Bifurcation of CMC Clifford Tori in Euclidean Spheres

Joint work with Luis J. Alías

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Universidad de Murcia

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"In honor of *Marcos Dajczer* on his 60th birthday"

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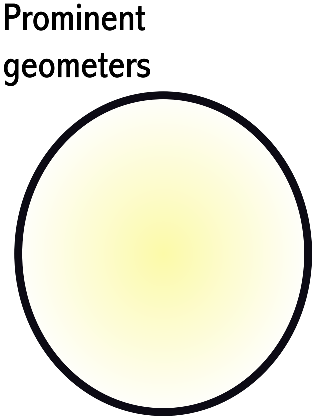
Happy birthday Marcos!

On the conference speakers



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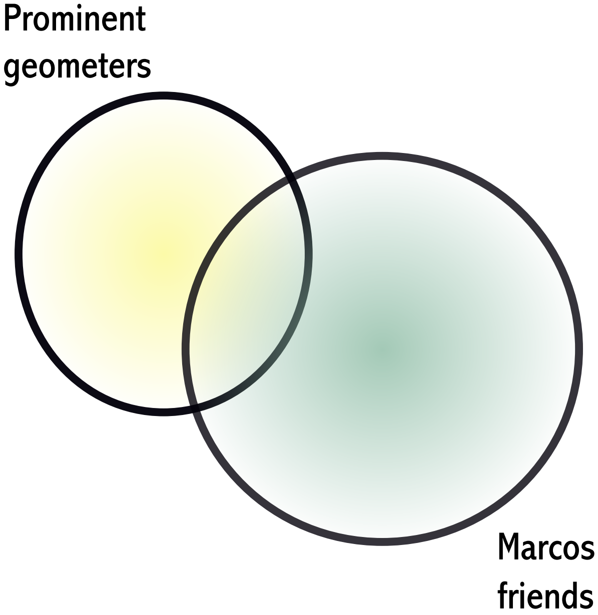

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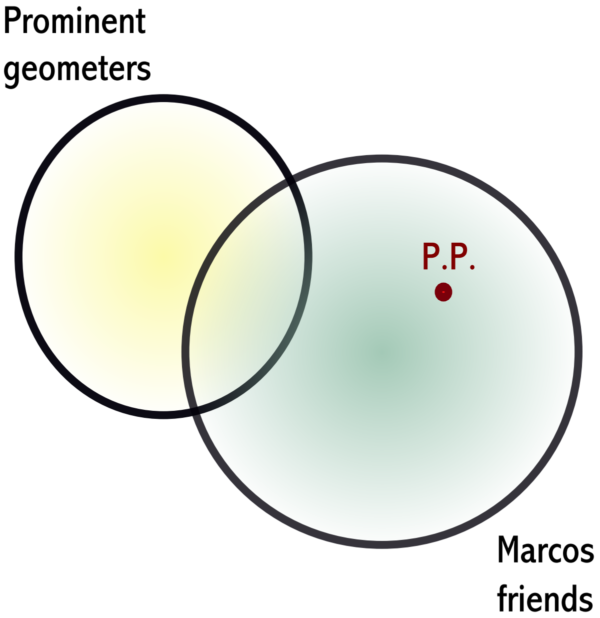


Outline of this talk.

CMC Clifford tori

$$1 \leq j < m, \qquad r \in]0, 1[$$

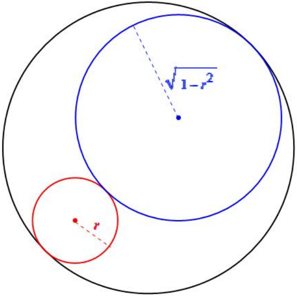
$$x_r^{j,m} : \mathbb{S}^j \times \mathbb{S}^{m-j} \longrightarrow \mathbb{S}^{m+1}$$

$$(p, q) \longmapsto \left(r \cdot p, \sqrt{1 - r^2} \cdot q\right)$$

William Kingdon Clifford

Constant mean curvature:

$$\boxed{H_r^{j,m} = \frac{mr^2 - j}{mr\sqrt{1 - r^2}}}$$



$r = \sqrt{\frac{j}{m}}$ *minimal* Clifford torus.

