O teorema de Gallai sobre caminhos disjuntos

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O teorema de Gallai[1] é uma generalização natural do teorema dos emparelhamentos máximos (Teorema 2.2.3 em Diestel[2]). Ele é a base de uma nova demonstração do teorema de Mader (Teorema 3.4.1 em Diestel) recentemente descoberta por Alex Schrijver (*A short proof of Mader's $\mathcal{S}$-paths theorem*)[3].

Estou tratando do assunto como se ele fosse a seção 2.4 do capítulo 2 do livro de Diestel (se bem que a seção deveria estar entre a 2.2 e a 2.3). A propósito, seria muito conveniente que Diestel substituisse a discussão no final da seção 2.2 (página 38) pelo seguinte corolário do Teorema 2.2.3:

Corollary 2.2.4 *Every graph G has a matching M and a disjoint collection $\mathcal{Y}$ of subsets of $V(G)$ such that no edge links two different members of $\mathcal{Y}$ and* $|M| = |V(G) \setminus \bigcup \mathcal{Y}| + \sum_{Y \in \mathcal{Y}} \lfloor \frac{1}{2}|Y| \rfloor$.

2.4 Gallai's theorem

Let A be a set of vertices of a graph G; we shall regard A as fixed throughout this section. An *A-path* is any path[4] whose ends are two distinct vertices in A and whose internal vertices are in $V(G) \setminus A$.[5] *A-path*

A collection $\mathcal{P}$ of A-paths is *disjoint* if no vertex of the graph belongs to more than one member of $\mathcal{P}$. A matching, for example, is essentially the same thing as a disjoint collection of $V(G)$-paths. *disjoint collection*

Our problem in this section is the characterization of maximum disjoint collections of A-paths. This will be done in terms of splits[6]. A *split* of G is just a disjoint collection of subsets of $V(G)$. The *A-width* of a split $\mathcal{Y}$ is the number *split* *width of a split*

$$\mathrm{wd}_A(\mathcal{Y}) := |\overline{\bigcup \mathcal{Y}}| + \sum_{Y \in \mathcal{Y}} \left\lfloor \frac{|Y \cap A|}{2} \right\rfloor,$$

[1] Tibor Gallai, Maximum-minimum Sätze und verallgemeinerte Faktoren von Graphen, *Acta Mathematica Academiae Scientiarum Hungaricae* 12 (1961), 131–173

[2] *Graph Theory*, 2nd. edition, Springer, 1999

[3] tomei conhecimento por intermédio de José Coelho de Pina Jr.

[4] by definition, paths have no repreated vertices

[5] we could drop the requirement that no internal vertex be in A, but this would have no practical consequence

[6] this term is not standard

where $\overline{U}$ denotes $V(G) \setminus U$. A split $\mathcal{Y}$ is *strong* if no edge of G links two different members of $\mathcal{Y}$. *strong split*

Lemma 2.4.1 *For any disjoint collection $\mathcal{P}$ of A-paths and any strong split $\mathcal{Y}$,*

$$|\mathcal{P}| \leq \mathrm{wd}_A(\mathcal{Y})\,.$$

Proof. Let $X := \overline{\bigcup \mathcal{Y}}$. Each A-path has a vertex in X or at least two vertices (its terminal vertices) in $Y \cap A$ for some Y in $\mathcal{Y}$. With each P in $\mathcal{P}$, we may associate *(1)* a vertex in $V(P) \cap X$ if this set is not empty or *(2)* a pair of vertices in $A \cap Y$, where Y is the appropriate member of $\mathcal{Y}$. Each vertex in X will have been associated with at most one member of $\mathcal{P}$, since $\mathcal{P}$ is disjoint. On the other hand, in each set of the form $Y \cap A$ a disjoint collection of pairs of vertices will have been defined; each of these pairs will have been associated with a different member of $\mathcal{P}$, since $\mathcal{P}$ is disjoint. The inequality $|\mathcal{P}| \leq \mathrm{wd}_A(\mathcal{Y})$ follows. □

The case $A = V(G)$ of the lemma has been studied in section 2.2: it amounts to the inequality $|M| \leq \mathrm{wd}_{V(G)}(\mathcal{Y})$, valid for any matching M and any strong split $\mathcal{Y}$.

Theorem 2.4.2 (Gallai 1961) *There exists a disjoint collection $\mathcal{P}$ of A-paths and a strong split $\mathcal{Y}$ such that* [3.4.1]

$$|\mathcal{P}| = \mathrm{wd}_A(\mathcal{Y})\,.$$

In other words, the A-width of a minimum-width strong split is equal to the maximum number of disjoint A-paths. This is a generalization of Theorem 2.2.3. That theorem asserts there exists a matching M and a strong split $\mathcal{Y}$ such that $|M| = \mathrm{wd}_{V(G)}(\mathcal{Y})$.

