A NOTE ON THE MORSE INDEX THEOREM FOR GEODESICS BETWEEN SUBMANIFOLDS IN SEMI-RIEMANNIAN GEOMETRY

PAOLO PICCIONE AND DANIEL V. TAUSK

ABSTRACT. The computation of the index of the Hessian of the action functional in semi-Riemannian geometry at geodesics with two variable endpoints is reduced to the case of a *fixed* final endpoint. Using this observation, we give an elementary proof of the Morse Index Theorem for Riemannian geodesics with two variable endpoints, in the spirit of the original Morse's proof. This approach reduces substantially the effort required in the proofs of the Theorem given in [1, 5, 10]. Exactly the same argument works also in the case of timelike geodesics between two submanifolds of a Lorentzian manifold. For the extension to the lightlike Lorentzian case, just minor changes are required and one obtains easily a proof of the focal index theorem presented in [8].

1. INTRODUCTION

A geodesic in a semi-Riemannian manifold $(\mathcal{M}, g)$ is a smooth curve $\gamma : [a, b] \longmapsto \mathcal{M}$ that is a stationary point for the action functional $f(z) = \frac{1}{2}\int_a^b g(\dot{z}, \dot{z})\ \mathrm{d}t$ defined in the set of paths z joining two given points of $\mathcal{M}$. If $(\mathcal{M}, g)$ is Riemannian, i.e., if g is positive definite, given one such critical point γ, the celebrated Morse Index Theorem relates some analytical properties of the second variation of f at γ with some geometrical properties of γ. More precisely, the *index* of Hess_f at γ, that gives the number of *essentially different* directions in which γ can be deformed to obtain a shorter curve, equals the number of conjugate points along γ counted with multiplicity, excluding the endpoints $\gamma(a)$ and $\gamma(b)$.

The Index Theorem opened a very active field of research for both geometers and analysts, and the original result of Morse was successively extended in several directions. Beem and Ehrlich extended the results to the case of timelike Lorentzian geodesics (see [3]) and to the lightlike Lorentzian case ([2, 3]). The case of a Riemannian geodesic with endpoints variable in two submanifolds of $\mathcal{M}$ has been treated by several authors, including Ambrose, Bolton and Kalish, (see [1, 5, 10], see also [17]). Following the approach of Kalish [10], Ehrlich and Kim have then proven in [8] the Morse Index Theorem for lightlike geodesics with endpoints varying on two spacelike submanifolds of a Lorentzian manifold. The case of spacelike geodesics in semi-Riemannian manifolds was treated by Helfer in [9], where an extension of the Index Theorem was proven in terms of the *Maslov* index of a curve, and by the introduction of a notion of *signature* for conjugate points. Edwards extended in [7] the Morse Index Theorem to the case of formally self-adjoint linear systems of ODE's, and Smale proved in [16] a general version of the Index Theorem for strongly elliptic operators on a Riemannian manifold.

The key point in the original Morse's proof of the theorem was the introduction of a function $i : [a,b] \longmapsto I\!\!N$ that gives the index of the form I_t, which is the Hessian Hess_f restricted to the geodesic $\gamma|_{[a,t]}$. Using a suitable subdivision of the interval $[a,b]$ and some geometrical arguments (see [12, 6]) Morse proved that i is non decreasing and left continuous, with discontinuities precisely at the conjugate points, and that the jump of i at each discontinuity point t_0 is given by the value of the multiplicity of the conjugate point $\gamma(t_0)$.

When passing to the case of variable endpoints, i.e., when one admits variations with curves having endpoints varying on two fixed submanifolds P and Q of $\mathcal{M}$, in which case a stationary point of f is a geodesic γ that is orthogonal to P and Q at its endpoints, some obstructions to the use of the original argument of Morse arise, due mainly to the fact that the restricted index form I_t does not detect the influence of the final manifold Q.

Ambrose [1] gave a proof of the Index Theorem that uses the subdivision argument, by introducing a family Q_t of *localized end-manifolds* along γ, constructed with the help of the geodesic flow of the normal bundle of P around $(\gamma(a), \dot\gamma(a))$. This construction leads to technical difficulties (see also [17]), due to the fact that the submanifold Q_t may lose dimension and differentiability. The proof of Bolton [5] also uses a subdivision argument, and it avoids the introduction of the manifolds Q_t, but it employs a restricted index function which is no longer nondecreasing.

The passage to a restricted index function is avoided in Kalish's proof of the Index Theorem in the variable endpoints case (see [10]). In this article, it is given an explicit direct sum decomposition of the space $\mathcal{H}^{(P,Q)} = B \oplus B^c_+ \oplus B^c_-$ of vector fields along γ which are everywhere orthogonal to γ and tangent to P and Q respectively at $\gamma(a)$ and $\gamma(b)$. The index theorem is deduced with a study of the sign of the index form in each of the three spaces; the definition of such decomposition is not very natural, and the remaining calculations are rather involved.

Ehrlich and Kim [8] have adapted Kalish's proof to the case of lightlike Lorentzian geodesics, where a suitable quotient space is used, in analogy with the null Morse Index Theorem of [2, 3].

The aim of this paper is to show that the proof of the Morse Index Theorem for geodesics with two variable endpoints is a simple adaptation of the classical proof for the fixed endpoints case, in the spirit of the original proof of Morse, which is well understood. To this goal, the key observation is that the case of a geodesic with final point varying on a submanifold Q can be deduced immediately from the case of a fixed final endpoint (see Theorem 2.7) by using a natural splitting of the space $\mathcal{H}^{(P,Q)}$. Moreover, we emphasize that the case of causal (nonspacelike) Lorentzian geodesics is essentially analogous to the Riemannian case.

We try to keep all the statements and proofs of the paper at the maximum level of generality; in particular, we present an approach that unifies the Riemannian and the causal Lorentzian case, obtaining a proof of all the results for Riemannian and causal Lorentzian geodesics at the same time. In Remark 2.9, among other things we observe that, in the Lorentzian lightlike case, the use of the quotient bundle employed in [2, 3, 8] is not really essential for the computation of the (non augmented) index, which allows to give an easier statement of the focal index theorem.