The Jacobi operator

$$J = -\Delta_r^{j,m} - m \cdot \mathrm{Ric}_{\mathbb{S}^{m+1}}(\vec{N}) - \left\| A_r^{j,m} \right\|^2$$

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multiplicity of ζ = sum of multiplicities of σ and ρ

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Proposition

There exists two monotone sequences $(r_i)_{i=1}^{\infty}$ and $(s_l)_{l=1}^{\infty}$, with

$$\lim_{l\to\infty} s_l = 0, \quad \text{and} \quad \lim_{i\to\infty} = 1,$$

where the *Morse index* of the CMC Clifford torus $x_r^{j,m}$ has a *jump*.

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Variational principle

x has *constant* mean curvature (CMC) iff *x* is a stationary point for the *area functional* restricted to embeddings of fixed *volume*.

Isometric congruence

Definition

$x_1, x_2 : M \longrightarrow N$ embeddings are *congruent* ($x_1 \cong x_2$) if there exists $\phi \in \mathrm{Diff}(M)$ and $\psi \in \mathrm{Iso}(N, g)$ such that $x_2 = \psi \circ x_1 \circ \phi^{-1}$.

$$\begin{tikzcd} M \arrow[r, "x_1"] \arrow[d, "\phi"'] & N \arrow[d, "\psi"] \\ M \arrow[r, "x_2"'] & N \end{tikzcd}$$

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Group actions:

- $\mathrm{Diff}(M)$ acts on the right (*free* action)
- $\mathrm{Iso}(N, g)$ acts on the left (action not free, but group *compact*)

Statement of the result

Theorem

$x_r^{j,m} : \mathbb{S}^j \times \mathbb{S}^{m-j} \longrightarrow \mathbb{S}^{m+1}$ *CMC Clifford torus,* $1 < j < m$*,* $r \in]0,1[$*.*

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$\exists$ *two sequences* $(r_i)_{i\in\mathbb{N}}$ *and* $(s_l)_{l\in\mathbb{N}}$ *such that:*

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- *the embeddings $x_{r_i}^{j,m}$ and $x_{s_l}^{j,m}$ are accumulation of pairwise non congruent CMC embeddings of $\mathbb{S}^j \times \mathbb{S}^{m-j}$ into $\mathbb{S}^{m+1}$, each of which is not congruent to any CMC Clifford torus.*

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For all other values of r, the CMC Clifford family is stable, i.e., if $x : \mathbb{S}^j \times \mathbb{S}^{m-j} \longrightarrow \mathbb{S}^{m+1}$ is a CMC embedding which is sufficiently close to some $x_r^{j,m}$, with $r \neq r_i$ and $r \neq s_l$, then x is congruent to some $x_r^{j,m}$.

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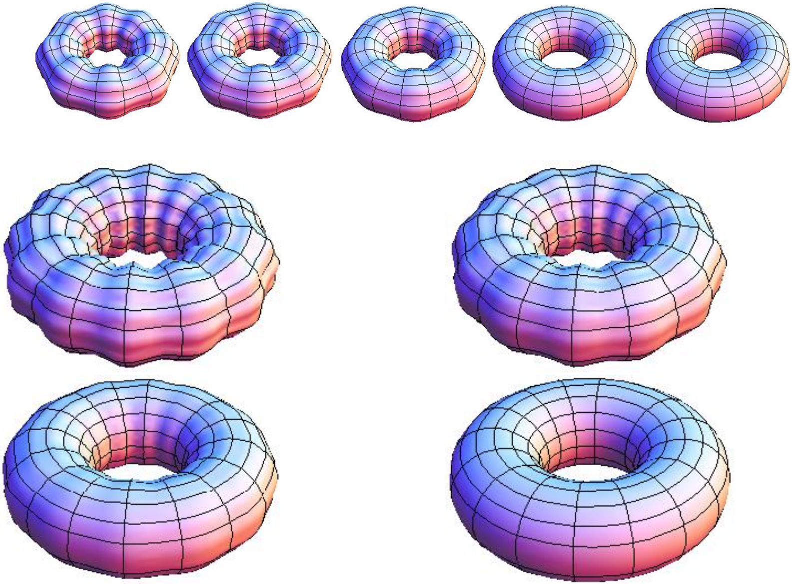
Observation. $r = \sqrt{\frac{j}{m}}$ (minimal) is *not* a bifurcation radius!

