Bifurcation of CMC Clifford Tori in Euclidean Spheres
Joint work with Luis J. Alías

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International Symposium on Differential Geometry “In honor of Marcos Dajczer on his 60th birthday”

IMPA, Rio de Janeiro, Brazil, August 2009
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Happy birthday Marcos!
On the conference speakers

Prominent geometers
On the conference speakers

Prominent geometers

Marcos friends
On the conference speakers

Prominent geometers

Marcos friends

P.P.
Outline of this talk.

1. CMC Clifford tori in spheres
2. Spectrum of the Jacobi operator
3. Statement of the result
4. Bifurcation
5. Abstract equivariant bifurcation result
6. The CMC variational problem
   - Area and volume functionals
   - Manifold of unparameterized embeddings
7. Local homological invariants
8. A fixed boundary CMC bifurcation problem
1 \leq j < m, \quad r \in ]0, 1[\]

\[ x^{j,m}_r : \mathbb{S}^j \times \mathbb{S}^{m-j} \longrightarrow \mathbb{S}^{m+1} \]

\[ (p, q) \longmapsto (r \cdot p, \sqrt{1 - r^2} \cdot q) \]

Constant mean curvature:

\[ H^{j,m}_r = \frac{mr^2 - j}{mr\sqrt{1 - r^2}} \]

\[ r = \sqrt{\frac{j}{m}} \text{ minimal Clifford torus.} \]
The Jacobi operator

\[ J = -\Delta_{r,m} - m \cdot \text{Ric}_{S^{m+1}}(\vec{N}) - \left\| A_{r,m} \right\|^2 \]
The Jacobi operator

\[ J = -\Delta^j,^m_r - m \cdot \text{Ric}_{S^{m+1}}(\tilde{N}) - \left\| A_r^{i,m} \right\|^2 \]

- \text{Ric}_{S^{m+1}}(\tilde{N}) \text{ Ricci curvature of } S^{m+1},
- \text{constant } \equiv 1
The Jacobi operator

\[ J = -\Delta_j^m - m \cdot \text{Ric}_{S^{m+1}}(\vec{N}) - \left\| A_{r}^{j,m} \right\|^2 \]

- **Ric}_{S^{m+1}}(\vec{N})** Ricci curvature of \( S^{m+1} \), constant \( \equiv 1 \)
- \( \left\| A_{r}^{j,m} \right\| \) norm of the second fundamental form, constant \( \equiv \frac{j}{r^2} + \frac{m-j}{1-r^2} \)
The Jacobi operator

\[ J = - \Delta^{j,m}_r - m \cdot \text{Ric}_{S^{m+1}}(\vec{N}) - \left\| A^{j,m}_r \right\|^2 \]

- \text{Ric}_{S^{m+1}}(\vec{N}) \text{ Ricci curvature of } S^{m+1}, \text{ constant } \equiv 1
- \left\| A^{j,m}_r \right\| \text{ norm of the second fundamental form, constant } \equiv \frac{j}{r^2} + \frac{m-j}{1-r^2}
- \Delta^{j,m}_r \text{ Laplacian of } S^j(r) \times S^{m-j}(\sqrt{1-r^2}).
The Jacobi operator

\[ J = -\Delta_r^{j,m} - m \cdot \text{Ric}_{\mathbb{S}^{m+1}}(\vec{N}) - \left\| A_r^{j,m} \right\|^2 \]

- \textbf{Ric}_{\mathbb{S}^{m+1}}(\vec{N}) \text{ Ricci curvature of } \mathbb{S}^{m+1}, \text{ constant } \equiv 1
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- \Delta_r^{j,m} \text{ Laplacian of } \mathbb{S}^j(r) \times \mathbb{S}^{m-j}(\sqrt{1-r^2}).

\[ \zeta \in \Sigma(\Delta_r^{j,m}) \iff \zeta = \sigma + \rho, \quad \sigma \in \Sigma(\Delta_r^j), \rho \in \Sigma(\Delta_{1-r^2}^{m-j}) \]
The Jacobi operator

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\[ \zeta \in \Sigma(\Delta^{j,m}_r) \iff \zeta = \sigma + \rho, \quad \sigma \in \Sigma(\Delta^j_r), \quad \rho \in \Sigma(\Delta^{m-j}_{1-r^2}) \]

multiplicity of \( \zeta \) = sum of multiplicities of \( \sigma \) and \( \rho \)
Spectrum of $J$

- $\Sigma(\Delta^J_r) = \{\sigma_1 < \sigma_2 < \cdots < \sigma_i < \ldots\}$, $\sigma_i = \frac{(i-1)(j+i-2)}{r^2}$
Spectrum of $J$

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- $\Sigma\left(\Delta^m_j \sqrt{1-r^2}\right) = \{\rho_1 < \rho_2 < \cdots < \rho_l < \cdots\}$, \quad \rho_l = \frac{(l-1)(m-j+l-2)}{1-r^2}$
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- $0 \in \Sigma(J) \iff \sigma_i + \rho_l - \left( \frac{j}{r^2} + \frac{m-j}{1-r^2} \right) = 0$
Spectrum of $J$

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$\sigma_2 + \rho_2 - \left(\frac{j}{r^2} + \frac{m-j}{1-r^2}\right) = 0$, \quad \text{multiplicity} = m + 1 + j(m - 1)$
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Other zeros of $\Sigma(J)$ are of the form:

$$\sigma_1 + \rho_l - \left( \frac{j}{r^2} + \frac{m-j}{1-r^2} \right) \quad \text{or} \quad \sigma_i + \rho_1 - \left( \frac{j}{r^2} + \frac{m-j}{1-r^2} \right).$$
Spectrum of $J$

