom}(\mathbb{R}^n \wedge \mathbb{R}^n; \mathbb{R}^n)}{\mathcal{A} (\text{hom}(\mathbb{R}^n; g))}$$

(8)

$p \mapsto [c_H]$
First-order structure function

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$$\rho \mapsto [c_H] \quad \quad \quad \quad \quad (8)$$

**Remarks**

- $c$ is an invariant of $B_G$. 

- For some $G$-structures, $c$ is identically zero, giving no information.

- In some cases, $c$ measures (obstruction to) integrability. For instance: almost symplectic structures, distributions, almost complex structures (Newlander-Nirenberg theorem).
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First prolongation of a $G$-structure

**Definition**

Choose $\mathcal{H} \subseteq \text{hom}(\mathbb{R}^n \wedge \mathbb{R}^n; \mathbb{R}^n)$ such that

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This choice induces a family of special frames on $B_G$

$$z : \mathbb{R}^n \times \mathfrak{g} \to T_p(B_G)$$

such that:
First prolongation of a $G$-structure

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These frames are actually a $G^{(1)}$-structure on $B_G$, called the *first prolongation of $B_G$* and denoted by $B_G^{(1)}$. 
**Prolongations of** $g$

**Definition**

To understand the group $G^{(1)}$, define the first prolongation of $g \subseteq \mathbb{R}^n \otimes (\mathbb{R}^n)^*$ by

\[
g^{(1)} := \left( g \otimes (\mathbb{R}^n)^* \right) \cap \left( \mathbb{R}^n \otimes S^2((\mathbb{R}^n)^*) \right)
\]  

**Remark**

$g^{(i)}$ may be realized as the space of multilinear symmetric functions $T: (\mathbb{R}^n)^{i+1} \to \mathbb{R}^n$ such that $\forall v_1, \ldots, v_i \in \mathbb{R}^n, v \mapsto T(v_1, \ldots, v_i, v) \in g$.  

---

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Prolongations of $g$

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$g^{(i)}$ may be realized as the space of multilinear symmetric functions $T: (\mathbb{R}^n)_{i+1} \to \mathbb{R}^n$ such that

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Definition

The group $G^{(1)} \subseteq GL(n + \dim(g))$ consists of all the matrices of the form

\[
\begin{pmatrix}
I_{\mathbb{R}^n} & 0 \\
T & I_g
\end{pmatrix},
\]

where $T \in g^{(1)}$. 

Remark

In general, the Lie algebra of $G^{(i)}$ is a representation of $g^{(i)}$. If $g^{(i)} = \{0\}$ for some $i$, then $G^{(i)} = \{1\}$. 

Prolongations of $B_G$
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Isometry group of Lorentz manifolds
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Paolo Piccione & Isometry group of Lorentz manifolds
Example

The group $O(n, \nu)$ is of finite type and order 1.
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$$\langle A(u)v, w \rangle_{\mathbb{R}^n_\nu} = \langle A(v)u, w \rangle_{\mathbb{R}^n_\nu} = -\langle v, A(u)w \rangle_{\mathbb{R}^n_\nu} =$$

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(12)
The semi-Riemannian isometry group

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Some other groups

- \( Sp(n) \), the symplectic group, is of \textit{infinite type}.
The semi-Riemannian isometry group

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Some other groups

- $Sp(n)$, the symplectic group, is of infinite type.
- $CO(n, \nu)$, the semi-Riemannian conformal group, is of finite type and order 2, for $n > 2$. 
The semi-Riemannian isometry group

Example

The group $O(n, \nu)$ is of finite type and order 1. To see this, let $u, v, w \in \mathbb{R}^n$ be any vectors, and let $A \in \mathfrak{so}(n, \nu)^{(1)}$. Then,

\[
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Some other groups

- $Sp(n)$, the symplectic group, is of infinite type.
- $CO(n, \nu)$, the semi-Riemannian conformal group, is of finite type and order 2, for $n > 2$.
- In general, $\forall k \in \mathbb{N} \exists n \in \mathbb{N}$ and $G \subseteq GL(n)$ s.t. $G$ is of finite type and has order $k$. 

Paolo Piccione Isometry group of Lorentz manifolds
Proposition 1

There is a sequence of principle fiber bundles

\[ M \xleftarrow{\pi} B_G \xleftarrow{\pi^1} B_G^{(1)} \xleftarrow{\pi^2} B_G^{(2)} \xleftarrow{\pi^3} \ldots \]

such that the structure group of \( B_G^{(i)} \) is \( G^{(i)} \). \( B_G^{(i)} \) is defined as \( \left( B_G^{(i-1)} \right)^{(1)} \).
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Definition

If \( g^{(i)} = \{0\} \) for some \( i \in \mathbb{N} \), we say that \( G \) (and any \( G \)-structure, for such \( G \)) is of finite type. The smallest such \( i \) is called the order of the \( G \)-structure.
Proposition 2

Let $\text{Aut}_G(M)$ be the group of automorphisms of $\pi : B_G \rightarrow M$. If $f \in \text{Aut}_G(M)$, then $f_\ast$ is an automorphism of $B_G^{(1)}$, that is, $(f_\ast)_\ast$ is a diffeomorphism of $B_G^{(1)}$ onto $B_G^{(1)}$. 
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Let $\text{Aut}_G(M)$ be the group of automorphisms of $\pi : B_G \to M$. If $f \in \text{Aut}_G(M)$, then $f_*$ is an automorphism of $B_G^{(1)}$, that is, $(f_*)_*$ is a diffeomorphism of $B_G^{(1)}$ onto $B_G^{(1)}$.