CMC tori bifurcation
picture



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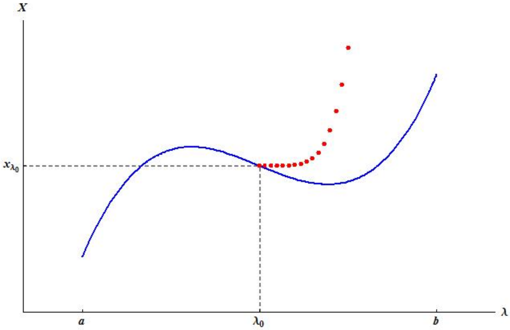

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Bifurcation occurs at *degenerate* critical points with *jumps* of the Morse index. In the equivariant case, bifurcation occurs at degenerate critical orbits where jumps of the *critical groups*.

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$$\boxed{\text{Accumulation of non congruent CMC embeddings}} \quad \leftrightsquigarrow \quad \boxed{\text{Constrained critical } G\text{-orbit bifurcation}}$$

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A set of axioms for equivariant constrained bifurcation — 1

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$\boxed{\text{Strong ellipticity}}$ $\overset{\text{Schauder's estimates}}{\Longrightarrow}$ $\boxed{\text{Fredholmness}}$

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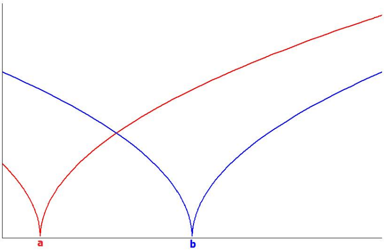

$f_a = |x-a|^\alpha$,
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for all $a \neq b$.

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D. Hilbert

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$\mathbf{H}_1$ = Sobolev space of H^1-sections of the normal bundle $x_0^\perp$

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$\int_M \nabla v_1 \cdot \nabla v_2$ inner product of H^1 $\rightsquigarrow$ *positive* isomorphism.

$\int_M \left[m \cdot \mathrm{Ric}_N(\vec{n}_{x_0}) + \|A\|^2\right] v_1 v_2$

does not contain derivatives $\rightsquigarrow$ *compact* operator

positive + compact = essentially positive

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Then, critical orbit bifurcation occurs at $r = \bar{r}$.

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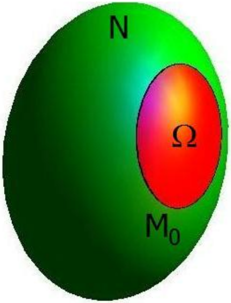

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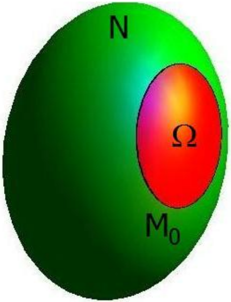

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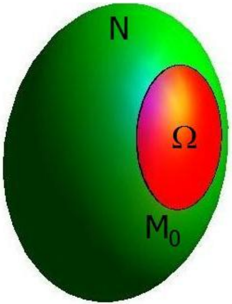

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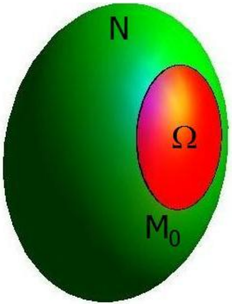

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Connected components of $N \setminus M_0$

Long exact reduced homology sequence:

$$
\begin{array}{ccccccc}
 & & & \text{onto} & & & \text{connected} \\
 & & & \Downarrow & & & \Downarrow \\
H_1(N) & \longrightarrow & H_1(N, N \setminus M_0) & \longrightarrow & \widetilde{H}_0(N \setminus M_0) & \longrightarrow & \widetilde{H}_0(N) = 0 \\
 & \Uparrow & \Uparrow & & \Uparrow & & \\
 & \text{either } 0 & H_1(M_0) \cong \mathbb{Z} & & \text{free} & & \\
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\end{array}
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Connected components of $N \setminus M_0$

Long exact reduced homology sequence:

$$
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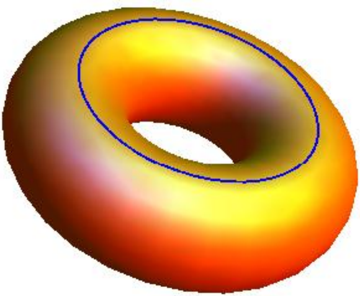
homologically non trivial embedding
its image is not a boundary



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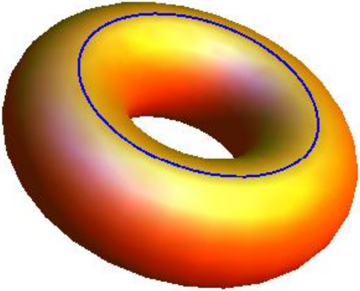

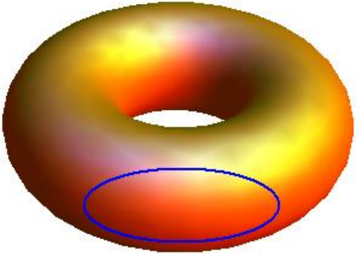

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A generalized *volume functional*

N non compact $\Longrightarrow \mathrm{vol}_g = \mathrm{d}\eta$ is exact.