It is also important to observe that the result of Theorem 2.7 applies to a great number of situations in semi-Riemannian geometry where the Morse Index Theorem may *not* work, like for instance in the case of spacelike geodesics in stationary Lorentzian manifolds (see Remark 2.10).

2. The Index Theorem

Let $(\mathcal{M}, g)$ be a semi-Riemannian manifold, $m = \dim(\mathcal{M})$, $P \subset \mathcal{M}$ be a smooth submanifold of $\mathcal{M}$ and $\gamma : [a, b] \longmapsto \mathcal{M}$ be a non constant geodesic in $\mathcal{M}$, with $\gamma(a) \in P$ and $\dot\gamma(a) \in T_{\gamma(a)}P^{\perp}$. We will say that γ is spacelike, timelike or lightlike according to $g(\dot\gamma, \dot\gamma)$ positive, negative or zero, respectively; by *causal* we will mean either timelike or lightlike.

Let ∇ denote the Levi–Civita connection of g and let

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

be the curvature tensor of ∇; moreover, for all $p \in P$ and all $n \in T_pP^{\perp}$, let $\mathcal{S}^P_n$ be the second fundamental form of P in the orthogonal direction n, which is the following symmetric bilinear form on T_pP:

$$\mathcal{S}^P_n(v_1, v_2) = g(n, \nabla_{v_1} V_2),$$

where V_2 is any extension of v_2 to a vector field tangent to P. Observe that we are *not* in principle making any non degeneracy assumption on P, but if the metric is non degenerate on T_pP then we can also define a linear map $\mathcal{S}^P_n : T_pP \longmapsto T_pP$ such that $g(\mathcal{S}^P_n(v_1), v_2) = \mathcal{S}^P_n(v_1, v_2)$.

Given a (piecewise) smooth vector field V along γ, we denote by V' the covariant derivative of V along γ; if V is piecewise smooth and $\tau \in [a, b]$, the symbols $V'(\tau^-)$ and $V'(\tau^+)$ will mean respectively the left and right limits of $V'(t)$ as $t \to \tau$.

If $(\mathcal{M}, g)$ is Lorentzian, i.e., if the index of g is 1, and γ is timelike, we have that $T_{\gamma(a)}P$ is spacelike, in the sense that the restriction of g to $T_{\gamma(a)}P$ is positive definite. More in general, the restriction of the metric g to the orthogonal space $\dot\gamma(t)^{\perp}$ is positive definite for all $t \in [a, b]$. If γ is lightlike, the restriction of the metric to the orthogonal space is just positive semi-definite (having a one dimensional kernel spanned by $\dot\gamma(t)$). However, if one assumes that $\dot\gamma(a) \notin T_{\gamma(a)}P$, then again $T_{\gamma(a)}P$ is spacelike.

Let $\tilde{\mathcal{H}}^P$ denote the vector space of all piecewise smooth vector fields V along γ such that $V(a) \in T_{\gamma(a)}P$ and let $\mathcal{H}^P$ be the subspace of $\tilde{\mathcal{H}}^P$ consisting of those V such that $g(V, \dot\gamma) \equiv 0$ and $V(b) = 0$; moreover, let $I^P : \tilde{\mathcal{H}}^P \times \tilde{\mathcal{H}}^P \longmapsto I\!R$ be the symmetric bilinear form given by:

(1)
$$I^P(V, W) = \int_a^b \Big[g(V', W') + g(R(\dot\gamma, V)\,\dot\gamma, W) \Big]\,\mathrm{d}t - \mathcal{S}^P_{\dot\gamma(a)}(V(a), W(a)).$$

Observe that if the submanifold P consists of just one point, the term involving its second fundamental form $\mathcal{S}^P_{\dot\gamma(a)}$ in (1) disappears. In this case we'll write just I instead of I^P.

Integration by parts on $g(V',W')$ gives yet another expression for I^P:

$$
\begin{aligned}
I^P(V,W) = \int_a^b g(R(\dot\gamma,V)\,\dot\gamma - V'',W)\;\mathrm{d}t\, + \\
+\, g(V'(b),W(b)) - g(V'(a),W(a)) - \mathcal{S}^P_{\dot\gamma(a)}(V(a),W(a)) + \\
+ \sum_{i=1}^{N-1} g(V'(t_i^-) - V'(t_i^+), W(t_i)),
\end{aligned}
\tag{2}
$$

where $a = t_0 < t_1 < \ldots < t_N = b$ is a partition of $[a,b]$ such that V is smooth in each interval $[t_i,t_{i+1}]$, $i = 0,1,\ldots,N-1$.

It is well known that γ is a stationary point for the action functional

$$f(z) = \frac{1}{2}\int_a^b g(\dot z,\dot z)\;\mathrm{d}t$$

defined in the set $\Omega_{P,\gamma(b)}$ of all piecewise smooth curves $z:[a,b]\to\mathcal{M}$ joining P and $\gamma(b)$. Under the viewpoint of Calculus of Variations and Global Analysis, the vector space $\mathcal{H}^P$ is a subspace of the *tangent space* of $\Omega_{P,\gamma(b)}$ at γ, and $I^P\big|_{\mathcal{H}^P}$ is the bilinear form given by the *second variation* of f at the stationary point γ. We will be concerned with the *index* of I^P in $\mathcal{H}^P$, defined as follows. If $\mathcal{K}$ is a vector subspace of $\tilde{\mathcal{H}}^P$, then the index $i(I^P,\mathcal{K})$ of I^P in $\mathcal{K}$ is the number:

$$\mathrm{ind}(I^P,\mathcal{K}) = \sup\{\dim(\mathcal{V}):\ \mathcal{V} \text{ subspace of } \mathcal{K} \text{ with } I^P\big|_{\mathcal{V}} < 0\},$$

and we set

$$\mathrm{ind}(I^P) = \mathrm{ind}(I^P,\mathcal{H}^P). \tag{3}$$

The number $\mathrm{ind}(I^P)$ will be called the *Morse Index* of γ.