Proposition 3

Let $\text{Aut}_{G(i)}(B_G^{(i-1)})$ be the group of automorphisms of $B_G^{(i)}$. The inclusions

$$\text{Aut}_G(M) \hookrightarrow \text{Aut}_{G(1)}(B_G) \hookrightarrow \text{Aut}_{G(2)}(B_G^{(1)}) \hookrightarrow \ldots$$

given by the map $*$ are group monomorphisms. Each image is closed in the $C^0$ topology.
Parallelisms

Notation

- $M$: connected manifold with a complete parallelism $\{X_1, \ldots, X_n\}$.
- $G$: the automorphism group of the correspondent 1-structure. Here, $G$ is given the $C^0$ topology.
- $\beta_x : G \rightarrow M$: action of $G$ on $x \in M$. 

Theorem 1

Given $x \in M$, the action $\beta_x : G \rightarrow M$ is a homeomorphism onto its image, the latter with the induced topology. Moreover, $\beta_x(G)$ is a closed submanifold of $M$, and the differentiable structure $G$ inherits from it makes $G$ a Lie group, now acting smoothly on $M$. This differentiable structure does not depend on $x$. 

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Isometry group of Lorentz manifolds
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Proof sketch

Step 1

Given $\nu = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, define the induced vector field

$$X_\nu(x) := \sum_{i=1}^{n} \lambda_i X_i(x).$$

Denote by $\exp(\nu)$ the time 1 flow of $X_\nu$, which is defined on a (possibly empty) open subset of $M$. 

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Isometry group of Lorentz manifolds
Step 1

Given \( v = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \), define the induced vector field

\[
X_v(x) := \sum_{i=1}^{n} \lambda_i X_i(x).
\]

Denote by \( \exp(v) \) the time 1 flow of \( X_v \), which is defined on a (possibly empty) open subset of \( M \). Given \( x, y \in M \), there exists \( v_1, \ldots, v_k \in \mathbb{R}^n \) such that

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y = \exp(v_1) \circ \cdots \circ \exp(v_k)(x).
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Using this, it’s easy to prove that \( G \) acts freely on \( M \) and that, given \( x \in M \) and \( g \in G \), if \( \{g_i\}_{i \in \mathbb{N}} \) is a sequence on \( G \) such that \( g_i(x) \rightarrow g(x) \), then \( g_i \rightarrow g \) in the compact-open topology.
Proof sketch

Step 2

In order to show that $\beta_x(G)$ is closed in $M$, consider a sequence $\{g_i(x)\}_{i \in \mathbb{N}}$ such that $g_i(x) \to q \in M$.

We introduce a Riemannian metric on $M$ such that $G$ becomes a subgroup of $\text{Iso}(M)$. Since Riemannian isometries are equicontinuous, an application of Arzelá-Ascoli yields a limit $g \in \text{Iso}(M)$ for the sequence $\{g_i(x)\}_{i \in \mathbb{N}}$. This $g$ actually belongs to $G$, since it will commute with the flows the form $\exp(v)$, for $v \in \mathbb{R}^n$. It is then easy to see that $g(x) = q$, so that $\beta_x(G)$ is closed.
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Proof sketch

Step 3

Consider the sets

\[ D_x := \{ v \in T_x M : \exp(sv) \text{ exists and belongs to } \beta_x(G), \forall s \in \mathbb{R} \} \]

One can show that \( D \) is actually a smooth, integrable distribution on \( M \).
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One can show that \( D \) is actually a smooth, integrable distribution on \( M \).

The maximal, connected, integral submanifolds of \( D \) that intersect \( \beta_x(G) \) are precisely the connected components of \( \beta_x(G) \). Thus, \( \beta_x(G) \) naturally inherits a differentiable structure from these submanifolds.
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The maximal, connected, integral submanifolds of \( D \) that intersect \( \beta_x(G) \) are precisely the connected components of \( \beta_x(G) \). Thus, \( \beta_x(G) \) naturally inherits a differentiable structure from these submanifolds.

As a consequence of this, we may give \( G \) a differentiable structure, using \( \beta_x \). This differentiable structure does not depend on \( x \in M \), and it makes \( G \) a Lie group, acting smoothly on \( M \). ■
Remark

Let $\pi : B_H \rightarrow N$ be a $H$-structure of finite type and order $k$. To conclude that $\text{Aut}_H(N)$ is a Lie group, we must apply Theorem 1 in the following context:

$$M = B(k-1)_H \quad G = \text{Aut}_H(k)(B(k-1)_H)$$

But $M$, being the total space of a fibre bundle whose structure group is possibly non-connected, may be non-connected itself. Thus, we need a more general version of Theorem 1.
Remark

Let $\pi : B_H \to N$ be a $H$-structure of finite type and order $k$.

In order to conclude that $\text{Aut}_H(N)$ is a Lie group, we must apply Theorem 1 in the following context:

- $M = B_H^{(k-1)}$
- $G = \text{Aut}_{H(k)}(B_H^{(k-1)})$

But $M$, being the total space of a fibre bundle whose structure group is possibly non-connected, may be non-connected itself.

Thus, we need a more general version of Theorem 1.
The non-connected case

**Lemma**

Let $G_0$ be a normal subgroup of a purely algebraic group $G$ such that $G/G_0$ is at most countable. Suppose $G_0$ is a Lie group and the maps $h \mapsto ghg^{-1}$ from $G_0$ to itself are smooth, for every $g \in G$.