1. $\Sigma(\Delta^j_r) = \{\sigma_1 < \sigma_2 < \cdots < \sigma_i < \ldots\}$, $\sigma_i = \frac{(i-1)(j+i-2)}{r^2}$

2. $\Sigma\left(\Delta^\frac{m-j}{\sqrt{1-r^2}}\right) = \{\rho_1 < \rho_2 < \cdots < \rho_l < \ldots\}$, $\rho_l = \frac{(l-1)(m-j+l-2)}{1-r^2}$

3. $0 \in \Sigma(J) \iff \sigma_i + \rho_l - \left(\frac{j}{r^2} + \frac{m-j}{1-r^2}\right) = 0$

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**Proposition**

There exists two monotone sequences $(r_i)_{i=1}^\infty$ and $(s_l)_{l=1}^\infty$, with

$$\lim_{l \to \infty} s_l = 0, \quad \text{and} \quad \lim_{i \to \infty} = 1,$$

where the *Morse index* of the CMC Clifford torus $x^{j,m}_r$ has a *jump*. 
Constant mean curvature variational problem

- $M^m$ compact oriented manifold
Constant mean curvature variational problem

- $M^m$ compact oriented manifold
- $(N^n, g)$ oriented Riemannian manifold
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- Mean curvature: $H_x = \text{tr}(2\text{nd fund. form})$
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$x : M \hookrightarrow N$ embedding

Mean curvature: $H_x = \text{tr}(2\text{nd fund. form})$

Variational principle

$x$ has constant mean curvature (CMC) iff $x$ is a stationary point for the area functional restricted to embeddings of fixed volume.
Isometric congruence

Definition

$x_1, x_2 : M \rightarrow N$ embeddings are congruent $(x_1 \cong x_2)$ if there exists $\phi \in \text{Diff}(M)$ and $\psi \in \text{Iso}(N, g)$ such that $x_2 = \psi \circ x_1 \circ \phi^{-1}$.

\[
\begin{array}{ccc}
M & \xrightarrow{x_1} & N \\
\downarrow \phi & & \downarrow \psi \\
M & \xrightarrow{x_2} & N
\end{array}
\]

commutes.
Isometric congruence

**Definition**

$x_1, x_2 : M \rightarrow N$ embeddings are *congruent* ($x_1 \cong x_2$) if there exists $\phi \in \text{Diff}(M)$ and $\psi \in \text{Iso}(N, g)$ such that $x_2 = \psi \circ x_1 \circ \phi^{-1}$.

If $x_1$ has CMC and $x_1 \cong x_2$, then $x_2$ has CMC!
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If $x_1$ has CMC and $x_1 \cong x_2$, then $x_2$ has CMC!

**Group actions:**

- $\text{Diff}(M)$ acts on the right *(free action)*
- $\text{Iso}(N, g)$ acts on the left *(action not free, but group compact)*
Statement of the result

**Theorem**

\[ x_{r}^{j,m} : S^{j} \times S^{m-j} \rightarrow S^{m+1} \]  

*CMC Clifford torus, 1 < j < m, r \in ]0, 1[.*
Statement of the result

**Theorem**

\[ x_r^{j,m} : S^j \times S^{m-j} \rightarrow S^{m+1} \text{ CMC Clifford torus, } 1 < j < m, r \in ]0, 1[. \]

\( \exists \) two sequences \( (r_i)_{i \in \mathbb{N}} \) and \( (s_l)_{l \in \mathbb{N}} \) such that:

- \( \lim_{i \rightarrow \infty} r_i = 1, \lim_{l \rightarrow \infty} s_l = 0 \)
Statement of the result

**Theorem**

\[ x_r^{j,m} : \mathbb{S}^j \times \mathbb{S}^{m-j} \longrightarrow \mathbb{S}^{m+1} \]

CMC Clifford torus, \( 1 < j < m, \ r \in ]0, 1[. \]

\( \exists \) two sequences \((r_i)_{i\in\mathbb{N}}\) and \((s_l)_{l\in\mathbb{N}}\) such that:

- \( \lim_{i\to\infty} r_i = 1, \lim_{l\to\infty} s_l = 0 \)
- the embeddings \( x_r^{j,m} \) and \( x_s^{j,m} \) are accumulation of pairwise non congruent CMC embeddings of \( \mathbb{S}^j \times \mathbb{S}^{m-j} \) into \( \mathbb{S}^{m+1} \), each of which is not congruent to any CMC Clifford torus.
Statement of the result

**Theorem**

\[ x^{j,m}_r : S^j \times S^{m-j} \rightarrow S^{m+1} \text{ CMC Clifford torus, } 1 < j < m, r \in ]0, 1[. \]

\[ \exists \text{ two sequences } (r_i)_{i \in \mathbb{N}} \text{ and } (s_l)_{l \in \mathbb{N}} \text{ such that:} \]

1. \[ \lim_{i \to \infty} r_i = 1, \lim_{l \to \infty} s_l = 0 \]

2. the embeddings \( x^{j,m}_{r_i} \) and \( x^{j,m}_{s_l} \) are accumulation of pairwise non congruent CMC embeddings of \( S^j \times S^{m-j} \) into \( S^{m+1} \), each of which is not congruent to any CMC Clifford torus.