A Jacobi field along γ is a smooth vector field J satisfying the linear equation $J'' - R(\dot\gamma,J)\,\dot\gamma = 0$. We say that J is a P-Jacobi field if it satisfies in addition:

$$J(a) \in T_{\gamma(a)}P, \tag{4}$$

and

$$g(J'(a),w) + \mathcal{S}^P_{\dot\gamma(a)}(J(a),w) = 0, \quad \text{for all } w \in T_{\gamma(a)}P. \tag{5}$$

If the metric is non degenerate on $T_{\gamma(a)}P$ we can rewrite (5) as

$$J'(a) + \mathcal{S}^P_{\dot\gamma(a)}(J(a)) \in T_{\gamma(a)}P^\perp.$$

In this case, a simple counting argument shows that the dimension of the vector space of P-Jacobi fields along γ is precisely equal to m and that the dimension of P-Jacobi fields satisfying $g(J,\dot\gamma) = 0$ is equal to $m-1$ (for P-Jacobi fields the condition $g(J,\dot\gamma) = 0$ is equivalent to $g(J'(a),\dot\gamma(a)) = 0$). Observe that if P is a point, then a P-Jacobi field is simply a Jacobi field J along γ such that $J(a) = 0$.

Two points $q_0 = \gamma(t_0)$ and $q_1 = \gamma(t_1)$, $t_0,t_1 \in [a,b]$, are said to be *conjugate* along γ if there exists a non null Jacobi field J along γ with $J(t_0) = 0$ and $J(t_1) =$

0. A point $q_0 = \gamma(t_0)$, $t_0 \in]a, b]$ is said to be a *P-focal point* along γ if there exists a non null P-Jacobi field J along γ such that $J(t_0) = 0$; the *geometrical multiplicity* $\mu^P(t_0)$ of a P-focal point $\gamma(t_0)$ is the dimension of the vector space of all P-Jacobi fields along γ that vanish at t_0. If $\gamma(t_0)$ is not P-focal, we set $\mu^P(t_0) = 0$.

It is well known that, if γ is either Riemannian or causal Lorentzian, and $\dot\gamma(a) \in T_{\gamma(a)}P^\perp \setminus T_{\gamma(a)}P$ (see Remark 2.6), then the set of P-focal points along γ is discrete, [1] hence finite. Namely, if $J_1, \dots, J_m$ is a linear basis for the space of P-Jacobi fields along γ and $E_1, \dots, E_m$ is a parallely transported orthogonal basis along γ, then the smooth function $r(t) = \det(g(J_i, E_j))$ has only simple zeroes on $[a, b]$, i.e., zeroes of finite multiplicity, exactly at those points $t_0 \in [a, b]$ such that $\gamma(t_0)$ is a P-focal point along γ (see for instance [13, Ex. 8, p. 299]). Similarly, for all $q_0 = \gamma(t_0)$, the set of points q_1 that are conjugate to q_0 along γ is finite.

We are interested in the kernel of the restriction of I^P to $\mathcal{H}^P$. To this aim, we introduce the spaces $\mathcal{N}$ and $\mathcal{J}_0$ as follows:

$$\mathcal{N} = \Big\{ f\dot\gamma : f : [a, b] \longmapsto I\!R \text{ piecewise smooth, } f(a) = f(b) = 0 \Big\};$$

(6)

$$\mathcal{J}_0 = \Big\{ P\text{-Jacobi fields } J \text{ along } \gamma : J(b) = 0 \Big\}.$$

If γ is lightlike we have $\mathcal{N} \subset \mathcal{H}^P$ and in fact $\mathcal{N}$ is contained in the kernel of I^P in $\mathcal{H}^P$, as follows directly from (1). We now compute this kernel in the case of Riemannian or causal Lorentzian geodesics.

Lemma 2.1. *Let $(\mathcal{M}, g)$ be either Riemannian or Lorentzian; in the latter case assume that γ is causal. The kernel of the restriction of the bilinear form I^P to $\mathcal{H}^P$ is equal to $\mathcal{J}_0$ if $(\mathcal{M}, g)$ is Riemannian or if $(\mathcal{M}, g)$ is Lorentzian and γ is timelike. If γ is lightlike and $\dot\gamma(a) \in T_{\gamma(a)}P^\perp \setminus T_{\gamma(a)}P$, this kernel is equal to $\mathcal{J}_0 \oplus \mathcal{N}$.*

Proof. Observe that a P-Jacobi field which vanishes at some instant on $]a, b]$ is automatically orthogonal to γ, so that we really have $\mathcal{J}_0 \subset \mathcal{H}^P$. If $V \in \mathcal{H}^P$ is in the kernel of (the restriction of) I^P, it follows from (2) and usual techniques of calculus of variations that $V'' - R(\dot\gamma, V)\,\dot\gamma$ is parallel to $\dot\gamma$ and that V satisfies equation (5). Since $V'' - R(\dot\gamma, V)\,\dot\gamma$ is also orthogonal to $\dot\gamma$, it follows that V is a Jacobi field, except for the case where γ is lightlike. In the latter case, we get $V'' - R(\dot\gamma, V)\,\dot\gamma = \varphi\dot\gamma$ for some function φ and therefore $V - f\dot\gamma$ is a Jacobi field, where f satisfies $f'' = \varphi$ and $f(a) = f(b) = 0$. Observe that $\mathcal{J}_0 \cap \mathcal{N} = \{0\}$ because $\dot\gamma(a) \notin T_{\gamma(a)}P$. □

The proof of the Index Theorem for Riemannian or causal Lorentzian geodesics with initial endpoint varying on a submanifold and fixed endpoint is a simple adaptation of the classical Morse proof of the Index Theorem in the case of fixed endpoints (see for instance [6, 12]). For the reader's convenience, we outline briefly such adaptation.