Then, $G$ admits a Lie group structure such that $G_0$ is an open Lie subgroup.
The non-connected case

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Then, $G$ admits a Lie group structure such that $G_0$ is an open Lie subgroup.

**Theorem 2**

Let $M$ be a smooth manifold of dimension $n$ such that

$$M = M_1 \cup \ldots \cup M_k,$$

where each $M_i$ is connected. If $M$ has complete parallelism, the automorphism group of the corresponding 1-structure is a Lie group with respect to the compact-open topology.
The non-connected case

Proof sketch

By Theorem 1, each $\text{Aut}_1(M_i)$ is a Lie group with respect to the compact-open topology, and the same is true for the product

$$G_0 := \prod_{i=1}^{k} \text{Aut}_1(M_i)$$
By Theorem 1, each $\text{Aut}_1(M_i)$ is a Lie group with respect to the compact-open topology, and the same is true for the product

$$G_0 := \prod_{i=1}^{k} \text{Aut}_1(M_i)$$

$G_0$ is a normal subgroup of $\text{Aut}_1(M)$, of index at most $k!$, satisfying the conditions stated in the previous Lemma. The topology obtained using the Lemma is precisely the compact-open topology. □
The non-connected case

Proof sketch

By Theorem 1, each $\text{Aut}_1(M_i)$ is a Lie group with respect to the compact-open topology, and the same is true for the product

$$G_0 := \prod_{i=1}^{k} \text{Aut}_1(M_i)$$

$G_0$ is a normal subgroup of $\text{Aut}_1(M)$, of index at most $k!$, satisfying the conditions stated in the previous Lemma. The topology obtained using the Lemma is precisely the compact-open topology. ■

Corollary

The automorphism group of a $G$-structure of finite type and order $k$ is a Lie group, with respect to the topology of $C^k$-uniform convergence over compact sets.
First examples of Lorentz isometries:

Constant curvature models
Zero curvature: Lorentz–Minkowski space

\[ M = \mathbb{R}^n + 1 = \left\{ (x_0, x_1, ..., x_n) : x_i \in \mathbb{R} \right\} \]

\[ g = \begin{pmatrix} \cdots \ \cdots \ \cdots \ \cdots \\
-1 & 1 & 0 & 0 \\
1 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & -1 \end{pmatrix} \]

Linear isometries:

\[ O(1, n) = \left\{ T \in \text{GL}(n+1) : T^t L = L \right\} \text{ non compact} \]

Transitive actions

On spheres:

\[ S_c = \left\{ v \in \mathbb{R}^{n+1} : g(v, v) = c \right\}, \quad c \in \mathbb{R} \]

On planes:

\[ P^+ = \left\{ P \subset \mathbb{R}^{n+1}, \text{dim}(P) = 2, P \text{ spacelike} \right\} \]

\[ P_0^+ = \left\{ P \subset \mathbb{R}^{n+1}, \text{dim}(P) = 2, P \text{ lightlike} \right\} \]

\[ P^- = \left\{ P \subset \mathbb{R}^{n+1}, \text{dim}(P) = 2, P \text{ timelike} \right\} \]

On subspaces with fixed \( g \)-signature.
Zero curvature: Lorentz–Minkowski space

\[ M = \mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) : x_i \in \mathbb{R}\} \]
Zero curvature: Lorentz–Minkowski space

- $M = \mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$

- $g = \langle L_{1,n}(\cdot), \cdot \rangle$, $L = \begin{pmatrix} -1 & & 0 \\ & 1 & \vdots \\ 0 & \ddots & 1 \end{pmatrix}$
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\[ O(1, n) = \{ T \in \text{GL}(n + 1) : T^t L_{1,n} T = L_{1,n} \} \text{ non compact.} \]
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**Transitive actions**

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Zero curvature: Lorentz–Minkowski space

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- On subspaces with fixed $g$-signature.
The full isometry group of Lorentz–Minkowski space

\[ \text{Iso}(\mathbb{R}^{n+1}, g_{LM}) = \mathbb{R}^{n+1} \rtimes O(1, n) \]

Observations:

- \( O(1, n) \) has four connected components.

Proof of ***.

If \( T \in \text{Iso}(\mathbb{R}^{n+1}, g_{LM}) \) fixes 0, then it sends lines through 0 into lines through 0 (because they are geodesics), and therefore \( T \) is linear.

Homogeneous space structure:

Stabilizer of \( e_n = (0, 0, \ldots, 0, 1) \):

\[ O(1, n-1) \sim (\mathbb{R}^{n+1} \rtimes O(1, n))/O(1, n-1) \]
The full isometry group of Lorentz–Minkowski space

$$\text{Iso}(\mathbb{R}^{n+1}, g_{LM}) = \mathbb{R}^{n+1} \rtimes \text{O}(1, n)$$

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Homogeneous space structure:

Stabilizer of $$e^n = (0, 0, \ldots, 0, 1)$$: $$\text{O}(1, n - 1)$$

$$\left(\mathbb{R}^{n+1} \rtimes \text{O}(1, n)\right)/\text{O}(1, n - 1)$$
The full isometry group of Lorentz–Minkowski space

\[ \text{Iso}(\mathbb{R}^{n+1}, g_{LM}) = \mathbb{R}^{n+1} \rtimes O(1, n) \]

\( \mathbb{R}^{n+1} = \) translations
The full isometry group of Lorentz–Minkowski space

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\[O(1, n) = \text{linear isometries}\]

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Paolo Piccione  Isometry group of Lorentz manifolds
The full isometry group of Lorentz–Minkowski space

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**Homogeneous space structure:**

---
The full isometry group of Lorentz–Minkowski space

\[ \text{Iso}(\mathbb{R}^{n+1}, g_{LM}) = \mathbb{R}^{n+1} \ltimes O(1, n) \]

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**Observe:** \( O(1, n) \) has **four** connected components.