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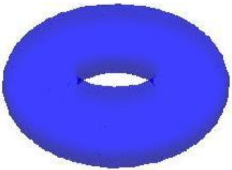

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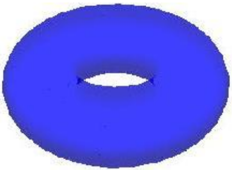

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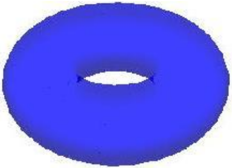

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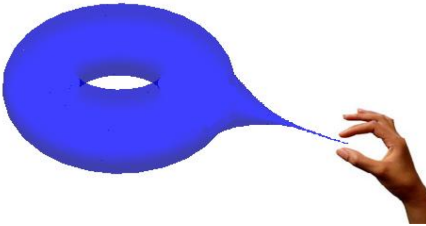

no more compact!

The geometric structure of $\mathrm{Emb}(M, N)/\mathrm{Diff}(M)$

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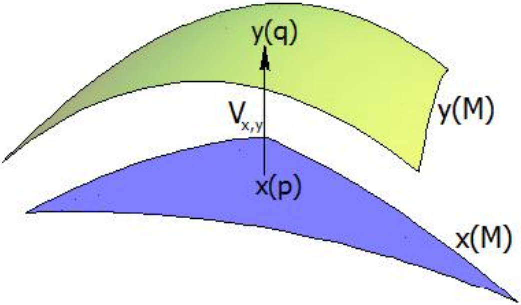

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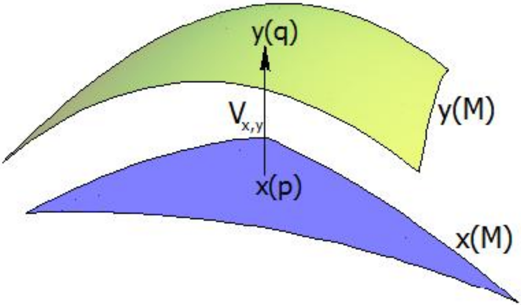

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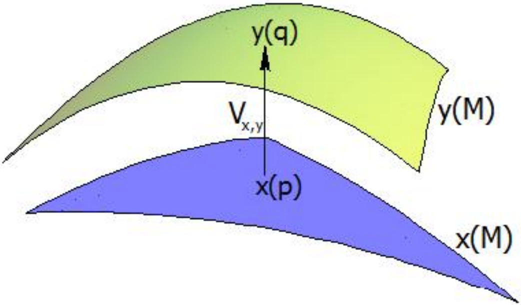

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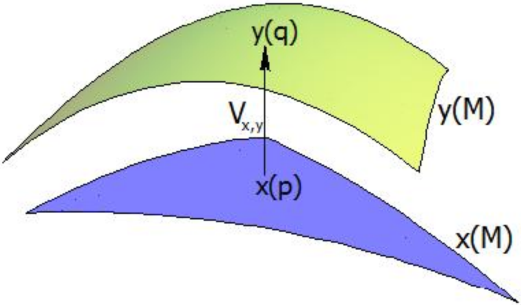

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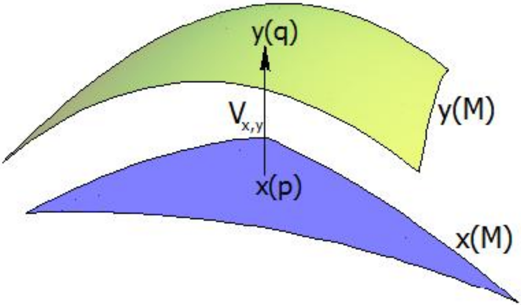