[1] As proved in [9], along a spacelike Lorentzian geodesic, or more in general along a semi-Riemannian geodesic, the conjugate points may accumulate.

We start with the following:

Lemma 2.2. *Let $J_1, J_2, \dots, J_n$ be any family of P-Jacobi fields (not necessarily linearly independent) and $\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_n$ be real piecewise smooth functions on $[a,b]$. Then,*

$$I^P\Big(\sum_{i=1}^n \phi_i \cdot J_i, \sum_{j=1}^n \psi_j \cdot J_j\Big) = \int_a^b g\Big(\sum_{i=1}^n \phi_i' \cdot J_i, \sum_{j=1}^n \psi_j' \cdot J_j\Big)\,\mathrm{d}t + \\ + g\Big(\sum_{i=1}^n \phi_i(b) \cdot J_i'(b), \sum_{j=1}^n \psi_j(b) \cdot J_j(b)\Big). \tag{7}$$

Proof. It is a simple computation that uses the Jacobi equation, formulas (5), (1) and the fact that, for P-Jacobi fields J_i and J_j, one has $g(J_i', J_j) = g(J_i, J_j')$. □

For Riemannian or causal Lorentzian geodesics, the above Lemma gives immediately the following Corollary:

Corollary 2.3. *Let $(\mathcal{M}, g)$ be either Riemannian or Lorentzian; in the latter case assume that γ is causal and that $\dot\gamma(a) \in T_{\gamma(a)}P^\perp \setminus T_{\gamma(a)}P$. Suppose there are no P-focal points along γ. Let $V, J \in \tilde{\mathcal{H}}^P$ be vector fields orthogonal to γ, with J a P-Jacobi field and such that $V(b) = J(b)$. Then $I^P(V,V) \ge I^P(J,J)$. In the Riemannian and timelike Lorentzian case equality holds if and only if $V = J$, and in the lightlike Lorentzian case it holds if and only if $V - J \in \mathcal{N}$.*

Proof. Set $k = \dim(P)$. For $i = 1, \dots, k$, choose Jacobi fields J_i such that the vectors $J_i(a)$ are a basis of $T_{\gamma(a)}P$ and such that $J_i'(a) = -\mathcal{S}^P_{\dot\gamma(a)}(J_i(a))$. For $i = k+1, \dots, m-1$, choose Jacobi fields J_i such that $J_i(a) = 0$ and the vectors $J_i'(a)$ form a basis of $T_{\gamma(a)}P^\perp \cap \dot\gamma(a)^\perp$. If γ is lightlike choose $J_{m-1}'(a) = \dot\gamma(a)$. Then, the J_i's form a basis of the space of P-Jacobi fields orthogonal to γ. Now, we can write $V = \sum_{i=1}^{m-1} f_i J_i$, for piecewise smooth functions f_i.

For, define $\bar J_i = J_i$ for $i = 1, \dots, k$ and $\bar J_i(t) = J_i(t)/(t-a)$, $\bar J_i(a) = J_i'(a)$, for $i = k+1, \dots, m-1$. The absence of P-focal points along γ and the fact that, under the hypothesis that $\dot\gamma(a) \in T_{\gamma(a)}P^\perp \setminus T_{\gamma(a)}P$, $T_{\gamma(a)}\mathcal{M} = T_{\gamma(a)}P \oplus T_{\gamma(a)}P^\perp$, imply that the vectors $\bar J_i(t)$ are a basis for $\dot\gamma(t)^\perp$ for $t \in [a,b]$.

Now, we have $J = \sum_{i=1}^{m-1} c_i J_i$, where $c_i = f_i(b)$. The desired inequality follows directly from the Lemma 2.2 (equality implies that all f_i are constant, except for f_{m-1}, in the lightlike case). □

We give the following definition:

Definition 2.4. A partition $a = t_0 < t_1 < \ldots < t_N = b$ of $[a,b]$ is said to be *normal* if the following conditions are satisfied:

(a) for all $i \ge 1$ and all $t \in]t_i, t_{i+1}]$, the point $\gamma(t)$ is not conjugate to $\gamma(t_i)$ along γ;
(b) for all $t \in]t_0, t_1]$, the point $\gamma(t)$ is not P-focal along γ.

If γ is either Riemannian or causal Lorentzian and $\dot\gamma(a) \in T_{\gamma(a)}P^\perp \setminus T_{\gamma(a)}P$, since the set of P-focal points along γ is finite, it is easy to see that there exists

$\delta > 0$ such that every partition $t_0, \ldots, t_N$ of $[a,b]$ with $t_{i+1} - t_i \le \delta$ for all i is normal. Namely, in order to (b) be satisfied, one can take δ to be the Lebesgue number of a covering of γ by *totally normal neighborhoods* (see Ref. [6]).

Given a normal partition, we define the following two subspaces of $\mathcal{H}^P$:

$$\begin{aligned} &\mathcal{H}_0^P = \Big\{ V \in \mathcal{H}^P : V(t_i) = 0, \ \forall\, i \ge 1 \Big\}; \\ &\mathcal{H}_J^P = \Big\{ V \in \mathcal{H}^P : V\big|_{[t_i,t_{i+1}]} \text{ is Jacobi } \forall\, i \ge 1, \text{ and } V\big|_{[t_0,t_1]} \text{ is } P\text{-Jacobi} \Big\}. \end{aligned} \tag{8}$$

Observe that there exists an isomorphism:

$$\phi : \mathcal{H}_J^P \longmapsto \bigoplus_{i=1}^{N-1} \dot\gamma(t_i)^\perp \tag{9}$$

given by setting $\phi(V) = \big(V(t_1), V(t_2), \ldots, V(t_{N-1})\big)$. Namely, since $\gamma(t_i)$ and $\gamma(t_{i+1})$ are non conjugate for $i \ge 1$, then $V\big|_{[t_i,t_{i+1}]}$ is uniquely determined by the boundary values $V(t_i)$ and $V(t_{i+1})$; moreover, since $\gamma(t_1)$ is not P-focal, then $V\big|_{[t_0,t_1]}$ is uniquely determined by the value $V(t_1)$.