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**Homogeneous space structure:**

Stabilizer of \( e_n = (0, 0, \ldots, 0, 1) \): \( O(1, n - 1) \)
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Homogeneous space structure:
Stabilizer of \( e_n = (0, 0, \ldots, 0, 1) \): \( O(1, n - 1) \)

\[ (\mathbb{R}^{n+1}, g_{LM}) \cong (\mathbb{R}^n \ltimes O(1, n))/O(1, n - 1) \]
Positive curvature model: de Sitter space

\( n \)-dimensional de Sitter space: \((dS^n, g_{dS^n})\)
Positive curvature model: de Sitter space

$n$-dimensional de Sitter space: $(dS^n, g_{dS^n})$

\[
dS^n = \left\{ x \in \mathbb{R}^{n+1} : \langle L_1, nx, x \rangle = 1 \right\}
\]
Positive curvature model: de Sitter space

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dS^n = \left\{ x \in \mathbb{R}^{n+1} : \langle L_1, n x, x \rangle = 1 \right\}
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Homogeneous space structure:
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Negative curvature model: anti-de Sitter space

$n$-dimensional Anti de Sitter space: $(AdS^n, g_{AdS^n})$
Negative curvature model: anti-de Sitter space

$n$-dimensional Anti de Sitter space: \((AdS^n, g_{AdS^n})\)

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AdS^n = \left\{ x \in \mathbb{R}^{n+1} : \langle L_{2,n-1} x, x \rangle = -1 \right\}
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Negative curvature model: anti-de Sitter space

$n$-dimensional Anti de Sitter space: \((\text{AdS}^n, g_{\text{AdS}^n})\)

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\text{AdS}^n = \left\{ x \in \mathbb{R}^{n+1} : \langle L_{2,n-1}x, x \rangle = -1 \right\}
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Homogeneous space structure:
Stabilizer of $e_1 = (1, 0, \ldots, 0)$: $O(1, n-1)$

$$(AdS^n, g_{AdS^n}) \cong O(2, n-1)/O(1, n-1)$$
Lack of compactness of $\text{Iso}(M, g)$

Unlike Riemannian isometries, Lorentzian isometries:

- need not be *equicontinuous*
- may generate *chaotic dynamics* on the manifold
Lack of compactness of $\text{Iso}(M, g)$

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**Example**

Dynamics of Lorentzian isometries can be of *Anosov type*, evocative of the fact that in General Relativity one can have contractions in time and expansion in space.
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**Example**

Dynamics of Lorentzian isometries can be of *Anosov type*, evocative of the fact that in General Relativity one can have contractions in time and expansion in space.

- $q$ Lorentzian quadratic form in $\mathbb{R}^n$,
  $\text{Iso}(\mathbb{R}^{n+1}, q) = O(q) \cong O(n, 1)$ non compact.
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**Example**

Dynamics of Lorentzian isometries can be of *Anosov type*, evocative of the fact that in General Relativity one can have contractions in time and expansion in space.

- $q$ Lorentzian quadratic form in $\mathbb{R}^n$, $\text{Iso}(\mathbb{R}^{n+1}, q) = O(q) \cong O(n, 1)$ non compact.
- The *orthogonal frame bundle* $\text{Fr}(M, g)$ has non compact fibers. $\text{Iso}(M, g)$ is identified topologically with any of its orbits in $\text{Fr}(M, g)$. 
An easy examples of non compactness (D’Ambra)

Consider

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$$

satisfying:

- $A$ diagonalizable on $\mathbb{R}$ (for instance, $b = c$):
  - Eigenvalues: $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 > 1$, $\lambda_2 = \lambda_1 - 1$
  - $e_1, e_2$ eigenvectors

Define a Lorentz metric $g$ on $\mathbb{R}^2$ by setting

$$g(e_1, e_1) = g(e_2, e_2) = 0, g(e_1, e_2) = 1.$$ 

$g$ induces a (flat) metric on the torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$

$A$ is an isometry of $(\mathbb{R}^2, g)$ and also of $(T^2, g)$.

$\text{Iso}(T^2, g)$ is not compact, since $A_k \to \infty$ as $k \to +\infty$ ($\lambda_k \to \infty$).
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Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ satisfying:

- $A$ diagonalizable on $\mathbb{R}$ (for instance, $b = c$):
- Eigenvalues: $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 > 1, \lambda_2 = \lambda_1^{-1}$
- $e_1, e_2$ eigenvectors

Define a Lorentz metric $g$ on $\mathbb{R}^2$ by setting $g(e_1, e_1) = g(e_2, e_2) = 0, g(e_1, e_2) = 1$.

- $g$ induces a (flat) metric on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$
- $A$ is an isometry of $(\mathbb{R}^2, g)$ and also of $(\mathbb{T}^2, g)$
- $\text{Iso}(\mathbb{T}^2, g)$ is not compact, since $A^k \to \infty$ as $k \to +\infty$ ($\lambda_1^k \to \infty$)
A slightly more elaborate example

A quadratic form on $\mathbb{R}^n$ induces a flat metric on $T^n = \mathbb{R}^n / \mathbb{Z}^n$.