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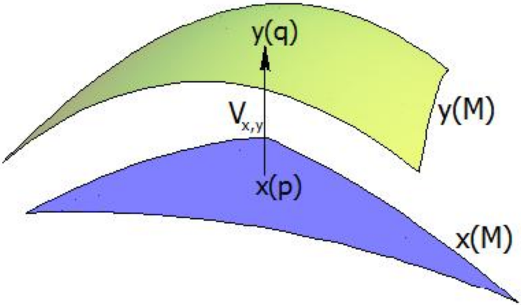

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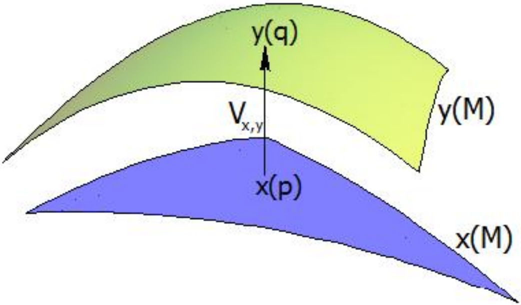

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- If $f : \mathrm{Emb}(M, N)$ is a smooth map which is invariant by $\mathrm{Diff}(M)$, then $f \circ \Phi_x$ is smooth for all x.

Lack of differentiability of composition functions

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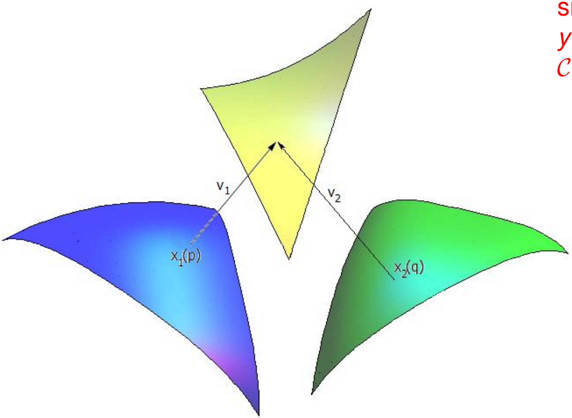

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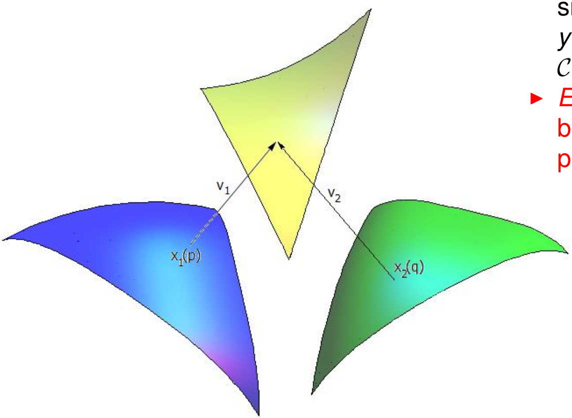

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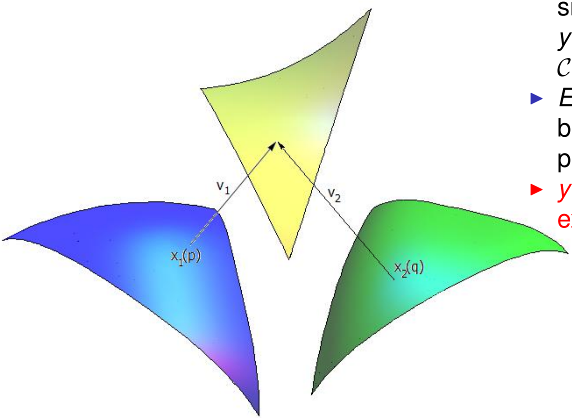

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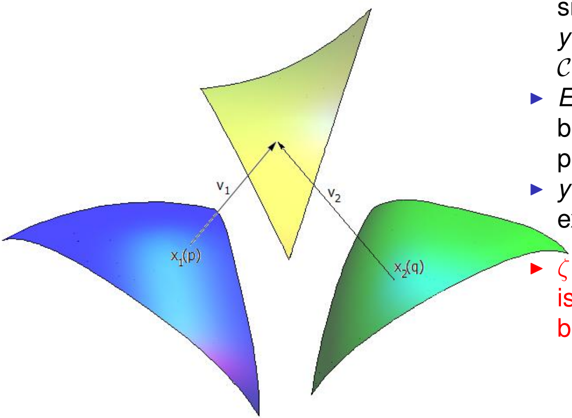