For all other values of \( r \), the CMC Clifford family is stable, i.e., if \( x : S^j \times S^{m-j} \rightarrow S^{m+1} \) is a CMC embedding which is sufficiently close to some \( x^{j,m}_{r_i} \), with \( r \neq r_i \) and \( r \neq s_l \), then \( x \) is congruent to some \( x^{j,m}_{r_i} \).
Statement of the result

**Theorem**

\[ x_r^{j,m} : S^j \times S^{m-j} \to S^{m+1} \] CMC Clifford torus, \( 1 < j < m, r \in ]0, 1[. \)

\[ \exists \text{ two sequences } (r_i)_{i \in \mathbb{N}} \text{ and } (s_l)_{l \in \mathbb{N}} \text{ such that:} \]

1. \( \lim_{i \to \infty} r_i = 1, \lim_{l \to \infty} s_l = 0 \)
2. the embeddings \( x_{r_i}^{j,m} \) and \( x_{s_l}^{j,m} \) are accumulation of pairwise non congruent CMC embeddings of \( S^j \times S^{m-j} \) into \( S^{m+1} \), each of which is not congruent to any CMC Clifford torus.

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**Observation.** \( r = \sqrt{\frac{j}{m}} \) (minimal) is *not* a bifurcation radius!
CMC tori bifurcation picture
Generalities on variational bifurcation

General bifurcation setup:

- \( \mathcal{M} \) differentiable manifold (possibly \( \dim = \infty \))
Generalities on variational bifurcation

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- $\mathcal{M}$ differentiable manifold (possibly $\dim = \infty$)
- $f_\lambda : \mathcal{M} \to \mathbb{R}$ family of smooth functionals, $\lambda \in [a, b]$
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- $\lambda \mapsto x_\lambda \in \mathcal{M}$ smooth curve of critical points: $df_\lambda(x_\lambda) = 0$ for all $\lambda$.

Bifurcation definition:
- Bifurcation at $\lambda_0 \in [a, b]$ if $\exists \lambda_n \to \lambda_0$ and $x_n \to x_{\lambda_0}$ as $n \to \infty$, with:
  - (a) $df_\lambda(x_n) = 0$ for all $n$;
  - (b) $x_n \neq x_{\lambda_n}$ for all $n$.
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Definition

*Bifurcation at* $\lambda_0 \in ]a, b[$ *if* $\exists \lambda_n \to \lambda_0$ and $x_n \to x_{\lambda_0}$ as $n \to \infty$, with:

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*as* $n \to \infty$, with:

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\[ x \]

$\lambda_0$

$[a, b]$
Equivariant bifurcation

Assume:

- $G$ Lie group acting on $\mathcal{M}$
- $f_\lambda$ is $G$-invariant for all $\lambda$

Note: the orbit $G \cdot x_\lambda$ consists of critical points.
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**Definition**

*Orbit bifurcation* at $\lambda_0 \in ]a, b[$ if $\exists \lambda_n \to \lambda_0$ and $x_n \to x_{\lambda_0}$ as $n \to \infty$, with:

(a) $df_{\lambda_n}(x_n) = 0$ for all $n$;
(b) $G \cdot x_n \neq G \cdot x_{\lambda_n}$ for all $n$. 

Standard bifurcation theory requires a quite involved variational setup: differentiability, Palais–Smale, Fredholmness...

Bifurcation occurs at degenerate critical points with jumps of the Morse index. In the equivariant case, bifurcation occurs at degenerate critical orbits where jumps of the critical groups.
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Standard bifurcation theory requires a quite involved variational setup: differentiability, Palais–Smale, Fredholmness...

Bifurcation occurs at *degenerate* critical points with *jumps of the Morse index*. In the equivariant case, bifurcation occurs at degenerate critical orbits where *jumps of the critical groups*.
Introduce a set $\text{Emb}(M, N)$ of embeddings $M \hookrightarrow N$. (which regularity?)
A scheme for the proof of main result

- Introduce a set $\text{Emb}(M, N)$ of embeddings $M \hookrightarrow N$. (which regularity?)

- Manifold structure on the quotient $\mathcal{M} = \text{Emb}(M, N)/\text{Diff}(M)$. 

- Volume and area functionals on $\mathcal{M}$ (invariant by $G$).

- CMC embeddings $M \hookrightarrow N \leftrightarrow$ constrained critical points of Area with fixed Volume.

- Accumulation of non-congruent CMC embeddings $\leftrightarrow$ Constrained critical $G$-orbit bifurcation.
A scheme for the proof of main result

- Introduce a set $\text{Emb}(M, N)$ of embeddings $M \hookrightarrow N$. (which regularity?)
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\[
\begin{array}{c|c}
\text{CMC embeddings} & \text{constrained critical points of Area with fixed Volume.} \\
M \hookrightarrow N & \\
\end{array}
\]

Accumulation of non congruent CMC embeddings $\leftrightarrow$ Constrained critical $G$-orbit bifurcation
(A1) $\mathcal{M}$ smooth manifold modeled on a separable Banach space $X$;
(A1) \( M \) smooth manifold modeled on a separable Banach space \( X \);

(A2) \( G \) compact (connected) Lie group acting continuously on \( M \);
(A1) $\mathcal{M}$ smooth manifold modeled on a separable Banach space $X$;

(A2) $G$ compact (connected) Lie group actioning *continuously* on $\mathcal{M}$;

(A3) $A : \mathcal{M} \to \mathbb{R}$ smooth $G$-invariant function;
(A1) $\mathcal{M}$ smooth manifold modeled on a separable Banach space $\mathbf{X}$;
(A2) $G$ compact (connected) Lie group actioning continuously on $\mathcal{M}$;
(A3) $A : \mathcal{M} \to \mathbb{R}$ smooth $G$-invariant function;
(A4) $V : \mathcal{M} \to \mathbb{R}$ smooth $G$-invariant function without critical points;
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(A5) orbits of critical points of $f_\lambda = A + \lambda \cdot V$ smooth submanifolds of $\mathcal{M}$. 