"Linear" isometry group of $(T^n, q)$:

$$O(q, \mathbb{Z}) = GL(n, \mathbb{Z}) \cap O(q)$$

Full isometry group:

$$O(q, \mathbb{Z}) \ltimes T^n$$

for generic $q$, $O(q, \mathbb{Z}) = \{1\}$

if $q$ is rational, then $O(q, \mathbb{Z})$ is big (a lattice in $O(q)$, by Harich–Chandra–Borel thm)

when $q$ is not rational, many intermediate situations, can be co-compact.

Isometry group of Lorentz manifolds
A slightly more elaborate example

- $q$ quadratic form on $\mathbb{R}^n$
A slightly more elaborate example

- $q$ quadratic form on $\mathbb{R}^n$
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Homogeneous Lorentz Geometries

A Lie group, $H \subset G$ closed subgroup, a homogeneous space $G/H$ is a Lorentz geometry if the action of $G$ on $G/H$ preserves a Lorentzian metric tensor.

A compact manifold $M$ is locally modeled by $(G, G/H)$ (or, $M$ is a realization of $(G, G/H)$ if there exists an atlas of charts of $M$ taking values in an open subset with transition maps in $G$.

In this case, all $G$-invariant objects on $G/H$ pass to $M$.

$(G, G/H)$ is maximal if $\not\exists H' \supset G$ acting on $G/H$. W. Thurston (1983): classification of maximal Riemannian 3-geometries that admit a compact realization.

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Isometry group of Lorentz manifolds
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Examples of Lorentzian 3-geometries

Lorentz Minkowski:
\[(\mathbb{R}^3, g_{LM}) = (\mathbb{R}^3 \rtimes O(1,2)) / O(1,2)\]

de Sitter:
\[(dS^3, g_{dS}) = O(1,3) / O(1,2)\]

By a result of E. Calabi (1963), it does not have compact realizations (only finite groups can act properly on dS\(n\)).

anti de Sitter:
\[(AdS^3, g_{AdS}) = O(2,2) / O(1,2)\]

Alternative description:
\[SL(2,\mathbb{R}) \times \mathbb{R}\]

Lorentz–Heisenberg geometry
Lorentz–Sol geometry

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Isometry group of Lorentz manifolds
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- Lorentz–Heisenberg geometry

- **Lorentz–Sol\(_3\) geometry**
The solvable case: $\text{SL}(2, \mathbb{R})$

- \(G = \text{SL}(2, \mathbb{R})\) semi-simple Lie group, \(\dim(G) = 3\)

- Lorentzian Killing form on \(\text{sl}(2, \mathbb{R})\):
  \[\langle A, B \rangle = \text{tr}(AB)\]

- Bi-invariant Lorentz metric \(g\) on \(\text{SL}(2, \mathbb{R})\)

- Iso\(0(\text{SL}(2, \mathbb{R}))\) = \(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})/Z \sim \text{O}(2, 2)\)

- \(Z\) group generated by \((-I_2, -I_2)\)

- \(\text{Iso}(M, g)\) isometric to \(\text{AdS}_3\)

- \(\Gamma \subset \text{SL}(2, \mathbb{R})\) co-compact lattice (i.e., discrete subgroup with \(M = G/\Gamma\) compact), then \((M, g)\) is a compact homogeneous Lorentz manifold

- \(\text{Iso}(0(M, g)) = \text{PSL}(2, \mathbb{R})\) (non compact)
The solvable case: $\text{SL}(2, \mathbb{R})$

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- $(SL(2, \mathbb{R}), g)$ isometric to $AdS^3$
The solvable case: \( SL(2, \mathbb{R}) \)

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- \( (SL(2, \mathbb{R}), g) \) isometric to \( AdS^3 \)
- \( \Gamma \subset SL(2, \mathbb{R}) \) co-compact lattice (i.e., discrete subgroup with \( M = G/\Gamma \) compact), then \( (M, g) \) is a compact homogeneous Lorentz manifold
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- *Lorentzian* Killing form on $\mathfrak{sl}(2, \mathbb{R})$: $\langle A, B \rangle = \text{tr}(AB)$
- bi-invariant Lorentz metric $g$ on $SL(2, \mathbb{R})$
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- $\text{Iso}_0(M, g) = PSL(2, \mathbb{R})$ (non compact)
The nilpotent case: $\text{Heis}^3$

$G = \text{Heis}^3 = \{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \}$

$\text{heis}^3 = \text{span}\{X, Y, Z\}$, $X \in z$, $[Z, Y] = X$ up to automorphisms, $\exists$ three left-invariant Lorentz metrics on $\text{Heis}^3$:

- $g(X, X) \to$ flat metric
- $g(X, X) = -1$, $X \perp \text{span}\{Y, Z\}$ $\Rightarrow$ Riemannian-like $g(X, X) = 1$, $X$ orthogonal to the Lorentz plane spanned by $Y$ and $Z$,
- $g(Y, Z) = 1$ $\Rightarrow$ Lorentz–Heisenberg geometry

Isometry group:

$\mathbb{R} \ltimes \text{Heis}^3$ (4-dim)

$\mathbb{R}$: 1-parameter group of automorphisms that fix $X$.
The nilpotent case: $\text{Heis}^3$