This shows that $\mathcal{H}_0^P \cap \mathcal{H}_J^P = \{0\}$ and that $\mathcal{H}_0^P + \mathcal{H}_J^P = \mathcal{H}^P$, hence we have:

$$\mathcal{H}_0^P \oplus \mathcal{H}_J^P = \mathcal{H}^P. \tag{10}$$

We are ready to prove the Morse Index Theorem for Riemannian or causal Lorentzian geodesics with variable initial point:

Theorem 2.5. *Let $(\mathcal{M}, g)$ be either Riemannian or Lorentzian, P a smooth submanifold of $\mathcal{M}$ and $\gamma : [a,b] \longmapsto \mathcal{M}$ a geodesic (causal, if (M,g) is Lorentzian) with $\gamma(a) \in P$ and $\dot\gamma(a) \in T_{\gamma(a)}P^\perp \setminus T_{\gamma(a)}P$. Then,* $\mathrm{ind}(I^P) = \sum_{t_0 \in\,]a,b[} \mu^P(t_0) < +\infty$.

Proof. For $[\alpha,\beta] \subset [a,b]$, let $I_{[\alpha,\beta]}$ be the bilinear form (1) for the restricted geodesic $\gamma|_{[\alpha,\beta]}$ (omitting the term involving $\mathcal{S}^P_{\dot\gamma(a)}$); if $\alpha = a$, then we set $I^P_{[\alpha,\beta]}$ to be just the bilinear form (1) for the restricted geodesic $\gamma|_{[\alpha,\beta]}$. For $t \in]a,b]$ let's write $i(t) = \mathrm{ind}(I^P_{[a,t]})$; observe that $i(b) = \mathrm{ind}(I^P)$. The function $i : [a,b] \longmapsto I\!N$ is non decreasing (if $t < s$ we can regard $I^P_{[a,t]}$ as a restriction of $I^P_{[a,s]}$, by extending vector fields on $[a,t]$ to $[a,s]$ defining them to be zero on $[t,s]$).

We show that $i(t)$ is piecewise constant and left-continuous on $[a,b]$, and that $i(t^+) - i(t^-) = \mu^P(t)$ for all $t \in]a,b[$.

Let $t \in]a,b]$ be fixed and choose a normal partition $t_0, \ldots, t_N$ of $[a,b]$ such that $t \in]t_i, t_{i+1}[$ for some $i \ge 1$ (we allow $t = t_{i+1}$ if $t = b$ and we set $i = N-1$). Let's denote by $\mathcal{H}_J^P([a,t])$ and $\mathcal{H}_0^P([a,t])$ the spaces defined in (8), replacing the interval $[a,b]$ by $[a,t]$ (and using the normal partition $t_0, \ldots, t_i, t$ of $[a,t]$).

We observe that the direct sum (10) (for the interval $[a,t]$) is $I^P_{[a,t]}$-orthogonal, i.e., $I^P_{[a,t]}(V_0, V_J) = 0$ for all $V_0 \in \mathcal{H}_0^P([a,t])$ and $V_J \in \mathcal{H}_J^P([a,t])$, which follows directly from (2).

Next, we claim that $I^P_{[a,t]}\big|_{\mathcal{H}^P_0([a,t])} \geq 0$. To check this, just observe that for $V \in \mathcal{H}^P_0([a,t])$ we have:

$$I^P_{[a,t]}(V,V) = I^P_{[t_0,t_1]}(V,V) + \sum_{j=1}^{i-1} I_{[t_j,t_{j+1}]}(V,V) + I_{[t_i,t]}(V,V).$$

The claim now follows from Corollary 2.3, by taking the Jacobi field $J = 0$.

It follows that $i(t) = \mathrm{ind}(I^P_{[a,t]}) = \mathrm{ind}(I^P_{[a,t]}, \mathcal{H}^P_J([a,t]))$; Observe that as in (9) the space $\mathcal{H}^P_J([a,t])$ is isomorphic to the space $\mathcal{H}_*$ defined by:

$$\mathcal{H}_* = \bigoplus_{j=1}^{i} \dot\gamma(t_j)^\perp,$$

and we'll call this isomorphism $\phi_t : \mathcal{H}^P_J([a,t]) \longmapsto \mathcal{H}_*$.

If $s \in]a,b]$ is sufficiently close to t or, more precisely, if $s \in]t_i, t_{i+1}]$, the arguments above can be repeated by replacing t with s (observe, in particular, that the space $\mathcal{H}_*$ obtained will be the same). We can use the isomorphism ϕ_s between $\mathcal{H}^P_J([a,s])$ and $\mathcal{H}_*$ to define a symmetric bilinear form I_s on $\mathcal{H}_*$ corresponding to $I^P_{[a,s]}$. Clearly $i(s) = \mathrm{ind}(I_s)$.

We have now a one parameter family I_s of symmetric bilinear forms on the (fixed) finite dimensional space $\mathcal{H}_*$ and it's not difficult to see that I_s depends continuously (actually, smoothly) on s.[2]

Let's consider the decomposition $\mathcal{H}_* = \mathcal{H}^+_* \oplus \mathcal{H}^-_* \oplus \mathcal{H}^0_*$, where I_t is positive (respectively, negative) definite on $\mathcal{H}^+_*$ (respectively, $\mathcal{H}^-_*$) and $\mathcal{H}^0_*$ is the kernel of I_t. We can also assume that this decomposition is I_t-orthogonal (this is just the Sylvester inertia Theorem). The dimension of $\mathcal{H}^-_*$ is $i(t)$.

