A result on Lusternik-Schnirelman theory

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Abstract

We present an expanded proof of Proposition 9.2.10 of [PT88].

We begin with a few definitions.

Definition. Let X be a topological space and $A \subseteq X$. The **Lusternik-Schnirelman category** of A in X, $\operatorname{cat}(A, X)$, is the least integer n such that A can be covered by n closed subsets of X each of which is contractible in X. If no such n exists, $\operatorname{cat}(A, X) = \infty$.

Definition. Let X be a topological space. The **covering dimension** of X, $\dim X = n$, is the least integer n such that each open cover of X has an open refining cover $\{U_a\}$ such that $U_{a_1} \cap \cdots \cap U_{a_m} = \varnothing$ whenever $m > n + 1$ (and the sets are distinct), i.e. at most $n + 1$ open sets intersect at a time. If no such n exists, $\dim X = \infty$.

We move on to a lemma; proof found in [Pal66a, p.7].

Lemma. *Let X be a Hausdorff and paracompact topological space. Then for any open cover $\{U_\alpha\}_{\alpha \in A}$ of X there exists a locally finite open refinement $\{G_{i\beta}\}_{\beta \in B_i}$, $i = 1, 2, \ldots$, such that $G_{i\beta} \cap G_{i\beta'} = \varnothing$ if $\beta \neq \beta'$.*

Moreover if $\dim X = n < \infty$ then we can assume that $B_i = \varnothing$ for $i > n + 1$, i.e. $i = 1, 2, \ldots, n + 1$.

Proof. By an initial refinement, we may assume that a given cover $\{U_a\}_{a \in A}$ is locally finite and take a partition of unity $\{\phi_a\}$ subordinate to it. For each $i = 1, 2, \ldots$, we define

$$B_i := \{\beta : \beta = \{a_1, a_2, \ldots, a_i\} \subseteq A\},$$

the set of sets of i indexes of A. Then for each $\beta \in B_i$, define

$$G_{i\beta} := \{x \in X : \phi_a(x) > 0, \forall a \in \beta, \text{ and } \phi_a(x) > \phi_b(x), \forall a \in \beta, \forall b \notin \beta\}.$$

We shall prove that this is the locally finite open refinement we seek.

(a) ($G_{i\beta}$ is open). Let $x \in G_{i\beta}$. Note that $\phi_a(x) > 0$ is an open condition (i.e. there is a neighbourhood of x where this is satisfied). Furthermore, there are only finitely many ϕ_b such that $\phi_b(x) > 0$, therefore finitely many conditions $\phi_a(x) > \phi_b(x)$. Then take the (finite) intersection of the formed neighbourhoods; since this is clearly in $G_{i\beta}$, x is an interior point, so $G_{i\beta}$ is open.

(b) ($\{G_{i\beta}\}$ covers X). For any $x \in X$, let $\{a_1, \ldots, a_k\}$ be the indexes such that $\phi_a(x) > 0$, ordered so that $\phi_{a_1}(x) = \cdots = \phi_{a_j}(x) > \phi_{a_{j+1}}(x) \geq \cdots \geq \phi_{a_k}(x)$. Then $x \in G_{j\{a_1,\ldots,a_j\}}$.

(c) ($\{G_{i\beta}\}$ is a refinement). Because of the condition $\phi_a(x) > 0$ in the definition, we have $G_{i\beta} \subseteq \bigcap_{a\in\beta} \operatorname{supp}\phi_a \subseteq U_a$, for some $a \in \beta \subseteq A$.

(d) ($\{G_{i\beta}\}$ is locally finite). For $x \in X$, take V a neighbourhood of x which intersects only finitely many U_a from our original (locally finite) covering, and let $A' = \{a \in A : V \cap U_a \neq \varnothing\}$. Note that $|A'| < \infty$. Then, for some $G_{i\beta}$, either $\beta \subseteq A'$ or $\beta \nsubseteq A'$. If $\beta \nsubseteq A'$, there is an index a such that: (i) $a \in \beta$, hence $G_{i\beta} \subseteq \operatorname{supp}\phi_a \subseteq U_a$; (ii) $a \notin A'$ and $V \cap U_a = \varnothing$. Therefore $G_{i\beta} \cap V = \varnothing$. Else, if $\beta \subseteq A'$, $G_{i\beta}$ may intersect V, but since A' is a finite set there are only finitely many such β, and the $G_{i\beta}$ form a locally finite collection.