$G = \text{Heis}^3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

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Isometry group of Lorentz manifolds
The nilpotent case: $\text{Heis}^3$

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- $\text{heis}^3 = \text{span}\{X, Y, Z\}, \ X \in 3, \ [Z, Y] = X$

- up to automorphisms, $\exists$ three left-invariant Lorentz metrics on $\text{Heis}^3$: 
The nilpotent case: $\text{Heis}^3$

- $G = \text{Heis}^3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

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The nilpotent case: $\text{Heis}^3$

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- up to automorphisms, $\exists$ three left-invariant Lorentz metrics on Heis$^3$:
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  - $g(X, X) = -1, \ X \perp \text{span}\{Y, Z\} \rightarrow$ Riemannian-like
  - $g(X, X) = 1, \ X$ orthogonal to the Lorentz plane spanned by $Y$ and $Z, \ g(Y, Z) = 1 \rightarrow$ Lorentz–Heisenberg geometry

Isometry group:

$\mathbb{R} \ltimes \text{Heis}^3$ (4-dim)

$\text{maximal geometry}$
The nilpotent case: $\text{Heis}^3$

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- Isometry group: $\mathbb{R} \ltimes \text{Heis}^3$ (4-dim)
  $\mathbb{R}$: 1-parameter group of automorphism that fix $X$. 

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Isometry group of Lorentz manifolds
The nilpotent case: $\text{Heis}^3$

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- Isometry group: $\mathbb{R} \rtimes \text{Heis}^3$ (4-dim)
  - $\mathbb{R}$: 1-parameter group of automorphism that fix $X$.

- maximal geometry
The solvable case: $\text{Sol}^3$

\[ G = \text{Sol}^3 = \{ (e^{-z}x_0, e^{-z}y_0, 0) \}, \quad x, y, z \in \mathbb{R} \]

3-dim solvable group $\text{sol}^3 = \text{span}\{X, Y, Z\}$, 
\[ [X, Y] = 0, \quad [X, Z] = X, \quad [Y, Z] = -Y \]

$[\text{sol}^3, \text{sol}^3] = \text{span}\{X, Y\}$

$g$ left-invariant Lorentz metric on $G$:
\[ g(Y, Y) = 1, \quad g(Y, Z) = 0, \quad g(X, Z) = 1 \]

Direct computation:
\[ R(Y, Z) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \] (non flat!)

Isometry group:
\[ \mathbb{R} \ltimes \text{Heis}^3, \quad \text{dim} = 4 \]

Obs.: $G$ admits also left invariant flat metrics

Maximal geometry
The solvable case: $\text{Sol}^3$

- $G = \text{Sol}^3 = \left\{ \begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}$

3-dim solvable group
The solvable case: $\text{Sol}^3$

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3-dim solvable group

- $\text{sol}^3 = \text{span}\{X, Y, Z\}$, $[X, Y] = 0$, $[X, Z] = X$, $[Y, Z] = -Y$
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- $[\mathfrak{sol}^3, \mathfrak{sol}^3] = \text{span}\{X, Y\}$


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3-dim solvable group

- $\mathfrak{sol}^3 = \text{span}\{X, Y, Z\}$, $[X, Y] = 0$, $[X, Z] = X$, $[Y, Z] = -Y$

- $[\mathfrak{sol}^3, \mathfrak{sol}^3] = \text{span}\{X, Y\}$

- $g$ left-invariant Lorentz metric on $G$: $\text{span}\{X, Y\} = X^\perp$
The solvable case: $\text{Sol}^3$

- $G = \text{Sol}^3 = \left\{ \left( \begin{array}{ccc} e^{-z} & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{array} \right) \right\} x, y, z \in \mathbb{R}$
  
 3-dim solvable group

- $\mathfrak{sol}^3 = \text{span}\{X, Y, Z\}$, $[X, Y] = 0$, $[X, Z] = X$, $[Y, Z] = -Y$

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Isometry group of Lorentz manifolds
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- maximal geometry
Classification of homogeneous 3-manifolds

Theorem (S. Dumitrescu, A. Zeghib, 2010)

If $(G, G/H)$ is a maximal Lorentzian (non-Riemannian) 3-geometry, then it is isometric to one of the following:

- Lorentz–Minkowski (isometry group of dim=6)
- Anti de Sitter (isometry group of dim=6)
- Lorentz–Heis $^3$ (isometry group of dim=4)
- Lorentz–Sol $^3$ (isometry group of dim=4)

Analogies with the Riemannian case

- no models with 5-dim isometry group

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- no positive curvature models
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The isometry group of a compact Lorentz manifold

\((M, g)\) compact Lorentz manifold.

Theorem (D'Ambra, Inventiones 1988)
If \((M, g)\) is analytic and simply connected, then \(\text{Iso}(M, g)\) is compact.