In the CMC variational problem:

$\mathcal{M} = \text{Emb}(\mathcal{M}, \mathcal{N}) / \text{Diff}(\mathcal{M})$

$X = \text{space of sections of some vector bundle over } \mathcal{M}$

$G = (\text{connected component of 1 of }) \text{isometry group of } (\mathcal{N}, g)$

$A = \text{area functional}$, $V = \text{volume functional}$

$\lambda = \text{mean curvature (up to a factor)}$
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A set of axioms for equivariant constrained bifurcation — 1

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(A1) $\mathcal{M}$ smooth manifold modeled on a separable Banach space $\mathbb{X}$;  
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\textbf{In the CMC variational problem:}

- $\mathcal{M} = \text{Emb}(M, N)/\text{Diff}(M)$
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A set of axioms for equivariant constrained bifurcation — 2
Hilbertization & Fredholmness A

For all \( \lambda \neq 0 \) and all \( x_0 \in \text{Crit}(f_{\lambda 0}) \),
there exists a neighborhood of \( x_0 \), a Banach space \( Y \),
a Hilbert space \( H_0 \), with continuous dense inclusions:
\[ X \hookrightarrow \rightarrow Y \hookrightarrow \rightarrow H_0, \]
and a map \( F : \lambda_0 - \varepsilon, \lambda_0 + \varepsilon \times U \rightarrow Y \) such that:
\[ df_{\lambda}(x)v = \langle F(\lambda, x), v \rangle_{H_0} \]
\[ \partial F/\partial x(\lambda_0, x_0) : X \rightarrow Y \text{ Fredholm of index 0}. \]

(HF-A) implies:
(a) local Palais–Smale condition for \( f_{\lambda} \);
(b) manifold structure of critical orbits near nondegenerate ones, via
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(HF-A) **gradient map for** $f_{\lambda}$: For all $\lambda_0$ and all $x_0 \in \text{Crit}(f_{\lambda_0})$, $\exists U$ neighborhood of $x_0$, a Banach space $Y$, a Hilbert space $H_0$, with continuous dense inclusions:

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Hilbertization & Fredholmness A

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\text{div} \left( \frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \right)
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Jacobi operator.

\[
\frac{\partial F}{\partial x}
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Strong ellipticity \( \Rightarrow \) Schauder’s estimates \( \Rightarrow \) Fredholmness
Regularity of embeddings

For Morse theoretical and Fredholmness questions it would be desirable to have a Hilbert manifold structure:

\[ H^1 \] lacks regularity of weak solutions of CMC equation

\[ H^k, k > 1 \]: Hessian not Fredholm (compact!)

Banach manifold structure:

\[ C^\infty \]: not Banach, only Frechet

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\[ C^k, \alpha, k \geq 2, \alpha \in [0,1] \]: almost fine, but not separable!

\[ f_a = |x - a|^{\alpha}, f_b = |x - b|^{\alpha}, \text{dist}_{0,\alpha}(f_a, f_b) \geq 2 \] for all \( a \neq b \).
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\]
For all $\lambda \neq 0$ and all $x_0 \in \text{Crit}(f_\lambda)$, there exists a Hilbert space $H_1$, with $X \subset H_1$, such that $d^2 f_\lambda(x_0)$ extends to an essentially positive bounded symmetric bilinear form on $H_1$:

$$d^2 f_\lambda(x_0)(v_1, v_2) = \langle P_\lambda, x_0 v_1, v_2 \rangle_{H_1}$$

For $P_\lambda, x_0 : H_1 \to H_1$ self-adjoint.

(HF-B) is used:

(a) to compute Morse index of $x_0$ (sum of dimension of negative eigenspaces of $P_\lambda, x_0$)

(b) to compute local homological invariants of critical orbits, via Morse Lemma.
(HF-B) **Fredholm Hessian for \( f_\lambda \):** For all \( \lambda_0 \) and all 
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\[
d^2 f_{\lambda_0}(x_0)(v_1, v_2) = \langle P_{\lambda_0, x_0} v_1, v_2 \rangle_{H_1}
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- \( P_{\lambda_0, x_0} : H_1 \to H_1 \) self-adjoint
- \( \Sigma_{\text{ess}}(P_{\lambda_0, x_0}) \subset ]0, +\infty[. \)
A set of axioms for equivariant constrained bifurcation — 3
Hilbertization & Fredholmness B