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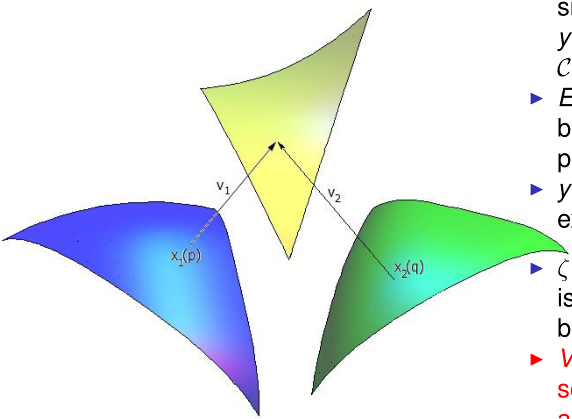

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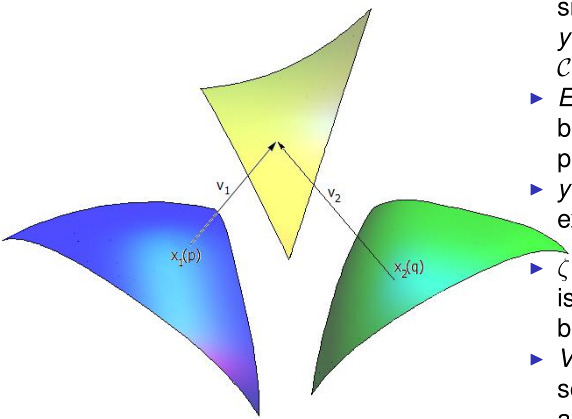

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$$h_{V_1} = \pi_2 \circ \zeta \circ V_1,$$

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- The G-orbit of any x smooth (in particular, of any CMC embedding) is smooth in local charts.

Stabilizer and orbit of CMC Clifford tori

- $x_r^{j,m}\big(\mathbb{S}^j \times \mathbb{S}^{m-j}\big) = \mathbb{S}^j(r) \times \mathbb{S}^{m-j}\big(\sqrt{1-r^2}\big) \subset \mathbb{S}^{m+1}(1)$

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RECALL $\Longrightarrow$ $\boxed{\text{nondegenerate critical orbits for } r \neq r_i, s_l}$

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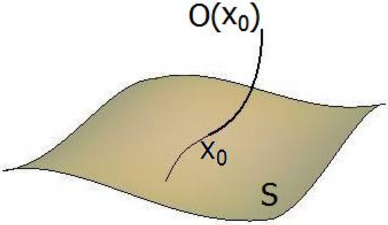
Then:

$$\boxed{\mathfrak{H}_k\big(\mathcal{O}(x_0), \mathbb{Z}_2\big) \cong H_{k-\mu}\big(\mathcal{O}(x_0), \mathbb{Z}_2\big)}$$ (shifted homology)

Proof of Proposition

- Existence of *slice* S for the action of G



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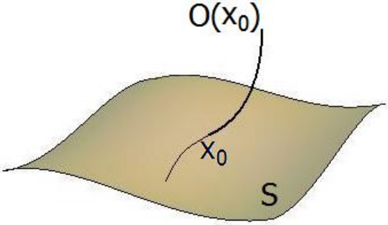

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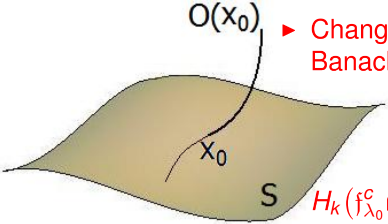

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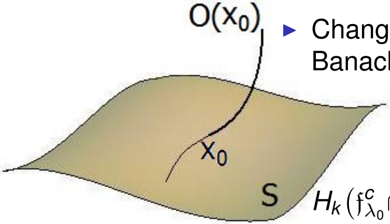

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- Excision + *Leray–Hirsch theorem* (homology of fiber bundles):

$$\mathfrak{H}_k\big(\mathcal{O}(x_0),\mathbb{Z}_2\big) \cong \bigoplus_{i=0}^{\dim\mathcal{O}(x_0)} H_i\big(\mathfrak{f}^c_{\lambda_0}\cap S, (\mathfrak{f}^c_{\lambda_0}\cap S)\setminus\{x_0\};\mathbb{Z}_2\big)\otimes H_{\dim\mathcal{O}(x_0)-i}\big(\mathcal{O}(x_0);\mathbb{Z}_2\big).$$