Proof of Theorem 2.4.2. Let G' and G'' be two mutually disjoint copies of G. For each vertex v of G, let v' and v'' denote the corresponding vertices in G' and G'' respectively. Let A' and A'' be the copies of A in G' and G''. Let $V := V(G)$, $V' := V(G')$, and $V'' := V(G'')$. A', A''

Let E be the set of all pairs $v'v''$ with v a vertex of G and let C the set of all pairs of the form $v'w''$ such that vw is an edge of G. Now define E

$$H := ((G' \cup G'') + E + C) - A''\,.$$

Note that for each edge vw of G the vertices v', w', v'', and w'' are pairwise adjacent in H. If $v'w'$ is an edge of H then so are $v'w''$ and $v''w'$ (unless $w'' \in A''$ or $v'' \in A''$), and if $v'w''$ is an edge of H with $v \neq w$ then also $v'w'$ and $v''w''$ are edges of H (unless $v'' \in A''$).

Let $\mathcal{Y}$ be a minimum-width strong split of H. Theorem 2.2.3 guarantees that (2.2.3)

$$|M| = \mathrm{wd}_{V(H)}(\mathcal{Y})$$

for some matching M in H. Choose the split $\mathcal{Y}$, among all those having minimum $M, \mathcal{Y}$

width, so that $\bigcup \mathcal{Y}$ is maximal. Then, for each v in $V \setminus A$,

$$\text{vertices } v' \text{ and } v'' \text{ are either both in } X \text{ or both not in } X, \tag{2.1}$$

where $X := \overline{\bigcup \mathcal{Y}}$. In order to prove this claim, suppose for a while that there exists a vertex v in $V \setminus A$ such that $v' \in X$ but $v'' \notin X$. Then $v'' \in Y$ for some Y in $\mathcal{Y}$. Now let $\tilde{Y} := Y \cup \{v'\}$ and $\tilde{\mathcal{Y}} := (\mathcal{Y} \setminus \{Y\}) \cup \{\tilde{Y}\}$. Observe that

$$\mathrm{wd}_{V(H)}(\tilde{\mathcal{Y}}) \leq \mathrm{wd}_{V(H)}(\mathcal{Y})\,.$$

Hence, the maximality of $\bigcup \mathcal{Y}$ implies that the split $\tilde{\mathcal{Y}}$ is not strong, i.e., H has an edge $v'w'$ with w' in $V' \setminus (X \cup Y)$ or an edge $v'w''$ with w'' in $V'' \setminus (X \cup Y)$. In the first case, $v''w'$ is an edge of H and it connects Y to some other member of $\mathcal{Y}$, which is impossible because $\mathcal{Y}$ is strong. In the second case, $v''w''$ is an edge of H and it connects Y to some other member of $\mathcal{Y}$, which is again impossible. This contradiction proves claim (2.1) in case $v' \in X$. The proof in case $v'' \in X$ is similar.

Let $X' := X \cap V'$ and $X'' := X \cap V''$. By virtue of (2.1), $|X' \setminus A'| = |X''|$. Hence $|X| = |X' \cap A'| + 2|X' \setminus A'|$. X', X''

For each Y in $\mathcal{Y}$, let $Y' := Y \cap V'$ and $Y'' := Y \cap V''$. For each v in $V \setminus A$, by virtue of (2.1), either Y contains both v' and v'' or none of these two vertices. Hence, $|Y' \setminus A'| = |Y''|$, and therefore $|Y| = |Y' \cap A'| + 2|Y' \setminus A'|$. The collection $\mathcal{Y}'$ of all Y' is a strong split of G' and Y', Y'' $\mathcal{Y}'$

$$\mathrm{wd}_{A'}(\mathcal{Y}') = \mathrm{wd}_{V(H)}(\mathcal{Y}) - |V' \setminus A'|\,.$$