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A set of axioms for equivariant constrained bifurcation — 3
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**HF-B**  
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For the CMC problem:

\[ H_1 = \text{Sobolev space of } H^1\text{-sections of the normal bundle } x_0^\perp \]

\[
d^2 f_{\lambda_0}(x_0)(v_1, v_2) = \int_M \nabla v_1 \cdot \nabla v_2 - \left[ m \cdot \text{Ric}_N(\vec{n}_{x_0}) + \|A\|^2 \right] v_1 v_2
\]

\[
\int_M \nabla v_1 \cdot \nabla v_2 \quad \text{inner product of } H^1 \quad \leadsto \quad \text{positive isomorphism.}
\]

\[
\int_M \left[ m \cdot \text{Ric}_N(\vec{n}_{x_0}) + \|A\|^2 \right] v_1 v_2
\]

\[
\text{does not contain derivatives} \quad \leadsto \quad \text{compact operator}
\]

\[
\text{positive + compact} = \text{essentially positive}
\]
Theorem

In the variational setup (A1)—(A5), satisfying (HF–A) + (HF–B), assume:

(B) \[ \mathbb{C}^1 \text{-maps:} \]

\[ [a, b] \ni r \mapsto \lambda_r \in \mathbb{R}, \text{ with } \lambda_r' > 0; \]

(B2) \[ [a, b] \ni r \mapsto x_r \in M, \text{ with } d_{f_{\lambda_r}}(x_r) = 0 \forall r. \]

(C) The connected component of the identity of the stabilizer of \( x_r \) does not depend on \( r \).

(D1) For \( r \neq \bar{r} \), \( O(x_r, f_{\lambda_r}) \) is a nondegenerate critical orbit.

(D2) For \( \varepsilon > 0 \) small, Morse index \( (x_{\bar{r} - \varepsilon}) \neq \text{Morse index} (x_{\bar{r} + \varepsilon}) \).

Then, critical orbit bifurcation occurs at \( r = \bar{r} \).
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Given $C_1$-maps:

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The connected component of the identity of the stabilizer of $x r$ does not depend on $r$.

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Constrained orbit bifurcation theorem

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Area and volume of an embedding

- $M^m$ compact oriented manifold
- $(N, g)$ oriented Riemannian manifold, $\mathrm{vol}_g$ volume form.
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$$Volume(x) = \int_\Omega vol_g$$
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- $x : M \to N$ embedding
- $x^*(g)$ pull-back metric
- $\text{vol}^* = \text{vol}(x^*(g))$.

\[
\text{Area}(x) = \int_M \text{vol}^*
\]

Assume $x(M) = M_0 = \partial \Omega$, i.e. $N \setminus M_0$ has 2 connected components.

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\text{Volume}(x) = \int_{\Omega} \text{vol}_g
\]

# of connected components of $N \setminus M_0$
= rank($\tilde{H}_0(N \setminus M_0)$)
Area and volume of an embedding

- $M^m$ compact oriented manifold
- $(N, g)$ oriented Riemannian manifold, $\text{vol}_g$ volume form.
- $n = m + 1$
- $x : M \rightarrow N$ embedding
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$\# \text{ of connected components of } N \setminus M_0$ = $\text{rank}(\tilde{H}_0(N \setminus M_0)) \iff \text{reduced homology}$
Connected components of $N \setminus M_0$

Long exact reduced homology sequence:

\[
\begin{align*}
H_1(N) & \twoheadrightarrow H_1(N, N \setminus M_0) \twoheadrightarrow \tilde{H}_0(N \setminus M_0) \rightarrow \tilde{H}_0(N) = 0 \\
\uparrow & \quad \uparrow \quad \uparrow \\
\text{onto} & \quad \text{connected} & \\
\text{either 0} & \quad \text{by excision} & \\
\text{either 0} & \quad \text{onto} & \\
H_1(M_0) & \cong \mathbb{Z} & \text{free}
\end{align*}
\]
Connected components of $N \setminus M_0$

Long exact reduced homology sequence:

$$
\begin{align*}
H_1(N) & \longrightarrow H_1(N, N \setminus M_0) \overset{\text{onto}}{\longrightarrow} \tilde{H}_0(N \setminus M_0) \overset{\text{connected}}{\longrightarrow} \tilde{H}_0(N) = 0 \\
\uparrow \text{either 0} & \quad \uparrow H_1(M_0) \cong \mathbb{Z} \quad \uparrow \text{free} \\
\text{or onto} & \quad \text{by excision}
\end{align*}
$$

**Proposition**

$N \setminus M_0$ has 2 connected components $\iff$ $H_1(N) \longrightarrow H_1(N, N \setminus M_0)$ is zero
Connected components of $N \setminus M_0$

homologically non trivial embedding
its image is not a boundary
Connected components of $N \setminus M_0$

the picture

homologically non trivial embedding
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homologically trivial
A generalized *volume functional*

\[ N \text{ non compact} \implies \text{vol}_g = d\eta \text{ is exact.} \]
A generalized \textit{volume functional}

\[ N \text{ non compact} \implies \text{vol}_g = d\eta \text{ is exact.} \]

Set \[\text{Volume}(x) = \int_M x^*(\eta)\]
A generalized volume functional

If non compact $\implies \text{vol}_g = d\eta$ is exact.

Set $\text{Volume}(x) = \int_M x^*(\eta)$

**Note.** If $x(M) = \partial \Omega$, then:

$$\text{Volume}(x) = \int_M x^*(\eta) = \int_{\partial \Omega} \eta \quad \text{by Stokes' theorem} \quad = \int_{\Omega} d\eta = \int_{\Omega} \text{vol}_g = \text{Volume}(\Omega).$$
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If \( N \) is compact, pick \( p \in N \setminus x(M) \) and replace \( N \) with \( N \setminus \{p\} \).
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Remove a point $\implies$ no more compact!
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The geometric structure of $\text{Emb}(M, N)/\text{Diff}(M)$
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- $x : M \rightarrow N$ smooth ($C^\infty$) embedding
The geometric structure of \( \text{Emb}(M, N)/\text{Diff}(M) \)

- \( x : M \rightarrow N \) smooth \((C^\infty)\) embedding
- \( x\perp \) normal bundle of \( x \)

**Proposition**

The map \( \Phi_x : y \mapsto V_{x,y} \) is a local chart for \( \text{Emb}(M, N)/\text{Diff}(M) \) with domain a neighborhood of \( x \) and taking values in a neighborhood of the null sections of the Banach space \( \Gamma_{k,\alpha}(x\perp) \).