Since the decomposition $\mathcal{H}^P_0([a,t]) \oplus \mathcal{H}^P_J([a,t])$ is orthogonal with respect to $I^P_{[a,t]}$, we know that the kernel of the restriction of $I^P_{[a,t]}$ to $\mathcal{H}^P_J([a,t])$ (which corresponds to $\mathcal{H}^0_*$ by the isomorphism ϕ_t) is just the intersection of $\mathcal{H}^P_J([a,t])$ and the kernel of $I^P_{[a,t]}$, the last one being given by Lemma 2.1. Observe that $\mathcal{J}_0 \subset \mathcal{H}^P_J([a,t])$ and denote by $\mathcal{J}_*$ the subspace of $\mathcal{H}_*$ which corresponds to $\mathcal{J}_0$, i.e., $\mathcal{J}_* = \phi_t(\mathcal{J}_0)$. In the lightlike Lorentzian case, write also $\mathcal{N}_* = \phi_t(\mathcal{N} \cap \mathcal{H}^P_J([a,t]))$.

Observe that $\mathcal{N}_*$ is just the set of i-tuples of vectors which are parallel to $\dot\gamma$, so that $\mathcal{N}_*$ doesn't change if we replace t by s in its definition, and therefore $\mathcal{N}_*$ is also contained in the kernel of I_s.

We see now that $\mathcal{H}^0_* = \mathcal{J}_*$, except for the lightlike Lorentzian case where $\mathcal{H}^0_* = \mathcal{J}_* \oplus \mathcal{N}_*$. The dimension of $\mathcal{J}_*$ is just the multiplicity $\mu^P(t)$ of $\gamma(t)$ as a P-focal point.

By the continuous dependence of I_s on s we see that for $\epsilon > 0$ sufficiently small and $s \in]t-\epsilon, t+\epsilon[$, I_s is negative definite on $\mathcal{H}^-_*$ so that $i(s) \geq i(t)$. For $s \in]t-\epsilon, t]$ we have also $i(s) \leq i(t)$ so that $i(s) = i(t)$, i.e., i is constant on

[2]To prove this fact, one uses equation (2) to write a expression for I^P on piecewise Jacobi fields and observes that the integral vanishes. Thus, formula (2) reduces to a finite sum, and the conclusion follows from the theorem on smooth dependence on the initial data for the solutions of the Jacobi differential equation.

$]t-\epsilon,t]$. This finishes the proof that i is left continuous. From now on we suppose $t<b$.

The same continuity argument show that for some $\epsilon>0$, we have that I_s is positive definite on $\mathcal{H}_*^+$ for $s\in[t,t+\epsilon[$ (and positive semi-definite on $\mathcal{H}_*\oplus\mathcal{N}_*$ for γ lightlike), so that $i(s)$ is bounded above by the codimension of $\mathcal{H}_*^+$ (or $\mathcal{H}_*^+\oplus\mathcal{N}_*$, respectively). If $\gamma(t)$ is not a P-focal point this codimension equals $i(t)$ so that $i(s)=i(t)$ for $s\in]t-\epsilon,t+\epsilon[$.

Finally, if $\gamma(t)$ is a P-focal point, by the above argument we only obtain the inequality $i(s)\le i(t)+\mu_\gamma^P(t)$. We'll show below that for $s\in]t,t_{i+1}]$ and for $V=(v_1,\ldots,v_i)\in\mathcal{H}_*$ we have $I_s(V,V)\le I_t(V,V)$, the inequality being strict if $v_i\ne 0$ (or if v_i is not parallel to $\dot\gamma$, in case γ is lightlike). But this hypothesis on v_i holds if $V\in\mathcal{J}_*$ and $V\ne 0$, observing that the corresponding vector field $\phi_t^{-1}(V)$ on $\mathcal{H}_J^P([a,t])$ is an unbroken Jacobi field. We conclude then that $I_s(V,V)<0$ for nonzero $V\in\mathcal{J}_*$ and hence for all nonzero $V\in\mathcal{H}_*^-\oplus\mathcal{J}_*$, which implies that I_s is negative definite on this space and $i(s)\ge i(t)+\mu^P(t)$.

We are now left with the proof of the inequality $I_s(V,V)\le I_t(V,V)$. Towards this goal, let $V_1\in\mathcal{H}_J^P([a,t])$ and $V_2\in\mathcal{H}_J^P([a,s])$ be the vector fields corresponding to $V\in\mathcal{H}_*$, i.e., $V_1=\phi_t^{-1}(V)$ and $V_2=\phi_s^{-1}(V)$. Extend V_1 to zero on $[t,s]$. Then, $I_t(V,V)=I^P_{[a,s]}(V_1,V_1)$ and $I_s(V,V)=I^P_{[a,s]}(V_2,V_2)$. The vector fields V_1 and V_2 differ at the most in the interval $[t_i,s]$. The restriction of V_1 to $[t_i,t]$ is the only Jacobi field such that $V_1(t_i)=v_i$ and $V_1(t)=0$, while the restriction of V_2 to $[t_i,s]$ is the only Jacobi field such that $V_2(t_i)=v_i$ and $V_2(s)=0$. We have:

$$I_t(V,V)-I_s(V,V)=I_{[t_i,s]}(V_1,V_1)-I_{[t_i,s]}(V_2,V_2).$$

We now apply Corollary 2.3 to the geodesic $\gamma|_{[t_i,s]}$ (with starting and ending points interchanged), for the Jacobi field V_2, vector field V_1 and submanifold equal to the point $\{\gamma(s)\}$. For the strict inequality we need the hypothesis that $v_i\ne 0$ (respectively, v_i not parallel to $\dot\gamma$, in the lightlike Lorentzian case), since this implies that V_1 is not Jacobi in $[t_i,s]$ (respectively, does not differ from a Jacobi field by a multiple of $\dot\gamma$, in the lightlike Lorentzian case). This concludes the proof. □

Remark 2.6. If (M,g) is Lorentzian, then the case that $\dot\gamma(a)\in T_{\gamma(a)}P\cap T_{\gamma(a)}P^\perp$ may happen only when γ is lightlike and P is a degenerate submanifold at $\gamma(a)$, i.e., the restriction of g to $T_{\gamma(a)}P$ is degenerate. Observe that in this case the thesis of Theorem 2.5 is clearly false. For instance, if $\mathcal{M}=I\!R^2$ and g is the flat Minkowski metric $\mathrm{d}x^2-\mathrm{d}t^2$, P is the diagonal $x=t$ and γ is any segment contained in P, then every point of γ is P-focal.

