INTRODUCTION TO SYMPLECTIC MECHANICS

Symplectic mechanics is a very active and thriving area of research in pure mathematics with ramifications in many other areas such as topology, differential geometry, and mathematical physics. It has its origin in the pionneering work of Lagrange and Hamilton on celestial mechanics. In addition to be a full-blown autonomous mathematical theory, symplectic mechanics has many applications in theoretical and mathematical physics. For instance, recent advances in theoretical physics have shown that the theory of Hamiltonian periodic orbits plays a crucial role in the understanding of quantum systems with a classical analogue. Symplectic mechanics has also profited from recent advances in symplectic topology, especially from the notion of symplectic capacity, whose definition is made possible by Gromov's non-squeezing theorem (1985), which shows that symplectomorphisms are much more than just volume-preserving diffeomorphisms if $n > 1$.

The aim of this course is to give an introduction to the theory of Hamiltonian systems from the symplectic point of view with an emphasis on the periodic orbits occurring in these systems (including the study of Maslov indices and their variants, such as the Conley-Zehnder index). It was already noted by Poincaré that periodic orbits of dynamical systems play a very privileged role in, for instance, celestial mechanics.

Prerequisites

A good knowledge of the theory of differential equations and of advanced calculus (real analysis and basic topology) are required. No prior knowledge of mechanics is required, but of course some literacy in elementary physics (e.g. Newton's second law $F = ma$) would be helpful, if only for the sake of motivation.

This course is organized as follows:

1. Symplectic forms; the notion of symplectomorphism. The groups $\mathrm{Sp}(n)$ and $\mathrm{Symp}(n)$.

2. Hamilton's equations; Hamiltonian vector fields. Symplectic forms and intrinsic formulation of the Hamiltonian formalism. The extended phase space.

3. Integrability and Lagrangian manifolds.

4. The variational equation associated to a Hamiltonian system and stability questions.

5. The symplectic and Lagrangian Maslov indices. Periodic Hamiltonian orbits: a few existence theorems.

6. The group Ham(n) of Hamiltonian symplectomorphisms; Banyaga's theorem.
7. The notion of symplectic capacity; Gromov's non-squeezing theorem. Application to quantization.
8. The Kashiwara-Wall and Leray indices; application to the Conley-Zehnder index of symplectic paths.

References

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