The above charts are continuously compatible, but not differentiably.

If \( f : \text{Emb}(M, N) \) is a smooth map which is invariant by \( \text{Diff}(M) \), then \( f \circ \Phi_x \) is smooth for all \( x \).

Thus, area and volume are smooth in local charts.
The geometric structure of $\Emb(M, N)/\Diff(M)$

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- $\exists! V_{x,y} \in \Gamma^{k,\alpha}(x^\perp)$ s.t. $y = \exp^\perp V_{x,y}$ (up to a reparameterization)
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- The above chart are *continuously* compatible, but not *differentiably*.
- If $f : \text{Emb}(M, N)$ is a smooth map which is invariant by $\text{Diff}(M)$, then $f \circ \Phi_x$ is smooth for all $x$. 
Lack of differentiability of composition functions

\( A, B, C \) smooth manifolds, \( F : B \to C \) map of class \( C^k \).
Lack of differentiability of composition functions

$A, B, C$ smooth manifolds, $F : B \to C$ map of class $C^k$.

$$C^k(A, B) \xrightarrow{L_F} C^k(A, C) \quad \text{left composition}$$

$$f \xrightarrow{} F \circ f$$
Lack of differentiability of composition functions

A, B, C smooth manifolds, $F : B \to C$ map of class $C^k$.

\[
\begin{align*}
\mathcal{C}^k(A, B) & \xrightarrow{L_F} \mathcal{C}^k(A, C) & \text{left composition} \\
f & \rightarrow F \circ f
\end{align*}
\]

\[
\text{d}L_f = L_{dF} \implies \text{if } F \notin C^{k+1}, L_F \text{ not differentiable!}
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\(A, B, C\) smooth manifolds, \(F : B \rightarrow C\) map of class \(C^k\).

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C^k(A, B) \xrightarrow{L_F} C^k(A, C)
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left composition

\[
f \rightarrow F \circ f
\]

\[
dL_f = L_{dF} \quad \Rightarrow \quad \text{if } F \notin C^{k+1}, L_F \text{ not differentiable!}
\]

\[
C^k(A, B) \times C^k(C, A) \rightarrow C^k(C, B)
\]

not differentiable!

\[
(f_1, f_2) \rightarrow f_1 \circ f_2
\]

\[
\text{Diff}^k(M) \xrightarrow{\text{inv}} \text{Diff}^k(M) \quad \text{not differentiable!}
\]

\[
\phi \rightarrow \phi^{-1}
\]
\( \text{Emb}(M, N)/\text{Diff}(M) \) is not a differentiable manifold!

\[
\begin{align*}
\text{―} & \quad x_1, x_2 : M \to N
good smooth embeddings, \\
y : M \to N \\
C^{k, \alpha} \text{ embedding}
\end{align*}
\]
Emb($M$, $N$)/Diff($M$) **is not a differentiable manifold!**

- $x_1, x_2 : M \to N$ smooth embeddings, $y : M \to N$ $C^{k,\alpha}$ embedding
- $E_i = x_i^\perp$ normal bundle, $\pi_i : E_i \to M$ projection, $i = 1, 2$
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- \( V_2' = \zeta \circ V_1 \) not a section of \( E_2 \). Needs and an adjustment.
\( \text{Emb}(M, N)/\text{Diff}(M) \text{ is not a differentiable manifold!} \)

- \( x_1, x_2 : M \to N \)
  smooth embeddings,
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  and an \emph{adjustment}:

\[
\begin{align*}
  h_{V_1} &= \pi_2 \circ \zeta \circ V_1, \\
  V_2 &= \zeta \circ V_1 \circ h_{V_1}^{-1} \leftarrow \text{not differentiable if } V_1 \notin C^{k+1}
\end{align*}
\]
Proposition

- $\text{Emb}(M, N)/\text{Diff}(M)$ does not have a \textit{natural} differentiable structure, i.e., such that $\text{Emb}(M, N) \to \text{Emb}(M, N)/\text{Diff}(M)$ is a smooth surjection.
Proposition

- $\text{Emb}(M, N)/\text{Diff}(M)$ does not have a natural differentiable structure, i.e., such that $\text{Emb}(M, N) \rightarrow \text{Emb}(M, N)/\text{Diff}(M)$ is a smooth surjection.

- The action of $G = \text{Iso}(N, g)$ on $\text{Emb}(M, N)/\text{Diff}(M)$ is continuous.
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- The action of $G = \text{Iso}(N, g)$ on $\text{Emb}(M, N)/\text{Diff}(M)$ is continuous.