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The identity component \(\text{Iso}_0(M, g)\) is direct product:

\[A \times K \times H\]

- A is abelian
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- H is locally isomorphic to:
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Action of $S^1$ on the Lie algebra $\mathfrak{heis}$:

- Positivity conditions on the eigenvalues $\implies$ existence of bi-invariant Lorentz metrics
- Arithmetic conditions $\implies$ existence of lattices.
On a (false) conjecture

- $(M, g)$ compact Lorentz manifold
- $K$ Killing field of $(M, g)$, $p \in M$, $g(K_p, K_p) < 0$
- the 1-parameter group of isometries generated by $K$ is pre-compact in $\text{Iso}_0(M, g)$
- in this situation, $\text{Iso}(M, g)$ has a non empty open cone of vectors that generate a precompact 1-parameter subgroup of $\text{Iso}(M, g)$
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**Conjecture:** Given a connected $G$ Lie group, $\mathfrak{g} = \text{Lie}(G)$, if $\mathfrak{g}$ has a non empty open cone of vectors $v$ such that $t \mapsto \exp(t \cdot v)$ is precompact in $G$, then $G$ is compact.
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**Counterexample:** $G = \text{SL}(2, \mathbb{R})$
Theorem

Let $G$ be a connected Lie group, $K \subset G$ a maximal compact subgroup and $\mathfrak{k} \subset \mathfrak{g}$ their Lie algebras. Let $\mathfrak{m}$ be an $\text{Ad}_K$-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then, $\mathfrak{g}$ has a non empty open cone of vectors that generate precompact $1$-parameter subgroups of $G$ if and only if there exists $v \in \mathfrak{k}$ such that the restriction $\text{ad}_v : \mathfrak{m} \rightarrow \mathfrak{m}$ is an isomorphism.
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Proof of the algebraic criterion

Proof

\[ \text{Proof} \]

\[ C = \{ v \in g : \exp(tv) \text{ is precompact} \} \]

\[ C \subset k', k' = \text{Lie}(K') \]

\[ \text{all maximal compact subgroups are conjugated} \]

\[ \Rightarrow C = \text{Ad}_G(k) \]

\[ F : G \times k \to g, F(g, v) = \text{Ad}_g(v) \]

\[ C = \text{Im}(F) \]

\[ \text{has non empty interior iff} F \]

\[ \text{has maximal rank at some point (Sard)} \]

\[ \text{by equivariance, iff it has maximal rank at some point } (e, v) \]

\[ dF(e, v)(g, k) = [g, v] + k = [m, v] + k. \]
Proof

\[ C = \{ v \in g : \exp(tv) \text{ is precompact} \} \]
Proof of the algebraic criterion

**Proof**

- \( \mathcal{C} = \{ v \in \mathfrak{g} : \exp(tv) \text{ is precompact} \} \)
- \( \mathcal{C} \subset \mathfrak{k}' \), \( \mathfrak{k}' = \text{Lie}(K') \), \( K' \subset G \) maximal compact

All maximal compact subgroups are conjugated.

\[ C = \text{Ad}_G(k) \]

\( F: G \times k \to \mathfrak{g} \), \( F(g, v) = \text{Ad}_g(v) \)

\( C = \text{Im}(F) \) has non empty interior iff \( F \) has maximal rank at some point (Sard).

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- \( dF_{(e,v)}(g, \mathfrak{k}) = [g, v] + \mathfrak{k} = [m, v] + \mathfrak{k}. \)
Corollary 1

Let \((M, g)\) be a compact Lorentz manifold that has a Killing vector field which is timelike somewhere. Then, \(\text{Iso}_0(M, g)\) is compact unless it contains a group locally isomorphic to \(\text{SL}(2, \mathbb{R})\) or to an oscillator group.
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- Assume \(\text{Iso}_0(M, g) = \mathfrak{h} + \alpha + \mathfrak{c}\), \(\mathfrak{h}\)=Heisenberg, \(\alpha\)=abelian, \(\mathfrak{c}\)=compact semi-simple
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- can assume \(A\) simply connected
- \(\mathfrak{h} + \mathfrak{a}\) nilpotent \(\implies\) for no \(v \in \mathfrak{h} + \mathfrak{a}\) the map \(\text{ad}_v\) is injective.
Corollary 2

If \((M, g)\) admits a somewhere timelike Killing vector field, then the two conditions are *mutually exclusive*:

(a) \(\text{Iso}_0(M, g)\) is not compact;
(b) \(\text{Iso}(M, g)\) has infinitely many connected components.
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**Proof.** Use Corollary 1 and Zeghib’s classification:

If \(\text{Iso}_0(M, g)\) contains a group locally isomorphic to \(\text{SL}(2, \mathbb{R})\) or to an oscillator group then:

- \(\text{Iso}(M, g)\) has only a finite number of connected components;
- \(M\) is not simply connected.
Compact Lorentz manifolds with *large* isometry group

**Definition**

\[ \rho : \Gamma \rightarrow \text{GL}(\mathcal{E}) \text{ representation}. \]
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\[ \rho : \Gamma \rightarrow \text{GL}(\mathcal{E}) \] representation. Then, \( \rho \) is said to be:

- of Riemannian type if it preserves some positive definite inner product on \( \mathcal{E} \);
- of post-Riemannian type if it preserves some positive semi-definite inner product on \( \mathcal{E} \) with kernel of dimension 1.

Obs.: \( \rho : \Gamma \rightarrow \text{GL}(\mathcal{E}) \) of Riemannian type \( \iff \rho(\Gamma) \) precompact.

Proposition

\( (M, g) \) compact Lorentz manifold. If the conjugacy action of \( \Gamma = \text{Iso}(M, g) / \text{Iso}^0(M, g) \) on \( \text{Iso}^0(M, g) \) is not of post-Riemannian type, then \( \text{Iso}^0(M, g) \) has a timelike orbit in \( M \), and \( \text{Iso}(M, g) \) has infinitely many connected components.
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The Gauss map

Killing fields

$\mathfrak{iso}(M, g) \ni v \mapsto K^v \in \text{Kill}(M, g)$.