We now want to extend the Morse Index Theorem to the case of two variable endpoints. To this end, we now assume that P and Q are smooth submanifolds of $\mathcal{M}$, and that $\gamma:[a,b]\longmapsto\mathcal{M}$ is a geodesic with $\gamma(a)\in P$, $\dot\gamma(a)\in T_{\gamma(a)}P^\perp$, $\gamma(b)\in Q$ and $\dot\gamma(b)\in T_{\gamma(b)}Q^\perp$.

We denote by $\mathcal{H}^{(P,Q)}$ the vector space of all piecewise smooth vector fields V along γ, with $g(V,\dot\gamma)\equiv 0$, $V(a)\in T_{\gamma(a)}P$ and $V(b)\in T_{\gamma(b)}Q$. Moreover, we will consider the symmetric bilinear form $I^{(P,Q)}$ on $\mathcal{H}^{(P,Q)}$, given by:

$$I^{(P,Q)}(V,W) = I^P(V,W) + \mathcal{S}^Q_{\dot\gamma(b)}(V(b),W(b)). \tag{11}$$

Let $\mathcal{J}^Q$ denote the subspace of $\mathcal{H}^{(P,Q)}$ consisting of P-Jacobi fields, and let $\mathcal{A}$ be the symmetric bilinear form on $\mathcal{J}^Q$ obtained by the restriction of $I^{(P,Q)}$. Then, it is easily computed from (1) using integration by parts:

$$\mathcal{A}(J_1,J_2) = \mathcal{S}^Q_{\dot\gamma(b)}(J_1(b),J_2(b)) + g(J_1'(b),J_2(b)), \quad J_1,J_2 \in \mathcal{J}^Q.$$

Moreover, for $t \in [a,b]$, we introduce the space $\mathcal{J}[t]$:

$$\mathcal{J}[t] = \Big\{ J(t) : J \text{ is } P\text{-Jacobi} \Big\} \subset T_{\gamma(t)}\mathcal{M};$$

observe that, for $t \in \,]a,b]$, $\gamma(t)$ is not P-focal if and only if $\mathcal{J}[t] = T_{\gamma(t)}\mathcal{M}$.

We can now state and prove the following extension of the Morse Index Theorem for geodesics between submanifolds:

Theorem 2.7. *Let $(\mathcal{M},g)$ be a semi-Riemannian manifold, P,Q submanifolds of $\mathcal{M}$ and $\gamma : [a,b] \longmapsto \mathcal{M}$ be a geodesic such that $\gamma(a) \in P$, $\dot\gamma(a) \in T_{\gamma(a)}P^\perp$, $\gamma(b) \in Q$ and $\dot\gamma(b) \in T_{\gamma(b)}Q^\perp$. Assume that $\mathcal{J}[b] \supset T_{\gamma(b)}Q$. Let $\mathcal{V}$ be a subspace of $\mathcal{H}^{(P,Q)}$ that contains the space $\mathcal{J}^Q$ of P-Jacobi fields along γ in $\mathcal{H}^{(P,Q)}$. Then, $\mathrm{ind}(I^{(P,Q)},\mathcal{V}) = \mathrm{ind}(I^P,\mathcal{H}^P \cap \mathcal{V}) + \mathrm{ind}(\mathcal{A},\mathcal{J})$.*

Proof. The space $\mathcal{H}^P$ is given by the subspace of $\mathcal{H}^{(P,Q)}$ consisting of those vector fields V such that $V(b) = 0$; moreover, the restriction of $I^{(P,Q)}$ to $\mathcal{H}^P$ is precisely I^P. Defining $\mathcal{J}_0$ as in formula (6), let $\mathcal{J}_1$ be any subspace of $\mathcal{J}^Q$ such that $\mathcal{J}^Q = \mathcal{J}_0 \oplus \mathcal{J}_1$. Clearly, $\mathcal{H}^{(P,Q)} = \mathcal{H}^P \oplus \mathcal{J}_1$, because $\mathcal{J}[b] \supset T_{\gamma(b)}Q$; moreover, from (11) it follows immediately that this decomposition is $I^{(P,Q)}$-orthogonal, i.e., $I^{(P,Q)}(V,J) = 0$ for all $V \in \mathcal{H}^P$ and all $J \in \mathcal{J}_1$. Since $\mathcal{V}$ contains $\mathcal{J}_1$, then $\mathcal{V} = (\mathcal{V} \cap \mathcal{H}^P) \oplus \mathcal{J}_1$. Hence, $\mathrm{ind}(I^{(P,Q)},\mathcal{V}) = \mathrm{ind}(I^P,\mathcal{H}^P \cap \mathcal{V}) + \mathrm{ind}(\mathcal{A},\mathcal{J}_1)$. To conclude the proof, we simply observe that $\mathrm{ind}(\mathcal{A},\mathcal{J}_1) = \mathrm{ind}(\mathcal{A},\mathcal{J})$, because $\mathcal{J}_0 \subset \mathrm{Ker}(\mathcal{A})$. □

Remark 2.8. One can consider suitable Hilbert space completions $\bar{\mathcal{H}}^P$ and $\bar{\mathcal{H}}^{(P,Q)}$ of the spaces $\mathcal{H}^P$ and $\mathcal{H}^{(P,Q)}$ with respect to an H^1-Sobolev norm. Then, the bilinear forms I^P and $I^{(P,Q)}$ extend uniquely to bounded symmetric bilinear forms on these Hilbert spaces. Observe that a bounded symmetric bilinear form on a Hilbert space and its restriction to any dense subspace have the same index. Using a Hilbert space approach, Theorem 2.5 can be proven alternatively by means of the spectral theory for compact self-adjoint operators (see [11, Theorem 5.9.3] for an idea of such a proof).