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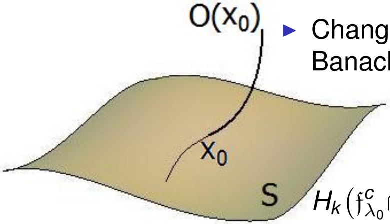

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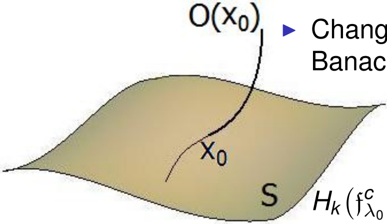

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Proof of main result concluded

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- Joint project with Jorge Herbert de Lira and Levi Lopes de Lima: study bifurcation and symmetry breaking of CMC Clifford tori in *Berger spheres* $\mathbf{S}_B^{2n+1}$. (1-parameter family of rotationally symmetric CMC embeddings $\mathbb{S}^1 \times \mathbb{S}^{2n-2} \hookrightarrow \mathbf{S}_B^{2n+1}$).

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Work in progress with Miyuki Koiso and Bennet Palmer

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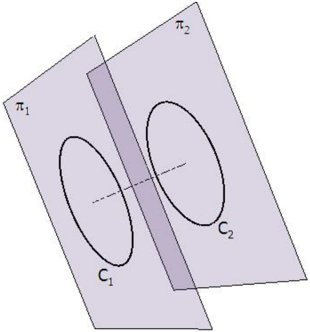

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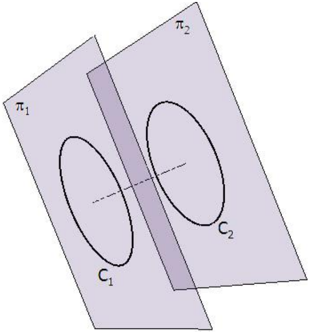

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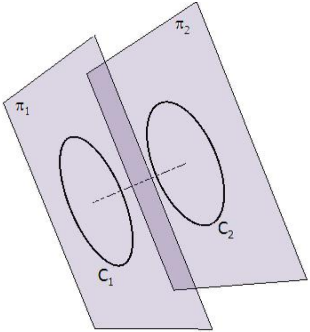

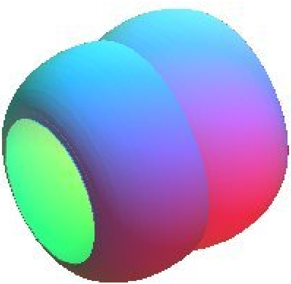

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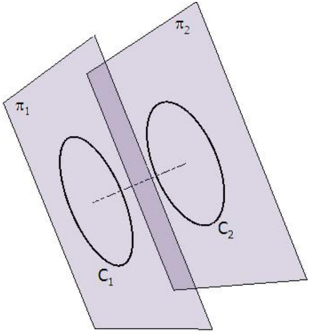

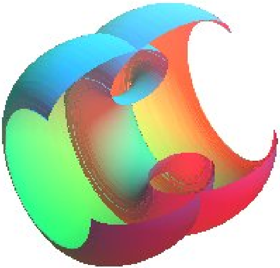

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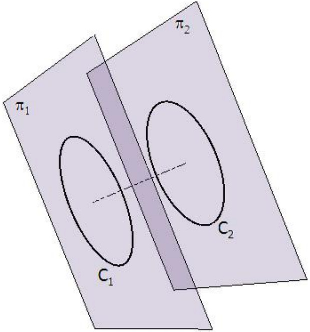

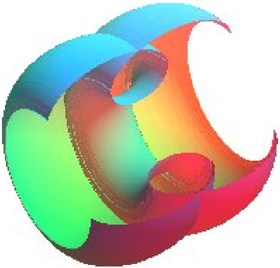

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- the only rotationally symmetric solutions are nodoids $\Longrightarrow$ break of symmetry. **QED**

Essential bibliography

- L. J. ALÍAS, A. BRASIL, O. PERDOMO, *On the stability index of hypersurfaces with constant mean curvatures in spheres*, Proc. Am. Math. Soc. **135**, no. 11 (2007), 3685–3693.
- L. J. ALÍAS, P. PICCIONE, *Bifurcation of constant mean curvature tori in Euclidean spheres*, preprint 2009, arXiv:0905.2128.
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That's all folks, thanks for the attention!

Luis Alías

back

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