- The $G$-orbit of any $x$ smooth (in particular, of any CMC embedding) is smooth in local charts.
Stabilizer and orbit of CMC Clifford tori

\[ x_r^{j,m}(S^j \times S^{m-j}) = S^j(r) \times S^{m-j}(\sqrt{1 - r^2}) \subset S^{m+1}(1) \]
Stabilizer and orbit of CMC Clifford tori

- \( x_r^{j,m}(S^j \times S^{m-j}) = S^j(r) \times S^{m-j}(\sqrt{1 - r^2}) \subset S^{m+1}(1) \)
- \( \begin{pmatrix} O(j + 1) & 0 \\ 0 & O(m - j + 1) \end{pmatrix} \subset \text{stab}([x_r^{j,m}]) \) (may not be equal!)
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**Corollary**

\( \text{stab}_{0}([x_{r}^{j,m}]) = \begin{pmatrix} \text{SO}(j + 1) & 0 \\ 0 & \text{SO}(m - j + 1) \end{pmatrix} \).
Stabilizer and orbit of CMC Clifford tori

- \( x_r^{j,m}(S^j \times S^{m-j}) = S^j(r) \times S^{m-j}(\sqrt{1 - r^2}) \subset S^{m+1}(1) \)

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\text{stab}_0([x_r^{j,m}]) = \begin{pmatrix} \text{SO}(j + 1) & 0 \\ 0 & \text{SO}(m - j + 1) \end{pmatrix}.
\]

\[
\dim(\mathcal{O}(x_r^{j,m})) = \dim(\text{SO}(m + 2)) - \dim(\text{SO}(j + 1)) - \dim(\text{SO}(m - j + 1)) = m + 1 + j(m - 1)
\]
Stabilizer and orbit of CMC Clifford tori

\[ x_r^{j,m}(S^j \times S^{m-j}) = S^j(r) \times S^{m-j}(\sqrt{1 - r^2}) \subset S^{m+1}(1) \]

\[ \left( \begin{array}{cc} O(j + 1) & 0 \\ 0 & O(m - j + 1) \end{array} \right) \subset \text{stab}([x_r^{j,m}]) \text{ (may not be equal!)} \]

\[ \left( \begin{array}{cc} \text{SO}(j + 1) & 0 \\ 0 & \text{SO}(m - j + 1) \end{array} \right) \text{ is a maximal connected subgroup of } \text{SO}(m + 2) \]

**Corollary**

\[ \text{stab}_0([x_r^{j,m}]) = \left( \begin{array}{cc} \text{SO}(j + 1) & 0 \\ 0 & \text{SO}(m - j + 1) \end{array} \right). \]

\[ \text{dim}(O(x_r^{j,m})) = \text{dim}(\text{SO}(m + 2)) - \text{dim}(\text{SO}(j + 1)) - \text{dim}(\text{SO}(m - j + 1)) = m + 1 + j(m - 1) \]

**RECALL**

nondegenerate critical orbits for \( r \neq r_i, s_i \)
Local homological invariants

Let $x_0 \in M$ be a critical point of $f_{\lambda_0}$, $c = f_{\lambda_0}(x_0)$ be its critical orbit, and $f_c_{\lambda_0} = f^{-1}_{\lambda_0}(c) = \{x \in M : f_{\lambda_0}(x) \leq c\}$ be the closed sublevel.

**Definition** $k$-th critical group $\text{H}_k(O(x_0), \mathbb{Z}_2)$ is the $k$-th relative homology space $\text{H}_k(f_c_{\lambda_0}, f_{\lambda_0}O(x_0), \mathbb{F})$ ($\mathbb{F}$ coefficient field).

**Proposition** Assume $O(x_0)$ nondegenerate, $\mu = \text{Morse index}(x_0)$, Axioms (A1)–(A5), and Axiom (HF–B). Then $\text{H}_k(O(x_0), \mathbb{Z}_2) \cong \text{H}_{k-\mu}(O(x_0), \mathbb{Z}_2)$ (shifted homology).
Local homological invariants

- \( x_0 \in \mathcal{M} \) critical point of \( f_{\lambda_0} \), \( c = f_{\lambda_0}(x_0) \)
Local homological invariants

- $x_0 \in \mathcal{M}$ critical point of $f_{\lambda_0}$, $c = f_{\lambda_0}(x_0)$
- $G \cdot x_0 = \mathcal{O}(x_0)$ critical orbit
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**Definition**

$k$-th critical group $\mathcal{H}_k(\mathcal{O}(x_0))$ is the $k$-th relative homology space $H_k(f^c_{\lambda_0}, f^c_{\lambda_0} \setminus \mathcal{O}(x_0); F)$ (F coefficient field)
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Assume:
- $\mathcal{O}(x_0)$ nondegenerate
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Assume:

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Then:

$$\mathcal{H}_k(\mathcal{O}(x_0), \mathbb{Z}_2) \cong H_{k-\mu}(\mathcal{O}(x_0), \mathbb{Z}_2)$$

(shifted homology)
Proof of Proposition

- Existence of slice $S$ for the action of $G$
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- Chang’s approach to Morse theory on Banach spaces (Axiom (HF–B)):

$$H_k((f_{\lambda_0}^c \cap S, (f_{\lambda_0}^c \cap S) \setminus \{x_0\}; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } k = \mu \\ 0 & \text{if } k \neq \mu. \end{cases}$$
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- Excision + \textit{Leray–Hirsch theorem} (homology of fiber bundles):

  \[
  \mathcal{H}_k(\mathcal{O}(x_0), Z_2) \cong \bigoplus_{i=0}^{\dim \mathcal{O}(x_0)} H_i(f_{\lambda_0}^c \cap S, (f_{\lambda_0}^c \cap S) \setminus \{x_0\}; Z_2) \otimes H_{\dim \mathcal{O}(x_0) - i}(\mathcal{O}(x_0); Z_2). \]
Proof of Proposition

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Corollary

Jump of the Morse index $\implies$ jump of the critical groups.
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Corollary

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Proof of main result concluded
A few CMC bifurcation problems to think about

- Continuity/smoothness of bifurcating branch.
- Break of symmetry.