Here is the proof of this equality, with k denoting the number of Y in $\mathcal{Y}$ for which $|Y|$ is odd, and therefore also the number of Y' in $\mathcal{Y}'$ for which $|Y' \cap A'|$ is odd:

$$\begin{aligned}
\mathrm{wd}_{V(H)}(\mathcal{Y}) &= |X| - \frac{k}{2} + \sum_{Y \in \mathcal{Y}} \frac{|Y|}{2} \\
&= |X' \cap A'| + 2|X' \setminus A'| - \frac{k}{2} + \sum_{Y' \in \mathcal{Y}'} \frac{|Y' \cap A'| + 2|Y' \setminus A'|}{2} \\
&= |X'| + |X' \setminus A'| - \frac{k}{2} + \sum_{Y'} \frac{|Y' \cap A'|}{2} + \sum_{Y'} |Y' \setminus A'| \\
&= |X'| + \sum_{Y'} \left\lfloor \frac{|Y' \cap A'|}{2} \right\rfloor + |V' \setminus A'| \\
&= \mathrm{wd}_{A'}(\mathcal{Y}') + |V' \setminus A'|\,,
\end{aligned}$$

Now we must convert matching M into a large disjoint collection of A'-paths. Let E' be the set of edges in E incident to $V' \setminus A'$; of course E' is a matching in H with exactly $|V' \setminus A'|$ edges. Since $|M| = \mathrm{wd}_{V(H)}(\mathcal{Y}) = \mathrm{wd}_{A'}(\mathcal{Y}') + |E'|$, the graph $H[E' \cup M]$ has at least $\mathrm{wd}_{A'}(\mathcal{Y}')$ components with more edges in M than in E'. Each such component is a path whose first and last edges are in M. The ends of such path must be both in A'; this is only possible if its first and last edges have both ends in

V' and therefore each of its edges in M has both ends in V' or both in V''. Hence, each of these paths has the form

$$v'_0 \; v'_1 \; v''_1 \; v''_2 \; v'_2 \; v'_3 \; v''_3 \; v''_4 \; \ldots \; v''_{k-1} \; v''_k \; v'_k \; v'_{k+1} \, ,$$

where v_0 and v_{k+1} are in A and $v_1, \ldots, v_k$ are in $V \setminus A$. Of course

$$v'_0 \; v'_1 \; v'_2 \; v'_3 \; v'_4 \; \ldots \; v'_k \; v'_{k+1}$$

is an A'-path in G'. Moreover, such paths are pairwise disjoint. So, we have a disjoint collection of

$$\mathrm{wd}_{A'}(\mathcal{Y}')$$

A'-paths in G'. Since G' is isomorphic to G, our result follows. □

An immediate consequence of Lemma 2.4.1 and Theorem 2.4.2 is the minimax identity $\max |\mathcal{P}| = \min \mathrm{wd}_A(\mathcal{Y})$, where $\max$ is taken over all disjoint collections $\mathcal{P}$ of A-paths and $\min$ is taken over all strong splits $\mathcal{Y}$.

Exercises

1. Let G' and G'' be two mutually disjoint copies of a graph G. For each vertex v of G, let v' and v'' denote the corresponding vertices in G' and G'' respectively. Let E denote the set of all pairs of the form $v'v''$ with v in $V(G)$. Let C denote the set of all pairs of the form $v'w''$ such that vw is an edge of G. Finally, let

$$H := (G' \cup G'') + E + C \, .$$

Describe, in detail, the components of the graph $H[M \cup E]$, where M is a matching in H. Now suppose one of the components of $H[M \cup E]$ is a path of the form

$$v'_1 \; v''_1 \; v''_2 \; v'_2 \; v'_3 \; v''_3 \; v''_4 \; v'_4 \; \ldots \; v'_{k-1} \; v'_k \; v''_k$$

and show that $v'_1 \; v'_2 \; v'_3 \; v'_4 \; \ldots \; v'_{k-1} \; v'_k$ is a path in G'.

2. Describe an algorithm for deciding whether a split $\mathcal{Y}$ is strong.

3. From Lemma 2.4.1, show that $|\mathcal{P}| \leq \lfloor \frac{1}{2}|A| \rfloor$ for any disjoint collection $\mathcal{P}$ of A-paths.

4. Give the terms ($\mathcal{P}$ and $\mathcal{Y}$) of Gallai's theorem in case $|A| \leq 1$. Repeat for $|A| = 2$. Repeat for $|A| = 3$.

5. Complete the proof of claim (2.1), i.e., show that there is no v in $V \setminus A$ such that $v' \notin X$ but $v'' \in X$.

6. Show that Theorem 2.4.2 could have been stated as follows: There exists a disjoint collection $\mathcal{P}$ of A-paths and a set X of vertices such that

$$|\mathcal{P}| = |X| + \sum_{C} \left\lfloor \frac{|V(C) \cap A|}{2} \right\rfloor ,$$

where the sum is taken over all components C of $G - X$.