Remark 2.9. If $(\mathcal{M},g)$ is Riemannian and $\mathcal{V} = \mathcal{H}^{(P,Q)}$, then Theorems 2.5 and 2.7 give as a particular case the Index Theorem of [10, p. 342] and the older versions of the Morse Index Theorem presented in [1, 5]. In [8] it was briefly mentioned the fact that results analogous to the Riemannian case could apply to the Lorentzian timelike case. As to the lightlike case, in References [2, 3, 8] the authors consider the index of $I^{(P,Q)}$ in the quotient space $\mathcal{H}^{(P,Q)}/\mathcal{N}$ (recall formula (6)); in this situation, $\mathcal{N}$ is contained in the kernel of $I^{(P,Q)}$. By a simple linear algebra argument one proves that the index of a bilinear form in a quotient space by a subspace of its kernel is the same as the index of the form in the original space. Hence, Theorems 2.5 and 2.7 generalize the results of [2, 3, 8].

Remark 2.10. The result of Theorem 2.7 becomes significant when the subspace $\mathcal{V}$ of $\mathcal{H}^{(P,Q)}$ is chosen in such a way that $\mathrm{ind}(I^P, \mathcal{H}^P \cap \mathcal{V})$ is finite; observe that $\mathrm{ind}(\mathcal{A}, \mathcal{J})$ is always finite. If one considers geodesics in semi-Riemannian manifolds with metric of index greater or equal to 2, or spacelike geodesics in Lorentzian manifolds, then $\mathrm{ind}(I^P, \mathcal{H}^P)$ is in general infinite (see Refs. [4, 9] for further results in this direction). Nevertheless, the restriction to suitable subspaces may yield the finiteness of the index, and, possibly, weaker versions of the Morse Index Theorem may apply. For instance (see Ref. [15]), let's consider the case of a stationary Lorentzian manifold $(\mathcal{M}, g)$, i.e., a Lorentzian manifold endowed with a timelike Killing vector field Y. Let γ be a spacelike geodesic; we consider for simplicity the case that the initial manifold P reduces to a point. The Killing vector field Y induces the conservation law $g(\dot\gamma, Y) \equiv C_\gamma$ for all geodesic γ; then, one can consider only variations γ_s of γ such that $g(\dot\gamma_s, Y) \equiv C_s$, and the corresponding variational field $V = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\gamma_s$ belongs to the space:

$$\mathcal{V} = \Big\{ V : \exists\, C_V \in I\!R \text{ such that } g(V', Y) - g(V, Y') \equiv C_V \Big\}.$$

It is a simple observation that $\mathcal{V}$ contains all the Jacobi fields along γ; moreover, using the Sobolev Embedding Theorem one proves that the bilinear form I^P is given by a self-adjoint operator T on the closure of $\mathcal{V} \cap \mathcal{H}^P$ in a suitable Sobolev space completion of $\mathcal{H}^P$, where T is a *compact perturbation* of the identity (see Ref. [11]). Thus, $\mathcal{V} \cap \mathcal{H}^{(P,Q)}$ satisfies the hypothesis of Theorem 2.7 and it is such that $\mathrm{ind}(I^P, \mathcal{V} \cap \mathcal{H}^P)$ is finite. The question of whether such index equals the geometrical index of γ remains still unanswered.

REFERENCES

[1] W. Ambrose, *The Index Theorem in Riemannian Geometry*, Ann. Math. **73**, vol. 1 (1961), 49–86.

[2] J. Beem, P. Ehrlich, *A Morse Index Theorem for Null Geodesics*, Duke Math. J. **46** (1979), 561–569.

[3] J. K. Beem, P. E. Ehrlich, K. L. Easly, *Global Lorentzian Geometry*, Marcel Dekker, Inc., New York and Basel, 1996.

[4] V. Benci, F. Giannoni, A. Masiello, *Some Properties of the Spectral Flow in Semiriemannian Geometry*, J. Geom. Phys. **27** (1998), 267–280.

[5] J. Bolton, *The Morse Index Theorem in the case of Two Variable Endpoints*, J. Diff. Geom. **12** (1977), 567–581.

[6] M. do Carmo, *Riemannian Geometry*, Birkhäuser, Boston, 1992.

[7] H. M. Edwards, *A Generalized Sturm Theorem*, Ann. of Math. **80** (1964), 22–57.

[8] P. E. Ehrlich, S. Kim, *A Focal Index Theorem for Null Geodesics*, J. Geom. Phys. **6**, n. 4 (1989), 657–670.

[9] A. D. Helfer, *Conjugate Points on Spacelike Geodesics or Pseudo-Self-Adjoint Morse-Sturm-Liouville Systems*, Pacific J. Math. **164**, n. 2 (1994), 321–340.

[10] D. Kalish, *The Morse Index Theorem where the Ends are Submanifolds*, Trans. Am. Math. Soc. **308**, n. 1 (1988), 341–348.

[11] A. Masiello, *Variational Methods in Lorentzian Geometry*, Pitman Research Notes in Mathematics **309**, Longman, London 1994.
[12] J. Milnor, *Morse Theory*, Princeton Univ. Press, Princeton, 1969.
[13] B. O'Neill, *Semi–Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.
[14] L. N. Patterson, *On the Index Theorem*, Amer. J. Math. **85** (1963), 271–297.
[15] F. Giannoni, P. Piccione, *An Intrinsic Approach to the Geodesical Connectedness of Stationary Lorentzian Manifolds*, Commun. An. Geom. **7**, n. 1 (1999), 157–197.
[16] S. Smale, *On the Morse Index Theorem*, J. Math. Mech. **14** (1965), 1049–1056.
[17] T. Takahashi, *Correction to "The Index Theorem in Riemannian Geometry" by W. Ambrose*, Ann. of Math. **80** (1964), 538–541.

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE SÃO PAULO, BRAZIL
E-mail address: piccione@ime.usp.br
URL: http://www.ime.usp.br/~piccione

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE SÃO PAULO, BRAZIL
E-mail address: tausk@ime.usp.br