Joint project with Jorge Herbert de Lira and Levi Lopes de Lima: study bifurcation and symmetry breaking of CMC Clifford tori in Berger spheres $S^n + 1$. (1-parameter family of rotationally symmetric CMC embeddings $S^1 \times S^n \rightarrow S^n + 1$.)
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- Joint project with Jorge Herbert de Lira and Levi Lopes de Lima: study bifurcation and symmetry breaking of CMC Clifford tori in Berger spheres $S_B^{2n+1}$. (1-parameter family of rotationally symmetric CMC embeddings $S^1 \times S^{2n-2} \leftrightarrow S_B^{2n+1}$).
A fixed boundary CMC bifurcation problem

Work in progress with Miyuki Koiso and Bennet Palmer

Fix parallel planes $\pi_1$ and $\pi_2$ in $\mathbb{R}^3$. $C_1 \subset \pi_1$, $C_2 \subset \pi_2$, circles with the same radius.

$\exists$ a 1-parameter family of nodoids $N_t$, $t \in \mathbb{R}$, intercepting $\pi_i$ on $C_i$, $i = 1, 2$.

At a discrete set $\{t_k\}$ $k \in \mathbb{Z}$ of values of the parameter $t$, $N_t$ is tangent to both $\pi_i$. 
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- at a discrete set $(t_k)_{k \in \mathbb{Z}}$ of values of the parameter $t$, $N_t$ is tangent to both $\pi_i$. 
Theorem

At $t = t_k$ there is bifurcation of CMC surfaces satisfying the given boundary condition.

Proof:

The normal field $\vec{N} = \nu_1 \vec{e}_1 + \nu_2 \vec{e}_2 + \nu_3 \vec{e}_3$ is tangent to the planes, then $\nu_1$ and $\nu_2$ satisfy the boundary conditions:

$\nu_i |_{\partial N_t} \equiv 0$.

The only rotationally symmetric solutions are nodoids, which explains the break of symmetry.

QED
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At $t = t_k$ there is bifurcation of CMC surfaces satisfying the given boundary condition. Bifurcating branch consists of *non rotationally symmetric* surfaces (break of symmetry).
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A similar result hold for a more general variational problem (Anisotropic CMC).
Bifurcation of fixed boundary CMC surfaces

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**Proof:**

- \( \tilde{N} = \nu_1 \cdot \tilde{e}_1 + \nu_2 \cdot \tilde{e}_2 + \nu_3 \cdot \tilde{e}_3 \) normal field
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- $\tilde{N} = \nu_1 \cdot \tilde{e}_1 + \nu_2 \cdot \tilde{e}_2 + \nu_3 \cdot \tilde{e}_3$ normal field

- $\nu_1$ and $\nu_2$ are *Jacobi fields*
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- When $N_t$ is tangent to the planes, then $\nu_1$ and $\nu_2$ satisfy the *boundary conditions*: $\nu_i|_{\partial N} = 0$. 

QED
Bifurcation of fixed boundary CMC surfaces

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At $t = t_k$ there is bifurcation of CMC surfaces satisfying the given boundary condition. Bifurcating branch consists of non rotationally symmetric surfaces (break of symmetry). A similar result hold for a more general variational problem (Anisotropic CMC).

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- $\tilde{N} = \nu_1 \cdot \tilde{e}_1 + \nu_2 \cdot \tilde{e}_2 + \nu_3 \cdot \tilde{e}_3$ normal field
- $\nu_1$ and $\nu_2$ are Jacobi fields (corresponding to translations in the direction $\tilde{e}_i$)
- When $\mathcal{N}_t$ is tangent to the planes, then $\nu_1$ and $\nu_2$ satisfy the boundary conditions: $\nu_i|_{\partial \mathcal{N}_i} \equiv 0$.
- Direct analysis of the spectrum of $J$, $\nu_1$ and $\nu_2$ determine a jump of the Morse index $\implies$ bifurcation.
Theorem

At \( t = t_k \) there is bifurcation of CMC surfaces satisfying the given boundary condition. Bifurcating branch consists of \textit{non rotationally symmetric} surfaces (break of symmetry).
A similar result hold for a more general variational problem (\textit{Anisotropic CMC}).

Proof:

- \( \vec{N} = \nu_1 \cdot \vec{e}_1 + \nu_2 \cdot \vec{e}_2 + \nu_3 \cdot \vec{e}_3 \) normal field
- \( \nu_1 \) and \( \nu_2 \) are \textit{Jacobi fields}
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- When \( \mathcal{N}_t \) is tangent to the planes, then \( \nu_1 \) and \( \nu_2 \) satisfy the \textit{boundary conditions}: \( \nu_i|_{\partial \mathcal{N}_i} \equiv 0 \).
- Direct analysis of the spectrum of \( J \), \( \nu_1 \) and \( \nu_2 \) determine a \textit{jump of the Morse index} \( \Longrightarrow \) \textit{bifurcation}.
- the only rotationally symmetric solutions are nodoids \( \Longrightarrow \) break of symmetry.

QED
Essential bibliography


Special thanks to Renato Ghini Bettiol for helping me with the pictures.

That's all folks, thanks for the attention!
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