$K^v$ infinitesimal generator of $t \mapsto \exp(tv)$.
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**Killing fields**

\( \mathcal{I}_{\text{so}}(M, g) \ni \nu \mapsto K^\nu \in \text{Kill}(M, g) \).

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\[ \Phi \in \text{Iso}(M, g) \mapsto \Phi^*(K^\nu) = K^{\text{Ad}_\Phi}(\nu) \]
The Gauss map

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\[ \Phi \in \text{Iso}(M, g) \implies \Phi_* (K^{\nu}) = K^{\text{Ad}_\Phi}(\nu) \]

Gauss map:

\[ \mathcal{G} : M \longrightarrow \text{Sym}(\mathcal{I}_\text{Is}(M, g)) \]

\[ \mathcal{G}_p(\nu, \omega) = g(K^{\nu}(p), K^{\omega}(p)) \]
The Gauss map

Killing fields

\[ \mathcal{I}_0(M, g) \ni \nu \mapsto K^\nu \in \text{Kill}(M, g). \]

\( K^\nu \) infinitesimal generator of \( t \mapsto \exp(tv) \)

\[ \Phi \in \text{Iso}(M, g) \implies \Phi^*(K^\nu) = K^{\text{Ad}_\Phi}(\nu) \]

Gauss map:

\[ G : M \longrightarrow \text{Sym}(\mathcal{I}_0(M, g)) \]

\[ G_p(\nu, \omega) = g(K^\nu(p), K^\omega(p)) \]

Proposition

If the action of \( \Gamma \) on \( \text{Iso}_0(M, g) \) is not of post-Riemannian type, then \( \text{Iso}_0(M, g) \) has somewhere timelike orbits.

**Proof:** Use \( \xi(\nu, \omega) = \int_M G_p(\nu, \omega) \, dp \).
Paradigmatic example

In $\mathbb{R}^n$, it induces a flat Lorentz metric on $T^n = \mathbb{R}^n / \mathbb{Z}^n$.

Linear isometry group of $T^n$: $O(q, \mathbb{Z}) = \text{GL}(n, \mathbb{Z}) \cap O(q)$.

Full isometry group: $O(q, \mathbb{Z}) \rtimes T^n$.

For generic $q$, $O(q, \mathbb{Z})$ is trivial.

If $q$ is rational, by Harish-Chandra-Borel theorem, $O(q, \mathbb{Z})$ is big in $O(q)$.

When $q$ is not rational, many intermediate situations occur.

Complicated dynamics of hyperbolic elements $A \in O(q, \mathbb{Z})$: they may have Salem numbers in their spectrum.

Theorem (P. P., A. Zeghib): Compact Lorentzian manifolds with large isometry groups are essentially built up by tori.
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**Theorem (P.P., A. Zeghib)**

*Compact Lorentzian manifolds with large isometry groups are essentially built up by tori.*
Theorem

Let \((M, g)\) be a compact Lorentz manifold that has a somewhere timelike Killing vector field, and whose isometry group \(\text{Iso}(M, g)\) has infinitely many connected components. Then:

- \(\text{Iso}_0(M, g)\) contains a torus \(\mathbb{T}^d\) endowed with a Lorentz form \(q\), such that \(\Gamma\) is a subgroup of \(O(q, \mathbb{Z})\);
- up to finite cover, \(M\) is:
  - either a direct product \(\mathbb{T}^d \times N\), with \(N\) compact Riemannian manifold
  - or an amalgamated metric product \(\mathbb{T}^d \times_{S^1} L\), where \(L\) is a lightlike manifold with an isometric \(S^1\)-action.
Amalgamated metric products

Amalgamated product $X \times_{S^1} Y$:

$Z = (X \times Y) / S^1$ diagonal action.

Assume $X$ Lorentzian, $Y$ Riemannian (or lightlike), and action of $S^1$ isometric.

Identify $T(x_0, y_0) Z$ with $T_{x_0} X \times \{S^1 - \text{orbit through } y_0\} \perp$.

Endow $T(x_0, y_0) Z$ with the induced metric (Lorentzian).

Long exact homotopy sequence of the fibration $X \times Y \to (X \times Y) / S^1$:

$Z \sim \pi_1(S^1) \to \pi_1(X) \times \pi_1(Y) \to \pi_1(Z) \to \pi_0(S^1) \sim \{1\}$

Proposition

If $\pi_1(X) \times \pi_1(Y)$ is not cyclic, then $(X \times Y) / S^1$ is not simply connected.
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**Amalgamated product** $X \times_{S^1} Y$:

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Two interesting consequences

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Assume $\text{Iso}(M, g)$ non compact. If there is a somewhere timelike Killing vector field, then there is an everywhere timelike Killing vector field.
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Assume $\text{Iso}(M, g)$ non compact. If there is a somewhere timelike Killing vector field, then there is an everywhere timelike Killing vector field.

**Theorem**

If $(M, g)$ admits a somewhere timelike Killing vector field and $M$ is simply connected, then $\text{Iso}(M, g)$ is compact.
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Theorem

If $(M, g)$ admits a somewhere timelike Killing vector field and $M$ is simply connected, then $\text{Iso}(M, g)$ is compact.

Proof.

When $\text{Iso}_0(M, g)$ contains a group locally isomorphic to $\text{SL}(2, \mathbb{R})$ or to an oscillator group use Zeghib’s classification.

When $\text{Iso}(M, g)$ has infinitely many connected components, use the structure result.
That’s all.
That’s all.
THANKS
That’s all.

THANKS

See you all at GeLo??2013!!!