(e) ($G_{i\beta} \cap G_{i\beta'} = \varnothing$ if $\beta \neq \beta'$). Let $\beta = \{a, \dots\}$ and $\beta' = \{a', \dots\}$ such that $a \notin \beta'$ and $a' \notin \beta$. If $x \in G_{i\beta}$, then $\phi_a(x) > \phi_{a'}(x)$, and if $x \in G_{i\beta'}$, $\phi_{a'}(x) > \phi_a(x)$. Since both cannot be true at the same time, $G_{i\beta} \cap G_{i\beta'} = \varnothing$.

Finally, if $\dim X = n < \infty$, we could have also taken our initial covering collection to be such that $U_{a_1} \cap \cdots \cap U_{a_m} = \varnothing$ whenever $m > n+1$, and we would have $G_{i\beta} \subseteq \bigcap_{k=0}^{i} U_{a_k} = \varnothing$ for $i > n+1$. We could then label the $G_{i\beta}$ with $i = 1, \dots, n+1$. □

Now we come to our result. Note that a closed subspace of a Hausdorff paracompact space is also Hausdorff and paracompact.

Proposition. *If M is a connected Banach manifold and $X \subseteq M$ is a closed subset, then* $\operatorname{cat}(X, M) \leq \dim X + 1$.

Proof. Let $\{O_a\}_{a\in A}$ be a cover of X by X-open sets, each of which is contractible in M [1]. If $\dim X = \infty$, the result follows trivially. Else, if $\dim X = n < \infty$, by the lemma above there is a collection $\{G_{i\beta}\}$, $i = 1, \dots, n+1$, $\beta \in B_i$, that is a locally finite open refinement to $\{O_a\}$ such that $G_{i\beta} \cap G_{i\beta'} = \varnothing$ whenever $\beta \neq \beta'$. Since $G_{i\beta} \subseteq O_a$ for some $a \in A$, $G_{i\beta}$ is contractible in M, and since M is arc-connected, then $G_i := \bigcup_{\beta\in B_i} G_{i\beta}$ is contractible in M [2].

Now let $\{U_{i\beta}\}$ be a cover of X by X-open sets such that $\overline{U_{i\beta}} \subseteq G_{i\beta}$ [3], then define

$$X_i := \bigcup_{\beta\in B_i} \overline{U_{i\beta}}, \quad i = 1, \dots, n+1.$$

Each $X_i \subseteq G_i$ is contractible, and clearly $X = \bigcup X_i$. Since there are $(n+1)$ such X_i, it remains only to prove that they are closed. Let (x_n) be a sequence in X_i converging to $x \in X$. There is a neighbourhood of x which contains the whole sequence from some point onward, and since $\{U_{i\beta}\}$ is locally finite, the sequence must converge on some $\overline{U_{i\beta}}$, hence on X_i. □

[1]For the discussion of why such a cover exists, we refer the reader back to the source [Pal66b, p.15], the paragraph before Theorem 6.3. The original result is proved for ANR (absolute neighbourhood retract) spaces, of which Banach manifolds are a particular case (see [Pal66a, pp.2-3]).

[2]Contract each $G_{i\beta}$ to a point (which is well defined since they are disjoint), take disjoint neighbourhoods around these points (if they accumulate somewhere there may be one neighbourhood for distinct points), and contract those to the same point on M.

[3]We could take, say, a partition of unity $\{\phi_{i\beta}\}$ subordinate to $\{G_{i\beta}\}$ and then take $U_{i\beta} = \phi_{i\beta}^{-1}(\mathbb{R} \setminus \{0\})$, $\overline{U_{i\beta}} = \operatorname{supp}\phi_{i\beta} \subseteq G_{i\beta}$.

References

[Pal66a] Richard S. Palais. Homotopy theory on infinite dimensional manifolds. *Topology*, 1966.

[Pal66b] Richard S. Palais. Lusternik-Schnirelman theory on Banach manifolds. *Topology*, 1966.

[PT88] Richard S. Palais and Chuu-Lian Terng. *Critical point theory and submanifold geometry*. Lecture Notes in Mathematics. Springer-Verlag, 1988.